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Stringy Jacobi fields in Morse theory

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We consider the variation of the surface spanned by closed strings in a spacetime manifold. Using the Nambu-Goto string action, we induce the geodesic surface equation and the geodesic surface deviation equation which yields a Jacobi field, and we define the index form of a geodesic surface as in the case of point particles to discuss conjugate strings on the geodesic surface.

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I. INTRODUCTION

It is well known that string theory [1,2] is one of the best candidates for a consistent quantum theory of gravity to yield a unification theory of all four basic forces in nature. In D-brane models [2], closed strings represent gravitons propagating on a curved manifold, while open strings describe gauge bosons such as photons, or matter attached on the D-branes. Moreover, because the two ends of an open string can always meet and connect, forming a closed string, there are no string theories without closed strings.

On the other hand, the supersymmetric quantum mechanics has been exploited by Witten [3] to discuss the Morse inequalities [4–6]. The Morse indices for a pair of critical points of the symplectic action function have also been investigated based on the spectral flow of the Hessian of the symplectic function [7], and on the Hilbert spaces the Morse homology [8] has been considered to discuss the critical points associated with the Morse index [9]. The string topology was initiated in the seminal work of Chas and Sullivan [10]. Using the Morse theoretic techniques, Cohen in Ref. [11] constructs string topology operations on the loop space of a manifold and relates the string topology operations to the counting of pseudoholomorphic curves in the cotangent bundle. He also speculates about the relation between the Gromov-Witten invariant [12] of the cotangent bundle and the string topology of the underlying manifold. Recently, the Jacobi fields and their eigenvalues of the Sturm-Liouville operator associated with the particle geodesics on a curved manifold have been investigated [13], to relate the phase factor of the partition function to the eta invariant of Atiyah [14,15].

In this paper, we will exploit the Nambu-Goto string action to investigate the geodesic surface equation and the geodesic surface deviation equation associated with a Jacobi field. The index form of a geodesic surface will also be discussed for the closed strings on the curved manifold.

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In Sec. II, the string action will be introduced to investigate the geodesic surface equation in terms of the world sheet currents associated with τ and σ world sheet coordinate directions. By taking the second variation of the surface spanned by closed strings, the geodesic surface deviation equation will be discussed for the closed strings on the curved manifold. In Sec. III, exploiting the orthonormal gauge, the index form of a geodesic surface will also be investigated together with breaks on the string tubes. The geodesic surface deviation equation in the orthonormal gauge will be exploited to discuss the Jacobi field on the geodesic surface.

II. STRINGY GEODESIC SURFACES IN MORSE THEORY

In analogy to the relativistic action of a point particle, the action for a string is proportional to the area of the surface spanned in spacetime manifold M by the evolution of the string. In order to define the action on the curved manifold, let (M, g_{ab}) be an n-dimensional manifold associated with the metric g_{ab}. Given g_{ab}, we can have a unique covariant derivative ∇_a satisfying [6] $\nabla_a g_{bc} = 0$, $\nabla_a \omega^b = \partial_a \omega^b + \Gamma^b{}_{ac}\omega^c$, and

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_c = R_{abc}{}^d\omega_d. \tag{2.1}$$

We parametrize the closed string by two world sheet coordinates τ and σ, and then we have the corresponding vector fields $\xi^a = (\partial/\partial\tau)^a$ and $\zeta^a = (\partial/\partial\sigma)^a$. The Nambu-Goto string action is then given by [1,2,16]

$$S = -\iint d\tau d\sigma f(\tau, \sigma) \tag{2.2}$$

where the coordinates τ and σ have ranges $0 \le \tau \le T$ and $0 \le \sigma \le 2\pi$, respectively, and

$$f(\tau, \sigma) = [(\xi\cdot\zeta)^2 - (\xi\cdot\xi)(\zeta\cdot\zeta)]^{1/2}. \tag{2.3}$$

We now perform an infinitesimal variation of the tubes $\gamma_\alpha(\tau, \sigma)$ traced by the closed string during its evolution in order to find the geodesic surface equation from the least action principle. Here we impose the restriction that the length of the string circumference is τ independent. Let the vector field $\eta^a = (\partial/\partial\alpha)^a$ be the deviation vector which represents the displacement to an infinitesimally nearby tube, and let Σ denote the three-dimensional submanifold spanned by the tubes $\gamma_\alpha(\tau, \sigma)$. We then may choose τ, σ, and α as coordinates of Σ to yield the commutator relations,

$$\begin{aligned} \mathcal{L}_\xi \eta^a &= \xi^b \nabla_b \eta^a - \eta^b \nabla_b \xi^a = 0, \\ \mathcal{L}_\zeta \eta^a &= \zeta^b \nabla_b \eta^a - \eta^b \nabla_b \zeta^a = 0, \\ \mathcal{L}_\xi \zeta^a &= \xi^b \nabla_b \zeta^a - \zeta^b \nabla_b \xi^a = 0. \end{aligned} \tag{2.4}$$

Now we find the first variation as follows [17]:

$$\begin{aligned} \frac{dS}{d\alpha} &= \iint d\tau d\sigma \eta_b (\xi^a \nabla_a P^b_\tau + \zeta^a \nabla_a P^b_\sigma) \\ &\quad - \int d\sigma P^b_\tau \eta_b |^{\tau=T}_{\tau=0} - \int d\tau P^b_\sigma \eta_b |^{\sigma=2\pi}_{\sigma=0}, \end{aligned} \tag{2.5}$$

where the world sheet currents associated with τ and σ directions are, respectively, given by [17]

$$\begin{aligned} P^a_\tau &= \frac{1}{f}[(\xi \cdot \zeta)\zeta^a - (\zeta \cdot \zeta)\xi^a], \\ P^a_\sigma &= \frac{1}{f}[(\xi \cdot \zeta)\xi^a - (\xi \cdot \xi)\zeta^a]. \end{aligned} \tag{2.6}$$

Using the endpoint conditions $\eta^a(0) = \eta^a(T) = 0$ and periodic condition $\eta^a(\sigma + 2\pi) = \eta^a(\sigma)$, we have the geodesic surface equation [17]

$$\xi^a \nabla_a P^b_\tau + \zeta^a \nabla_a P^b_\sigma = 0, \tag{2.7}$$

and the constraint identities [17]

$$\begin{aligned} P_\tau \cdot \zeta &= 0, \qquad P_\tau \cdot P_\tau + \zeta \cdot \zeta = 0, \\ P_\sigma \cdot \xi &= 0, \qquad P_\sigma \cdot P_\sigma + \xi \cdot \xi = 0. \end{aligned} \tag{2.8}$$

Let $\gamma_\alpha(\tau, \sigma)$ denote a smooth one-parameter family of geodesic surfaces: for each $\alpha \in \mathbf{R}$, the tube γ_α is a geodesic surface parametrized by affine parameters τ and σ. For an infinitesimally nearby geodesic surface in the family, we then have the following geodesic surface deviation equation:

$$\begin{aligned} &\xi^b \nabla_b (\eta^c \nabla_c P^a_\tau) + \zeta^b \nabla_b (\eta^c \nabla_c P^a_\sigma) \\ &\quad + R_{bcd}{}^a (\xi^b P^d_\tau + \zeta^b P^d_\sigma)\eta^c \equiv (\Lambda\eta)^a = 0. \end{aligned} \tag{2.9}$$

For a small variation η^a, our goal is to compare $S(\alpha)$ with $S(0)$ of the string. The second variation $d^2S/d\alpha^2(0)$ is then needed only when $dS/d\alpha(0) = 0$. Explicitly, the second variation is given by

$$\begin{aligned} \frac{d^2S}{d\alpha^2}\bigg|_{\alpha=0} &= - \iint d\tau d\sigma [(\eta^c \nabla_c P^b_\tau)(\xi^a \nabla_a \eta_b) \\ &\quad + (\eta^c \nabla_c P^b_\sigma)(\zeta^a \nabla_a \eta_b) \\ &\quad - R_{acb}{}^d (\xi^a P^b_\tau + \zeta^a P^b_\sigma)\eta^c \eta_d] \\ &\quad - \int d\sigma P^b_\tau \eta^a \nabla_a \eta_b |^{\tau=T}_{\tau=0} \\ &\quad - \int d\tau P^b_\sigma \eta^a \nabla_a \eta_b |^{\sigma=2\pi}_{\sigma=0}. \end{aligned} \tag{2.10}$$

Here the boundary terms vanish for the fixed endpoint and the periodic conditions, even though on the geodesic surface we have breaks which we will explain later. After some algebra using the geodesic surface deviation equation, we have

$$\frac{d^2S}{d\alpha^2}\bigg|_{\alpha=0} = \iint d\tau d\sigma \eta_a (\Lambda\eta)^a. \tag{2.11}$$

III. JACOBI FIELDS IN ORTHONORMAL GAUGE

The string action and the corresponding equations of motion are invariant under reparametrization $\tilde{\sigma} = \tilde{\sigma}(\tau, \sigma)$ and $\tilde{\tau} = \tilde{\tau}(\tau, \sigma)$. We then have gauge degrees of freedom so that we can choose the orthonormal gauge as follows [17]:

$$\xi \cdot \zeta = 0, \qquad \xi \cdot \xi + \zeta \cdot \zeta = 0, \tag{3.1}$$

where the plus sign in the second equation is due to the fact that $\xi \cdot \xi$ is timelike and $\zeta \cdot \zeta$ is spacelike. Note that the gauge fixing (3.1) for the world sheet coordinates means that the tangent vectors are orthonormal everywhere up to a local scale factor [17]. In this parametrization the world sheet currents (2.6) satisfying the constraints (2.8) are of the form

$$P^a_\tau = -\xi^a, \qquad P^a_\sigma = \zeta^a. \tag{3.2}$$

The geodesic surface equation and the geodesic surface deviation equation read

$$-\xi^a \nabla_a \xi^b + \zeta^a \nabla_a \zeta^b = 0 \tag{3.3}$$

and

$$\begin{aligned} &-\xi^b \nabla_b (\xi^c \nabla_c \eta^a) + \zeta^b \nabla_b (\zeta^c \nabla_c \eta^a) \\ &\quad - R_{bcd}{}^a (\xi^b \xi^d - \zeta^b \zeta^d)\eta^c = (\Lambda\eta)^a = 0. \end{aligned} \tag{3.4}$$

We now restrict ourselves to strings on a constant scalar curvature manifold such as S^n. We take an ansatz that on this manifold the string shape on the geodesic surface γ_0 is the same as that on a nearby geodesic surface γ_α at a given time τ. We can thus construct the variation vectors $\eta^a(\tau)$ as vectors associated with the centers of the string of the two nearby geodesic surfaces at the given time τ. We then introduce an orthonormal basis of spatial vectors e^a_i ($i = 1, 2, \ldots, n-2$) orthogonal to ξ^a and ζ^a and parallelly propagated along the geodesic surface. The geodesic surface deviation equation (3.4) then yields for i, $j = 1, 2, \ldots, n-2$,

$$\frac{d^2\eta^i}{d\tau^2} + (R_{\tau j \tau}{}^i - R_{\sigma j \sigma}{}^i)\eta^j = 0. \tag{3.5}$$

The value of η^i at time τ must depend linearly on the initial data $\eta^i(0)$ and $\frac{d\eta^i}{d\tau}(0)$ at $\tau = 0$. Since by construction $\eta^i(0) = 0$ for the family of geodesic surfaces, we must have

$$\eta^i(\tau) = A^i{}_j(\tau)\frac{d\eta^j}{d\tau}(0). \tag{3.6}$$

Inserting (3.6) into (3.5) we have the differential equation for $A^i{}_j(\tau)$,

$$\frac{d^2A^i{}_j}{d\tau^2} + (R_{\tau k\tau}{}^i - R_{\sigma k\sigma}{}^i)A^k{}_j = 0, \qquad (3.7)$$

with the initial conditions

$$A^i{}_j(0) = 0, \qquad \frac{dA^i{}_j}{d\tau}(0) = \delta^i{}_j. \qquad (3.8)$$

Note that in (3.7) we have the last term originated from the contribution of string property.

Next we consider the second variation equation (2.10) under the above restrictions,

$$\frac{d^2S}{d\alpha^2}\bigg|_{\alpha=0} = \iint d\tau d\sigma\Big(\frac{d\eta^i}{d\tau}\frac{\eta_i}{d\tau} - (R_{\tau j\tau}{}^i - R_{\sigma j\sigma}{}^i)\eta^j\eta_i\Big). \qquad (3.9)$$

We define the index form I_γ of a geodesic surface γ as the unique symmetric bilinear form I_γ: $T_\gamma \times T_\gamma \rightarrow \mathbf{R}$ such that

$$I_\gamma(V, V) = \frac{d^2S}{d\alpha^2}\bigg|_{\alpha=0} \qquad (3.10)$$

for $V \in T_\gamma$. From (3.9) we can easily find

$$I_\gamma(V, W) = \iint d\tau d\sigma\Big(\frac{dW^m}{d\tau}\frac{dV_m}{d\tau} - (R_{\tau j\tau}{}^m - R_{\sigma j\sigma}{}^m)W^jV_m\Big). \qquad (3.11)$$

If we have breaks $0 = \tau_0 < \cdots < \tau_{k+1} = T$, and the restriction of γ to each set $[\tau_{i-1}, \tau_i]$ is smooth, then the tube γ is piecewise smooth. The variation vector field V of γ is always piecewise smooth. However, $dV/d\tau$ will generally have a discontinuity at each break τ_i $(1 \leq i \leq k)$. This discontinuity is measured by

$$\Delta\frac{dV}{d\tau}(\tau_i) = \frac{dV}{d\tau}(\tau_i^+) - \frac{dV}{d\tau}(\tau_i^-), \qquad (3.12)$$

where the first term derives from the restrictions $\gamma|[\tau_i, \tau_{i+1}]$ and the second from $\gamma|[\tau_{i-1}, \tau_i]$. If γ and $V \in T_\gamma$ have breaks $\tau_1 < \cdots < \tau_k$, we have

$$\sum_{i=0}^{k}\int_{\tau_i}^{\tau_{i+1}}\frac{d}{d\tau}\Big(V_m\frac{dW^m}{d\tau}\Big)d\tau = -\sum_{i=0}^{k}V_m\Delta\frac{dW^m}{d\tau}(\tau_i) \qquad (3.13)$$

to yield

$$I_\gamma(V,W) = -\iint d\tau d\sigma V^m\Big(\frac{d^2W^m}{d\tau^2} + (R_{\tau j\tau}{}^m - R_{\sigma j\sigma}{}^m)W^j\Big) \qquad (3.14)$$

$$-\sum_{i=0}^{k}\int d\sigma V_m\Delta\frac{dW^m}{d\tau}(\tau_i). \qquad (3.15)$$

Here note that if we do not have breaks, (3.9) yields

$$\frac{d^2S}{d\alpha^2}\bigg|_{\alpha=0} = -\iint d\tau d\sigma\eta_i\Big(\frac{d^2\eta^i}{d\tau^2} + (R_{\tau j\tau}{}^i - R_{\sigma j\sigma}{}^i)\eta^j\Big). \qquad (3.16)$$

A solution η^a of the geodesic surface deviation equation (3.5) is called a Jacobi field on the geodesic surface γ. A pair of strings $p, q \subset \gamma$ defined by the centers of the closed strings on the geodesic surface is then conjugate if there exists a Jacobi field η^a which is not identically zero but vanishes at both strings p and q. Roughly speaking, p and q are conjugate if an infinitesimally nearby geodesic surface intersects γ at both p and q. From (3.6), q will be conjugate to p if and only if there exists nontrivial initial data: $d\eta^i/d\tau(0) \neq 0$, for which $\eta^i = 0$ at q. This occurs if and only if $\det A^i{}_j = 0$ at q, and thus $\det A^i{}_j = 0$ is the necessary and sufficient condition for a conjugate string to p. Note that, between conjugate strings, we have $\det A^i{}_j \neq 0$ and thus the inverse of $A^i{}_j$ exists. Using (3.7) we can easily see that

$$\frac{d}{d\tau}\Big(\frac{dA_{ij}}{d\tau}A^i{}_k - A_{ij}\frac{dA^i{}_k}{d\tau}\Big) = 0. \qquad (3.17)$$

In addition, the quantity in parentheses in (3.17) vanishes at p, since $A^i{}_j(0) = 0$. Along a geodesic surface γ, we thus find

$$\frac{dA_{ij}}{d\tau}A^i{}_k - A_{ij}\frac{dA^i{}_k}{d\tau} = 0. \qquad (3.18)$$

If γ is a geodesic surface with no string conjugate to p between p and q, then $A^i{}_j$ defined above will be nonsingular between p and q. We can then define $Y^i = (A^{-1})^i{}_j\eta^j$ or $\eta^i = A^i{}_jY^j$. From (3.16) and (3.18), we can easily verify that

$$\frac{d^2S}{d\alpha^2}\bigg|_{\alpha=0} = \iint d\tau d\sigma\Big(A_{ij}\frac{dY^j}{d\tau}\Big)^2 \geq 0. \qquad (3.19)$$

Locally γ minimizes the Nambu-Goto string action, if γ is a geodesic surface with no string conjugate to p between p and q.

On the other hand, if γ is a geodesic surface but has a conjugate string r between strings p and q, then we have a nonzero Jacobi field J^i along γ which vanishes at p and r. Extend J^i to q by putting it to zero in $[r, q]$. Then $dJ^i/d\tau(r^-) \neq 0$, since J^i is nonzero. But $dJ^i/d\tau(r^+) = 0$ to yield

$$\Delta\frac{dJ^i}{d\tau}(r) = -\frac{dJ^i}{d\tau}(r^-) \neq 0. \qquad (3.20)$$

We choose any $k^i \in T_\gamma$ such that

$$k_i\Delta\frac{dJ^i}{d\tau}(r) = c, \qquad (3.21)$$

with a positive constant c. Let η^i be $\eta^i = \epsilon k^i + \epsilon^{-1} J^i$ where ϵ is some constant; then we have

$$I_\gamma(\eta, \eta) = \epsilon^2 I_\gamma(k, k) + 2I_\gamma(k, J) + \epsilon^{-2} I_\gamma(J, J). \quad (3.22)$$

By taking ϵ small enough, the first term in (3.22) vanishes and the third term also vanishes due to the definition of the Jacobi field and (3.15). Substituting (3.21) into (3.15) we have $I_\gamma(k, J) = -2\pi c$ and thus

$$\frac{d^2 S}{d\alpha^2}\bigg|_{\alpha=0} = -4\pi c, \quad (3.23)$$

which is negative definite. From the above arguments, we conclude that, given a smooth timelike tube γ connecting two strings $p, q \subset M$, the necessary and sufficient condition that γ locally minimizes the surface of the closed string tube between p and q over smooth one-parameter variations is that γ is a geodesic surface with no string conjugate to p between p and q. It is also interesting to see that, on S^n, the first nonminimal geodesic surface has $n - 1$ conjugate strings as in the case of the point particle. Moreover, on the Riemannian manifold with the constant sectional curvature K, the geodesic surfaces have no conjugate strings for $K < 0$ or $K = 0$, while conjugate strings occur for $K > 0$ [18].

IV. CONCLUSIONS

The Nambu-Goto string action has been introduced to study the geodesic surface equation in terms of the world sheet currents associated with τ and σ directions. By constructing the second variation of the surface spanned by closed strings, the geodesic surface deviation equation has been discussed for the closed strings on the curved manifold.

Exploiting the orthonormal gauge, the index form of a geodesic surface has been defined together with breaks on the string tubes. The geodesic surface deviation equation in this orthonormal gauge has been derived to find the Jacobi field on the geodesic surface. Given a smooth timelike tube connecting two strings on the manifold, the condition that the tube locally minimizes the surface of the closed string tube between the two strings over smooth one-parameter variations has also been discussed in terms of the conjugate strings on the geodesic surface.

In the Morse theoretic approach to the string theory, one could consider the physical implications associated with geodesic surface congruences and their expansion, shear, and twist. It would also be desirable for the string topology and the Gromov-Witten invariant to be investigated by exploiting the Morse theoretic techniques. These works are in progress and will be reported elsewhere.

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