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Waves and solitons in the two-family Calogero model

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We analyze soliton solutions in the two-family Calogero model. There are two types of solutions, a one-soliton–antisoliton solution and a wave solution. It is shown that there is no finite number of solitons at finite distances in the limit when the period of wave solutions tends to infinity.

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I. INTRODUCTION

The multispecies Calogero model has been extensively studied in the quantum mechanical [1–4] and in the collective-field theory context [5–9]. It represents the generalization of the ordinary Calogero model with identical particles to the Calogero model with nonidentical particles. This can be achieved by allowing particles to have different masses and different couplings to each other [10–12]. In all of these generalizations the interaction parameters and masses satisfy some specific relations in order to avoid the cumbersome three-body interactions. It has been shown quite recently [3,4] that the two-species Calogero models are exactly solvable if the interaction strength between particles of the first and the second family λ_{12} is equal to -1. As far as the other interesting parameter sector $\lambda_{12} = 1$ of the same model is concerned, in [7] it was shown that the Hamiltonian of the Hermitian matrix model can be interpreted as a two-species Calogero model, whose families are connected by duality. In the next paper [8], the same authors studied the solitonic structure of the aforementioned duality-based matrix model. The multi-vortex solutions of the coupled Bogomol'nyi-Prasad-Sommerfeld equations were interpreted as giant gravitons [13]. On the other hand [5,6], we have studied the same BPS equations in the collective-field variant of the Calogero model with distinguishable particles and no multivortex solutions were found. We think that a more detailed analyses of coupled BPS equations is welcome, not only to present comparison with earlier investigations but also to unveil the full spectrum of all possible solitonic solutions. Therefore, it is natural to ask if our collective-field approach admits vortexlike solutions and, in this case, which are their properties. Our approach follows closely that developed in [5] for constructing soliton solutions in the two-species Calogero model except that now we demonstrate explicitly that the problem can be reduced to two decoupled free one-family Calogero systems. In this brief note we will prove that in the two-species Calogero model, there are no solutions describing the finite number of solitons, mutually at finite distances, contrary to the claims in [7,8].

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II. BPS EQUATIONS AND THEIR SOLUTIONS

The Hamiltonian [5–9] for the two-family Calogero model in the collective-field formulation is

$$\begin{aligned} H = &\int dx \frac{\rho_1(x)}{2m_1}\left[(\partial_x \pi_1(x))^2 + \left(\frac{\lambda_1 - 1}{2}\frac{\partial_x \rho_1}{\rho_1} + \fint dy \frac{\lambda_1\rho_1(y) + \rho_2(y)}{x-y}\right)^2\right] \\ &+ \int dx \frac{\rho_2(x)}{2m_2}\left[(\partial_x \pi_2(x))^2 + \left(\frac{\lambda_2 - 1}{2}\frac{\partial_x \rho_2}{\rho_2} + \fint dy \frac{\lambda_2\rho_2(y) + \rho_1(y)}{x-y}\right)^2\right], \end{aligned} \tag{1}$$

up to some singular terms which do not contribute in the leading order in the $\frac{1}{N_1}$ and $\frac{1}{N_2}$ expansions. The collective fields ρ_1 and ρ_2 are normalized as

$$\int dx \rho_1(x) = N_1, \qquad \int dx \rho_2(x) = N_2, \tag{2}$$

where N_1 and N_2 are large numbers of particles in the first and the second family, respectively, and $\pi_1(x)$ and $\pi_2(x)$ represent the corresponding canonical momenta. In (1) we imposed the restriction that there be no three-body interactions which requires [1,2,5,9]

$$\frac{\lambda_1}{{m_1}^2} = \frac{\lambda_2}{{m_2}^2} = \frac{1}{m_1 m_2}. \tag{3}$$

In addition, we restrict ourselves to the so-called dual variant of the model in which the coupling parameters are related by

$$\lambda_1 \lambda_2 = 1. \tag{4}$$

To find the ground-state energy, we assume that the corresponding densities are static. Since their momenta are vanishing, the leading part of the Hamiltonian (1) in the $\frac{1}{N_1}$ and $\frac{1}{N_2}$ expansions is given by the effective potential

$$V_{\rm eff} = \int dx \frac{\rho_1(x)}{2m_1}\left(\frac{\lambda_1 - 1}{2}\frac{\partial_x \rho_1}{\rho_1} + \fint dy \frac{\lambda_1 \rho_1(y) + \rho_2(y)}{x-y}\right)^2 + \int dx \frac{\rho_2(x)}{2m_2}\left(\frac{\lambda_2 - 1}{2}\frac{\partial_x \rho_2}{\rho_2} + \fint dy \frac{\lambda_2 \rho_2(y) + \rho_1(y)}{x-y}\right)^2. \tag{5}$$

The potential (5) is positive semidefinite and its contribution to the ground-state energy vanishes if there exist positive solutions of the coupled BPS equations

$$\frac{\lambda_1 - 1}{2}\frac{\partial_x \rho_1}{\rho_1} + \fint dy \frac{\lambda_1 \rho_1(y) + \rho_2(y)}{x-y} = 0, \tag{6}$$

$$\frac{\lambda_2 - 1}{2}\frac{\partial_x \rho_2}{\rho_2} + \fint dy \frac{\lambda_2 \rho_2(y) + \rho_1(y)}{x-y} = 0. \tag{7}$$

By substracting BPS Eqs. (6) and (7) and having in mind the condition (4), we easily get

$$\rho_1(x)\rho_2(x) = c. \tag{8}$$

Let us now extend our construction of a one-soliton–anti-soliton solution [5] and obtain a general solution of two coupled BPS Eqs. (6) and (7). It is more convenient to work with the simpler, auxiliary problem

$$\frac{\lambda_1 - 1}{2}\frac{\partial_x r_1}{r_1} + \lambda_1 \fint dy \frac{r_1(y)}{x-y} = 0, \tag{9}$$

$$\frac{\lambda_2 - 1}{2}\frac{\partial_x r_2}{r_2} + \lambda_2 \fint dy \frac{r_2(y)}{x-y} = 0. \tag{10}$$

These equations are in fact the BPS equations for two decoupled one-family Calogero systems described by the collective fields r_1 and r_2. Our construction of a general solution to the coupled system (6) and (7) is given by

$$\rho_1(x) = \alpha + r_1(x), \qquad \rho_2(x) = \frac{c}{\alpha} + r_2(x), \tag{11}$$

where α is some constant parameter. It is easy to verify that our construction (11) automatically satisfies the coupled BPS equations (6) and (7). To this end we use Eqs. (4) and (11) and rewrite ρ_1 and ρ_2 in terms of r_1 and r_2 as

$$\rho_1(x) = -\frac{c}{\alpha}\frac{r_1(x)}{r_2(x)}, \qquad \rho_2(x) = -\alpha\frac{r_2(x)}{r_1(x)}. \tag{12}$$

By differentiating these relations with respect to x and using Eqs. (9) and (10), we end up with the coupled BPS equations (6) and (7) for ρ_1 and ρ_2. In this way, we have reduced the two-family Calogero model to two one-family Calogero decoupled systems. We will prove this decoupling in Sec. III without relying on any devised ansatz for ρ_1 and ρ_2. To proceed further, it is useful to recall the following Hilbert transforms [14]

$$\fint dy \frac{1}{x-y}\frac{b}{b^2+y^2} = \pi\frac{x}{b^2+x^2}, \qquad b > 0, \tag{13}$$

$$\fint dy \frac{1}{x-y}\frac{\sinh u}{\cosh u - \cos ky} = \pi\frac{\sin kx}{\cosh u - \cos kx}, \quad u \geq 0, k > 0. \tag{14}$$

From these transforms, one easily finds only two types of solutions of the auxiliary BPS equations (9) and (10):

(1) Nonperiodic one-soliton solution

$$r(x) = \frac{\lambda - 1}{\pi\lambda}\frac{b}{x^2 + b^2}, \tag{15}$$

with the property

$$\int_{-\infty}^{\infty} dx r(x) = \frac{\lambda - 1}{\lambda}. \tag{16}$$

(2) Periodic, wave solution

$$r(x) = \frac{\lambda - 1}{2\pi\lambda}k\frac{\sinh u}{\cosh u - \cos kx}, \tag{17}$$

with arbitrary period $\frac{2\pi}{k}$ and with the property

$$\int_{-(\pi/k)}^{\pi/k} dx r(x) = \frac{\lambda - 1}{\lambda}. \tag{18}$$

Let us first analyze the nonperiodic solutions. According to Eq. (15) the solutions r_1 and r_2 are given by

$$r_1(x) = \frac{\lambda_1 - 1}{\pi\lambda_1}\frac{b}{x^2+b^2}, \qquad r_2(x) = \frac{\lambda_2 - 1}{\pi\lambda_2}\frac{a}{x^2+a^2}, \quad a, b > 0. \tag{19}$$

From Eq. (8) and the normalization conditions (2), we find

$$a = \frac{N_1(\lambda_1 - 1)}{\alpha\pi N_2(1 - {N_1}^2{\lambda_1}^2/{N_2}^2)}, \quad b = \frac{(1-\lambda_1)}{\lambda_1\alpha\pi(1 - {N_2}^2/{\lambda_1}^2{N_1}^2)}, \quad c = \frac{N_1 N_2}{L^2}, \quad \alpha = \frac{N_1}{L} \equiv \rho_{1,0}, \quad \frac{c}{\alpha} = \frac{N_2}{L} \equiv \rho_{2,0}. \tag{20}$$

It is important to note that the numbers of particles N_1, N_2 and the length of the system L are simultaneously taken to infinity keeping the particle densities $\rho_{1,0}$ and $\rho_{2,0}$ fixed. We see that in the limit $|x| \to \infty$, $r_1(x)$ and $r_2(x)$ vanish, while the $\rho_1(x)$ and $\rho_2(x)$ reduce to constants α and $\frac{c}{\alpha}$, respectively. For $\lambda_1 < 1$, the first soliton ρ_1 behaves like the hole in the condensate α and the second one ρ_2 behaves like the particle above the condensate $\frac{c}{\alpha}$. It is interesting to observe that in the singular limit $c \to 0$ [5], the first soliton ρ_1 reduces to the "vortex" profile

$$\rho_1(x) = \frac{1-\lambda_1}{\pi\lambda_1 b}\frac{x^2}{x^2+b^2}, \quad (21)$$

while the second soliton ρ_2 transforms into the sharp delta-function profile

$$\rho_2(x) = (1-\lambda_1)\delta(x). \quad (22)$$

Let us now apply our general construction (11) to the periodic solutions of the form (14) with the same period $\frac{2\pi}{k}$. We obtain

$$r_1(x) = \frac{\lambda_1 - 1}{2\pi\lambda_1}k\frac{\sinh u_1}{\cosh u_1 - \cos kx}, \quad (23)$$

$$r_2(x) = \frac{\lambda_2 - 1}{2\pi\lambda_2}k\frac{\sinh u_2}{\cosh u_2 - \cos kx}, \quad (24)$$

where $u_1 > u_2 \geq 0$. Note that k is a free arbitrary parameter. From Eq. (8) we find

$$\alpha = \frac{1-\lambda_1}{2\pi\lambda_1}k\frac{\sinh u_1}{\cosh u_1 - \cosh u_2} > 0, \quad (25)$$

$$\frac{c}{\alpha} = \frac{1-\lambda_1}{2\pi}k\frac{\sinh u_2}{\cosh u_1 - \cosh u_2} > 0, \quad (26)$$

$$\frac{c}{\lambda_1\alpha^2} = \frac{\sinh u_2}{\sinh u_1}, \quad (27)$$

whereas from Eqs. (11) it follows

$$\alpha = \rho_{1,0} + \frac{1-\lambda_1}{2\pi\lambda_1}k, \qquad \frac{c}{\alpha} = \rho_{2,0} - \frac{1-\lambda_1}{2\pi}k. \quad (28)$$

Using Eqs. (25)–(28), we obtain

$$\coth\frac{u_1+u_2}{2} = 2 + \frac{\lambda_1\rho_{1,0} - \rho_{2,0}}{(1-\lambda_1)k}2\pi, \qquad \coth\frac{u_1-u_2}{2} = \frac{\lambda_1\rho_{1,0} + \rho_{2,0}}{(1-\lambda_1)k}2\pi. \quad (29)$$

Let us now discuss some interesting cases. For $u_2 \to 0$ and finite k and u_1, the soliton solutions are

$$\rho_1(x) = \alpha + \frac{\lambda_1 - 1}{2\pi\lambda_1}k\frac{\sinh u_1}{\cosh u_1 - \cos kx} = \alpha\frac{\sin^2\frac{kx}{2}}{\sinh^2\frac{u_1}{2} + \sin^2\frac{kx}{2}}, \quad (30)$$

$$\rho_2(x) = (1-\lambda_1)\sum_{i\in Z}\delta\left(x - \frac{2\pi}{k}i\right), \quad (31)$$

where

$$(1-\lambda_1)k = 2\pi\rho_{2,0}, \qquad \alpha = \frac{1-\lambda_1}{2\pi\lambda_1}k\coth\frac{u_1}{2}. \quad (32)$$

The solution ρ_1, in fact, represents the large-amplitude stationary wave with vanishing minimum density. The other solution ρ_2, is given by the sum of an infinite number of δ-function contributions with the finite period $\frac{2\pi}{k}$. It is interesting to observe that the product $\rho_1(x)\rho_2(x)$ vanishes. Let us now investigate another interesting case in which parameter k goes to zero. Taking the limit $k \to 0$, we must simultaneously take $u_1 \to 0$ in order to have finite α (for finite u_1, α goes to zero, leading to an unacceptable solution $\rho_1 = 0$). For $\frac{u_1}{k} = b$ finite, we obtain only one nonperiodic solution

$$\rho_1(x) = \alpha\frac{x^2}{x^2+b^2}, \qquad \rho_2(x) = (1-\lambda_1)\delta(x). \quad (33)$$

This is the same solution as already obtained, (21) and (22). Note that the limiting procedure survives only one δ-function contribution in $\rho_2(x)$, namely, that with $i = 0$. Of course, this is to be expected, since the period $\frac{2\pi}{k} \to \infty$ and one obtains only one nonperiodic solution (33). In other words, all of the other copies of the basic profile (33) escape to infinity. Consequently, there are no solutions describing the finite number of vortices, mutually at finite distances, contrary to the claims in [7,8]. The reason for this is quite clear. While we are considering the BPS equations on the whole real line, the authors of [7,8] solve them on the finite interval L with periodic boundary conditions. This enables them to find the periodic solutions with a finite number of solitons. However, in extrapolating their solutions to the whole real line, i. e. by taking the limit $L \to \infty$, the authors of [7,8] overlooked the fact that the mutual distances (L/n) of the n solitons also go to infinity. Consequently, all multisoliton solutions simply disappear in that limit. The only solutions that survive are the one-soliton–antisoliton solution and stationary wave solutions.

III. PROOF OF DECOUPLING AND CONNECTION WITH MOVING SOLITON SOLUTIONS

Let us now prove in general that the two-family Calogero model can be reduced (at least in the static sector) to two decoupled one-family Calogero systems. To this end, we insert the ρ_2 from Eq. (8) into the Eq. (6) and get

$$\frac{\lambda_1 - 1}{2}\partial_x\rho_1(x) + \lambda_1\rho_1(x)\fint dy\frac{\rho_1(y)}{x-y} + c\rho_1(x)\fint dy\frac{1}{\rho_1(y)}\frac{1}{x-y} = 0. \quad (34)$$

At this point we make use of the identity

$$\fint\fint dxdyf(x)g(y)\left(\frac{1}{x-y}\frac{1}{x-z} + \frac{1}{y-x}\frac{1}{y-z} + \frac{1}{z-x}\frac{1}{z-y}\right) = \pi^2(f(z)g(z) - f_0g_0), \quad (35)$$

where the average densities f_0 and g_0 satisfy

$$\int dx(f(x) - f_0) = \int dx(g(x) - g_0) = 0. \quad (36)$$

Choosing $f = g = \rho_1$, we can rewrite the Hilbert transform of Eq. (34) as

$$\frac{\lambda_1 - 1}{2}\partial_x \fint dy \frac{\rho_1(y)}{x-y} + \frac{\lambda_1}{2}\Big[\Big(\fint dy \frac{\rho_1(y)}{x-y}\Big)^2 - \pi^2\rho_1(x)^2 + \pi^2{\rho_{1,0}}^2\Big] + c\fint dz \frac{\rho_1(z)}{x-z}\fint dy \frac{1}{\rho_1(y)}\frac{1}{z-y} = 0. \quad (37)$$

By using the identity (35) for $f = g = \frac{1}{\rho_2}$ and then for $f = \frac{1}{\rho_1}$ and $g = \rho_2$, we can completely get rid of all integrals involving $\frac{1}{\rho_1}$. The final step of the proof is given by differentiation of Eq. (6) with respect to x. Then Eq. (37) reduces, after some algebra, to the second order, nonlinear integro-differential equation for ρ_1:

$$\frac{\pi^2{\lambda_1}^2}{2}(\rho_1^2 - \rho_{1,0}^2) - \frac{(\lambda_1-1)^2}{4}\partial_x\Big(\frac{\partial_x\rho_1}{\rho_1}\Big) - \frac{(\lambda_1-1)^2}{8}\Big(\frac{\partial_x\rho_1}{\rho_1}\Big)^2 - \lambda_1(\lambda_1-1)\fint dy\frac{\partial_y\rho_1(y)}{x-y} + \frac{c^2\pi^2}{2}\Big(\frac{1}{\rho_1^2} - \frac{1}{\rho_{1,0}^2}\Big) = 0. \quad (38)$$

This is nothing but the one-family Calogero model variational equation for a localized constant profile configuration, propagating at speed v_1 given by

$$v_1 = \pm\frac{c\pi}{\rho_{1,0}}. \quad (39)$$

The analogous equation for ρ_2 and the corresponding velocity v_2 can be obtained from Eqs. (38) and (39) simply by permuting the family indices $1 \leftrightarrow 2$. The two-family Calogero model can thus be thought of as a system of two separated noninteracting Calogero families. The periodic and nonperiodic solutions of Eq. (38) were already found in [15–18]. For example, the nonperiodic soliton solution reads

$$\rho_1(x) = \rho_{1,0} + \frac{\lambda_1 - 1}{\pi\lambda_1}\frac{s}{x^2+s^2}, \quad (40)$$

where s denotes the parameter given by

$$s = \frac{\lambda_1(\lambda_1-1)\pi\rho_{1,0}}{v_1^2 - {\lambda_1}^2\pi^2{\rho_{1,0}}^2}. \quad (41)$$

By substituting velocity v_1 from Eq. (39) into Eq. (41), we obtain the solution (40) which coincides with the form (11) and relation (15). The same conclusion can be reached concerning the periodic solutions of Eq. (38).

IV. CONCLUSION

In this paper, we have shown that our general construction of the solutions of the coupled BPS equations in the two-family Calogero model leads to the decoupled BPS equations for one-family Calogero models. This has enabled us to find all the solutions of the original BPS equations which, in fact, reduce only to one-soliton–anti-soliton solution and periodic, stationary wave solutions. Finally, we have shown that in the singular limit, when the period of solutions tends to infinity, there are no solutions describing the finite number of solitons (vortices), mutually at finite distances.

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