

**Open string descriptions of spacelike singularities in two dimensional string theory**

Sumit R. Das\* and Luiz H. Santos†

*Department of Physics and Astronomy, University of Kentucky, Lexington, Kentucky 40506, USA*

(Received 16 April 2007; published 8 June 2007)

The matrix model formulation of two dimensional string theory has been shown to admit time dependent classical solutions whose closed string duals are geodesically incomplete space-times with spacelike boundaries. We investigate some aspects of the dynamics of fermions in one such background. We show that even though the background solution appears pathological, the time evolution of the system is smooth in terms of open string degrees of freedom, viz. the fermions. In particular, an initial state of fermions evolves smoothly into a well-defined final state over an infinite open string time interval, while the time perceived by closed strings appears to end abruptly. We outline a method of calculating fermion correlators exactly using symmetry properties. The result for the two-point function is consistent with the semiclassical picture.

DOI: [10.1103/PhysRevD.75.126001](https://doi.org/10.1103/PhysRevD.75.126001)

PACS numbers: 11.25.Pm

**I. INTRODUCTION**

Recently several examples of spacelike and null singularities in string theory have been analyzed using holographic dual formulations. These include backgrounds in the matrix model formulation of two dimensional string theory which lead to spacelike boundaries in the closed string interpretation [1], ten dimensional backgrounds which admit a matrix theory type formulation [2–5], and deformations of *AdS* space-times which admit a dual gauge theory description [6,7]. In all these examples, (which build on a large body of earlier work on strings on time dependent background [8]), the low energy space-time description breaks down at the singularity as expected. However in each case the open string dual appears to be well defined. This realizes a long held belief that near singularities the usual notions of space and time have to be abandoned and replaced by some more fundamental structure: in these cases this structure is provided by the open string dual.

The examples in two dimensional string theory are particularly significant in this respect since the dual matrix model reduces to a model of free fermions in an external inverted harmonic potential and therefore in some sense solvable. In recent years it has been realized that the matrix model/string theory connection is in fact open-closed duality just like the *AdS/CFT* correspondence [9]. The matrix degrees of freedom—and hence the fermionic eigenvalues—are the open strings, while the closed string description is most conveniently provided by collective field theory [10,11]. The collective field is a bosonization of the fermion field. Nontrivial time dependent classical solutions of the collective field theory correspond to time dependent Fermi surfaces. Such backgrounds have been studied for quite a while [12]. In [13–15] a class of such backgrounds were studied as toy models of cosmology.

In [1] it was shown that there is a class of backgrounds whose closed string interpretation involve space-times with a spacelike boundary. In these latter backgrounds, the time of the matrix model—which we call the open string time—runs over the full range  $-\infty < t < \infty$ . However, the time in terms of which the collective field fluctuations define a relativistic theory—which we call the closed string time—stops abruptly, thus forming a spacelike  $I^+$ . This  $I^+$  is however not an asymptotic region since the coupling is nonvanishing, though there is a part of  $I^+$  where the coupling becomes arbitrarily weak.

At first sight, it appears that the spacelike singularity is caused by the fact that in this background the eigenvalue space shrinks to zero at late times so that closed strings have no space to propagate. The same fact might also suggest that there is no well defined scattering problem. The analysis of [1] however showed that the “space” on which the closed string modes propagate is related to the space of eigenvalues in a nontrivial time dependent way. As a result, this *closed string space* is still of infinite extent at arbitrarily late times and there is indeed a well-posed scattering problem. This conclusion is based on the quadratic part of the action for collective field fluctuations. Since a large part of  $I^+$  is strongly coupled, one might worry that the effects of nonzero coupling may significantly modify this picture.

In this paper we revisit this model from the point of view of the fermionic theory so that the couplings of the collective field are treated in an exact fashion. We first study the classical evolution of small ripples on the Fermi sea and derive an equation which determines the shape of the ripple at late times in terms of the initial state, analogous to the scattering equation for perturbations around the ground state [16,17]. We find that initial pulses which are approximately localized at early or late retarded time scatter into final pulses which are localized on the (spacelike)  $I^+$  at large values of the closed string spatial coordinate  $q$ —as expected from linearized collective field theory. This is consistent, since for large  $q$  the coupling is weak. For small

\*Electronic address: [das@pa.uky.edu](mailto:das@pa.uky.edu),†Electronic address: [luiz.santos@uky.edu](mailto:luiz.santos@uky.edu)

retarded times, the scattered pulse is still localized, although its location is shifted compared to the expectations of free collective theory.

We then establish a general relation between *exact* fermion correlators in the time dependent background and those in the ground state. This enables a calculation of these correlators in terms of those obtained in [18]. We illustrate this by an exact calculation of the eigenvalue density. The result again shows that the picture based on collective field theory is reliable in the expected regime.

The present paper deals with one particular background in [1]. The methods and results are, however, expected to be similar for the other backgrounds discussed in that paper.

In Sec. II, we review the background solution and the behavior of collective field fluctuations at the linearized level described in [1]. Section III describes some aspects of the *classical* dynamics of small ripples on the Fermi sea using the exact fermion equations of motion, following [16,17]. This is used in Sec. IV to derive a scattering equation for such ripples riding on the time dependent background. In Sec. V we obtain profiles of the scattered pulse for an initial Gaussian pulse by numerically investigating the scattering equations. In Sec. VI we outline a general method for calculating *exact* correlation functions of fermions in such backgrounds, utilizing properties of  $W_\infty$  symmetry and obtain an explicit expression for the eigenvalue expectation value.

## II. $c = 1$ MATRIX MODEL AND 2D STRING THEORY

In this section we review the way (approximate) relativistic space-time appears in the  $c = 1$  matrix model [19].

The dynamical variable of the model is a single  $N \times N$  matrix  $M_{ij}(t)$  and there is a constraint which restricts the states to be singlets. In the singlet sector, and in the double scaling limit [19], matrix quantum mechanics reduces to a theory of an infinite number of fermions with the single particle Hamiltonian given by

$$H = \frac{1}{2}[p^2 - x^2], \quad (1)$$

where we have adopted conventions in which the string scale  $\alpha' = 1$  for the bosonic string and  $\alpha' = \frac{1}{2}$  in the Type 0B string. The Fermi energy in this rescaled problem will be denoted by  $-\mu$ .

In the classical limit, the system is equivalent to an incompressible Fermi fluid in phase space. The ground state is the static Fermi profile

$$(x - p)(x + p) = 2\mu. \quad (2)$$

The dual closed string theory is best obtained by rewriting the theory in terms of the collective field  $\phi(x, t)$  which is defined as the density of eigenvalues of the original matrix.

$$\partial_x \phi(x, t) = \frac{1}{N} \text{Tr} \delta(M(t) - x \cdot I). \quad (3)$$

At the classical level the action of the collective field is given by

$$S = N^2 \int dx dt \left[ \frac{1}{2} \frac{(\partial_t \phi)^2}{(\partial_x \phi)} - \frac{\pi^2}{6} (\partial_x \phi)^3 - \left( \mu - \frac{1}{2} x^2 \right) \partial_x \phi \right]. \quad (4)$$

This is of course a theory in  $1 + 1$  dimensions, the spatial dimension arising out of the space of eigenvalues.

Fermi seas with quadratic profiles appear as classical solutions to collective field theory in the appropriate limit.<sup>1</sup> The space-time which is generated may be obtained by looking at the dynamics of fluctuations of the collective field around the classical solution. Expanding around an *arbitrary* classical solution  $\phi_0(x, t)$

$$\phi(x, t) = \phi_0(x, t) + \frac{1}{N} \varphi(x, t). \quad (5)$$

The action for these fluctuations at the quadratic level may be written as

$$S_\varphi^{(2)} = \frac{1}{2} \int dt dx \sqrt{g} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi, \quad (6)$$

where  $\mu, \nu = t, x$ . The line element determined by  $g_{\mu\nu}$  is conformal to

$$ds^2 = -dt^2 + \frac{(dx + \frac{\partial_t \phi_0}{\partial_x \phi_0} dt)^2}{(\pi \partial_x \phi_0)^2}. \quad (7)$$

Therefore, *regardless of the classical solution*, the spectrum is always a massless scalar in one space dimension given by  $x$ . The metric can be determined only up to a conformal factor. However, as we will see below, the global properties of the space-time can be determined from the nature of the classical solution.

The classical interaction Hamiltonian is purely cubic when expressed in terms of the fluctuation field  $\varphi$  and its canonically conjugate momentum  $\Pi_\varphi$ ,

$$H_\varphi^{(3)} = \int dx \left[ \frac{1}{2} \Pi_\varphi^2 \partial_x \varphi + \frac{\pi^2}{6} (\partial_x \varphi)^3 \right]. \quad (8)$$

### A. The ground state and its fluctuations

The ground state (2) is a quadratic profile and the classical solution is

<sup>1</sup>Quadratic profiles are those Fermi surfaces  $f(x, p) = 0$  where  $p$  appears at most quadratically. Profiles which are not quadratic do not correspond to *classical* solutions of collective field theory [20]. Rather they are highly quantum states of the collective theory in which quantum dispersions do not vanish in the classical limit [21].

$$\partial_x \phi_0 = \frac{1}{\pi} \sqrt{x^2 - 2\mu}, \quad \partial_t \phi_0 = 0. \quad (9)$$

Around the ground state, the metric (7) is given by

$$ds^2 = -dt^2 + \frac{dx^2}{x^2 - 2\mu}. \quad (10)$$

The perturbative fluctuations live in the region  $|x| > \sqrt{2\mu}$  and the field  $\varphi$  satisfies Dirichlet boundary condition at the “mirrors” given by  $x = \pm\sqrt{2\mu}$ . The fields on the “left” and “right” side are decoupled at the perturbative level. The physics of these fields is made transparent by choosing Minkowskian coordinates  $(\sigma, \tau)$  which in this case are

$$t = \tau, \quad x = \pm\sqrt{2\mu} \cosh \sigma. \quad (11)$$

In these coordinates

$$ds^2 = -d\tau^2 + d\sigma^2. \quad (12)$$

The field  $\varphi$  may be now thought of being made of *two* fields,  $\varphi_{S,A}(x, t)$  each of which live in the region  $x > 0$

$$\varphi_{S,A}(x, t) = \frac{1}{2}[\varphi(x, t) \pm \varphi(-x, t)]. \quad (13)$$

In terms of the Minkowskian coordinates, solutions to the linearized equations are plane waves  $\varphi_{S,A} \sim e^{-i\omega(t \pm \sigma)} \varphi_{S,A}(\omega)$  and these Fourier components are related to the two space-time fields—the tachyon  $T$  and the axion  $C$ —which appear in the standard formulation of Type 0B string theory [22]:

$$\begin{aligned} T(\omega) &= (\pi/2)^{-i\omega/8} \frac{\Gamma(i\omega/2)}{\Gamma(-i\omega/2)} \varphi_S(\omega) \\ C(\omega) &= (\pi/2)^{-i\omega/8} \frac{\Gamma((1+i\omega)/2)}{\Gamma((1-i\omega)/2)} \varphi_A(\omega). \end{aligned} \quad (14)$$

In any case, the space-time generated is quite simple. The Penrose diagram is that of two dimensional Minkowski space with a mirror at  $\sigma = 0$ , as shown in Fig. 1. The fluctuations are massless particles which come in from  $I_{L,R}^-$ , get reflected at the mirror, and arrive at  $I_{L,R}^+$ .

Recall that we are working in string units. The transforms (14) imply that the position space fields are related by a transform which is nonlocal at the string scale. Therefore points on the Penrose diagram should be thought of as smeared over the string scale. But then, this should be true of any Penrose diagram drawn in a string theory.

In terms of the Minkowskian coordinates the interaction Hamiltonian becomes

$$H_3 = \int d\sigma \frac{1}{2\sinh^2 \sigma} \left[ \frac{1}{2} \tilde{\Pi}_\varphi^2 \partial_\sigma \varphi + \frac{\pi^2}{6} (\partial_\sigma \varphi)^3 \right]. \quad (15)$$

The canonically conjugate momentum  $\tilde{\Pi}_\varphi$  satisfies the standard commutator  $[\varphi(\sigma), \tilde{\Pi}_\varphi(\sigma')] = i\delta(\sigma - \sigma')$ . The interactions therefore vanish at  $\sigma = \infty$  and are strong at  $\sigma = 0$ —this gives rise to a nontrivial wall S-matrix.

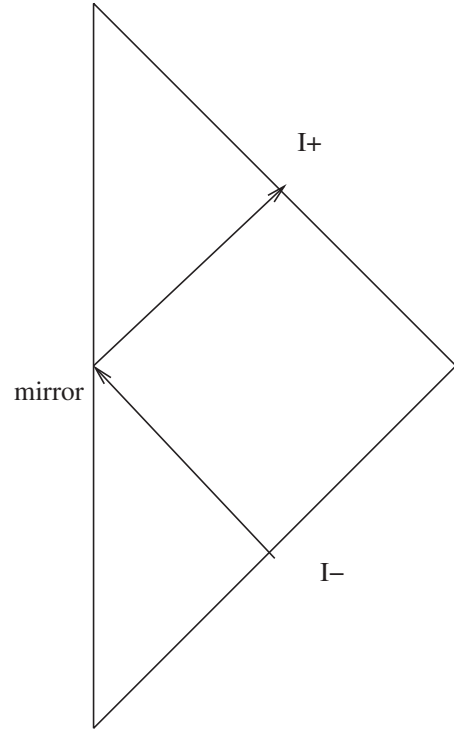


FIG. 1. Penrose diagram of space-time produced by ground state solution showing an incoming ray getting reflected at the mirror.

In the Type 0B string theory interpretation, the Penrose diagram has to be folded across the center.

As emphasized above, the metric is determined only up to a conformal transformation. So long as the conformal transformation is nonsingular, this is sufficient to draw Penrose diagrams. A conformal transformation would, however, mix up the space  $\sigma$  and time  $\tau$  and it would appear that this leads to an ambiguity. The special property of the space and time coordinates defined above is that the *interaction Hamiltonian is time-independent* with this choice and a conformal transformation would destroy this property. This makes the physics transparent and easy to compare with string theory results. Of course a different coordinatization with a time dependent Hamiltonian is physically equivalent and should be compared with the string theory results in an appropriately chosen gauge.

## B. Time dependent Fermi surfaces

The theory has an infinite number of symmetries, the  $W_\infty$  symmetries. At the classical level, the generators of these symmetries in fermion phase space are given by

$$W_{rs} = e^{(r-s)t}(x-p)^r(x+p)^s. \quad (16)$$

For  $r \neq s$  these generators do not commute with the Hamiltonian. Therefore starting with the ground state, one can obtain exact time dependent solutions by the action

of these generators [15]. We will be particularly interested in generators  $W_{r0}$  and  $W_{0r}$ . These generate the transformations with parameters  $\lambda_{\pm}$

$$(x \pm p) \rightarrow (x \pm p) + \lambda_{\pm} e^{\pm rt} (x \mp p)^{r-1} \quad (17)$$

and leads to the following Fermi surfaces

$$x^2 - p^2 + \lambda_- e^{-rt} (x + p)^r + \lambda_+ e^{rt} (x - p)^r + \lambda_+ \lambda_- (x^2 - p^2)^{r-1} = 2\mu. \quad (18)$$

Formally, the state of the fermion system is related to the ground state  $|\mu\rangle$  by

$$|\lambda\rangle = \exp[i\lambda W]|\mu\rangle, \quad (19)$$

where  $W$  denotes the  $W_{\infty}$  charge which generates this solution. However this state is not normalizable and therefore not contained in the Hilbert space of the model. Rather, this corresponds to a deformation of the Hamiltonian of the theory to

$$H' = e^{-i\lambda W} H e^{i\lambda W}. \quad (20)$$

### C. The closing hyperbola solution

In this paper we will concentrate on the ‘‘closing hyperbola solution’’ found in [1]. This is the solution generated by the action of  $W_{20}$ , i.e. with  $\lambda_- = 0$  and  $\lambda_+ < 0$ . In this case we can choose the origin of time and choose  $\lambda_+ = -1$ . Furthermore  $x$  and  $p$  may be rescaled to set  $\mu = 1/2$ . In the rest of the paper we will stick to these choices. The classical collective field is then

$$\begin{aligned} \partial_x \phi_0 &= \frac{1}{\pi(1 + e^{2t})} \sqrt{x^2 - (1 + e^{2t})} \\ \partial_t \phi_0 &= -\frac{x e^{2t}}{1 + e^{2t}} \partial_x \phi_0. \end{aligned} \quad (21)$$

This will be called the ‘‘closing hyperbola’’ solution shown and explained in Fig. 2.

The (approximately) relativistic space-time perceived by the fluctuations around this classical solution can be once again best seen in Minkowskian coordinates  $q, \tau$  in terms of which the quadratic action is

$$S^{(2)} = \int dq \int d\tau [(\partial_{\tau} \varphi)^2 - (\partial_q \varphi)^2] \quad (22)$$

and the interactions are independent of  $\tau$ . These coordinates are related to the original coordinates  $(t, x)$  by the relations

$$x = -\frac{\cosh q}{\sqrt{1 - e^{2\tau}}}, \quad e^t = \frac{e^{\tau}}{\sqrt{1 - e^{2\tau}}}. \quad (23)$$

We have restricted our attention to one side of the inverted harmonic potential ( $x < 0$ ) and the range of  $q$  can be chosen to be

$$0 \leq q \leq \infty. \quad (24)$$

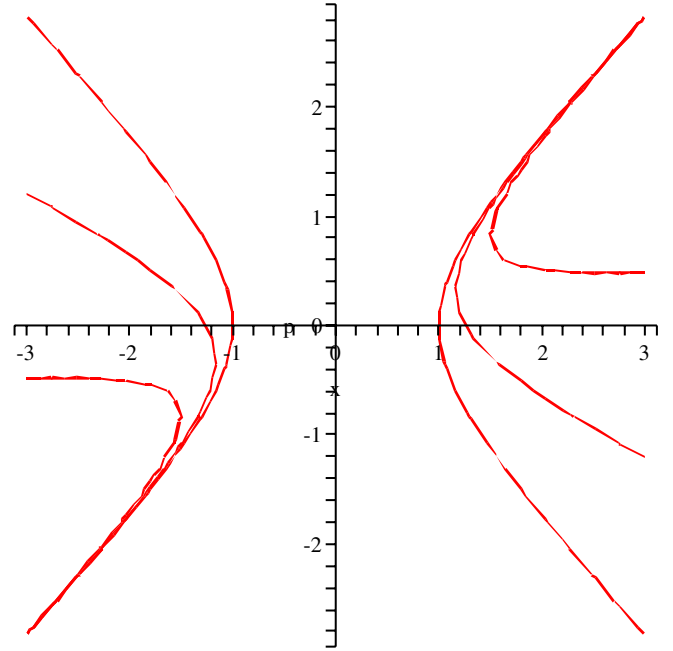


FIG. 2 (color online). The closing hyperbola solution. At late times the hyperbola closes on into itself draining all the fermions.

To describe the other side we need to have another patch of the  $(q, \tau)$  coordinates. In the rest of the paper we will restrict to one side since for this background the physics of the other side is identical.<sup>2</sup> The interaction Hamiltonian is then given by

$$H_3 = \int dq \frac{1}{2 \sinh^2 q} \left[ \frac{1}{2} \tilde{\Pi}_{\varphi}^2 \partial_q \varphi + \frac{\pi^2}{6} (\partial_q \varphi)^3 \right]. \quad (25)$$

The Eqs. (23) immediately show that as  $-\infty < t < \infty$  the time  $\tau$  has the range  $-\infty < \tau < 0$ . Since the dynamics of the matrix model ends at  $t = \infty$  the resulting space-time appears to be geodesically incomplete with a spacelike boundary  $I^+$  at  $\tau = 0$ . This boundary is, however, not an asymptotic region since the coupling is generally nonvanishing here except for  $q \rightarrow \infty$ .

The edge of the Fermi sea is at  $q = 0$ , which forms a timelike reflecting boundary. If we ignore the couplings of the collective field theory, fluctuations coming in from  $I^-$  along  $q_+ = q + \tau = \tau_0$  will get reflected by the mirror at  $q = 0$  so long as  $\tau_0 < 0$  and hit the spacelike boundary  $I^+$  at  $q = -\tau_0$ . For  $\tau_0 > 0$  this ray cannot reach the mirror before time ends—rather it directly hits the spacelike boundary at  $q = \tau_0$ . The Penrose diagram with these two classes of rays is shown in Fig. 3. Note, however that *in x space the ray always turns around at some point*, as may be seen from the change of variables (23).

<sup>2</sup>This is not adequate for some of the other solutions described in [1].

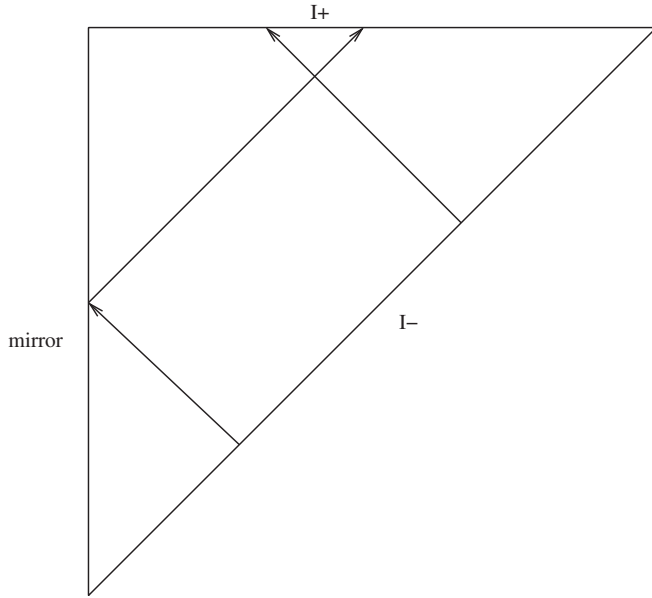


FIG. 3. Penrose diagram for small fluctuations around the closing hyperbola solution showing two classes of null rays.

In fact, in the fermion picture the Fermi sea gets drained out and it appears that at  $t \rightarrow \infty$  there is no Fermi sea at all, whereas in the bosonic picture there is a spacelike  $I^+$ . What is happening is that at  $t = \infty$ , the entire spacelike boundary  $\tau = 0$  is at  $|x| = \infty$ , where the coordinate  $q$  runs over its full range  $0 \leq q \leq \infty$ . It is therefore clear that the scattering problem has to be formulated in the  $(q, \tau)$  space rather than in the  $(x, t)$  space.

Since the bosonic coupling is nonvanishing on the  $I^+$ , one might worry that interaction effects might substantially modify this picture. To address these issues we next analyze the dynamics of small ripples on the Fermi sea directly in the fermion picture, thus taking into account the bosonic interactions in an *exact* fashion.

### III. CLASSICAL DYNAMICS OF RIPPLES ON THE FERMION SEA

We now set up the exact equations for the dynamics of fluctuations around an arbitrary time dependent solution with a quadratic profile, following the method of [16]. Consider a point on the deformed Fermi surface labeled by a parameter  $\alpha$ . The dynamics of this point is then given by

$$\begin{aligned} x(t) &= -a(\alpha) \cosh(t - \alpha), \\ p(t) &= -a(\alpha) \sinh(t - \alpha). \end{aligned} \quad (26)$$

This implies

$$x^2(t) - p^2(t) = -a^2(\alpha), \quad x(t) + p(t) = -a(\alpha)e^{t-\alpha}. \quad (27)$$

Let this point be at some location  $x = x_1$  at some time  $t_1$ . If

this point returns again to this location at some later time  $t_2$ , we must have  $p(t_1) = -p(t_2)$ . Therefore (26) shows that

$$t_1 - \alpha = \alpha - t_2. \quad (28)$$

Note that we have assumed that  $t_2 > t_1$  and this can happen only if

$$|p(t_1)| < |x_1| \quad p(t_1) > 0. \quad (29)$$

If the first inequality is not satisfied the point goes over to the other side of the potential. If the second inequality is violated, the point never returns to the same value of  $x$  at a later time.

Using the equations of motion to eliminate the parameter  $\alpha$ , the ripple can be described by a function  $P(x, t)$ . Then the equation  $p(t_1) = -p(t_2)$  provides a *scattering equation* for the ripple

$$P(x_1, t_1) = -P(x_1, t_2). \quad (30)$$

Combining (27) and (28) we get

$$\begin{aligned} t_2 - t_1 &= 2(\alpha - t_1) = 2(t_2 - \alpha) = \log \frac{x_1 + P(x_1, t_2)}{x_1 - P(x_1, t_2)} \\ &= \log \frac{x_1 - P(x_1, t_1)}{x_1 + P(x_1, t_1)}. \end{aligned} \quad (31)$$

In this paper we will be interested in the evolution of a pulse around the Fermi surface given by the closing hyperbola solution (21). This is given by a Fermi surface

$$x^2 - p^2 - e^{2t}(x - p)^2 = 1 \quad (32)$$

which may be solved to obtain the value of momentum at a point  $x$  at time  $t$

$$\bar{P}_{\pm}(x, t) = \frac{xe^{2t}}{1 + e^{2t}} \pm \frac{\sqrt{x^2 - (1 + e^{2t})}}{1 + e^{2t}}. \quad (33)$$

Expressed in terms of the coordinates  $(q, \tau)$  defined in (23) these become

$$\bar{P}_{\pm}(q, \tau) = -\frac{\cosh q e^{2\tau}}{\sqrt{1 - e^{2\tau}}} \pm \sinh q \sqrt{1 - e^{2\tau}}. \quad (34)$$

We will call the solution  $P_+$  the upper branch of the Fermi sea, and  $P_-$  the lower branch of the Fermi sea.

Consider a ripple which is a perturbation of the upper branch  $\bar{P}_+(x, t)$  at early times. Since the Fermi surface is itself time dependent, at late times the ripple can appear either as a perturbation of the lower branch or the upper branch, depending on the initial condition. To determine which branch the final ripple ends up in, it is sufficient to consider a point exactly on the Fermi surface. The motion of a generic point may be written as

$$x(\tau_0, t) = -(\cosh \tau_0 e^t + \frac{1}{2} e_0^{\tau} e^{-t}) \quad (35)$$

where  $\tau_0$  parametrizes the particular point. It is clear from the solution (33) that the upper and lower branch meet at

$$x = -\sqrt{1 + e^{2t}}. \quad (36)$$

This means that if

$$x(\tau_0, t) < -\sqrt{1 + e^{2t}} \quad (37)$$

for all times, the initial point remains in the upper branch. From (35) the condition (37) implies

$$\sinh^2 \tau_0 e^{2t} + \frac{1}{4} e^{2\tau_0} e^{-2t} + (e_0^\tau \cosh \tau_0 - 1) > 0. \quad (38)$$

A sufficient condition for this to happen is

$$\tau_0 > 0. \quad (39)$$

It is straightforward to see that this is also a necessary condition, by explicitly analyzing the trajectory equation for small  $\tau_0 < 0$ .

To understand the significance of this condition it is useful to express the trajectory in terms of the coordinates  $(q, \tau)$  defined in (21). In terms of  $\tau$ , (35) simply becomes

$$x(\tau_0, \tau) = -\frac{\cosh(\tau_0 - \tau)}{\sqrt{1 - e^{2\tau}}}. \quad (40)$$

In other words the trajectory is simply described by

$$q + \tau = \tau_0. \quad (41)$$

In other words,  $\tau_0$  is the retarded time in the closed string interpretation and gives us two distinct situations for the scattering: for  $\tau_0 < 0$ , the scattered pulse is a ripple on the lower branch, while for  $\tau_0 > 0$  it remains on the upper branch. This is in exact correspondence with the behavior of an excitation of the collective field theory at the linearized level discussed in the previous section. In that case, the excitation got reflected by the mirror at  $q = 0$  for the retarded time  $\tau_0 < 0$ , while for  $\tau_0 > 0$  the excitation never reaches the mirror. However, the behavior of the pulse discussed in this section is *exact* at the classical level and therefore includes effects of the interactions in collective field description.

#### IV. THE SCATTERING EQUATION

We would like to understand how an initial small fluctuation produced around the closing hyperbola solution background evolves in time. Perturbing around the classical solution we define fluctuation fields  $\eta_\pm(x, t)$  as follows

$$P_\pm(x, t) = \bar{P}_\pm(x, t) \pm \eta_\pm(x, t). \quad (42)$$

The initial time will be taken to be localized near some large negative value of  $x = -x_1$  at an early time  $t \rightarrow -\infty$ . In terms of the  $q, \tau$  variables defined in (23) this means

$$q_1 \rightarrow \infty \quad \tau_1 \rightarrow -\infty \quad (43)$$

with some finite value of the retarded time

$$q_1 + \tau_1 = \tau_0 = \text{finite}. \quad (44)$$

The aim is to find the behavior of the pulse at  $t = t_2 = \infty$  at

some finite value of  $q_2$

$$\tau_2 \rightarrow \infty \quad q_2 = \text{finite}. \quad (45)$$

Now let us study both cases separately:

##### A. The case $\tau_0 < 0$

For this case, the scattered pulse reaches the lower branch of the Fermi surface and the scattering equations will be

$$t_1 = t_2 - 2(t_2 - \alpha) = t_2 - \ln \frac{x_1 + P_-(t_2, x_1)}{x_1 - P_-(t_2, x_1)}, \quad (46)$$

where we have used (31). The condition  $p(t_1) = -p(t_2)$  which describes scattering becomes

$$P_-(x_1, t_2) = -P_+(x_1, t_1). \quad (47)$$

Using the fluctuation fields this is

$$\eta_-(x_1, t_2) = \eta_+(x_1, t_1) + [\bar{P}_+(x_1, t_1) + \bar{P}_-(x_1, t_2)]. \quad (48)$$

A straightforward calculation using (34) yields

$$[\bar{P}_+(x_1, t_1) + \bar{P}_-(x_1, t_2)] = \frac{1}{\sqrt{1 - e^{2\tau_1}}} \left[ -e^{-q_1} + e^{q_1 + 2\tau_1} + \frac{\cosh q_2}{\cosh q_1} (1 - e^{2\tau_1}) \right]. \quad (49)$$

In deriving this we have used the fact that the points  $q_2$  and  $q_1$  refer to the same value of  $x = x_1$  albeit at different times, so that (23) yields

$$x_1 = -\frac{\cosh q_1}{\sqrt{1 - e^{2\tau_1}}} = -\frac{\cosh q_2}{\sqrt{1 - e^{2\tau_2}}}. \quad (50)$$

In the limit (43)–(45), the right-hand side of (49) vanishes as  $e^{\tau_1}$ . Therefore the scattering equation simply becomes

$$\eta_-(x_1, t_2) = \eta_+(x_1, t_1). \quad (51)$$

It is useful to define new fields  $\Phi_\pm(q, t)$  by the relations

$$\eta_\pm(x, t) = \frac{\sqrt{1 - e^{2\tau}}}{\sinh q} \Phi_\pm(q, \tau). \quad (52)$$

The motivation for introducing these factors is the following. The perturbations  $\eta_\pm$  are related to the collective field  $\varphi$  in a complicated way. However at the linearized level these relationships become

$$\eta_\pm(x, t) = \frac{\sqrt{1 - e^{2\tau}}}{\sinh q} (\partial_\tau \pm \partial_q) \varphi. \quad (53)$$

Since  $\varphi$  is a massless field in  $(q, \tau)$  space at the linearized level, it is clear that in the same approximation  $\Phi_\pm$  are *chiral* fields. However, this linearized approximation is valid only at large  $q$ . In our scattering problem, the initial pulse is at  $q_1 \sim \infty$  and therefore we can take

$$\Phi_+(q_1, \tau_1) = \Phi_+(\tau_1 + q_1). \quad (54)$$

Since the final pulse is at finite  $q_2$ ,  $\Phi_-(q_2, \tau_2)$  is generally a function of both  $q_2$  and  $\tau_2$ .

In terms of  $\Phi_{\pm}$  the scattering equation becomes

$$\Phi_{\text{out}}(q_2, \tau_2) = \Phi_-(q_2, \tau_2) = \frac{\tanh q_2}{\tanh q_1} \Phi_+(q_1, \tau_1). \quad (55)$$

In the limits (43)–(45) this simplifies to

$$\Phi_{\text{out}}(q_2, \tau_2) = \Phi_-(q_2, \tau_2) = \tanh q_2 \Phi_+(\tau_1 + q_1). \quad (56)$$

We now obtain an expression for the time delay (46) when (43)–(45) holds. Using the Eqs. (33) and (50) repeatedly we get

$$\begin{aligned} \Delta &= \frac{x_1 + P_-(x_1, t_2)}{x_1 - P_-(x_1, t_2)} = \frac{x_1 + \bar{P}_-(x_1, t_2) - \eta_-(x_1, t_2)}{x_1 - \bar{P}_-(x_1, t_2) + \eta_-(x_1, t_2)} \\ &= \frac{2 \cosh^2 q_1}{(1 - e^{2\tau_1}) \cosh q_2 (e^{-q_2} - \frac{\Phi_-(\tau_2, q_2)}{\sinh q_2})} - 1. \end{aligned} \quad (57)$$

This expression simplifies considerably when the conditions (43)–(45) hold.

$$\ln \Delta \approx 2q_1 - \ln \left[ 2 \cosh q_2 \left( e^{-q_2} - \frac{\Phi_-(q_2, 0)}{\sinh q_2} \right) \right] \quad (58)$$

so that the time delay in this limit becomes

$$t_1 = t_2 - 2q_1 + \ln[1 + e^{-2q_2} - 2 \coth q_2 \Phi_-(q_2, 0)]. \quad (59)$$

In the limits defined in (43)–(45) it is easy to see that

$$\begin{aligned} \tau_1 + q_1 \sim t_1 + q_1 = t_2 - q_1 + \ln[1 + e^{-2q_2} \\ - 2 \coth q_2 \Phi_-(q_2, 0)]. \end{aligned} \quad (60)$$

In the same limit we can again use the definitions of  $(q, \tau)$  and the relation (50) to show

$$t_2 - q_1 \sim -\ln(2 \cosh q_2). \quad (61)$$

Therefore the scattering equation (56) becomes

$$\Phi_-(q_2, 0) \approx (\tanh q_2) \Phi_+ \left( \ln \frac{1 + e^{-2q_2} - 2 \coth q_2 \Phi_-(q_2, 0)}{2 \cosh q_2} \right). \quad (62)$$

In terms of the redefined scattered pulse

$$\Psi_{\text{out}}(q_2, 0) = \coth q_2 \Phi_-(q_2, 0) \quad (63)$$

we have the simple scattering equation

$$\Psi_{\text{out}}(q_2, 0) \approx \Phi_+ \left( \ln \frac{1 + e^{-2q_2} - 2 \Psi_{\text{out}}(q_2, 0)}{2 \cosh q_2} \right). \quad (64)$$

At this point, one could be worried that the expression inside the logarithm in (62) can become negative for certain values of  $(q_2, \tau_2)$ . However we will show hereafter that the condition of the existence of scattering automatically rules out the possibility of a singular behavior. The

proof of this assertion goes as follows. From (31) we can see that as  $t \rightarrow -\infty$ ,  $P_+(x, t) \rightarrow \sqrt{x^2 - 1}$  and therefore  $P_+(x, t) = \bar{P}_+(x, t) + \eta_+(x, t) = \sqrt{x^2 - 1} + \eta_+(x, t)$ . The time delay equation then implies

$$e^{t_1 - t_2} = \frac{y - \sqrt{y^2 - 1} - \eta_+}{y + \sqrt{y^2 - 1} + \eta_+} \quad (65)$$

where we have defined the positive quantity  $y \equiv -x_1$ . Since the left-hand side of this equation is always a positive quantity, consistency requires

$$y - \sqrt{y^2 - 1} - \eta_+ > 0. \quad (66)$$

Using the definition of  $\Phi_+$  in (52) and the fact that at  $\tau_1 \rightarrow -\infty$ , (50) implies  $x_1 = -\cosh q_1$  this becomes

$$\cosh q_1 - \sinh q_1 - \frac{1}{\sinh q_1} \Phi_+ > 0 \quad (67)$$

which implies (neglecting terms of order  $e^{-2q}$ )

$$\Phi_+ < \frac{1}{2}. \quad (68)$$

The basic scattering equation (56) then implies

$$2 \coth q_2 \Phi_- < 1. \quad (69)$$

This relation implies that

$$1 + e^{-2q_2} - 2 \coth q_2 \Phi_- > 0 \quad (70)$$

which proves the consistency of our scattering equation.

For  $q_2 \gg 0$  the scattering equation (62) becomes

$$\Phi_-(q_2, 0) \approx \Phi_+(-q_2 + \ln[1 - 2\Phi_-(q_2, 0)]). \quad (71)$$

To lowest order in  $\Phi_-$ , the scattered pulse is therefore peaked at the value of  $|\tau_0|$ , exactly as in linearized collective field theory. This is expected since both the incident and the scattered pulses are in the weak coupling region.

For  $q_2 \sim 0$ , the quantity  $\Psi_{\text{out}}$  satisfies to lowest order

$$\Psi_{\text{out}}(q_2, 0) = \Phi_+(\ln[1 - \Psi_{\text{out}}(q_2, 0)]). \quad (72)$$

## B. The case $\tau_0 > 0$

For  $\tau_0 > 0$  the final pulse at  $t_2 \rightarrow \infty$  or  $\tau_2 \rightarrow 0$  remains on the upper branch of the Fermi surface. This implies a modification of the various formulas in the previous subsection. Basically we have to make the replacement

$$P_-(x_1, t_2) \rightarrow P_+(x_1, t_2), \quad \eta_-(x_1, t_2) \rightarrow -\eta_+(x_1, t_2). \quad (73)$$

In particular, the time delay equation (46) gets modified to

$$t_1 = t_2 - 2(t_2 - \alpha) = t_2 - \ln \frac{x_1 + P_+(t_2, x_1)}{x_1 - P_+(t_2, x_1)} \quad (74)$$

while the basic scattering equation becomes

$$P_+(x_1, t_2) = -P_+(x_1, t_1). \quad (75)$$

In a way entirely analogous to the derivation of (49), the quantity  $[\bar{P}_+(x_1, t_1) + \bar{P}_+(x_1, t_2)]$  now vanishes exponentially fast so that the scattering equation reduces to

$$\eta_+(x_1, t_2) = -\eta_+(x_1, t_1). \quad (76)$$

The remaining steps to the final scattering equation are also identical, leading to

$$\Phi_{\text{out}}(q_2, 0) \equiv \Phi_+(\tau_2, q_2) = -\tanh q_2 \Phi_+(\tau_1 + q_1). \quad (77)$$

Once again in the limit (43)–(45) the expression for the time delay simplifies, which now leads to, instead of (60)

$$\tau_1 + q_1 \sim t_1 + q_1 = t_2 - q_1 + \ln[1 + e^{2q_2} - 2 \coth q_2 \Phi_+(\tau_2, q_2)]. \quad (78)$$

Using (61) the final equation which yields the scattered pulse at late times becomes, instead of (62)

$$\begin{aligned} \Phi_{\text{out}}(q_2, 0) &= \Phi_+(q_2, 0) \\ &\approx -(\tanh q_2) \Phi_+ \left( \ln \frac{1 + e^{2q_2} - 2 \coth q_2 \Phi_+}{2 \cosh q_2} \right). \end{aligned} \quad (79)$$

## V. BEHAVIOR OF THE SCATTERED PULSE

In this section we numerically investigate the behavior of the scattered pulse for a given initial pulse for various values of  $\tau_0$ . We start with a Gaussian pulse which is centered at a retarded time equal to  $\tau_0$ , i.e. the function  $\Phi_+$  is

$$\Phi_+(w) = A \exp \left[ -\frac{(w - \tau_0)^2}{a^2} \right] \quad (80)$$

with some constant  $A$  which is small, so that the condition (68) is satisfied. We then numerically find the final scattered pulse at  $t \rightarrow \infty$  or equivalently  $\tau = 0$  using (62) or (79).

At the level of linearized collective field theory, the behavior of the scattered pulse is described at the end of Sec. II and depicted in Fig. 3. This shows that at this level of approximation, the scattered pulse is also a Gaussian which is peaked at  $q = |\tau_0|$ . For  $\tau_0 < 0$  this happens due to a reflection from the mirror. Such a reflection would invert the pulse—however we have defined the quantity  $\Phi_-$  above to incorporate this. For  $\tau_0 > 0$  there is no reflection—this means the  $\Phi_{\text{out}}$  now has the opposite sign of  $\Phi_+$ .

This behavior is exactly what we observe in the exact solution. Figure 4 shows the initial pulse, while the Figs. 5 and 6 are the scattered pulses for  $\tau_0 = -2$  and  $\tau_0 = 2$  respectively. These are reasonably large values of  $|\tau_0|$ . Note the scattered pulse is also centered around  $q = -2$  and  $q = 2$  respectively. This is expected since for large

### INITIALPULSE

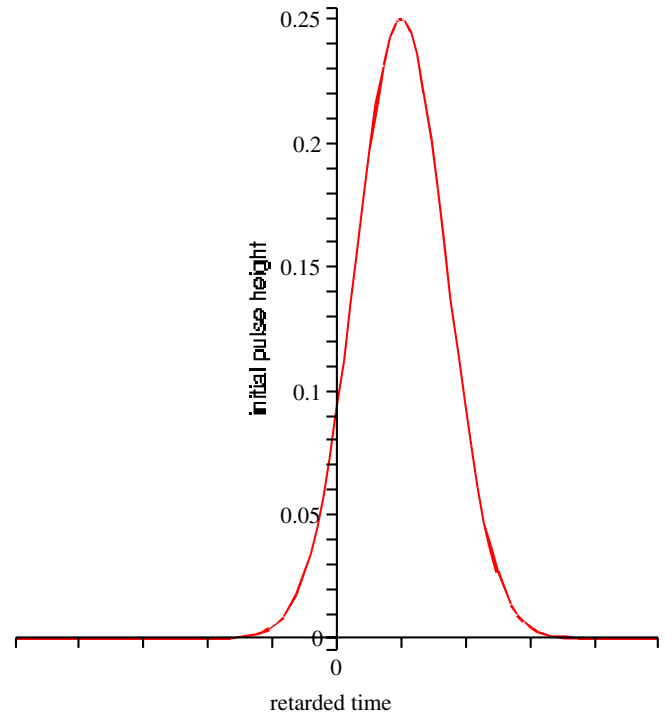


FIG. 4 (color online). The initial pulse, i.e.  $\Phi_+$  as a function of the retarded time.

values of  $q$  on the  $\tau = 0$  surface, the collective field theory is reasonably weakly coupled and the linearized approxi-

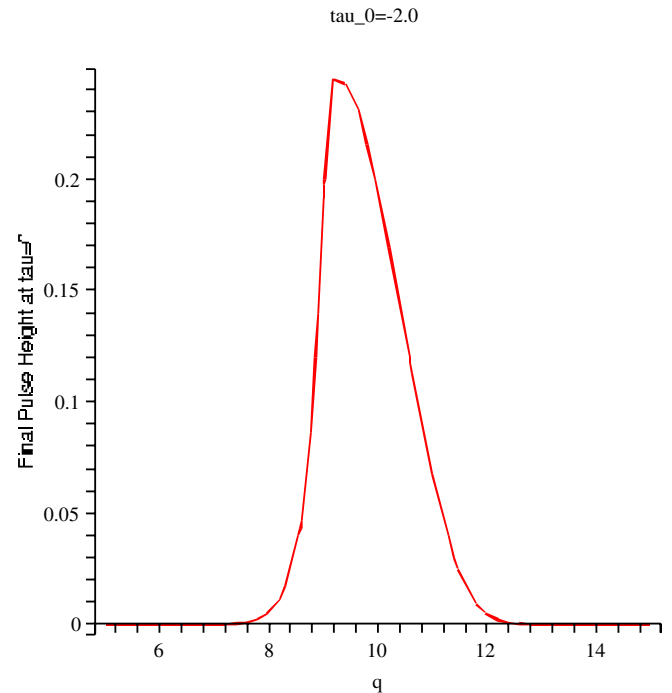


FIG. 5 (color online). The scattered pulse for  $\tau_0 = -2$ .



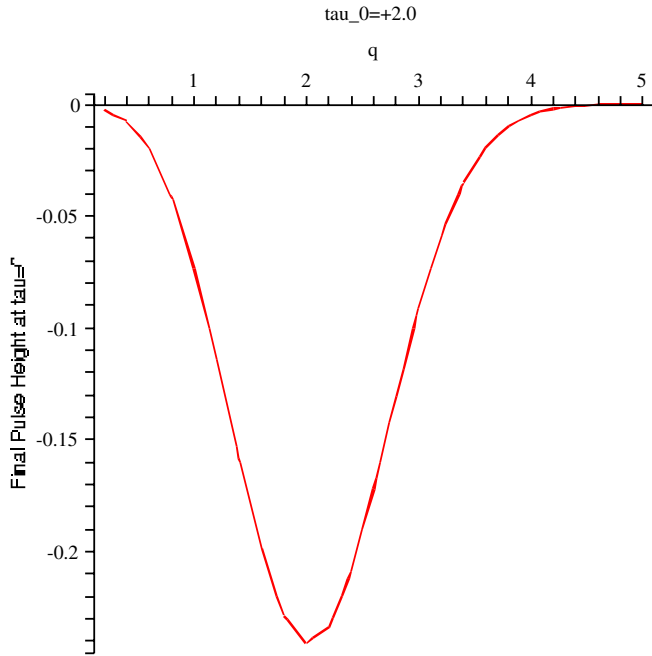


FIG. 6 (color online). The scattered pulse for  $\tau_0 = 2$ .

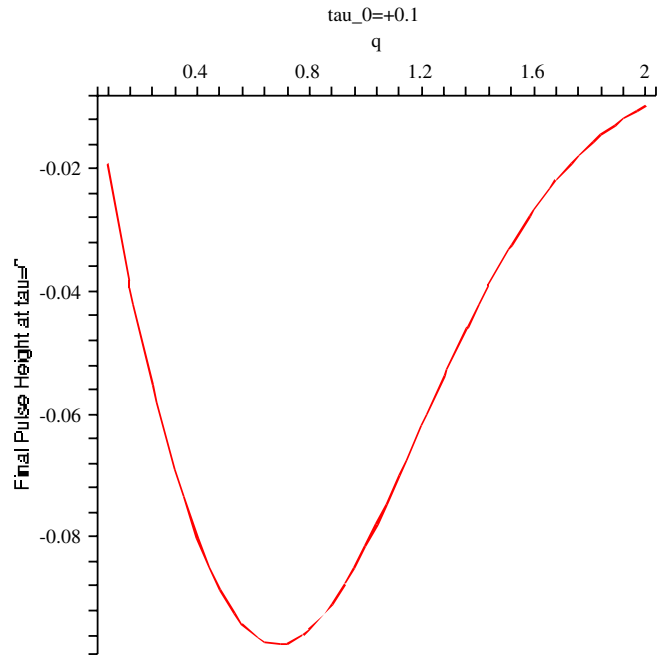


FIG. 8 (color online). The scattered pulse for  $\tau_0 = 0.1$ .

mation reliable. Furthermore, as we would expect there is not much deformation of the pulse.

Figures 7 and 8 show the scattered pulse for *small* values of  $|\tau_0|$ .

Unlike the previous cases, the scattered pulses are centered at values of  $q > |\tau_0|$ . This is then the effect of nonlinearities in the collective field description which are

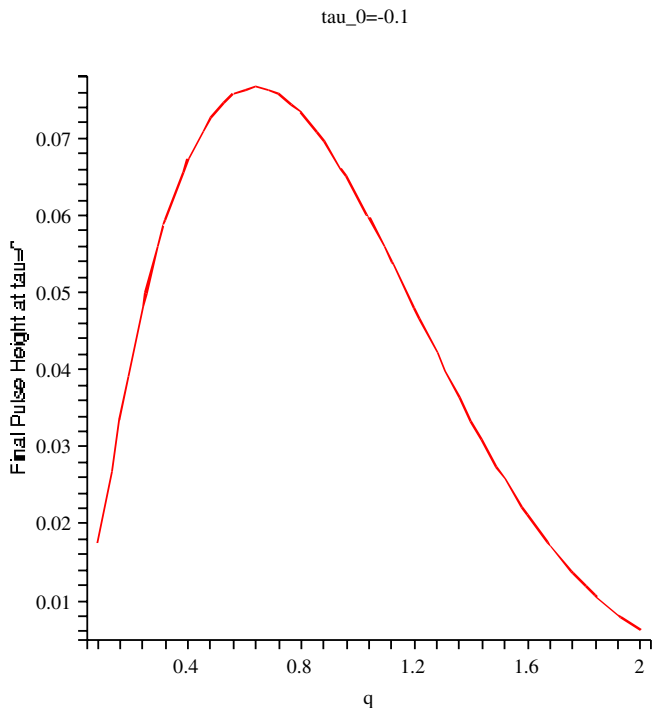


FIG. 7 (color online). The scattered pulse for  $\tau_0 = -0.1$ .

expected to be strong at small values of  $q$ . This trend becomes more pronounced as we go to smaller values of  $|\tau_0|$

The main point is that the scattering problem is well defined for all values of  $\tau_0$  and the scattered pulse is smooth, always vanishing at  $q = 0$ . In the original space  $x$  provided by the eigenvalues of the matrix, the background Fermi sea gets completely drained out and ceases to exist for any finite  $x$ . It might appear from this that the scattering problem is pathological since the pulses do not seem to have any space to move in at infinitely late times. Our discussion shows that this is not the space in which the scattering problem has to be formulated. In the closed string space,  $q$ , scattering makes perfect sense and smooth. While the closed string time ends at  $\tau = 0$  the real time evolution of the ripple on the Fermi sea is over a complete time range, and the fact that  $I^+$  is not weakly coupled does not pose any problem for the scattering data.

### VI. FERMION CORRELATORS IN TIME DEPENDENT BACKGROUNDS

In this section, we will outline a method to obtain *exact* fermionic correlators in the nontrivial time dependent background. We will calculate the two-point function explicitly and show that there is no pathological behavior at late times. This is something we already know at the classical level from the previous section. However the following considerations take into account all quantum corrections.

The strategy is to use the fact that these time dependent solutions are obtained from the ground state by the action

of nondiagonal  $W_\infty$  transformations, as described by the Eq. (20). As discussed extensively in [23,24], these symmetry generators can be obtained from the basic quantity

$$W(\alpha, \beta, t) = \frac{1}{2} \int dx e^{i\alpha x} \psi^\dagger(x + \beta/2, t) \psi(x - \beta/2, t). \quad (81)$$

In terms of  $W(\alpha, \beta, t)$ , the charges are  $W_{r,s}$ :

$$W_{rs} = e^{-(r-s)t} \int d\alpha d\beta F_{rs} \left( \frac{\alpha - \beta}{\sqrt{2}}, \frac{\alpha + \beta}{\sqrt{2}} \right) W(\alpha, \beta, t) \quad (82)$$

where

$$F_{rs}(a, b) = 2 \cos\left(\frac{ab}{2}\right) (i\partial_a)^r (i\partial_b)^s \delta(a) \delta(b). \quad (83)$$

The basic quantity in the fermionic theory is the Wigner operator

$$u(x, p) \equiv \int dy e^{ipy} : \psi^\dagger(x - y/2) \psi(x + y/2) :. \quad (84)$$

Under the  $W_\infty$  transformations this quantity changes by the basic commutator [24]

$$[W(\alpha, \beta, t), u(x, p)] = \frac{1}{2} e^{i(\alpha x - \beta p)} (u(x - \beta/2, p - \alpha/2) - u(x + \beta/2, p + \alpha/2)). \quad (85)$$

Using these expressions, we can calculate the change of the phase space density operator  $u(x, p, t)$  under a finite transformation

$$u(x, p) \rightarrow u'(x, p) = \exp(-i\lambda W_{rs}) u(x, p, t) \exp(i\lambda W_{rs}). \quad (86)$$

Now consider a time dependent background which is described by a non-normalizable state

$$|\lambda\rangle = e^{i\lambda W_{rs}} |\mu\rangle, \quad (87)$$

where  $|\mu\rangle$  denotes the ground state. Then we have the identity

$$\langle \lambda | u(x_1, p_1) u(x_2, p_2) \cdots | \lambda \rangle = \langle \mu | u'(x_1, p_1) u'(x_2, p_2) \cdots | \mu \rangle. \quad (88)$$

Since the operator

$$: \psi^\dagger(x_1) \psi(x_2) : \quad (89)$$

can be expressed in terms of the operator  $u(x, p)$  we can use (88) to calculate correlators like

$$\langle \lambda | : \psi^\dagger(x_1) \psi(y_1) : : \psi^\dagger(x_2) \psi(y_2) : \cdots | \lambda \rangle \quad (90)$$

in terms of the correlators in the ground state. The latter have been calculated exactly in [18], which may be then used to calculate the correlators in the nontrivial background.

Essentially the same philosophy was used in [14] for the draining Fermi sea solution. However in that case, the corresponding  $W_\infty$  transformation did not involve a non-trivial transformation of the momentum variable. This meant that the correlators of density operators  $\psi^\dagger \psi$  in the

nontrivial background can be written in terms of correlators of density operators in the ground state. In the cases of our interest, we need correlators of higher moments of the phase space density.

In the following we will perform an explicit computation of the expectation value of the density operator in the closing hyperbola solution.

### A. Density operator in the closing hyperbola solution

The density operator  $\rho$  is

$$\rho(x, t) = : \psi^\dagger(x, t) \psi(x, t) : = \frac{1}{2\pi} \int dp u(x, p, t). \quad (91)$$

The state in the fermionic theory which corresponds to the closing hyperbola solution is

$$|\lambda\rangle = e^{i\lambda W_{02}} |\mu\rangle. \quad (92)$$

For  $r = 0, s = 2$  the relations (82) and (83) become

$$W_{02} = -e^{2t} (\partial_\alpha + \partial_\beta)^2 W(\alpha, \beta, t) |_{\alpha=\beta=0}. \quad (93)$$

A straightforward calculation using (85) then yields

$$[u(x, p), W_{02}] = e^{2t} (-i)(x - p) (\partial_x + \partial_p) u(x, p). \quad (94)$$

The form of this commutator immediately shows that a finite transformation of the Wigner operator itself is

$$\exp(-i\lambda W_{02}) u(x, p) \exp(i\lambda W_{02}) = e^{\lambda e^{2t}(x-p)(\partial_x + \partial_p)} u(x, p) = u(x', p'), \quad (95)$$

where we have defined

$$x' = x + \lambda e^{2t}(x - p) \quad p' = p + \lambda e^{2t}(x - p). \quad (96)$$

This is what one would expect at the classical level since these are precisely the transformations in the single particle phase space. What we have shown, however, is that the result is exact at the quantum level.

The above results show that the expectation value of the density operator in the closing hyperbola state  $|\lambda\rangle$  is given in terms of the two-point fermion correlator in the ground state  $|\mu\rangle$  by

$$\langle \lambda | \rho(x) | \lambda \rangle = \frac{1}{2\pi} \int dy dp e^{ip'y} \left\langle \mu \left| \psi^\dagger \left( x' - \frac{y}{2} \right) \times \psi \left( x' + \frac{y}{2} \right) \right| \mu \right\rangle \quad (97)$$

with  $x', p'$  given by (96).

In [18] the two-point fermion correlator was determined in the ground state as<sup>3</sup>

<sup>3</sup>In [18] the single particle Hamiltonian is taken as  $h = p^2 - \frac{1}{4}x^2$  which differs from ours by rescaling of  $p$  and  $x$ . The formula (98) differs from that in [18] since takes into account this rescaling as well as a rescaling of the fermion field necessary to preserve the correct anticommutation relation.

$$\langle \psi^\dagger(x_1)\psi(x_2) \rangle_\mu = \sqrt{2}i \int_{-\infty}^{+\infty} \frac{dq}{2\pi} \int_0^{\text{sgn}(q)\infty} ds \frac{e^{-sq+is\mu}}{(-4\pi i \sinh s)^{1/2}} e^{\mathcal{H}(x_1, x_2)} \quad (98)$$

where

$$\mathcal{H}(x_1, x_2) = -\frac{i}{2} \left( \frac{x_1^2 + x_2^2}{\tanh s} - 2 \frac{x_1 x_2}{\sinh s} \right). \quad (99)$$

Using (96) we therefore get

$$\langle \lambda | \rho(x, t) | \lambda \rangle = \sqrt{2} \frac{1}{2\pi} i \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} dy e^{ip'y} i \int_{-\infty}^{+\infty} \frac{dq}{2\pi} \times \int_0^{\text{sgn}(q)\infty} ds F(s) G(s, x', y) + (s \rightarrow -s) \quad (100)$$

where

$$F(s) = \frac{e^{-sq+is\mu}}{(-4i\pi \sinh s)^{1/2}} \quad (101)$$

$$G(s, x', y) = \exp \left[ -\frac{i}{2} \left( 2 \tanh \frac{s}{2} x'^2 + \coth \frac{s}{2} \frac{y^2}{2} \right) \right].$$

Performing the integration over  $y$  we get

$$\langle \lambda | \rho(x, t) | \lambda \rangle = \frac{\sqrt{2}}{2\pi} \left[ i \int \frac{ds}{2\pi s} \left( \frac{-4\pi i}{\coth s/2} \right)^{1/2} \times \frac{e^{is\mu}}{(-4i\pi \sinh s)^{1/2}} \times \int_{-\infty}^{+\infty} dp e^{-i \tanh(s/2)(x^2 - p^2)} + \text{c.c.} \right]. \quad (102)$$

Defining the functions

$$\begin{aligned} f(t) &= 1 + 2\lambda e^{2t} & h(t) &= -2\lambda e^{2t} \\ g(t) &= -(1 - 2\lambda e^{2t}) \end{aligned} \quad (103)$$

(96) yields

$$x'^2 - p'^2 = -\frac{1}{g(t)} x^2 + g \left( p + \frac{h(t)x}{g(t)} \right)^2 \quad (104)$$

so that integrating over  $p$  we get the final answer

$$\langle \lambda | \rho(x, t) | \lambda \rangle = 2\sqrt{2} \left[ i \left( \frac{-1}{g(t)} \right)^{1/2} \int_0^\infty \frac{ds}{2\pi s} \times \frac{e^{is\mu - (i \tanh(s/2)x^2 - g(t))}}{(-4i\pi \sinh s)^{1/2}} + \text{c.c.} \right]. \quad (105)$$

The density expectation value therefore satisfies the re-

markably simple relation

$$\langle \lambda | \rho(x, t) | \lambda \rangle = \frac{1}{(1 - 2\lambda e^{2t})^{1/2}} \times \left\langle \mu \left| \rho \left( \frac{x}{\sqrt{1 - 2\lambda e^{2t}}}, t \right) \right| \mu \right\rangle. \quad (106)$$

It is easy to check that this exact expression leads to the correct semiclassical answer. To do this, it is necessary to perform a derivative with respect to  $\mu$ ,

$$\partial_\mu \langle \lambda | \rho(x, t) | \lambda \rangle = -2\sqrt{2} \left[ \left( \frac{-1}{g(t)} \right)^{1/2} \int_0^\infty \frac{ds}{2\pi} \times \frac{e^{is\mu - (i \tanh(s/2)x^2 - g(t))}}{(-4i\pi \sinh s)^{1/2}} + \text{c.c.} \right] \quad (107)$$

and consider the limit  $s \approx 0$  in (107). This leads, after redefining  $z^2 = \frac{x^2}{-g(t)}$ , to:

$$\partial_\mu \langle \lambda | \rho(x, t) | \lambda \rangle = -2\sqrt{2} \left[ \left( \frac{-1}{g} \right)^{1/2} \int_0^\infty \frac{ds}{2\pi} \frac{1}{(-4\pi i)^{1/2}} \times \frac{i(\mu - \frac{z^2}{2})s}{\sqrt{s}} + \text{c.c.} \right]. \quad (108)$$

Integration over  $s$  then gives

$$\partial_\mu \langle \lambda | \rho(x, t) | \lambda \rangle = -\frac{1}{(-g)^{1/2}} \frac{1}{\pi \sqrt{z^2 - 2\mu}}. \quad (109)$$

To compare with the semiclassical answer (21) for the closing hyperbola, we need to set  $2\mu = 1$  and  $2\lambda = -1$ .<sup>4</sup> This yields

$$\left\langle -\frac{1}{2} | \rho(x, t) | -\frac{1}{2} \right\rangle = \frac{1}{\pi} \frac{\sqrt{x^2 - (1 + e^{2t})}}{1 + e^{2t}} \quad (110)$$

which is identical to (21).

The exact answer in fact corroborates our conclusions about the nature of the background based on the semiclassical solution. At large times, Eq. (106) shows that the density goes to zero at any finite  $x$ . However, as our previous sections show the physics at late times occurs at infinite values of  $|x|$  and finite values of the closed string coordinate  $q$ .

Our result for the one point function of the eigenvalue density does not reveal any pathological behavior. The scattering matrix is related to higher point functions which may be calculated using similar techniques. It is important to see whether these have any interesting behavior coming from nonperturbative effects.

<sup>4</sup>Note that the  $\lambda$  of this section is related to  $\lambda_+$  of Sec. II B by  $2\lambda = \lambda_+$

- [1] S. R. Das and J. L. Karczmarek, *Phys. Rev. D* **71**, 086006 (2005) [arXiv:hep-th/0412093]; S. R. Das, *Mod. Phys. Lett. A* **20**, 2101 (2005).
- [2] B. Craps, S. Sethi, and E. P. Verlinde, *J. High Energy Phys.* 10 (2005) 005 [arXiv:hep-th/0506180]; B. Craps, A. Rajaraman, and S. Sethi, *Phys. Rev. D* **73**, 106005 (2006) [arXiv:hep-th/0601062]; B. Craps, *Classical Quantum Gravity* **23**, S849 (2006) [arXiv:hep-th/0605199].
- [3] M. Li, *Phys. Lett. B* **626**, 202 (2005) [arXiv:hep-th/0506260]; M. Li and W. Song, *J. High Energy Phys.* 10 (2005) 073 [arXiv:hep-th/0507185]; B. Chen, *Phys. Lett. B* **632**, 393 (2006) [arXiv:hep-th/0508191]; J. H. She, *J. High Energy Phys.* 01 (2006) 002 [arXiv:hep-th/0509067]; M. Li and W. Song, *J. High Energy Phys.* 08 (2006) 089 [arXiv:hep-th/0512335]; H. Chen and B. Chen, *Phys. Lett. B* **638**, 74 (2006) [arXiv:hep-th/0603147]; T. Ishino and N. Ohta, *Phys. Lett. B* **638**, 105 (2006) [arXiv:hep-th/0603215]; H. Kodama and N. Ohta, *Prog. Theor. Phys.* **116**, 295 (2006) [arXiv:hep-th/0605179]; T. Ishino, H. Kodama, and N. Ohta, *Phys. Lett. B* **631**, 68 (2005) [arXiv:hep-th/0509173].
- [4] S. R. Das and J. Michelson, *Phys. Rev. D* **72**, 086005 (2005) [arXiv:hep-th/0508068]; **73**, 126006 (2006) [arXiv:hep-th/0602099].
- [5] D. Robbins and S. Sethi, *J. High Energy Phys.* 02 (2006) 052 [arXiv:hep-th/0509204]; E. J. Martinec, D. Robbins, and S. Sethi, *J. High Energy Phys.* 08 (2006) 025 [arXiv:hep-th/0603104].
- [6] S. R. Das, J. Michelson, K. Narayan, and S. P. Trivedi, *Phys. Rev. D* **75**, 026002 (2007) [arXiv:hep-th/0610053]; **74**, 026002 (2006) [arXiv:hep-th/0602107].
- [7] C. S. Chu and P. M. Ho, *J. High Energy Phys.* 04 (2006) 013 [arXiv:hep-th/0602054].
- [8] A partial list of references may be found in [6].
- [9] J. McGreevy and H. Verlinde, *J. High Energy Phys.* 12 (2003) 054 [arXiv:hep-th/0304224]; I. R. Klebanov, J. Maldacena, and N. Seiberg, *J. High Energy Phys.* 07 (2003) 045 [arXiv:hep-th/0305159]; A. Sen, *Mod. Phys. Lett. A* **19**, 841 (2004) [arXiv:hep-th/0308068].
- [10] A. Jevicki and B. Sakita, *Nucl. Phys.* **B165**, 511 (1980).
- [11] S. R. Das and A. Jevicki, *Mod. Phys. Lett. A* **5**, 1639 (1990).
- [12] D. Minic, J. Polchinski, and Z. Yang, *Nucl. Phys.* **B369**, 324 (1992); G. W. Moore and R. Plesser, *Phys. Rev. D* **46**, 1730 (1992) [arXiv:hep-th/9203060]; S. Y. Alexandrov, V. A. Kazakov, and I. K. Kostov, *Nucl. Phys.* **B640**, 119 (2002) [arXiv:hep-th/0205079].
- [13] J. L. Karczmarek and A. Strominger, *J. High Energy Phys.* 04 (2004) 055 [arXiv:hep-th/0309138].
- [14] J. L. Karczmarek and A. Strominger, *J. High Energy Phys.* 05 (2004) 062 [arXiv:hep-th/0403169].
- [15] S. R. Das, J. L. Davis, F. Larsen, and P. Mukhopadhyay, *Phys. Rev. D* **70**, 044017 (2004) [arXiv:hep-th/0403275].
- [16] J. Polchinski, *Nucl. Phys.* **B362**, 125 (1991).
- [17] M. Natsuume and J. Polchinski, *Nucl. Phys.* **B424**, 137 (1994) [arXiv:hep-th/9402156].
- [18] G. W. Moore, *Nucl. Phys.* **B368**, 557 (1992).
- [19] For reviews and references to the original literature see e.g. I. R. Klebanov, arXiv:hep-th/9108019; S. R. Das, arXiv:hep-th/9211085; A. Jevicki, arXiv:hep-th/9309115; P. H. Ginsparg and G. W. Moore, arXiv:hep-th/9304011; J. Polchinski, arXiv:hep-th/9411028; E. J. Martinec, arXiv:hep-th/0410136.
- [20] A. Dhar, G. Mandal, and S. R. Wadia, *Int. J. Mod. Phys. A* **8**, 3811 (1993) [arXiv:hep-th/9212027];.
- [21] S. R. Das and S. D. Mathur, *Phys. Lett. B* **365**, 79 (1996) [arXiv:hep-th/9507141]; S. R. Das, arXiv:hep-th/0401067.
- [22] T. Takayanagi and N. Toumbas, *J. High Energy Phys.* 07 (2003) 064 [arXiv:hep-th/0307083]; M. R. Douglas, I. R. Klebanov, D. Kutasov, J. Maldacena, E. Martinec, and N. Seiberg, arXiv:hep-th/0307195.
- [23] S. R. Das, A. Dhar, G. Mandal, and S. R. Wadia, *Int. J. Mod. Phys. A* **7**, 5165 (1992) [arXiv:hep-th/9110021]; *Mod. Phys. Lett. A* **7**, 71 (1992) [arXiv:hep-th/9111021]; **7**, 937 (1992); **7**, 2245(E) (1992) [arXiv:hep-th/9112052].
- [24] A. Dhar, G. Mandal, and S. R. Wadia, *Int. J. Mod. Phys. A* **8**, 325 (1993) [arXiv:hep-th/9204028]; *Mod. Phys. Lett. A* **7**, 3129 (1992) [arXiv:hep-th/9207011]; **8**, 3557 (1993) [arXiv:hep-th/9309028].