Topological low-temperature limit of Z(2) spin-gauge theory

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We study Z(2) lattice gauge theory on triangulations of compact d-manifolds. We reformulate the theory algebraically, describing it in terms of the structure constants of a bidimensional vector space $\mathcal{H}$ equipped with algebra and coalgebra structures, and prove that in the low-temperature limit $\mathcal{H}$ reduces to a Hopf algebra, in which case the theory becomes equivalent to a topological field theory. The degeneracy of the ground state is shown to be a topological invariant. This fact is used to compute the zeroth- and first-order terms in the low-temperature expansion of Z for arbitrary triangulations. In finite temperatures, the algebraic reformulation gives rise to new duality relations among classical spin models, related to changes of basis of $\mathcal{H}$.

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I. INTRODUCTION

The Z(2) spin-gauge theory we consider in this work is a lattice gauge theory, locally invariant under the Abelian discrete group Z(2), and one of the simplest examples of a gauge theory on the lattice, according to Wilson's ideas [1]. There are two main motivations for the study of this model. First, it is hoped that the understanding of this simple example will bring some insight into the physics of more realistic gauge theories. Second, it is known that in three dimensions this theory is dual to the 3D Ising model [2], an outstanding problem in statistical mechanics. We study Z(2) pure gauge theory on arbitrary finite triangulations of a compact d-manifold. Usually, one considers hypercubic lattices [3,4], but we do not follow this procedure here, in order to achieve a more general framework for the study of topological properties.

The theory is defined as follows. Let $\sigma_a = (-1)^a$, $a = 0, 1$, be gauge variables sitting at the links a of a triangulation L. Let $\omega(f)$ be the product of all gauge variables at the boundary of a particular face f. Then the action is $S = \beta\sum_f \omega(f)$, where β is the coupling constant of the theory, and the sum runs over all faces f of L. The partition function is given by $Z = \sum_{\{\sigma_a\}} \exp(S)$, the sum running over all gauge configurations $\{\sigma_a\}$. We will find it convenient to rewrite Z as a product of local Boltzmann weights, in the form

$$Z = \sum_{\{\sigma_a\}} \prod_f M(f), \tag{1}$$

where $M(f) = \exp[\beta\omega(f)]$. The low-temperature limit is obtained by letting β go to infinity.

As the lattice L is a triangulation, all its faces are triangular. Thus the weights $M(f)$ describe a three-spin interaction. We write them as M_{abc}, where a, b, c are the indices of the gauge variables σ_a at the boundary of f. It follows that $M_{abc} = \exp[(-1)^{a+b+c}\beta]$.

We give the theory an algebraic interpretation, in terms of a bidimensional algebra $\mathcal{H}$. For that, we follow prescriptions introduced by Chung, Fukuma, and Shapere (CFS) in [5], in the context of topological quantum field theory [6]. There, it is shown how to build up a field theory on a three-dimensional lattice from the knowledge of the structure constants of an algebra of interest. Such a procedure allows one to encode symmetries of the theory in symmetries of the algebra, and was first applied in the investigation of topological invariance in three dimensions. The main result found was that the lattice theory is topological whenever $\mathcal{H}$ is a Hopf algebra [5,7].

Here we show that Z(2) pure gauge theory can be reformulated as a CFS theory, for any dimension d. We display explicitly an algebra $\mathcal{H}$ which leads to Z(2) gauge theory through the use of the prescriptions of [5]. The algebra $\mathcal{H}$ depends on the value of β. In the low-temperature limit, the Hopf algebra axioms hold, up to the appearance of some extra factors in its defining relations. It follows that the theory is almost topological in this limit. In fact, we prove that $Z|_{\beta\to\infty}$ is essentially the product of a volume-dependent factor and a topological invariant, giving an example of a quasitopological field theory [8]. This decomposition leads to a solution of the leading terms in the low-temperature expansion of Z described in [9]. The zeroth-order term depends only on the degeneracy $\Xi(L)$ of the ground state. We prove that it is a topological invariant, and show how to compute it. The first-order correction depends on simple combinatorial factors. It is hoped that higher order terms can be dealt with in a similar fashion.

II. ALGEBRAIC REFORMULATION

Let us describe the algebra $\mathcal{H}$ associated to Z(2) gauge theory in the CFS formalism. Let $B = \{\phi_0, \phi_1\}$ be a basis of $\mathcal{H}$, and $B^\star = \{\psi^0, \psi^1\}$ its dual basis, for which $\psi^a(\phi_b) = \delta^a_b$. We define a linear product $M: \mathcal{H}\otimes\mathcal{H} \mapsto \mathcal{H}$, which

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sends the pair of vectors $u, v \in \mathcal{H}$ into $u \cdot v \in \mathcal{H}$, as usual. The products of the basis vectors read

$$\begin{aligned} \phi_0 \cdot \phi_0 &= \phi_1 \cdot \phi_1 = f\phi_0 + e\phi_1, \\ \phi_0 \cdot \phi_1 &= \phi_1 \cdot \phi_0 = e\phi_0 + f\phi_1, \end{aligned} \tag{2}$$

where $e = \rho^{-1}\sinh(x)$, $f = \rho^{-1}\cosh(x)$, and ρ and x are fixed by the coupling constant β through

$$e^{-2\beta} = \tanh(3x), \qquad \rho^6 = 2\sinh(6x). \tag{3}$$

The algebra has a unity $\iota = \rho\cosh(x)\phi_0 - \rho\sinh(x)\phi_1$, and is associative. Its structure constants are the coefficients $M^c_{ab} = \psi^c(\phi_a \cdot \phi_b)$ of the product tensor M.

Defining the trace $T \in \mathcal{H}^\star$ as the dual vector with coefficients $T_a = M^b_{ab}$, it follows that $T = 2f\psi^0 + 2e\psi^1$, and that

$$M_{abc} = T(\phi_a \cdot \phi_b \cdot \phi_c) = M^x_{ab}M^y_{xc}M^z_{yz}. \tag{4}$$

This formula shows how the weights of Z(2) gauge theory are related to the structure constants of the algebra $\mathcal{H}$: they are coefficients of a three-indexed covariant tensor built from the product tensor M alone.

The expression in Eq. (4) is an example of the CFS formalism, and the first step in our reformulation of the Z(2) theory. In this formalism, the weights M_{abc} are thought as determined by the algebra $\mathcal{H}$. If a different algebra was taken, there would be different local weights M_{abc} at the lattice faces, and another set of configurations a at the links. Thus, a different lattice field theory, is determined by the structure constants of the algebra. For the particular algebra $\mathcal{H}$ and basis $\mathcal{B}$ we defined, the formula happens to give the weights $M_{abc} = \exp[(-1)^{a+b+c}\beta]$ of Z(2) pure gauge theory, as can be checked. The problem of finding an algebra $\mathcal{H}$ which gives a specified set of weights M_{abc} may not be a trivial task. In our case it involved the solution of a system with eight equations, reduced to a single ninth degree equation in two variables.

We carry the algebraic reformulation further. We want to write the partition function Z in tensorial form. As it is a number, it shall be a scalar of $\mathcal{H}$. Therefore, some kind of contravariant tensor is needed, so that we can build contractions with M_{abc} in order to define scalars. For that purpose, we give $\mathcal{H}$ a coalgebra structure, to whose definition we now turn.

Let $\Delta\colon \mathcal{H} \mapsto \mathcal{H} \otimes \mathcal{H}$ be a coproduct on $\mathcal{H}$, whose action on basis vectors is given by

$$\Delta(\phi_a) = \phi_a \otimes \phi_a. \tag{5}$$

The coalgebra so defined has a co-unity $\varepsilon = \psi^0 + \psi^1$, and is coassociative. Its structure constants are the coefficients $\Delta^{bc}_a = (\psi^b \otimes \psi^c)(\Delta(\phi_a)) = \delta^b_a\delta^c_a$ of the tensor Δ.

In analogy with Eq. (4), we define the n-indexed contravariant tensors $\Delta^{a_1a_2\ldots a_{n-1}a_n}$ by

$$\Delta^{a_1a_2\ldots a_{n-1}a_n} = \Delta^{x_1x_2}_{x_1}\Delta^{a_1x_3}_{x_2}\Delta^{a_2x_4}_{x_3}\ldots\Delta^{a_{n-1}a_n}_{x_n}. \tag{6}$$

For the coalgebra defined by Eq. (5), they read

$$\Delta^{a_1a_2\ldots a_{n-1}a_n} = \delta_{a_1,a_n}\delta_{a_2,a_n}\ldots\delta_{a_{n-1},a_n}, \tag{7}$$

where the δ's are Kronecker deltas.

The expression in Eq. (6) is another formula of the CFS formalism. It defines local weights which are assigned to links. In their formalism, the local configurations are not assigned to links or vertices, as usual. Instead, if a link a is shared by n faces, then there are n configurations a_i, $i = 1, 2, \ldots, n$, sitting at the intersections of the link and the faces. The link is thought of not as a thin line, but as a sort of "hinge": an object with n strips emerging from a central line, where faces are glued (see Ref. [5]). There is a configuration a_i at each strip. The coefficients $\Delta^{a_1a_2\ldots a_{n-1}a_n}$ are local weights assigned to hinges, determined by the configurations at its strips. Different coalgebras lead to different hinge weights and, therefore, to distinct lattice field theories.

The partition function is defined as a scalar written with contractions of indices of hinge and face weights. Consider the outer product of all hinge and face weights in the triangulation L. For each hinge strip, there is a configuration a, which appears as a lower index in a face weight, and as an upper index in some hinge weight. We contract this pair of indices. Then we do the same for all hinge strips. A scalar is obtained, which is defined as the partition function Z of the theory. In the original work of CFS, there is also an antipode operator $S\colon \mathcal{H} \mapsto \mathcal{H}$, which interferes in the contractions, and is used to define orientations. The rule we gave here is equivalent to setting S equal to the identity, $S^b_a = \delta^b_a$. This special case is enough for our purposes.

Now it just remains to note that the special form of the coproduct given in Eq. (5) is just the one needed in order to recover the interpretation of configurations at links. For, according to Eq. (7), in this case the hinge weights are zero, unless all its indices are equal. But then this index can be thought of as assigned to the link itself. Therefore, under such a condition, a global configuration is an assignment of an index $a = 0, 1$ to each link of L. There is a weight $\prod_f M_{abc}(f)$ for each configuration. The contractions of indices which define Z automatically implements the sum over configurations. For $\mathcal{H}$, the weights are simply $M_{abc} = \exp[(-1)^{a+b+c}\beta]$, and the result is Z(2) pure gauge theory's partition function, as written in Eq. (1). That completes our reformulation of the theory.

III. LOW-TEMPERATURE LIMIT

Let us turn to the low-temperature limit of the theory now. It is obtained by letting $\beta \to \infty$, or, equivalently, $x \to 0$, as shown by Eq. (3). The partition function diverges in this limit, $\lim_{x\to 0} Z = \infty$, and we want to understand how that happens, looking at the algebraic reformulation of the theory. The structure constants of $\mathcal{H}$ are not both well defined when x goes to zero. The coefficient e presents no problem, as $\lim_{x\to 0} e(x) = 0$, but for f we have

$\lim_{x\to 0} f(x) = \infty$. We will show that the divergences of the algebra coefficients are related to the divergence of the partition function itself in this limit. Such interpretation comes from a simple algebraic manipulation. A continuous change of basis of $\mathcal{H}$, parametrized by x, is used to isolate the singular part of Z, which is then found to be a function of f.

Note that the partition function Z, being a scalar of $\mathcal{H}$, does not change its value if a different basis B' is chosen to write the link and face weights of the theory. But the weights itself do change, and thus we have a different lattice field theory with the same partition function. For instance, consider the change of basis, which we will use in the low-temperature limit, given by the scaling transformation

$$\xi_0 = f^{-1}\phi_0, \qquad \xi_1 = f^{-1}\phi_1. \tag{8}$$

In the new basis $B' = \{\xi_0, \xi_1\}$, the structure constants of $\mathcal{H}$ are given by

$$\begin{aligned} \xi_0 \cdot \xi_0 = \xi_1 \cdot \xi_1 = \xi_0 + \frac{e}{f}\xi_1, \\ \xi_0 \cdot \xi_1 = \xi_1 \cdot \xi_0 = \frac{e}{f}\xi_0 + \xi_1, \end{aligned} \tag{9}$$

and the coproduct is

$$\Delta(\xi_a) = f\xi_a \otimes \xi_a. \tag{10}$$

The hinge weights, determined by Eq. (6), are now given by

$$\Delta^{\prime a_1 a_2 \ldots a_{n-1} a_n} = f^n \delta_{a_1,a_n}\delta_{a_2,a_n}\ldots\delta_{a_{n-1},a_n}. \tag{11}$$

They still allow a link configuration interpretation of the theory. But now there is a local weight f^n at each link, which does not depend on the configuration at the link. The face weights given by Eq. (4) read

$$\begin{aligned} M'_{000} = M'_{011} = 2 + 6\left(\frac{e}{f}\right)^2, \\ M'_{001} = M'_{111} = 6\frac{e}{f} + 2\left(\frac{e}{f}\right)^3, \end{aligned} \tag{12}$$

and are cyclically symmetric.

The theory obtained with the basis B' can thus be summarized as follows. There are configurations $a = 0$, 1 at all lattice links. At each link, there is a local weight f^n, where n is the number of faces meeting at the link. On each face, there is a cyclically symmetric weight whose value is given by Eq. (12). The partition function Z is the sum over all link configurations of the product of all face weights in the lattice. This partition function has the same value as that of Z(2) pure gauge theory. This is an example of a new duality relation between lattice spin models, and we will discuss it later in this paper.

A simplification can be achieved if the contribution of the weights f^n is removed from Z. For any configuration, these weights taken together lead to a factor f^{3N_f}, where N_f is the total number of faces in the lattice. Thus if we define a new theory with face weights given by Eq. (12), but with no link weights, it will have a partition function $X = Zf^{-3N_f}$. Its CFS description is given by the algebra displayed in Eq. (9), with the coproduct of Eq. (5).

We consider the low-temperature limit of such a theory. This is enough for us, since a solution for X gives a solution for Z. But when $x \to 0$ ($\beta \to \infty$), the structure constants are:

$$\begin{aligned} \xi_0 \cdot \xi_0 = \xi_1 \cdot \xi_1 = \xi_0, \qquad \xi_0 \cdot \xi_1 = \xi_1 \cdot \xi_0 = \xi_1, \\ \Delta(\xi_a) = \xi_a \otimes \xi_a. \end{aligned} \tag{13}$$

This product and coproduct, taken together with an antipode $S^a_b = \delta^a_b$, constitute a Hopf algebra, which can be recognized as the Z(2) group algebra.

A direct implication of the Hopf structure of the low-temperature limit of $\mathcal{H}$ given in Eq. (13) is that in three dimensions the partition function $X|_{\beta\to\infty}$ is a topological invariant. Thus its value does not depend on the details of the lattice, being determined by the topology of L alone. Actually, as we defined the theory, there is an extra factor yet. From Kuperberg's work [7], we have that

$$X|_{\beta\to\infty} = \Xi(L)2^{2N_t}2^{N_v-1}, \tag{14}$$

where $\Xi(L)$ is Kuperberg's topological invariant for the Z(2) group algebra, and N_t and N_v are the total number of tetrahedra and vertices in L, respectively.[1] Then we can write the partition function of three-dimensional Z(2) lattice gauge theory in the low-temperature limit as

$$Z|_{\beta\to\infty} = \Xi(L)(2f^3)^{2N_t}2^{N_v-1}, \tag{15}$$

i.e., as the product of a topological invariant $\Xi(L)$, a volume-dependent factor $(2f^3)^{2N_t}$, and a factor 2^{N_v-1} which counts gauge equivalent configurations. The factor $(2f^3)^{2N_t}$ gives the temperature dependence of Z as β approaches ∞. This factor diverges at zero temperature, and is the singular part of $Z|_{\beta\to\infty}$. For a given β, the value of $Z|_{\beta\to\infty}$ depends only on N_t, the discrete version of the volume. The remaining factor 2^{N_v-1} is quite irrelevant, and could well be eliminated by gauge fixing.

The proof of Kuperberg's theorem which leads to Eq. (15) is based on methods of combinatorial topology which are valid only in three dimensions. Yet the validity of the equation is not restricted to dimension 3. In fact, we will show while discussing the interpretation of the invariant $\Xi(L)$ that this result can be restated in a form which is

[1] Kuperberg's theory is equivalent to CFS theory, but is defined on Heegard diagrams D, instead of on lattices. For any Hopf algebra $\mathcal{H}$, the partition function is $Z(\mathcal{H}) = \#(\mathcal{D}, \mathcal{H}) \times (\dim\mathcal{H})^{n_u+n_v-g(S)}$, where $\#(\mathcal{D}, \mathcal{H})$ is a topological invariant, g is the genus of $\mathcal{D}$, and n_u, n_v are the number of upper and lower curves of $\mathcal{D}$, respectively. In the language of triangulations, this formula reduces to $Z(\mathcal{H}) = \Xi(L, \mathcal{H})2^{2N_t}2^{N_v-1}$, where $\Xi(L, \mathcal{H}) = \#(\mathcal{D}, \mathcal{H})$.

valid for any dimension d. A new approach will be required for that purpose. In Kuperberg's work, a Heegard diagram description [10] of the lattice model is used in order to prove that the partition function is invariant under elementary transformations which generate all homeomorphisms of 3-manifolds (compact and orientable). In lower dimensions, and especially for $d < 4$, the requirement of invariance under a set of basic homeomorphisms is usually an efficient method to study topological invariants (see [11,12], for example). In higher dimensions, however, the growing number of elementary transformations makes this combinatoric approach cumbersome. Then the methods of algebraic topology should be applied. In order to prove that a quantity is a topological invariant, one may try to write it in terms of known invariants coming from homology or cohomology theories, for example. We will see in the next section that this can be done for the low-temperature partition function $Z|_{\beta\to\infty}$ in any dimension d.

The formula in Eq. (15), in addition to the algebraic reformulation of the theory, is the main result of this paper. Next we will discuss it further, and show how to use the equation in explicit calculations. In particular, we will give solutions for the zeroth and first-order terms in the low-temperature expansion of Z.

IV. APPLICATIONS

A. Interpretation of the invariant $\Xi(L)$

Let us give an interpretation for the topological invariant $\Xi(L)$. For that, consider the low-temperature limit of the weights M'_{abc} which define X. As $\lim_{x\to 0}(e/f) = 0$, then we can write

$$M'_{000}(= M'_{011})|_{\beta\to\infty} = 2, \qquad M'_{001}(= M'_{111})|_{\beta\to\infty} = 0. \tag{16}$$

These formulae have a very simple meaning. The weight of a gauge configuration is the product of all local weights M'_{abc}. According to Eq. (16), it is nonzero, in the low-temperature limit, only if $w(f) = 1$ at all faces of L. In this case, it is equal to 2^{N_f}. Suppose there are A_0 such gauge configurations. Then $X|_{\beta\to\infty} = 2^{N_f}A_0$. From Eq. (14), it follows that in three dimensions

$$\Xi(L) = \frac{A_0}{2^{N_v-1}}, \tag{17}$$

so that in this case $\Xi(L)$ is the number of equivalence classes of gauge configurations for which $w(f) = 1, \forall f \in L$. This condition selects gauge configurations with minimal energy, that is, ground state configurations. Thus $\Xi(L)$ is the degeneracy of the ground state on the lattice L, and we have just proved that it is a topological invariant.

The determination of the degeneracy of the ground state is the first step in any low-temperature expansion of the theory, and can be severely simplified due to topological symmetry. The evaluation of A_0 is naturally a hard combinatorial problem. One must enumerate all gauge configurations on L, check one by one if the ground state condition is satisfied, and count the number of occurrences. We give an alternative procedure. There is a topological field theory which performs this counting. Its partition function is $X|_{\beta\to\infty} = 2^{N_f}A_0$, and one can take the simplest lattice L' homeomorphic to L in order to evaluate it, as topological symmetry implies that $X(L) = X(L')$. For the usual topologies, L' will be a lattice with a small number of tetrahedra, and the calculation of $X|_{\beta\to\infty}$ is feasible (a small finite sum). Then we get a solution for A_0, namely, $A_0 = 2^{-N_f}X|_{\beta\to\infty}$.

In the case of a generic dimension d, the topological invariance of $\Xi(L)$ can be proved in an alternative way with more traditional methods of algebraic topology. This is done in Appendix A. There it is shown that the number of equivalence classes of gauge configurations for which $\omega(f) = 1$ at all faces f of a lattice L is equal to the number of elements in the first cohomology group of L with Z_2-coefficients, denoted $H^1(L, Z_2)$, which is a well-known topological invariant. In other words, it is proved that

$$A_0 = 2^{N_v-1}\#H^1(L, Z_2), \tag{18}$$

where $\#H^1(L, Z_2)$ is the number of elements in $H^1(L, Z_2)$. Comparing Eqs. (17) and (18), we find that in three dimensions the invariant $\#H^1(L, Z_2)$ is equal to Kuperberg's invariant $\Xi(L)$ evaluated with the Z(2) group algebra, a result which agrees with [7]. In the language of lattice gauge theory, this invariant corresponds to the degeneracy of the ground state, as discussed. Furthermore, Eq. (18) allows us to write the low-temperature partition function in any dimension d as

$$Z|_{\beta\to\infty} = (2f^3)^{N_f}2^{N_v-1}\#H^1(L, Z_2).$$

This is the generalization of Eq. (15) to arbitrary dimension d. The low-temperature partition function is the product of the topological invariant $\#H^1(L, Z_2)$, a temperature-dependent factor $(2f^3)^{N_f}$, whose value is fixed at each temperature by the number of faces in the triangulation, and an additional factor which counts gauge transformations.

Now consider the low-temperature expansion of Z as given in [9]. At zero temperature, only ground state configurations are accessible. These configurations give the zeroth-order approximation of Z, which reads

$$Z^{(0)} \equiv A_0 e^{\beta N_f} = 2^{N_v-1}e^{\beta N_f}\#H^1(L, Z_2). \tag{19}$$

The first-order correction can also be solved. It consists of contributions coming from elementary excitations of the lattice. Pick a ground state configuration. There is a weight e^{β} at each face. Now let a be any link, and σ_a the local gauge variable at it. Invert its sign, $\sigma_a \mapsto -\sigma_a$. A new gauge configuration is obtained, for which $w(f) = 1$ at all faces, except for those meeting a. The contributions for the first-order correction of Z come from these configurations.

We write it as

$$Z^{(1)} = A_0 \sum_a e^{\beta(N_f - 2n_a)}, \tag{20}$$

where n_a is the number of faces meeting the link a. The determination of the combinatorial factors n_a should not bring problems. For instance, for a hypercubic lattice $n_a = 2(d-1)$ is constant, and we have

$$Z^{(1)} = 2^{N_v - 1} N_l e^{\beta[N_f - 4(d-1)]} \# H^1(L, Z_2). \tag{21}$$

This is our solution for $Z^{(1)}$, and another example of a calculation based on the formalism we presented here.

B. Dualities

Before concluding the work, we would like to get back to Eqs. (8)–(12), and discuss the meaning of the change of basis presented there. The theory given by the original basis $\mathcal{B}$ is just Z(2) gauge theory, which can be thought of as a spin model. The spins σ_a are situated at the links of L, and there are Boltzman local weights M_{abc} at faces. After the change of basis, a new spin model is defined. The spins are still at the lattice links, but the face weights are M'_{abc}, and there are also link weights. Yet, the partition function is the same. Thus we have an example of a new duality between spin models, similar to those depicted in [2]. In the CFS formalism, there is one such duality for each change of basis of $\mathcal{H}$. If, beyond the scaling transformations studied in this work, more general changes of basis are considered, one may hope that a larger class of new dualities will arise. This method for studying dualities can be applied to any lattice theory which can be reformulated as a CFS theory.

The Z(2) lattice gauge theory studied in this paper is related by a Kramers-Wannier duality to the Ising model in three dimensions. It is interesting to observe that this duality can be obtained as a certain change of basis in $\mathcal{H}$, as demonstrated in Appendix B. There are also similar dualities relating Z(2) lattice gauge theory to different classes of statistical mechanics models in any dimension d, all of which can be reformulated as changes of basis in $\mathcal{H}$ as well. Therefore, these transformations are in fact a generalization of the classical Kramers and Wannier dualities as presented in [2].

V. CONCLUSION

We have shown in this paper that Z(2) lattice gauge theory can be reformulated as a CFS theory, giving it an algebraic interpretation in terms of a vector space $\mathcal{H}$ with algebra and coalgebra structure, and proved that the low-temperature limit of the theory is equivalent to a topological field theory. The reformulation is based on the application of mathematical prescriptions coming from topological field theories to the realm of lattice gauge theory. Calculations made in the low-temperature limit are considerably simplified when topological symmetry is taken into account, and can be evaluated for arbitrary triangulations. Moreover, we have found that any change of basis of $\mathcal{H}$ is related to some duality relation among spin models. This observation leads to an algebraic method for the investigation of dualities, which we shall develop elsewhere.

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APPENDIX A: PROOF THAT $\Xi(L) = H^1(L, Z_2)$

The link variables $\sigma_{ij} = \pm 1$, where the indices i, j refer to the vertices at the two ends of the link, can be rewritten as

$$\sigma_{ij} = e^{i\pi A_{ij}} = \pm 1,$$

where $A_{ij} \in \{0, 1\}$. Similarly, gauge transformations ϕ_i are in 1-to-1 correspondence with Z_2-valued fields λ_i on lattice sites:

$$\phi_i = e^{i\pi\lambda_i} = \pm 1.$$

The gauge transformation $\sigma_{ij} \to \phi_i \sigma_{ij} \phi_j^{-1}$ corresponds to

$$A_{ij} \to A_{ij} + \lambda_i - \lambda_j. \tag{A1}$$

The variables $\{A_{ij}\}$ and $\{\lambda_i\}$, with the gauge transformations (A1) and the condition $\omega(f) = 1, \ \forall f \in L$, have simple and natural interpretation in terms of Z_2-valued cochain structure of the lattice L. The A_{ij}'s form a Z_2-valued 1-cochain. i.e. a function A on the lattice links, given by

$$A([i, j]) = A_{ij},$$

with values in Z_2. Similarly, the λ_i's form a Z_2-valued 0-cochain, i.e. a function λ on the lattice sites, given by

$$\lambda([i]) = \lambda_i.$$

There is a "coboundary operator" d which maps p-cochains to $(p+1)$-cochains, given in terms of the boundary operator ∂ on $(p+1)$-cells of L by

$$d\alpha([i_1, \ldots, i_{p+1}] = \alpha(\partial[i_1, \ldots, i_{p+1}]) = \sum_{k=1}^{p+1} (-1)^k \alpha([i_1, \ldots, \hat{i}_k, \ldots, i_{p+1}]),$$

where $\hat{i}_k$ denotes an omitted vertex. Note that $d \circ d = 0$. The gauge transformation (A1) can now be expressed as

$$A \to A + d\lambda.$$

To interpret the condition $\omega(f) = 1, \ \forall f \in L$, use the notation $f = [i, j, k]$ where the sites i, j, k are the vertices of the face f. Then

$$\omega([i, j, k]) = \sigma_{ij}\sigma_{jk}\sigma_{ki} = e^{i\pi(A_{ij}+A_{jk}+A_{ki})}.$$

Noting that $A_{lm} = -A_{lm} \bmod 2$, we have (mod 2)

$$A_{ij} + A_{jk} + A_{ki} = A(\partial[i, j, k])] = dA([i, j, k]),$$

hence $\omega(f) = e^{i\pi dA(f)}$. Thus $\omega(f) = 1 \Leftrightarrow dA(f) = 0$, and therefore

$$\omega(f) = 1 \forall f \Leftrightarrow dA = 0$$

as a Z_2-valued 2-cochain. The collection of Z_2-valued 1-cochains A satisfying $dA = 0$ modulo transformations $A \rightarrow A + d\lambda$ is precisely $H^1(L, Z_2)$, the first cohomology group of the cochain complex. It is a fundamental result in algebraic topology that $H^1(L, Z_2)$ is independent of the lattice L used to discretize a space M, and one therefore denotes it $H^1(M, Z_2)$. In particular, the number of elements in $H^1(L, Z_2)$ is independent of the triangulation L. This completes the argument. Note that there was no restriction on the dimension of the spacetime manifold M.

APPENDIX B: KRAMERS-WANNIER DUALITY AS A CHANGE OF BASIS IN $\mathcal{H}$

In the three-dimensional Ising model, there are spin variables σ_i at the vertices i of the lattice, and an energy term $E_l = -\beta_I\sigma_i\sigma_j$ for each link l, where β_I is the inverse temperature, and the indices i, j refer to the vertices at the two ends of the link. There corresponds to this term a local link weight

$$W_l = \exp(\beta_I\sigma_i\sigma_j). \tag{B1}$$

The partition function is

$$Z_{IS} = \sum_{\{\sigma_i\}} \prod_l W_l,$$

where the sum runs over all spin configurations on links of L, and the product runs over all links l of the lattice. This model can be reformulated as having spin variables at links. For that purpose, define the link variables

$$\sigma_l = \sigma_i\sigma_j, \tag{B2}$$

in terms of which the local weights are $W_l = \exp(\beta\sigma_l) = \pm 1$. The product of link spins σ_l over any closed circuit is always 1, as the spin associated with a vertex crossed in the circuit appears twice in the product. In particular, the product $\omega(f)$ of the spins at the boundary of any face f is always 1. Now we can rewrite the partition function as

$$Z_{IS} = 2\sum_{\{\sigma_l\}} \prod_l W_l,$$

where the sum runs over all configurations for which $\omega(f) = 1$ at all faces. The factor 2 accounts for the fact that two distinct vertex configurations $\{\sigma_i\}$ are related to each link configuration $\{\sigma_l\}$, as reversing all spins $\sigma_i \in \{\sigma_i\}$ does not change $\{\sigma_l\}$.

Now consider the change of basis E of $\mathcal{H}$ given by

$$E = \begin{pmatrix} \frac{1}{2}\rho e^{-x} & \frac{1}{2}\rho e^{-x} \\ \frac{1}{2}\rho e^{x} & -\frac{1}{2}\rho e^{x} \end{pmatrix}, \tag{B3}$$

and denote the new basis by $\mathcal{B}' = \{\xi_0, \xi_1\}$. The products of the vector basis are given by

$$\xi_0^2 = \rho^{-1}e^{x}\xi_0, \qquad \xi_1^2 = \rho^{-1}e^{-x}\xi_1,$$
$$\xi_0 \cdot \xi_1 = \xi_0 \cdot \xi_1 = 0,$$

and the trace is

$$T(\xi_0) = \rho^{-1}\exp(x), \qquad T(\xi_1) = \rho^{-1}\exp(-x).$$

The coproduct is given by

$$\Delta(\xi_0) = \xi_0 \otimes \xi_0 + \xi_1 \otimes \xi_1$$
$$\Delta(\xi_1) = \xi_0 \otimes \xi_1 + \xi_1 \otimes \xi_0,$$

and the cotrace is $C = 2\xi_0$. Evaluating the local weights for this algebra and coalgebra we find

$$M'_{000} = \rho^{-3}e^{3x}, \qquad M'_{111} = \rho^{-3}e^{-3x} \tag{B4}$$

for the face weights, and

$$\Delta^{la_1a_2\cdots a_m} = 2\delta_{(-1)^{a_1+a_2+\cdots a_m},1} \tag{B5}$$

for the link weights.

The new weights of Eqs. (B4) and (B5) are interpreted in the dual lattice L' [2] as follows. In three dimensions, dualization interchanges the roles of faces and links: a face in L is related to a link in L', and a link in L to a face in L'. Thus the coefficients of M'_{abc} describe link weights in L'. From Eq. (B4), these are nonzero only if all spins meeting at the link are equal, i.e. if in M'_{abc}, we have $a = b = c$. Such a spin can be thought of as assigned to the link itself. The corresponding weights are, up to a common factor ρ^{-3}, given by

$$W_l = \exp(\sigma_l 3x), \tag{B6}$$

where σ_l is the spin at the link. Moreover, the face weights of Eq. (B5) are all equal to one or zero, up to a common factor 2. These restrict the set of spin configurations $\{\sigma_l\}$ to be considered in the sum over states: the nonzero weights correspond to those configurations for which the product of spins at the boundary of any face in L' is equal to 1, since

$$(-1)^{a_1+a_2+\cdots a_m} = 1 \Leftrightarrow \sigma_{a_1}\sigma_{a_2}\ldots\sigma_{a_m} = 1.$$

Collecting the results, we have found the following. The new model, obtained from the Z(2) pure gauge theory by the application of the transformation E of Eq. (B3), has spins σ_l at links in the dual lattice L', local link weights $W_l = \exp(\sigma_l 3x)$, and configurations restricted to those for which the product of spins at the boundary of any face is

equal to 1. According to previous discussions, this gives an Ising model in L'. Furthermore, comparing Eqs. (B1) and (B2) with Eq. (B6), we find that the inverse temperature of this Ising model is $\beta_I = 3x$. And from Eq. (3), it follows that

$$e^{-2\beta_g} = \tanh\beta_I,$$

which is the classical Kramers and Wannier duality relation for Z(2) pure gauge theory in three dimensions [2].

If the change of basis E defined in Eq. (B3) is applied for the weights of the d-dimensional Z(2) pure gauge theory, we find the same face and link weights given in Eqs. (B4) and (B5), since the same algebra $\mathcal{H}$ describes the model in any dimension. The interpretation of these weights on the dual lattice is changed, however. The dual of a link is a $(d-1)$-simplex, and the dual of a face is a $(d-2)$-simplex. Thus Eq. (B4) means that in the dual lattice there are configurations $\sigma = \pm 1$ at $(d-2)$-simplexes, and local weights $\exp(\sigma 3x)$ corresponding to these. The Eq. (B5) restricts again the set of allowed configurations. The weight at a $(d-1)$-simplex is nonzero only if the product of the spins at its boundary is 1, in which case the weight is equal to one. This condition is automatically satisfied if one thinks of the spins at the $(d-2)$-simplexes as obtained from another set of spin variables $s = \pm 1$ sitting at the $(d-3)$-simplexes of the triangulation, that is, if the spin $\sigma_{\Delta^{(d-2)}}$ at a simplex $\Delta^{(d-2)}$ is written as

$$\sigma_{\Delta^{(d-2)}} = \prod_{\partial\Delta^{(d-2)}} s_i,$$

where the s_i denote the spins at the $(d-3)$-faces of $\sigma_{\Delta^{(d-2)}}$. This prescription implies that in the product of all spins σ at the boundary of a $(d-1)$-simplex, each spin s of a $(d-3)$-face appears twice. Thus the product is equal to 1, as claimed. Moreover, there is a fixed number Q of s-configurations over $(d-3)$-faces corresponding to each σ-configuration at $(d-2)$-simplexes. That follows from the fact that two distinct s-configurations are always related by a local transformation

$$s_i \rightarrow s_i\psi_i,$$

where the variables $\psi_i = \pm 1$ define a Z(2)-valued field over $(d-3)$-simplexes. The effect of this transformation on the spins $\sigma_{\Delta^{(d-2)}}$ is given by

$$\sigma_{\Delta^{(d-2)}} \rightarrow \sigma_{\Delta^{(d-2)}} \prod_{\partial\Delta^{(d-2)}} \psi_i.$$

Therefore, two s-configurations define the same σ-configuration if and only if they are related by a transformation for which $\prod_{\partial\Delta^{(d-2)}}\psi_i = 1$ at all $(d-2)$-simplexes of the triangulation. We denote the number of such transformation by Q.

Thus we have found the following. The dual of Z(2) gauge theory on a d-dimensional triangulation is a model with spins $s_i = \pm 1$ sitting at $(d-3)$-simplexes, and local weights W_{d-2} at the $(d-2)$-simplexes, which are given by

$$W_{d-2}(\Delta^{(d-2)}) = \exp\left[\beta^\star \prod_{\partial\Delta^{(d-2)}} s_i\right],$$

where $\beta^\star = 3x$. The partition function $Z^\star$ of this model is related to that of Z(2) gauge theory by

$$QZ(\beta, L) = Z^\star(\beta^\star, L^\star), \tag{B7}$$

where $L^\star$ is the lattice dual to the original triangulation L. From Eq. (3), it follows that

$$e^{-2\beta} = \tanh\beta^\star. \tag{B8}$$

The expressions in Eqs. (B7) and (B8) correspond to the dualities of Wegner in any dimension d.

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