Neutral gluon polarization tensor in a color magnetic background at finite temperature

M. Bordag*

University of Leipzig, Institute for Theoretical Physics, Augustusplatz 10/11, 04109 Leipzig, Germany

V. Skalozub†

Dnepropetrovsk National University, 49050 Dnepropetrovsk, Ukraine

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In the framework of SU(2) gluodynamics, we derive the tensor structure of the neutral gluon polarization tensor in an Abelian homogeneous magnetic field at finite temperature and calculate it in the one-loop approximation in the Lorentz background field gauge. The imaginary time formalism and the Schwinger operator method are used. The latter is extended to the finite temperature case. The polarization tensor turns out to be nontransversal. It can be written as a sum of ten tensor structures with corresponding form factors. Seven tensor structures are transversal; three are not. We represent the form factors in terms of double-parametric integrals and the temperature sum which can be computed numerically. As applications we calculate the Debye mass and the magnetic mass of neutral gluons in the background field at high temperature. A comparison with the results of other authors is done.

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I. INTRODUCTION

The investigations of QCD at high temperature carried out in recent years have elucidated the important role of color magnetic fields. In Refs. [1,2] it was discovered in lattice simulations that sufficiently strong constant Abelian magnetic fields described by the potential of the form $A^a_\mu = B\delta_{\mu 2}\delta^{a3}$—where B is the field strength, a is the index of internal symmetry, μ is the Lorentz index—shift the deconfinement phase transition temperature T_c. In particular, it was shown that an increase in the field strength decreases the transition temperature and for sufficiently strong field strengths $T_c(B)$ can be equal to zero. On the other hand, in Refs. [3,4] from the analysis of lattice simulations and in Refs. [5–7] from perturbative resummations of daisy graphs in the background field at high temperature, it was found that Abelian chromomagnetic fields of order $gB \sim g^4T^2$, where g is the gauge coupling, are spontaneously created.

These results are of interest not only for QCD but also for problems of the early universe where strong magnetic fields of a different kind had likely been present. This was discussed in [8]; for a recent review see [9]. These discussions served as a motivation for investigations started in our recent papers, Refs. [10,11], with the goal to determine the operator structure of the gluon polarization tensor in the constant Abelian chromomagnetic background field at finite temperature and to find a representation which is, as much as possible, explicit and convenient. This representation should serve as a starting point for a deepened investigation within the framework of a self-consistent solution of the Schwinger-Dyson equation, i.e., for substantial resummations of the perturbative expansion, and, especially at medium and at high temperature, for further investigations of the quark-gluon plasma. As a preliminary step, the tensor structure of the gluon polarization tensor as well as the one-loop contributions to its form factors at zero temperature have been obtained and partially investigated therein.

A necessary prerequisite for a number of approaches is a proper knowledge of the perturbative expansion. In general, in QCD the perturbative expansion does not give directly physical results (even at high temperature). But it may serve as a starting point for resummations, for instance in a Schwinger-Dyson equation approach. Also one should remember the role the perturbative expansion played in the advent of QCD in conjunction with the operator product expansion and renormalization group equation. Interest in perturbative expansions comes also from the W bosons which formally differ only by their mass from gluons. There exists quite a number of perturbative calculations, especially for spin-1 particles in a magnetic field and at finite temperature. However, these are mostly restricted to special cases like specific asymptotic regions (like the static limit or the high temperature limit), projection onto on-shell states or specific gauge fixations. Also, in most cases these are technically very involved and represented in a form less covariant than necessary.

We remind the reader that in the presence of a magnetic background field it is meaningful to transform the gluon fields A^a_μ by means of $W^\pm_\mu = 1/\sqrt{2}(A^1 \pm iA^2_\pm)$, $A_\mu = A^3_\mu$ into the so-called "charged basis." These fields have the interpretation of being spin-1 fields with color charge ($W^\pm_\mu$) or being color neutral (A_μ). The behavior with respect to the color charge is exactly the same as with respect to an electric charge. For instance, the color charged gluons move on Landau levels in a color magnetic background

*Email address: Michael.Bordag@itp.uni-leipzig.de
†Email address: Skalozub@ff.dsu.dp.ua

whereas the color neutral ones do not interact directly with such background.

In the present paper we determine the tensor structure of the polarization tensor of neutral gluons in the magnetic field at finite temperature and calculate the tensor in the one-loop approximation. We use the Lorentz-Feynman covariant gauge. The results are represented in terms of tensor structures and the corresponding form factors. The tensor structures are determined by the symmetries left in the external field and they carry the Lorentz indices. The form factors depend on the invariant combinations of the external momenta and, of course, on the magnetic field and on the temperature. They are represented as integrals over two proper time parameters and the temperature sum. For the integration over the internal momentum the formalism developed by Schwinger [12] was used. It is extended to account for nonzero temperature. In this representation it is easy to calculate the form factors numerically or to calculate their asymptotic expansions for large and small temperature or magnetic field. We check our formulas by recalculating the known results at zero temperature or at zero magnetic field. For instance, as new results, we calculate the one-loop Debye mass and the magnetic mass of neutral gluons in the magnetic field.

The paper is organized as follows. In the next section we introduce the necessary notations and briefly review the basic formulas. In Secs. III and IV we derive the operator structure of the gluon polarization tensor and develop the calculation procedure to carry out the integration over internal momenta of Feynman diagrams in the field at finite temperature. Explicit formulas for the form factors in the form of two-parametric integrals are obtained in Sec. V. This is in contrast to the previous paper [10], where the tadpole diagrams were not discussed in detail and instead some arguments in fewer of the cancellation of this contributions in the total by the surface terms appearing in other parts of the polarization tensor were used. Here we consider this cancellation because of peculiarities appearing at finite temperature. In Sec. VI the transition to the zero field case is discussed. The Debye mass of the neutral gluon in the external field is calculated in Sec. VII. In the next section we calculate the mean values of the operator in the physical states of the transverse modes and show that the gluon "magnetic mass" in the field is zero in one-loop order and that the fictitious pole of the Green function is eliminated. The discussion of the results obtained and further prospects are given in Sec. IX.

Throughout the paper we use Latin letters $a, b, \ldots = 1, 2, 3$ for the color indexes and Greek letters $\lambda, \mu, \ldots = 1, \ldots, 4$ for the Lorentz indices. Summation over doubly appearing indices is assumed. All formulas are in the Euclidean formulation. We put all constants including the coupling equal to unity. Since the present work is a continuation of investigations that began in Ref. [10], we follow the notations, definitions, and calculation procedures used therein as close as possible.

II. BASIC NOTATIONS

In this section we collect the well-known basic formulas for SU(2) gluodynamics to set up the notations which we will use. We work in the Euclidean version. Dropping arguments and indices, the Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}(\partial_\mu A_\mu)^2 + \bar{\eta}\partial_\mu D_\mu \eta, \tag{1}$$

where ξ is the gauge fixing parameter and η is the ghost field. The action is $S = \int dx L$ and the generating functional of the Green functions is $Z = \int DA \exp(S)$. In the following we divide the gauge field $A_\mu^a(x)$ into the background field $B_\mu^a(x)$ and quantum fluctuations $Q_\mu^a(x)$,

$$A_\mu^a(x) = B_\mu^a(x) + Q_\mu^a(x). \tag{2}$$

The covariant derivative depending on a field A is

$$D_\mu^{ab}[A] = \frac{\partial}{\partial x^\mu}\delta^{ab} + \epsilon^{acb}A_\mu^c(x) \tag{3}$$

and the field strength is

$$F_{\mu\nu}^a[A] = \frac{\partial}{\partial x^\mu}A_\nu^a(x) - \frac{\partial}{\partial x^\nu}A_\mu^a(x) + \epsilon^{abc}A_\mu^b(x)A_\nu^c(x) \tag{4}$$

and

$$[D_\mu[A], D_\nu[A]]^{ab} = \epsilon^{acb}F_{\mu\nu}^c[A] \tag{5}$$

holds. For the field split into background and quantum parts we note

$$F_{\mu\nu}^a[B+Q] = F_{\mu\nu}^a[B] + D_\mu^{ab}[B]Q_\nu^b(x) - D_\nu^{ab}[B]Q_\mu^b(x) + \epsilon^{abc}Q_\mu^b(x)Q_\nu^c(x). \tag{6}$$

The square of it is

$$-\tfrac{1}{4}(F_{\mu\nu}^a[B+Q])^2 = -\tfrac{1}{4}(F_{\mu\nu}^a[B])^2 + Q_\nu^a D_\mu^{ab}[B]F_{\mu\nu}^b[B] - \tfrac{1}{2}Q_\mu^a K_{\mu\nu}^{ab}Q_\nu^b + \mathcal{M}_3 + \mathcal{M}_4. \tag{7}$$

The second term in the right-hand side (r.h.s.) is linear in the quantum field and disappears if the background fulfills its equation of motion which will hold in our case of a constant background field. The third term is quadratic in Q_μ^a and it defines the "free part" with the kernel

$$K_{\mu\nu}^{ab} = -\delta_{\mu\nu}D_\rho^{ac}[B]D_\rho^{cb}[B] + D_\mu^{ac}[B]D_\nu^{cb}[B] - 2\epsilon^{acb}F_{\mu\nu}^c[B]. \tag{8}$$

The interaction of the quantum field is represented by the vertex factors

$$\begin{aligned}\mathcal{M}_3 &= -\epsilon^{abc}(D_\mu^{ad}Q_\nu^d)Q_\mu^b Q_\nu^c,\\ \mathcal{M}_4 &= -\tfrac{1}{4}Q_\mu^a Q_\nu^a Q_\mu^b Q_\nu^b + \tfrac{1}{4}Q_\mu^a Q_\nu^b Q_\mu^a Q_\nu^b.\end{aligned} \tag{9}$$

The complete Lagrangian

$$\mathcal{L} = -\tfrac{1}{4}(F_{\mu\nu}^a[B+Q])^2 + \mathcal{L}_{\rm gf} + \mathcal{L}_{\rm gh} \tag{10}$$

consists of (7), the gauge fixing term (in the following we put $\xi = 1$)

$$\mathcal{L}_{\rm gf} = -\frac{1}{2\xi}(D^a_\mu[B]Q^a_\mu)^2 = \frac{1}{2\xi}Q^a_\mu D^{ac}_\mu[B]D^{cb}_\nu[B]Q^b_\nu, \tag{11}$$

and the ghost term

$$\mathcal{L}_{\rm gh} = \bar{\eta}^a D^{ac}_\mu[B](D^{cb}_\mu[B] + \epsilon^{cdb}Q^d_\mu)\eta^b. \tag{12}$$

These formulas are valid for an arbitrary background field. Now we turn to the specific background of an Abelian homogeneous magnetic field of strength B which is oriented along the third axis in both color and configuration spaces. An explicit representation of its vector potential is

$$B^a_\mu(x) = \delta^{a3}\delta_{\mu 1}x_2 B \tag{13}$$

and the corresponding field strength is

$$F^a_{ij} = \delta^{a3}F_{ij} = B\epsilon^{3ij}, \tag{14}$$

where only the spatial components ($i, j = 1, 2, 3$) are nonzero. Once the background is chosen to be Abelian it is useful to turn to the so-called charged basis,

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(Q^1_\mu \pm iQ^2_\mu), \qquad Q_\mu = Q^3_\mu, \tag{15}$$

with the interpretation of $W^\pm_\mu$ as color charged fields and Q_μ as a color neutral field. This is in parallel to electrically charged and neutral fields. Note also that Q_μ is real while $W^\pm_\mu$ are complex conjugated one to the other. In the following we will omit the word "color" when speaking about charged and neutral objects. The same transformation is done for the ghosts,

$$\eta^\pm_\mu = \frac{1}{\sqrt{2}}(\eta^1_\mu \pm i\eta^2_\mu), \qquad \eta_\mu = \eta^3_\mu. \tag{16}$$

A summation over the color indices turns into

$$Q^a_\mu Q^a_\nu = Q_\mu Q_\nu + W^+_\mu W^-_\nu + W^-_\mu W^+_\nu. \tag{17}$$

All appearing quantities have to be transformed into that basis. For the covariant derivative we obtain

$$D^{33}_\mu = \partial_\mu, \qquad D^{-+}_\mu = \partial_\mu - iB_\mu \equiv D_\mu, \qquad D^{+-}_\mu = \partial_\mu + iB_\mu \equiv D^*_\mu \tag{18}$$

where D^*_μ is the complex conjugate to D_μ. Starting from here we do not need to indicate the arguments in the covariant derivatives any longer.

Before proceeding with writing down the remaining formulas in the charged basis, it is useful to turn to momentum representation. This can be done in a standard way by the formal rules. It remains to define the signs in the exponential factors. We adopt the notation

$$Q \sim e^{-ikx}, \qquad W^- \sim e^{-ipx}, \qquad W^+ \sim e^{ip'x}. \tag{19}$$

In all the following calculations the momentum k will denote the momentum of a neutral line and the momenta p and p' will denote that of the charged lines whereby k and p are incoming and p' is outgoing. In these notations the covariant derivative D_μ acts on a W^-_μ and turns into

$$D_\mu = -i(i\partial_\mu + B_\mu) \equiv -ip_\mu. \tag{20}$$

Note that the components of the momentum p_μ do not commute,

$$[p_\mu, p_\nu] = iBF_{\mu\nu}. \tag{21}$$

In these notations the quadratic term of the action turns into

$$-\tfrac{1}{2}Q^a_\mu K^{ab}_{\mu\nu}Q^b_\nu = \tfrac{1}{2}Q_\mu K^{33}_{\mu\nu}Q_\nu + \tfrac{1}{2}W^+_\mu K^{-+}_{\mu\nu}W^-_\nu + \tfrac{1}{2}W^-_\mu K^{+-}_{\mu\nu}W^+_\nu \tag{22}$$

with

$$K^{33}_{\mu\nu} \equiv K_{\mu\nu}(k) = \delta_{\mu\nu}k^2 - k_\mu k_\nu \tag{23}$$

and

$$K^{-+}_{\mu\nu} \equiv K_{\mu\nu}(p) = \delta_{\mu\nu}p^2 - p_\mu p_\nu + 2iBF_{\mu\nu}. \tag{24}$$

We use the arguments k and p instead of the indices to indicate to which line $K_{\mu\nu}$ belongs. The third term in the r.h.s. of Eq. (22) is the same as the second one due to the complex conjugation rules. In the Feynman rules $(K^{33}_{\mu\nu})^{-1}$ is the line for neutral gluons and is denoted by a wavy line and $(K^{-+}_{\mu\nu})^{-1}$ is the line for charged gluons and is denoted by a directed solid line. We remark that these lines represent propagators in the background of the magnetic field. Frequently they are denoted by thick or double lines. Because we have in this paper no other lines the notation with ordinary (thin) lines is unique.

Here we note that the spectrum of the operator (24) in a constant magnetic field,

$$E^2_n = p^2_3 + B(2n + 1), \qquad n = -1, 0, 1, \ldots, \tag{25}$$

where p_3 is the momentum along the field direction $B = B_3$, contains the well-known tachyonic mode at $n = -1$ (see [13,14]). This state is a peculiarity of non-Abelian gauge fields as it is discussed in different aspects in the literature (see, for instance, Refs. [15–19] and references therein).

For later use we introduce here the set of eigenstates for the operator (23). For the color neutral states we take exactly the same polarizations $|k, s\rangle$ as known from electrodynamics,

$$|k,1\rangle_\mu = \frac{1}{h}\begin{pmatrix} -k_2 \\ k_1 \\ 0 \\ 0 \end{pmatrix}_\mu, \qquad |k,2\rangle_\mu = \frac{1}{kh}\begin{pmatrix} k_1k_3 \\ k_2k_3 \\ -h^2 \\ 0 \end{pmatrix}_\mu,$$
$$|k,3\rangle_\mu = \frac{1}{k}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ 0 \end{pmatrix}_\mu, \qquad |k,4\rangle_\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}_\mu \tag{26}$$

with $h = \sqrt{k_1^2 + k_2^2}$, $k = \sqrt{k_1^2 + k_2^2 + k_3^2}$. Here the polarizations $s = 1, 2$ describe the two transversal gluons ($k_\mu |k, s = 1, 2\rangle_\mu = 0$), $s = 3$ is the longitudinal one and $s = 4$, after rotation into Minkowski space, becomes the timelike one. For the transversal gluons

$$K_{\mu\nu}(k)|k, s = 1, 2\rangle_\nu = (k_4^2 + k^2)|k, s = 1, 2\rangle_\mu \tag{27}$$

holds.

After discussing the free part of the Lagrangian (7) and (10), in the "charged basis" we note that for the vertex factor $\mathcal{M}_3$ in (9) we obtain

$$\mathcal{M}_3 = W_\mu^- \Gamma_{\mu\nu\lambda} W_\nu^+ Q_\lambda \tag{28}$$

with

$$\Gamma_{\mu\nu\lambda} = \delta_{\mu\nu}(k - 2p)_\lambda + \delta_{\lambda\mu}(p + k)_\nu + \delta_{\lambda\nu}(p - 2k)_\mu. \tag{29}$$

The notations are shown in Fig. 1. It should be remarked that all graphs and combinatorial factors are exactly the same as in the well-known case without a magnetic field. On this level the only difference is in the meaning of the momentum p_μ which in our case depends on the background magnetic field; see Eq. (20).

The vertex factor describing the interaction of the neutral gluons with charged ghost fields is

$$\mathcal{M}_3^{\text{ghost}} = \eta^- \Gamma_\lambda^{q\eta} \eta^+ Q_\lambda \tag{30}$$

where

$$\Gamma_\lambda^{q\eta} = p'_\lambda = (p + k)_\lambda. \tag{31}$$

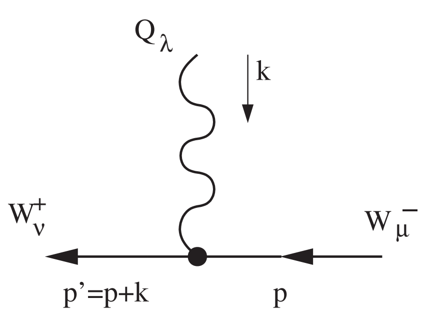

FIG. 1. Notations for the vertex $\mathcal{M}_3$.

We also need the four-particle vertices which are momentum independent and have the same form as at zero external field.

III. OPERATOR STRUCTURES

The neutral polarization tensor (PT) is denoted by $\Pi_{\lambda\lambda'}(k)$ where the argument k is an ordinary momentum. Its one-loop diagram representation is shown in Fig. 2. In this section we discuss its general tensor structure at zero and finite temperature. This was considered in Refs. [20,21]. As it was shown in Refs. [10,11] it is not transverse in a magnetic background field. This means that the condition $k_\lambda \Pi_{\lambda\lambda'}(k) = 0$ does not hold. This follows either from the Slavnov-Taylor identity for the gluon Green function or from an explicit one-loop calculation. So, we are left with the weaker condition

$$k_\lambda \Pi_{\lambda\lambda'}(k) k_{\lambda'} = 0. \tag{32}$$

At zero temperature, it can be combined with the remaining Lorentz symmetry which results in a dependence of $\Pi_{\lambda\lambda'}(k)$ on two vectors, l_λ and h_λ, and on the magnetic field.

We use the notation

$$l_\mu = \begin{pmatrix} 0 \\ 0 \\ k_3 \\ k_4 \end{pmatrix}, \qquad h_\mu = \begin{pmatrix} k_1 \\ k_2 \\ 0 \\ 0 \end{pmatrix}, \qquad d_\mu = \begin{pmatrix} k_2 \\ -k_1 \\ 0 \\ 0 \end{pmatrix},$$
$$F_{\mu\lambda} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{33}$$

The third vector is $d_\mu \equiv F_{\mu\nu}k_\nu$. From here and until Sec. VI we put the magnetic field strength B equal to unity. For the vectors $k_\lambda = l_\lambda + h_\lambda$ holds.

The general structure of $\Pi_{\lambda\lambda'}(k)$ at $T = 0$ allowed by (32) and the vectors l_λ and h_λ is determined by the set of tensor structures

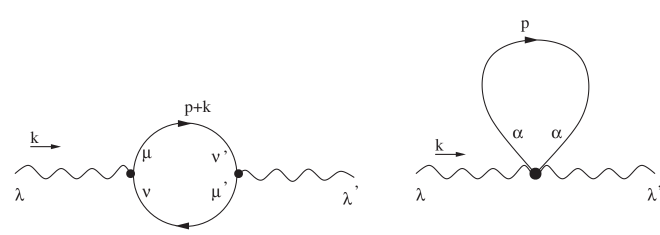

FIG. 2. The neutral polarization tensor.

$$\begin{aligned} T^{(1)}_{\lambda\lambda'} &= l^2\delta^{\parallel}_{\lambda\lambda'} - l_\lambda l_{\lambda'},\\ T^{(2)}_{\lambda\lambda'} &= h^2\delta^{\perp}_{\lambda\lambda'} - h_\lambda h_{\lambda'} = d_\lambda d_{\lambda'},\\ T^{(3)}_{\lambda\lambda'} &= h^2\delta^{\parallel}_{\lambda\lambda'} + l^2\delta^{\perp}_{\lambda\lambda'} - l_\lambda h_{\lambda'} - h_\lambda l_{\lambda'},\\ T^{(4)}_{\lambda\lambda'} &= i(l_\lambda d_{\lambda'} - d_\lambda l_{\lambda'}) + il^2 F_{\lambda\lambda'},\\ T^{(5)}_{\lambda\lambda'} &= h^2\delta^{\parallel}_{\lambda\lambda'} - l^2\delta^{\perp}_{\lambda\lambda'}, \qquad T^{(6)}_{\lambda\lambda'} = iF_{\lambda\lambda'} \end{aligned} \tag{34}$$

together with the identity $d_\lambda h_{\lambda'} - h_\lambda d_{\lambda'} = h^2 F_{\lambda\lambda'}$. Further we introduced the notations $\delta^{\perp}_{\mu\lambda} = \mathrm{diag}(1, 1, 0, 0)$ and $\delta^{\parallel}_{\mu\lambda} = \mathrm{diag}(0, 0, 1, 1)$. The first four operators are transversal, $k_\lambda T^{(i)}_{\lambda\lambda'} = T^{(i)}_{\lambda\lambda'} k_{\lambda'} = 0$ with $i = 1, 2, 3, 4$; the last two fulfill (32) only. These structures are composed in a simple way (proposed first, to our knowledge, in QED in Ref. [20]) by writing down all possible linear combinations from all structures which can be composed out of the vectors l_λ, h_λ, and d_λ and out of the matrices $\delta^{\parallel}_{\lambda\lambda'}$, $\delta^{\perp}_{\lambda\lambda'}$, and $F_{\lambda\lambda'}$. Note that higher powers of $F_{\mu\nu}$ are not independent, $F_{\mu\lambda}F_{\lambda\nu} = -\delta^{\perp}_{\mu\nu}$ and so on. By imposing the condition (32) and requiring Hermiticity, six independent linear combinations remain. The first three are chosen in such a way that their sum is just the transversal part of the kernel of the quadratic part of the action, Eq. (23),

$$T^{(1)}_{\lambda\lambda'} + T^{(2)}_{\lambda\lambda'} + T^{(3)}_{\lambda\lambda'} = K_{\lambda\lambda'}(k). \tag{35}$$

At finite temperature, an additional vector u_μ—the velocity of the thermostat—must be accounted for and used in the construction of the tensors $T^{(i)}$. Therefore new tensor structures appear. We chose them in the form

$$\begin{aligned} T^{(7)}_{\lambda\lambda'} &= (u_\lambda l_{\lambda'} + l_\lambda u_{\lambda'})(uk) - \delta^{\parallel}_{\lambda\lambda'}(uk)^2 - u_\lambda u_{\lambda'} l^2,\\ T^{(8)}_{\lambda\lambda'} &= (u_\lambda h_{\lambda'} + h_\lambda u_{\lambda'})(uk) - \delta^{\perp}_{\lambda\lambda'}(uk)^2 - u_\lambda u_{\lambda'} h^2,\\ T^{(9)}_{\lambda\lambda'} &= u_\lambda i d_{\lambda'} - i d_\lambda u_{\lambda'} + iF_{\lambda\lambda'}(uk),\\ T^{(10)}_{\lambda\lambda'} &= k^2\delta_{\lambda\lambda'} - \frac{u_\lambda u_{\lambda'}(k^2)^2}{(uk)^2}. \end{aligned} \tag{36}$$

Obviously the sum

$$\begin{aligned} T^{(7)}_{\lambda\lambda'} + T^{(8)}_{\lambda\lambda'} &= (u_\lambda k_{\lambda'} + k_\lambda u_{\lambda'})(uk) - \delta_{\lambda\lambda'}(uk)^2 - u_\lambda u_{\lambda'} k^2\\ &= B_{\lambda\lambda'} \end{aligned} \tag{37}$$

is equal to one of two transversal tensor structures commonly used at zero field [22]. In that case the first structure is given by the sum of tensors, Eqs. (23). Below, in actual calculations, we use the reference frame of the thermostat, so only one component of u_μ is nonzero: $u_\mu = (0, 0, 0, 1)_\mu$. We mention that the first two tensors in Eq. (36) are transversal and the other two satisfy the weaker condition (32).

We adopt the following way for the representation of our expressions. The dimensionality of the polarization tensor is implemented in the tensors $T^{(i)}$. To restore the dimensionality for the tensors in Eqs. (34) and (36), one has to multiply the operator $T^{(6)}$ by the factor B, and the operator $T^{(9)}$ by $\sqrt{B}$. The form factors are dimensionless functions of dimensionless momenta l^2, h^2, and temperature T. This means that, in fact, they are measured in units of B. To restore the correct dimensionality one has to replace $l^2 \to l^2/B$, $h^2 \to h^2/B$, and $T \to T/\sqrt{B}$. Correspondingly, the arguments of all functions appearing in the actual calculations are also dimensionless.

Knowing the operators (34) and (36), which may appear at zero and finite temperature, the polarization tensor can be represented in the form

$$\Pi_{\lambda\lambda'}(k) = \sum_{i=1}^{10} \Pi^{(i)}(k) T^{(i)}_{\lambda\lambda'}, \tag{38}$$

where the form factors $\Pi^{(i)}(k)$ depend on the external momentum k_μ through the variables l^2 and h^2 at zero temperature, and h^2, k_4, and k_3 at finite temperature. In the former case, the polarization tensor $\Pi_{\lambda\lambda'}(k)$ is real and symmetric in its indices, so the form factors $\Pi^{(4)}(k)$ and $\Pi^{(6)}(k)$ are zero. In the latter case all structures will contribute, in general.

At finite temperature, it makes sense to add to the set of tensors, Eqs. (34) and (36), which is complete, a structure of a special type,

$$u_\mu u_\nu \phi(k_4), \tag{39}$$

where the function $\phi(k_4)$ is nonzero for $k_4 = 0$ and $T \neq 0$ only (remember, k_4 takes discrete values). This term is obviously transversal on its own and it is a linear combination of $T^{(7)}_{\lambda\lambda'}$ and $T^{(8)}_{\lambda\lambda'}$ (at $k_4 = 0$). The reason to introduce this as an extra term is that it represents just the Debye mass; see Sec. VII.

IV. CALCULATION OF THE NEUTRAL POLARIZATION TENSOR

In this section we calculate the one-loop contribution to the PT at finite temperature. The imaginary time formalism is used. This means that in loops the integration over the momentum component p_4 is substituted by an infinite series in discrete values $p_4 = 2\pi NT$: $\int_{-\infty}^{+\infty}\frac{dp_4}{2\pi} f(p_4) \to T\sum_{N=-\infty}^{+\infty} f(2\pi NT)$.

The neutral polarization tensor has the following representation in momentum space (see Fig. 2):

$$\Pi_{\lambda\lambda'}(k) = T\sum_{N=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{d^3p}{(2\pi)^3}\Pi(p, p_4, k, k_4)_{\lambda\lambda'}, \tag{40}$$

where in the integrand we noted explicitly the dependence on the external momentum and the momentum inside loops. In what follows, as in Ref. [10], at the intermediate stage of computation we, for brevity, shall omit the general factors and the signs of integration and summation. That is, we relate these factors and operations with the momentum p appearing in the r.h.s. of the equations. Within this convention we write

$$\Pi_{\lambda\lambda'}(k) = \Gamma_{\mu\nu\lambda}G_{\mu\mu'}(p)\Gamma_{\mu'\nu'\lambda'}G_{\nu'\nu}(p-k) - p_\lambda G(p)(p-k)_{\lambda'}G(p-k) - (p-k)_\lambda G(p)p_{\lambda'}G(p-k) + G_{\lambda\lambda'}(p) + G_{\lambda'\lambda}(p) - 2\delta_{\lambda\lambda'}\,\mathrm{tr}G(p). \quad (41)$$

The second line gives the contribution from the ghost loops and the third and fourth ones are due to the tadpole diagram.

The vertex factor is given in Eq. (29). For a convenient grouping of terms it is useful to rearrange it,

$$\Gamma_{\mu\nu\lambda} = \underbrace{g_{\mu\nu}(k-2p)_\lambda}_{} + \underbrace{2(g_{\lambda\mu}k_\nu - g_{\lambda\nu}k_\mu)}_{} + \underbrace{g_{\lambda\mu}(p-k)_\nu + g_{\lambda\nu}p_\mu}_{} \equiv \Gamma^{(1)}_{\mu\nu\lambda} + \Gamma^{(2)}_{\mu\nu\lambda} + \Gamma^{(3)}_{\mu\nu\lambda}, \quad (42)$$

where in the last line a subdivision into three parts is done.

The propagators are given by

$$G(p) = \frac{1}{p^2} = \int_0^\infty ds e^{-sp^2}, \qquad G(p-k) = \frac{1}{(p-k)^2} = \int_0^\infty dt e^{-t(p-k)^2} \quad (43)$$

for the scalar lines and by

$$G_{\lambda\lambda'}(p) = \left(\frac{1}{p^2 + 2iF}\right)_{\lambda\lambda'} = \int_0^\infty ds e^{-sp^2}E^{-s}_{\lambda\lambda'}, \qquad G_{\lambda\lambda'}(p-k) = \left(\frac{1}{(p-k)^2 + 2iF}\right)_{\lambda\lambda'} = \int_0^\infty dt e^{-t(p-k)^2}E^{-t}_{\lambda\lambda'} \quad (44)$$

for the vector lines (in the Feynman gauge, $\xi = 1$) with

$$E^s_{\lambda\lambda'} \equiv (e^{2isF})_{\lambda\lambda'} = \delta^{\|}_{\lambda\lambda'} + iF_{\lambda\lambda'}\sinh(2s) + \delta^{\perp}_{\lambda\lambda'}\cosh(2s). \quad (45)$$

At zero temperature, the momentum integration can be carried out by means of Schwinger's algebraic procedure [12] and converted into an integration over two scalar parameters, s and t. Here we educe the known results in order to present their modifications at $T \neq 0$. The basic exponential is

$$\Theta = e^{-sp^2}e^{-t(p-k)^2} \quad (46)$$

and the integration over the momentum p is denoted by the average $\langle\ldots\rangle$. The following formulas hold:

$$\langle\Theta\rangle = \frac{\exp[-k(\frac{st}{s+t}\delta^{\|} + \frac{ST}{S+T}\delta^{\perp})k]}{(4\pi)^2(s+t)\sinh(s+t)}, \quad (47)$$

with $S = \tanh(s)$ and $T = \tanh(t)$, and

$$\langle p_\mu\Theta\rangle = \left(\frac{A}{D}k\right)_\mu\langle\Theta\rangle, \quad (48)$$

$$\langle p_\mu p_\nu\Theta\rangle = \left(\left(\frac{A}{D}k\right)_\mu\left(\frac{A}{D}k\right)_\nu - i\left(\frac{F}{D^\top}\right)_{\mu\nu}\right)\langle\Theta\rangle. \quad (49)$$

The notation $A \equiv E^t - 1$ and $D \equiv E^{s+t} - 1$ is used. Explicit formulas are

$$\frac{A}{D} = \delta^{\|}\frac{t}{s+t} - iF\frac{\sinh(s)\sinh(t)}{\sinh(s+t)} + \delta^{\perp}\frac{\cosh(s)\sinh(t)}{\sinh(s+t)} \quad (50)$$

along with

$$\frac{-2iFE^{-s}}{D^\top} = \frac{\delta_{\|}}{s+t} - iF\frac{\sinh(s-t)}{\sinh(s+t)} + \delta_{\perp}\frac{\cosh(s-t)}{\sinh(s+t)}, \quad (51)$$

where we dropped the indices. It should be remarked that all these matrices, i.e., E, F, D, and A, commute. In addition we need the relation

$$p(s)_\mu \equiv e^{-sp^2}p_\mu e^{sp^2} = E^s{}_{\mu\nu}p_\nu$$

for commuting a factor p_μ with the propagator $G(p)$,

$$p_\mu G_{\mu\mu'}(p) = G(p)p_{\mu'}. \quad (52)$$

Now we turn to the finite temperature case. Our goal is to account for the temperature dependence within the above representation in a natural way. Usually in the imaginary time formalism the summation over p_4 and the integration over three-momenta are carried out separately. To restore the equivalence of these variables and to make use of the formulas (47)–(49) we proceed in the following way. First we note that any function $f(p_4 = 2\pi NT)$ of p_4 can be written in the form

$$f(p_4 = 2\pi NT) = \int dp_4 f(p_4)\delta(p_4 - 2\pi NT) = \int dp_4 f(p_4)\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\lambda e^{i\lambda(p_4 - 2\pi NT)}, \quad (53)$$

where $\delta(x)$ is Dirac's delta function. Then we change the order of integration in variables λ and p_4 and fulfill the momentum integration as at zero temperature. The factor $1/(2\pi)$ coming from the delta function and the factor $1/(2\pi)^3$ coming from the three-momentum integration combine into the factor $1/(2\pi)^4$ appearing at zero temperature. The only new factor, $e^{i\lambda p_4}$, appears as another exponential in Eq. (46). Since the fourth component is not related to the magnetic field, the calculation is actually the same as at zero temperature. Remember that in the course of this calculation the factor $1/(2\pi)^4$ results in the factor $1/(4\pi)^2$ in the function $\langle\Theta(s,t)\rangle$ [12]. The integral over p_4 is Gaussian, and we obtain for the algebraic averaging (or bracket) procedure $\langle\ldots\rangle$ at $\lambda \neq 0$,

$$\langle\Theta(s,t,\lambda)\rangle_\lambda = \langle e^{i\lambda p_4}\Theta(s,t)\rangle = \langle\Theta(s,t)\rangle \exp\left(-\frac{\lambda^2}{4q} + i\lambda k_4 \frac{t}{q}\right), \quad (54)$$

where we marked the averaging procedure with λ dependence by the subscript λ. Here k_4 is the discrete fourth component of the external momentum and we introduced $q = s + t$ as a convenient variable. The expression in the angle brackets on the r.h.s., $\langle\Theta(s,t)\rangle$, is the zero temperature value given in Eq. (47). Below we will denote the function (54) by $\Theta(s,t,\lambda)$.

In the next step, we integrate the function $\Theta(s,t,\lambda)$ over λ and after the restoration of the sum over N we obtain the basic expression at finite temperature,

$$\begin{aligned}\langle\Theta(s,t)\rangle_T &= T\sum_{N=-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\lambda\,\Theta(s,t,\lambda)e^{-i\lambda(2\pi NT)} \\ &= T\sum_{N=-\infty}^{+\infty}\sqrt{4\pi q}\langle\Theta(s,t)\rangle \\ &\quad\times\exp\left(-k_4^2\frac{t^2}{q} + 2k_4 t(2\pi NT) - (2\pi NT)^2 q\right) \\ &\equiv \sum_{N=-\infty}^{+\infty}\Xi_T(s,t),\end{aligned} \quad (55)$$

where we introduced the notation $\Xi_T(s,t)$.

To obtain the result of the average procedure with the momentum p_4 entering, one has to differentiate Eq. (54) with respect to $i\lambda$ and then calculate the integral over λ. This can be done also by means of differentiation of Eq. (55) with respect to the parameter $b_N \equiv 2\pi NT$. In this way all integrals of interest can be computed.

The expressions with the spatial indices $i, j = 1, 2, 3$ are given by Eqs. (47)–(49), where the function (55) must be substituted. The average with one spatial component and p_4 is

$$\begin{aligned}\langle p_i p_4\Theta(s,t)\rangle_T &= T\sum_{N=-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\lambda e^{-i\lambda(2\pi NT)}\frac{k_4 + i\lambda/2}{q} \\ &\quad\times\left(\frac{A}{D}k\right)_i\Theta(s,t,\lambda) \\ &= \sum_{N=-\infty}^{+\infty}\left(\frac{A}{D}k\right)_i 2\pi NT\Theta(s,t,\lambda).\end{aligned} \quad (56)$$

The average procedure with p_4^2 results in the following expression:

$$\begin{aligned}\langle p_4^2\Theta(s,t)\rangle_T &= T\sum_{N=-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\lambda e^{-i\lambda(2\pi NT)} \\ &\quad\times\left(\left(\frac{k_4 + i\lambda/2}{q}\right)^2 + \frac{1}{2q}\right)\Theta(s,t,\lambda) \\ &= \sum_{N=-\infty}^{+\infty}\Xi_T(s,t)(2\pi NT)^2.\end{aligned} \quad (57)$$

Note as an interesting fact that the k_4 dependence in the last two equations comes in only through the exponential.

Thus, we collected the necessary integrals over the internal loop momentum which appear in the magnetic background field at finite temperature. These expressions are useful when the high temperature limit is investigated because, due to the exponential factor in Eq. (55), a few first terms in the series ($N = 0, 1, 2, \ldots$) give the leading contributions at $T \to \infty$. The factor $\sqrt{q}$ entering the integrals over the s and t parameters ensures the convergence at $s, t \to 0$. This corresponds to the superficial divergence degree. In the four dimensional theory the form factors have zero superficial divergence degree, hence the three dimensional theory which effectively appears in the high temperature expansion is ultraviolet finite. The convergence at infinity is due to the exponential factors except for the complications resulting from the tachyonic mode which will be discussed later. However, the low temperature limit is less trivial.

Now we consider another representation for the integrals, which is convenient at low temperature, and carry out the ultraviolet renormalization. For this purpose we make resummations of the form

$$\sum_{N=-\infty}^{+\infty} e^{(-zN^2+aN)} = \sqrt{\frac{\pi}{z}}\sum_{N=-\infty}^{+\infty} e^{(a+2\pi iN)^2/4z} \quad (58)$$

of the series in N in Eq. (55) with the parameters $a = 4\pi T k_4 t$ and $z = 4\pi^2 T^2 q$ chosen. In this case the dependence on k_4^2 in the exponential disappears and we obtain for $\langle\Theta(s,t)\rangle_T$

$$\begin{aligned}\langle\Theta(s,t)\rangle_T &= \sum_{N=-\infty}^{+\infty}\langle\Theta(s,t)\rangle\exp\left(-\frac{N^2}{4T^2 q} + i\frac{k_4 tN}{qT}\right) \\ &= \sum_{N=-\infty}^{+\infty}\Theta_T(s,t).\end{aligned} \quad (59)$$

We introduced the notation $\Theta_T(s,t)$ which is the basic function appearing in all form factors below.

In order to obtain the functions $\langle\Theta(s,t)p_4\rangle_T$ and $\langle\Theta(s,t)p_4^2\rangle_T$ we have to calculate the integrals over the variable λ with the powers of λ and λ^2 in Eqs. (56) and (57) by means of differentiation of the expression (59) with respect to the parameter $b_k = ik_4t/q$ 1 and 2 times, correspondingly. In this way we obtain, in the first case,

$$\langle p_4\Theta(s,t)\rangle_T = \sum_{N=-\infty}^{+\infty}\left(k_4\frac{t}{q}+\frac{iN}{2qT}\right)\langle\Theta(s,t)\rangle \times \exp\left(-\frac{N^2}{4qT^2}+\frac{ik_4Nt}{qT}\right). \tag{60}$$

This expression can be combined with the bracket operation for the spatial momenta p_i. In fact, the function $(\frac{A}{D}k)_\mu$ for $\mu = 4$ is equal to $(t/q)k_4$ which coincides with the first term in the brackets in the above equation. To account for the second term, we assume that the thermostat is at rest and therefore the vector u_μ has only one nonzero component, $u_\mu = \delta_{\mu 4}$. To write down Eq. (60) in short, we introduce the notation $\tilde{u}_\mu = \frac{iN}{2qT}u_\mu$. Then, the average operation with one internal momentum yields

$$\langle p_\mu\Theta(s,t)\rangle_T = \sum_{N=-\infty}^{+\infty}\left(\left(\frac{A}{D}k\right)_\mu + \tilde{u}_\mu\right)\Theta_T(s,t). \tag{61}$$

In the same manner, in the case of two internal momenta the result can be presented in the form

$$\langle p_\mu p_\nu\Theta(s,t)\rangle_T = \sum_{N=-\infty}^{+\infty}\left[\left(\left(\frac{A}{D}k\right)_\mu + \tilde{u}_\mu\right)\left(\left(\frac{A}{D}k\right)_\nu + \tilde{u}_\nu\right) - i\left(\frac{F}{D^\top}\right)_{\mu\nu}\right]\Theta_T(s,t). \tag{62}$$

Thus, we have calculated all integrals needed for what follows.

The expressions (47)–(49), (59), (61), and (62) solve the problem of the momentum integration in the Schwinger operator formalism at finite temperature. The results are expressed in terms of double-parametric integrals, like at zero temperature, and a sum over N which takes into account the temperature dependence.

So, all further steps of calculations necessary to obtain the expression for the polarization tensor actually coincide with those in Ref. [10]. Within this formalism, the polarization tensor becomes an expression of the type

$$\Pi_{\lambda\lambda'}(k) = \int_0^\infty\int_0^\infty dsdt\langle M_{\lambda\lambda'}(p,k)\Theta\rangle_T, \tag{63}$$

where in $M_{\lambda\lambda'}(p,k)$ we collected all factors appearing from the vertices and from the lines except for those which go into Θ. By using (61) and (62) the average (the momentum integration over p) can be transformed into

$$\langle M_{\lambda\lambda'}(p,k)\Theta\rangle_T = M_{\lambda\lambda'}(s,t)\Theta_T, \tag{64}$$

where now $M_{\lambda\lambda'}(s,t)$ collects all factors except for Θ_T.

As in Ref. [10], we break the whole polarization tensor into parts according to the division introduced in Eq. (42). We write

$$\Pi_{\lambda\lambda'}(k) = \sum_{i,j}\Pi^{ij}_{\lambda\lambda'}(k) + \Pi^{\text{ghost}}_{\lambda\lambda'}(k) + \Pi_{\lambda\lambda'}(k)^{\text{tadpole}} \tag{65}$$

with $i, j = 1, 2, 3$ and

$$\Pi^{ij}_{\lambda\lambda'}(k) = \Gamma^{(i)}_{\mu\nu\lambda}G_{\mu\mu'}(p)\Gamma^{(j)}_{\mu'\nu'\lambda}G_{\nu'\nu}(p-k) \tag{66}$$

including corresponding subdivisions of M.

To organize our calculation, we remind the reader that the PT is calculated in terms of double-parametric integrals in s and t. The function $\langle\Theta(s,t)\rangle$ (47) is symmetric whereas $\Theta_T(s,t)$ is not, because of the factor $e^{ik_4Nt/qT} = e^{2(\tilde{u}k)t}$ in the temperature dependent exponential. To restore the $s \leftrightarrow t$ symmetry, we write

$$e^{2(\tilde{u}k)t} = S_T + A_T \tag{67}$$

where

$$S_T = \tfrac{1}{2}(e^{(2\tilde{u}k)t} + e^{2(\tilde{u}k)s}), \qquad A_T = \tfrac{1}{2}(e^{2(\tilde{u}k)t} - e^{2(\tilde{u}k)s}). \tag{68}$$

Then the function $\Theta_T(s,t)$ can be split into symmetric and antisymmetric parts, $\Theta_T = \Theta^s_T + \Theta^a_T$, with

$$\Theta^s_T = \langle\Theta(s,t)\rangle S_T e^{-(N^2/4T^2q)}, \quad \Theta^a_T = \langle\Theta(s,t)\rangle A_T e^{-(N^2/4T^2q)}. \tag{69}$$

With these definitions introduced, the terms entering the integral in Eq. (63) have the following general structure. The expressions in $M_{\lambda\lambda'}(s,t)$ which are symmetric under $s \leftrightarrow t$ appear multiplied by the symmetric temperature factor Θ^s_T, and those which are antisymmetric appear multiplied by Θ^a_T. This observation gives us the possibility to make use of the results in Ref. [10], where the symmetric form factors for the operators $T^{(1)}$–$T^{(4)}$ were calculated. At finite temperature, they should be multiplied by the factor Θ^s_T in the total expression. The antisymmetric terms as well as the terms giving contributions to the remaining form factors must be calculated separately.

In fact, this procedure is rather simple and it is described in the previous work. To make the present paper self-contained, we educe the details of this calculation in the Appendix. In the main text, we restrict ourselves to the description of it in general words. For the first six operators, i.e., for those which do not contain the vectors u_μ, the terms appearing after the momentum integrations have the following forms. First,

$$\Pi_{\lambda\lambda'} = P_\lambda P^\top_{\lambda'} + a\delta^{\parallel}_{\lambda\lambda'} + b\delta^{\perp}_{\lambda\lambda'} + icF_{\lambda\lambda'}, \tag{70}$$

where P_λ is given in terms of the vectors (33),

$$P_\lambda = rl_\lambda + \alpha i d_\lambda + \beta h_\lambda, \tag{71}$$

and $P^\top_{\lambda'}$ is the transposed expression; r, α, β, a, b, c are some functions specific for different parts of $\Pi_{\lambda\lambda'}$.

The second type of expression has a slightly more complicated form,

$$\Pi_{\lambda\lambda'} = P_\lambda Q_{\lambda'} + Q^\top_\lambda P^\top_{\lambda'} + a\delta^{\parallel}_{\lambda\lambda'} + b\delta^{\perp}_{\lambda\lambda'} + icF_{\lambda\lambda'}, \tag{72}$$

with P_λ from (71) and

$$Q_\lambda = sl_\lambda + \gamma i d_\lambda + \delta h_\lambda, \tag{73}$$

where s, γ, δ are some other functions. Then, from the requirement that the weak transversality condition (32) holds, one can derive the expressions standing in front of the operators $T^{(1)}$–$T^{(6)}$. This procedure is efficient but, of course, not mandatory.

The form factors standing in front of the operators $T^{(7)}$–$T^{(10)}$ can be obtained after integration in accordance with Eqs. (61) and (62) as that part of the expressions which is proportional to one or to two powers of u_μ. To make clear the structure of the expressions which appear after the averaging procedure, we note that it results in the formal substitution of p_μ in the initial expressions by $p_\mu \to \tilde{p}_\mu + \tilde{u}_\mu = [(\frac{A}{D}k) + \tilde{u}]_\mu$ in the final ones. So, the u-dependent parts are easily determined. The necessary details for different parts of the polarization tensor are given in the next section.

We complete this section with the description of the renormalization procedure adopted. As it is well known, the divergent parts of the polarization tensor do not depend on temperature or field. So, each form factor can be written as follows:

$$\Pi^{(i)}(B,T) = [\Pi^{(i)}(B,T) - \Pi^{(i)}(B=0,T=0)] + \Pi^{(i)}(B=0,T=0). \tag{74}$$

Then, the expression in the brackets is finite whereas the last term is divergent and must be renormalized by the standard procedure in quantum field theory. In terms of the resummed series (59), (61), and (62) the term with $N=0$ just corresponds to the zero temperature case. So, the above procedure actually has relevance to these terms only. As a result, we obtain the renormalized polarization tensor at finite temperature in the presence of the field which is the object of interest.

V. CALCULATION OF THE FORM FACTORS

In this section we calculate the contributions to the form factors stemming from individual terms $\Gamma^{(i)}_{\mu\nu\lambda}$ ($i = 1, 2, 3$) introduced in Eq. (42). This was done in Ref. [10] at zero temperature. So, here we present mainly those parts of the calculations having relevance to the temperature dependence of these expressions. The first is

$$\Pi^{11}_{\lambda\lambda'} = (k-2p)_\lambda G_{\mu\mu'}(p)(k-2p)_{\lambda'} G_{\mu'\mu}(p-k) \tag{75}$$

and it transforms into

$$\begin{aligned} M^{11}_{\lambda\lambda'}(s,t)\Theta_T &= \langle (k-2p)_\lambda (k-2p(s))_{\lambda'} E^{-s}_{\mu\mu'} E^{-t}_{\mu'\mu}\Theta\rangle_T \\ &= \left[\left(\left(1-2\frac{A}{D}\right)k - 2\tilde{u}\right)_\lambda \left(\left(1-2E^s\frac{A}{D}\right)k - 2\tilde{u}\right)_{\lambda'} - 4i\left(\frac{E^{-s}F}{D^\top}\right)_{\lambda\lambda'} \mathrm{tr} E^{-s-t}\right]\Theta_T. \end{aligned} \tag{76}$$

We note the property $E^s\frac{A}{D} = (\frac{A}{D})^\top$. The trace is

$$\mathrm{tr}\, E^{-s-t} = 2[1+\cosh(2q)]. \tag{77}$$

Remember the variable $q = s+t$.

Here we also introduce the variable $\xi = s-t$ which is antisymmetric with respect to the replacement $s \leftrightarrow t$ and assume s and t to be replaced, $s = (q+\xi)/2$ and $t = (q-\xi)/2$.

Using the notation of Eq. (70) we define

$$\begin{aligned} P_\lambda &= \left(\left(1-2\frac{A}{D}\right)k\right)_\lambda \\ &= \frac{\xi}{q} l_\lambda + 2id_\lambda \frac{\sinh(s)\sinh(t)}{\sinh(q)} + h_\lambda \frac{\sinh(\xi)}{\sinh(q)} \\ &\equiv rl_\lambda + i\alpha d_\lambda + \beta h_\lambda \end{aligned} \tag{78}$$

and below we will express the obtained results for different form factors in terms of r, α, β.

The second part in the expression (76) has to be integrated by parts. This procedure at finite temperature requires a special explanation. We represent

$$\frac{-2iFE^{-s}}{D^\top} = \left(\frac{\partial}{\partial\xi}\right)\left[r\delta^\parallel - iF\frac{\cosh(\xi)}{\sinh(q)} + \delta^\perp\beta + C\right], \tag{79}$$

where C is a constant, i.e., it must be independent on ξ. We include it in the integration procedure in order to make use of this parameter (see below). The derivative with respect to ξ will be integrated by parts. We should note that expressions which are symmetric under an exchange of s and t, i.e., which depend on q only, are not affected in Eq. (79). So, in Θ_T we have to differentiate only the terms which depend on ξ,

$$\frac{\partial}{\partial\xi}\Theta_T = \left(\frac{1}{2}B_1 - 2(\tilde{u}k)\right)\Theta_T, \tag{80}$$

with the notation

$$B_1 \equiv rl^2 + \beta h^2. \tag{81}$$

Then, after integration by parts, the last term in Eq. (76) gives the contribution

$$-4i\left(\frac{E^{-s}F}{D^\top}\right)_{\lambda\lambda'} = 2(1+\cosh(2q))(2(\tilde{u}k) - B_1) \times \left[r\delta^\parallel - iF\frac{\cosh(\xi)}{\sinh(q)} + \delta^\perp\beta + C\right]\Theta_T, \tag{82}$$

which has to be added to the first part of the equation. It contains a term $4\tilde{u}_\mu\tilde{u}_\nu\Theta_T$ which is the quadratic in $\tilde{u}_\mu$. Now we choose the constant $C = -2\frac{\tilde{u}_\mu\tilde{u}_\nu}{(\tilde{u}k)}$ in a way that this term cancels and considerable simplifications in further calculations appear. In this way we arrive at the final expression at finite temperature. As concerns the surface term, it is canceled by the contribution of the tadpole diagram (see below).

Applying formula (A3) from the Appendix for the function (70) with the parameters $a = rB_1$ and $b = \beta B_1$, giving the contributions to the first six form factors, and gathering the factors at the u-dependent structures giving rise to the operators $T^{(7)}$–$T^{(10)}$, we obtain for $M^{11}_{\lambda\lambda'}$

$$M^{11}(s,t) = \{-r^2T^{(1)} + (\alpha^2 - \beta^2)T^{(2)} - r\beta T^{(3)} - \alpha r T^{(4)} + [B_1 \coth(q) - 2(\tilde{u}k)]T^{(6)} + 2(\tilde{u}k)[-rT^{(7)} - \beta T^{(8)} + \alpha T^{(9)}]\} 2(1 + \cosh(2q)). \quad (83)$$

This rather simple expression includes symmetric and antisymmetric terms with respect to $s \leftrightarrow t$. Since the integral in s, t is symmetric, actually, the factor Θ^s_T is present in the former terms, and Θ^a_T at the latter ones. This remark concerns all the expressions written below for other parts of $\Pi_{\lambda\lambda'}$.

The next contribution is M^{22}. From (42) we get

$$\Pi^{22}_{\lambda\lambda'} = 4(\delta_{\lambda\mu}k_\nu - \delta_{\lambda\nu}k_\mu)G_{\mu\mu'}(p)(\delta_{\lambda'\mu'}k_{\nu'} - \delta_{\lambda'\nu'}k_{\mu'})G_{\nu'\nu}(p-k) \quad (84)$$

and

$$M^{22}_{\lambda\lambda'}(s,t) = 4(E^{-s}_{\lambda\lambda'}(kE^{-t}k) + E^{t}_{\lambda\lambda'}(kE^{-s}k) - (E^{-s}k)_\lambda(E^{-t}k)_{\lambda'} - (E^{t}k)_\lambda(E^{s}k)_{\lambda'}). \quad (85)$$

Here we have an expression of the type (A4). The parameters are

$$r = 1, \quad \alpha = \sinh(2t), \quad \beta = \cosh(2t), \quad s = 1, \quad \gamma = \sinh(2s), \quad \delta = \cosh(2s), \quad (86)$$

and

$$a = -2l^2 - h^2(\cosh(2s) + \cosh(2t)),$$
$$b = -l^2(\cosh(2s) + \cosh(2t)) - 2h^2\cosh(2s)\cosh(2t).$$

Using formula (A6) from the Appendix we obtain

$$M^{22}(s,t) = 8T^{(1)} + 8\cosh(2(s+t))T^{(2)} + 4(\cosh(2s) + \cosh(2t))T^{(3)} - 4(\sinh(2s) - \sinh(2t))T^{(4)}. \quad (87)$$

Note that the last form factor is antisymmetric and does not contribute at zero temperature.

Now we consider Π^{12} and Π^{21}. From (42) we get

$$\Pi^{12}_{\lambda\lambda'} + \Pi^{21}_{\lambda\lambda'} = \delta_{\mu\nu}(k-2p)_\lambda G_{\mu\mu'}(p)2(\delta_{\lambda'\mu'}k_{\nu'} - \delta_{\lambda'\nu'}k_{\mu'}) \times G_{\nu'\nu}(p-k) + 2(\delta_{\lambda\mu}k_\nu - \delta_{\lambda\nu}k_\mu) \times G_{\mu\mu'}(p)\delta_{\mu'\nu'}(k-2p)_{\lambda'}G_{\nu'\nu}(p-k),$$

and further

$$(M^{12}_{\lambda\lambda'} + M^{21}_{\lambda\lambda'})\Theta_T = \langle 2\{(k-2p)_\lambda((E^{s+t} - E^{-s-t})k)_{\lambda'} + ((E^{s+t} - E^{-s-t})^\top k)_\lambda(k - 2p(s))_{\lambda'}\}\Theta\rangle_T. \quad (88)$$

We use the averages (61) and obtain

$$(M^{12}_{\lambda\lambda'} + M^{21}_{\lambda\lambda'})\Theta_T = 2\left\{\left(\left(1 - 2\frac{A}{D}\right)k - 2\tilde{u}\right)_\lambda (Qk)_{\lambda'} + (Q^\top k)_\lambda\left(\left(1 - 2\frac{A^\top}{D^\top}\right)k - 2\tilde{u}\right)_{\lambda'}\right\}\Theta_T.$$

This is an expression of the form of (A4) with P_λ from Eq. (78) and with additional u-dependent terms, where

$$Q \equiv E^{s+t} - E^{-s-t} = 2iF\sinh(2q). \quad (89)$$

Then from Eq. (A6) and accounting for the u-dependent structures, we find

$$M^{12}(s,t) + M^{21}(s,t) = 4\sinh(2q)\left[-2\alpha T^{(2)} + rT^{(4)} + (2(\tilde{u}k) - B_1)T^{(6)} - \frac{iN}{qT}T^{(9)}\right]. \quad (90)$$

Next we consider the contribution of Π^{33} together with the contribution from the ghosts, Π^{ghost}. We get

$$\Pi^{33} = (\delta_{\lambda\mu}(p-k)_\nu + \delta_{\lambda\nu}p_\mu)G_{\mu\mu'}(p)(\delta_{\lambda'\mu'}(p-k)_{\nu'} + \delta_{\lambda'\nu'}p_{\mu'})G_{\nu'\nu}(p-k). \quad (91)$$

We use the property of the propagator

$$p_\mu G_{\mu\mu'}(p) = G(p)p_{\mu'} \quad (92)$$

and obtain after simple calculation, using, for instance, the cyclic property of the trace,

$$\Pi^{33}_{\lambda\lambda'} = G_{\lambda\lambda'}(p)G(p-k)(p-k)^2 + p^2G(p)G_{\lambda'\lambda}(p-k) + p_\lambda G(p)(p-k)_{\lambda'}G(p-k) + (p-k)_\lambda G(p)p_{\lambda'}G(p-k).$$

In the first line on the r.h.s. one propagator collapses into a point by means of, e.g., $p^2G(p) = 1$ and the corresponding graph becomes a tadpole-like contribution which will be considered below separately together with the tadpole diagram. The last two lines are just equal to the contribution from the ghosts, second line in (41), with opposite sign and cancel each other. So we obtain

$$\Pi^{33} + \Pi^{\text{ghost}} = 0. \quad (93)$$

Let us turn to the Π^{13} and Π^{31} contributions. We start from

$$\Pi^{13}_{\lambda\lambda'} = \delta_{\mu\nu}(k-2p)_\lambda G_{\mu\mu'}(p)(\delta_{\lambda'\mu'}(p-k)_{\nu'} + \delta_{\lambda'\nu'}p_{\mu'})G_{\nu'\nu}(p-k),$$
$$\Pi^{31}_{\lambda\lambda'} = (\delta_{\lambda\mu}(p-k)_\nu + \delta_{\lambda\nu}p_\mu)G_{\mu\mu'}(p) \times \delta_{\mu'\nu'}(k-2p)_{\lambda'}G_{\nu'\nu}(p-k) \quad (94)$$

and arrive at

$$M^{13}_{\lambda\lambda'}(s,t) = -\langle (k-2p)_\lambda (E^{\mathrm{sy}}(k-2p(s)) + E^{\mathrm{as}}k)_{\lambda'}\Theta\rangle_T,$$
$$M^{31}_{\lambda\lambda'}(s,t) = -\langle (E^{\mathrm{sy}}(k-2p) - E^{\mathrm{as}}k)_\lambda (k-2p(s))_{\lambda'}\Theta\rangle_T,$$

where the notation

$$E^{\mathrm{sy}} = \tfrac{1}{2}(E^{s+t} + E^{-s-t}) = \delta^{\parallel} + \delta^{\perp}\cosh(2q),$$
$$E^{\mathrm{as}} = \tfrac{1}{2}(E^{s+t} - E^{-s-t}) = iF\sinh(2q) = \tfrac{1}{2}Q \quad (95)$$

is introduced. Remember that the contributions appearing after the averaging procedure can be obtained by the substitution $p_\mu \to \tilde{p}_\mu + \tilde{u}_\mu = [(\frac{A}{D}k) + \tilde{u}]_\mu$ in the final expressions.

Doing so we obtain, for the averages,

$$M^{13}_{\lambda\lambda'}(s,t) = \left\{ -\tilde{P}_\lambda \tilde{Q}_{\lambda'} + 4i\left(\frac{E^{\mathrm{sy}}E^{-s}F}{D^{\top}}\right)_{\lambda\lambda'}\right\}\Theta_T,$$
$$M^{31}_{\lambda\lambda'}(s,t) = \left\{ -\tilde{Q}^{\top}_\lambda \tilde{P}^{\top}_{\lambda'} + 4i\left(\frac{E^{\mathrm{sy}}E^{-s}F}{D^{\top}}\right)_{\lambda\lambda'}\right\}\Theta_T, \quad (96)$$

where $\tilde{P}_\lambda$ is given by Eq. (78) with the mentioned replacement being done, and $\tilde{Q}_\lambda = Q_\lambda - 2\tilde{u}_\lambda$ with

$$Q_{\lambda'} = \left(\left(E^{\mathrm{as}} + E^{\mathrm{sy}}\left(1 - 2\frac{A}{D}\right)^{\top}\right)k\right)_{\lambda'}$$
$$= rl_{\lambda'} + id_{\lambda'}(\sinh(2q) - \alpha\cosh(2q)) + \beta\cosh(2q)h_{\lambda'}$$
$$\equiv sl_{\lambda'} + \gamma id_{\lambda'} + \delta h_{\lambda'}. \quad (97)$$

The second contributions to the r.h.s. of both lines in Eq. (96) must be integrated by parts.

This expression differs from that in Eq. (79) by the factor E^{sy}. This results in the following replacements in Eqs. (79) and (82): $\beta \to \beta\cosh(2q) \equiv \delta$, $\frac{\cosh(\xi)}{\sinh(q)} \to \frac{\cosh(\xi)}{\sinh(q)}\cosh(2q) \equiv \tilde{\gamma}$. With these substitutions done and accounting for the overall factor 8, we calculate this contribution,

$$8i\frac{E^{\mathrm{sy}}E^{-s}F}{D^{\top}} = [2B_1 - 4(\tilde{u}k)](r\delta^{\parallel} + \delta\delta^{\perp} - i\tilde{\gamma}F). \quad (98)$$

Again, we have chosen the constant $C = -2\frac{\tilde{u}_\mu\tilde{u}_\nu}{(\tilde{u}k)}$ and omitted the surface term. The latter will be considered separately. The above expression gives the parameters a, b, c entering Eq. (A4) for this part. By using Eq. (A6) from the Appendix and the structure of the u-dependent tensors, we obtain

$$\Pi^{13} + \Pi^{31} = 2r^2T^{(1)} + 2\{[\beta^2 - \alpha^2]\cosh(2q) + \alpha\sinh(2q)\}T^{(2)} + r\beta(1+\cosh(2q))T^{(3)} - 2r\sinh(q)\sinh(\xi)T^{(5)}$$
$$+ [2(\tilde{u}k)(\alpha - \gamma + 2\tilde{\gamma}) + l^2r(\gamma - \alpha - 2\tilde{\gamma}) + h^2\beta(\gamma - \alpha\cosh(2q) - 2\tilde{\gamma})]T^{(6)}\frac{iN}{qT}$$
$$\times\left[2\frac{r}{k_4}T^{(7)} + \frac{\beta(1+\cosh(2q))}{k_4}T^{(8)} + r(\alpha-\gamma)T^{(4)} + (\gamma-\alpha)T^{(9)} + k_4\beta(1-\cosh(2q))T^{(10)}\right]. \quad (99)$$

Finally, we need Π^{23} and Π^{32}. Proceeding in the same way as before, we derive from (41) and (42)

$$\Pi^{23}_{\lambda\lambda'} = 2(\delta_{\lambda\mu}k_\nu - \delta_{\lambda\nu}k_\mu)G_{\mu\mu'}(p)(\delta_{\lambda'\mu'}(p-k)_{\nu'} + \delta_{\lambda'\nu'}p_{\mu'})G_{\nu'\nu}(p-k),$$
$$\Pi^{32}_{\lambda\lambda'} = 2(\delta_{\lambda\mu}(p-k)_\nu + \delta_{\lambda\nu}p_\mu)G_{\mu\mu'}(p) \times (\delta_{\lambda'\mu'}k_{\nu'} - \delta_{\lambda'\nu'}k_{\mu'})G_{\nu'\nu}(p-k), \quad (100)$$

which give

$$M^{23}_{\lambda\lambda'}(s,t)\langle\Theta\rangle = \langle 2\{-E^{-s}_{\lambda\lambda'}(kE^t(k-p(s))) - E^t_{\lambda\lambda'}(kE^{-s}p(s)) + (E^{-s}p(s))_\lambda(E^{-t}k)_{\lambda'} + (E^t(k-p(s)))_\lambda(E^sk)_{\lambda'}\}\Theta\rangle_T,$$
$$M^{32}_{\lambda\lambda'}(s,t)\langle\Theta\rangle = \langle 2\{-E^{-s}_{\lambda\lambda'}(kE^{-t}(k-p)) - E^t_{\lambda\lambda'}(pE^{-s}k)) + (E^{-s}k)_\lambda(E^{-t}(k-p))_{\lambda'} + (E^tk)_\lambda(E^sp)_{\lambda'}\}\Theta\rangle_T. \quad (101)$$

Now the average (61) must be used. As it was noted, this results in the substitution $p_\mu \to \tilde{p}_\mu + \tilde{u}_\mu$ in Eq. (101). For the operator $\tilde{p}(s)$ the matrix $(\frac{A^{\top}}{D^{\top}}k)$ must be substituted. In this way the u-dependent part can be calculated. The u-independent part can be rearranged according to

$$M^{23}_{\lambda\lambda'} + M^{32}_{\lambda\lambda'} = 2(A+B) \quad (102)$$

with

$$A = -2E^t_{\lambda\lambda'}\left(k\frac{E^t-1}{D}k\right) + (E^tk)_\lambda\left(\left(\frac{E^t-1}{D}\right)^{\top}k\right)_{\lambda'} + \left(\frac{E^t-1}{D}k\right)_\lambda(E^{-t}k)_{\lambda'},$$
$$B = -2E^{-s}_{\lambda\lambda'}\left(k\frac{E^s-1}{D}k\right) + (E^{-s}k)_\lambda\left(\frac{E^s-1}{D}k\right)_{\lambda'} + \left(\left(\frac{E^s-1}{D}\right)^{\top}k\right)_\lambda(E^sk)_{\lambda'}.$$

A and B have the structure of (A4). For A we obtain

$$r' = 1, \qquad \alpha' = \sinh(2t), \qquad \beta' = \cosh(2t)$$
$$s' = \frac{t}{q}, \qquad \gamma' = \frac{1}{2}\alpha, \qquad \delta' = \frac{\cosh(s)\sinh(t)}{\sinh(q)} \tag{103}$$

and

$$a' = -2\left(\frac{t}{q}l^2 + \delta' h^2\right), \qquad b' = \cosh(2t)a',$$
$$c' = \sinh(2q)a', \tag{104}$$

and for B

$$r'' = 1, \qquad \alpha'' = -\sinh(2s), \qquad \beta'' = \cosh(2s)$$
$$s'' = \frac{s}{q}, \qquad \gamma'' = -\gamma', \qquad \delta'' = \frac{\sinh(s)\cosh(t)}{\sinh(q)} \tag{105}$$

and

$$a'' = -2\left(\frac{s}{q}l^2 + \delta'' h^2\right), \qquad b'' = \cosh(2s)a'',$$
$$c'' = -\sinh(2s)a''. \tag{106}$$

For convenience, in the formulas (103) and (105) we used apostrophes for denoting similar parameters having the same structure in different parts of Eq. (A4).

Putting these contributions together we obtain with (A6)

$$\begin{aligned}\Pi^{23} + \Pi^{32} = 2\Big\{&-2T^{(1)} - 2\left[\frac{\cosh(2t)\cosh(s)\sinh(t) + \cosh(2s)\sinh(s)\cosh(t)}{\sinh(q)} + 2\cosh(\xi)\sinh(s)\sinh(t)\right]T^{(2)}\\ &-\left[1 + \frac{s\cosh(2s) + t\cosh(2t)}{q}\right]T^{(3)} + \frac{s\sinh(2s) - t\sinh(2t)}{q}T^{(4)} + \left[-1 + \frac{s\cosh(2s) + t\cosh(2t)}{q}\right]T^{(5)}\\ &+\left[\left(\frac{s\sinh(2s) - t\sinh(2t)}{q}\right)l^2 + 2\cosh(q)\sinh(\xi)h^2\right]T^{(6)}\\ &+2(\tilde{u}k)\left[-\sinh(q)\cosh(\xi)T^{(6)} - \frac{\sinh(q)\sinh(\xi)}{k_4}T^{(8)} - \sinh(q)\cosh(\xi)T^{(9)} + \sinh(q)\sinh(\xi)T^{(10)}\right]\Big\}.\end{aligned} \tag{107}$$

Now we consider the contribution of the tadpole diagram, the last line in Eq. (41). Accounting for the explicit form of the propagator,

$$G_{\mu\nu}(p) = \int_0^\infty dq e^{-qp^2} E^{-q}_{\mu\nu}, \tag{108}$$

and calculating $\mathrm{tr}G(p) = 2\int_0^\infty dq e^{-qp^2}(1 + \cosh(2q))$ and the bracketed averages, we obtain

$$\begin{aligned}\Pi^{\rm tadpole}_{\mu\nu} &= \langle G_{\mu\nu} + G_{\nu\mu} - 2\,\mathrm{tr}G\delta_{\mu\nu}\rangle_T\\ &= -2\frac{1}{(4\pi)^2}\sum_{N=-\infty}^{+\infty}\int_0^\infty dq e^{(-(N^2/4qT^2))}\\ &\quad\times\frac{[\delta^{\|}_{\mu\nu}(1 + 2\cosh(2q)) + \delta^{\perp}_{\mu\nu}(2 + \cosh(2q))]}{q\sinh(q)}.\end{aligned} \tag{109}$$

The contribution from the tadpole-like terms coming from $\Pi^{33}_{\mu\nu}$, Eq. (91), equals just the first two terms in Eq. (109). This adds the extra terms $2(\delta^{\|}_{\mu\nu} + \delta^{\perp}_{\mu\nu}\cosh(2q))$ into the numerator of the above expression and cancels the first and last terms in the total. Hence, the final contribution of the tadpoles and the tadpole-like terms is

$$\Pi^{\rm tadpole}_{\rm tot} = -4\frac{1}{(4\pi)^2}\times\sum_{N=-\infty}^{+\infty}\int_0^\infty dq e^{-(N^2/4qT^2)}\frac{\delta^{\|}\cosh(2q) + \delta^{\perp}}{q\sinh(q)}. \tag{110}$$

Now let us consider the surface contributions from the sum $\Pi^{\rm sf} = \Pi^{(11)} + \Pi^{(13)} + \Pi^{(31)}$:

$$\Pi^{\rm sf} = 4\int_{\rm surface} dsdt\left[\delta^{\|}\frac{\xi}{q}\cosh(2q) + \delta^{\perp}\frac{\sinh(\xi)}{\sinh(q)} - iF\frac{\cosh(\xi)}{\sinh(q)} - 2\frac{\tilde{u}\cdot\tilde{u}}{(\tilde{u}k)}\cosh(2q)\right]\Theta_T, \tag{111}$$

where the explicit expressions for the parameters r, β from Eq. (78) are substituted. To relate the integrals in q and s, t, we introduce an integration variable $v = \xi/q$ in place of q. In this case $\frac{\partial}{\partial\xi} = \frac{1}{q}\frac{\partial}{\partial v}$ and $\int_0^\infty\int_0^\infty dsdt \to \frac{1}{2}\times\int_0^\infty qdq\int_{-1}^{+1}dv$. Hence, the integration over ξ results in an integration by parts over v. In the new variables the above equation reads

$$\Pi^{\rm sf} = 2\int_0^\infty dq\Theta_T\left[\delta^{\|}v\cosh(2q) + \delta^{\perp}\frac{\sinh(qv)}{\sinh(q)} - iF\frac{\cosh(qv)}{\sinh(q)} - 2\frac{\tilde{u}\cdot\tilde{u}}{(\tilde{u}k)}\cosh(2q)\right]_{v=-1}^{v=+1}, \tag{112}$$

with

$$\Theta_T = \exp\left[2(\tilde{u}k)\frac{1}{2}q(1-v) - \frac{N^2}{4qT^2}\right]\langle\Theta\rangle, \tag{113}$$

where

$$\langle\Theta\rangle = \frac{1}{(4\pi)^2}\frac{e^{-k(\delta^{\parallel}(1/4)(1-v^2)+\delta^{\perp}(ST/S+T))k}}{q\sinh(q)} \tag{114}$$

and $\frac{ST}{S+T} = \frac{\tanh[\frac{1}{2}q(1-v)]\tanh[\frac{1}{2}q(1+v)]}{\tanh[\frac{1}{2}q(1-v)]+\tanh[\frac{1}{2}q(1+v)]}$. Hence it follows that

$$\begin{aligned}(\Theta_T)^{+1} &= \frac{1}{(4\pi)^2}\frac{e^{-(N^2/4qT^2)}}{q\sinh(q)},\\ (\Theta_T)_{-1} &= \frac{1}{(4\pi)^2}e^{iNk_4/T}\frac{e^{-(N^2/4qT^2)}}{q\sinh(q)}.\end{aligned} \tag{115}$$

After substitution of these functions in Eq. (112) we obtain

$$\begin{aligned}\Pi^{\mathrm{sf}} = 2\int_0^{\infty}dq\sum_{N=-\infty}^{+\infty}(\Theta_T)^{+1}\Big[&(\delta^{\parallel}\cosh(2q)+\delta^{\perp})\\ &\times(1+e^{iNk_4/T}) - \left(iF\coth(q) + 2\frac{\tilde{u}\cdot\tilde{u}}{(\tilde{u}k)}\cosh(2q)\right)\\ &\times(1-e^{iNk_4/T})\Big].\end{aligned} \tag{116}$$

Now we note that in the imaginary time formalism the external momentum k_4 is $k_4 = 2\pi n_k T$ with $n_k = 0, \pm1, \pm2, \ldots$. The phase factor in the exponential is $\frac{Nk_4}{T} = 2\pi N n_k$. So, the exponentials in the brackets equal 1. This results in the factor 2 for the first two terms and zero for the third term in the integrand. To find the last term we also note that for $k_4 \neq 0$ the denominator $(\tilde{u}k) \neq 0$ and the bracket is zero too. For $k_4 = 0$, we expand the exponential in a series and find $-2(1-e^{iNk_4/T})\frac{\tilde{u}\cdot\tilde{u}}{(\tilde{u}k)}|_{k_4=0} = -\frac{N^2}{qT^2}$. Thus, we come to the conclusion that the last term contributes in the static limit $k_4 = 0$ only.

As a final step, we take Eq. (116) and the tadpole contribution, Eq. (110), together. The terms in front of δ^{II} and $\delta^{\perp}$ do cancel in the total. So, the only contribution is the u-dependent part coming from Π^{sf}. We denote it as $\Pi^{\mathrm{sf}}_{\mathrm{tot}}$,

$$\Pi^{\mathrm{sf}}_{\mathrm{tot}} = -\frac{4}{(4\pi)^2}\int_0^{\infty}dq\sum_{N=-\infty}^{+\infty}\frac{\tilde{u}\cdot\tilde{u}}{(\tilde{u}k)}\frac{\cosh(2q)}{q\sinh(q)} \times e^{-(N^2/4qT^2)}[1-e^{iNk_4/T}]. \tag{117}$$

This term contributes in the static limit only and it is transversal on its own. This is because being multiplied by k_λ it is zero due to the difference in the square brackets, if $k_4 \neq 0$. The integrand is nonzero if $k_4 = 0$, but for the product $k_\lambda(\Pi^{\mathrm{sf}}_{\mathrm{tot}})_{\lambda\lambda'} = 0$ holds. In fact, it is the limiting expression of the tensor $T^{(8)}$ for $k_4 = 0$. However, we will consider this part separately for convenience. Thus, we collect all the contributions coming from individual parts.

Now let us gather them together to obtain the one-loop form factors $M^{(i)}(s,t)$ for the polarization tensor. We present the results as a list of explicit functions of the variables $q = s+t$ and $\xi = s-t$ with the notation $M_i \equiv M^{(i)}(s,t)$:

$$\begin{aligned}&M_1 = 4 - 2\left(\frac{\xi}{q}\right)^2\cosh(2q), \qquad M_2 = 4\frac{1-\cosh(q)\cosh(\xi)}{(\sinh(q))^2} - 2 + 8\cosh(q)\cosh(\xi),\\ &M_3 = -2\cosh(2q)\frac{\xi\sinh(\xi)}{q\sinh(q)} - 2 + 6\cosh(\xi)\cosh(q), \qquad M_4 = 2\frac{\xi}{q}\left(\sinh(2q) - \frac{\cosh(q)-\cosh(\xi)}{\sinh(q)}\right) - 6\cosh(q)\sinh(\xi),\\ &M_5 = -2 + 2\cosh(q)\cosh(\xi),\\ &M_6^{(1)} = 2\left[\frac{\xi}{q}\coth(q)(1-3(\sinh(q))^2) + \sinh(\xi)\cosh(q)\right]l^2 + 2\left[\frac{\sinh(\xi)}{\sinh(q)}\coth(q)(1-3(\sinh(q))^2) + 2\sinh(\xi)\cosh(q)\right]h^2,\\ &M_6^{(2)} = \frac{iN}{qT}k_4 2(\sinh(2q) - \coth(q)), \qquad M_7 = \frac{iN}{qT}\frac{1}{k_4}\frac{\xi}{q}(-2\cosh(2q)), \qquad M_8 = \frac{iN}{qT}\frac{1}{k_4}\left(-2\frac{\sinh(\xi)}{\sinh(q)} - 4\sinh(q)\sinh(\xi)\right),\\ &M_9 = \frac{iN}{qT}2\left[\frac{\cosh(q)-\cosh(\xi)}{\sinh(q)} - \sinh(2q) - 2\sinh(q)\cosh(\xi)\right], \qquad M_{10} = 0\end{aligned} \tag{118}$$

(note that M_6 consists of two parts, $M_6 = M_6^{(1)} + M_6^{(2)}$). The symmetric form factors have to be multiplied by Θ_T^s and the antisymmetric ones by Θ_T^a, when the integration over s, t is carried out. It is interesting that M_{10} comes out to be zero in one-loop order.

Thus, according to Eq. (38), we present the polarization tensor in the form

$$\Pi_{\lambda\lambda'}(k) = \sum_{i=1}^{9}T^{(i)}_{\lambda\lambda'}\Pi^{(i)}(k) + (\Pi^{\mathrm{sf}}_{\mathrm{tot}})_{\lambda\lambda'}, \tag{119}$$

where the last term is just the additional structure discussed at the end of Sec. III. It is given by Eq. (117). The form factors are given by

$$\Pi^{(i)}(k) = \sum_{N=-\infty}^{+\infty} \int_0^\infty \int_0^\infty ds dt\, M^{(i)}(s,t)\Theta_T \qquad (120)$$

as double-parametric integrals over the proper time parameters s, t and the temperature sum. This representation is the basic result of this paper. It preserves as much symmetries of the polarization tensor as possible.

The obtained expression for the polarization tensor can be used in various applications. In the next three sections we consider the zero field limit, the Debye mass, and the magnetic mass of gluons in the magnetic background field at high temperature.

VI. LIMIT OF ZERO BACKGROUND FIELD

In order to establish the link between our formalism and the usual one, let us consider the limit of a zero background field, $B = 0$. In our dimensionless variables this limit simply corresponds to the limit of q and ξ going to zero. In this case, most form factors and operators also go to zero. More precisely, we have for form factors

$$M_1 = M_2 = M_3 = 4 - 2\left(\frac{\xi}{q}\right)^2 \qquad (121)$$

and

$$M_7 = M_8 = \frac{iN}{qT}\frac{1}{k_4}\left(-2\frac{\xi}{q}\right). \qquad (122)$$

All the other operators or form factors are zero. In accordance with Eqs. (35) and (37) this means that we obtain two transversal operators with the correct form factors at finite temperature [22]. Hence, at $B = 0$, the polarization tensor can be written as

$$\begin{aligned}\Pi_{\lambda\lambda'}(k,T) = {}& \frac{1}{(4\pi)^2}\sum_{N=-\infty}^{+\infty}\int_0^\infty ds dt\Big[\frac{e^{-k^2(st/q)}}{q^2} \\ & \times e^{(2(\tilde{u}k)t-(N^2/4qT^2))}\left(4-2\left(\frac{\xi}{q}\right)^2\right)K_{\lambda\lambda'} \\ & - 2\left(\frac{iN}{qT}\frac{1}{k_4}\frac{\xi}{q}\right)B_{\lambda\lambda'}\Big] \\ & - \frac{4}{(4\pi)^2}\sum_{N=-\infty}^{+\infty}\int_0^\infty dq\frac{\tilde{u}_\lambda\tilde{u}_{\lambda'}}{(\tilde{u}k)}\frac{e^{-(N^2/4qT^2)}}{q^2} \\ & \times(1-e^{iNk_4/T}), \end{aligned} \qquad (123)$$

where we substituted the surface contribution, Eq. (117), at $B = 0$.

In the representation in terms of a series resummed according to Eq. (58), the value $N = 0$ corresponds to the zero temperature case. In Eq. (123) this is

$$\Pi^{(T=0)}_{\lambda\lambda'} = \frac{1}{(4\pi)^2}\int_0^\infty ds dt\frac{e^{-k^2(st/q)}}{q^2}\left(4-2\left(\frac{\xi}{q}\right)^2\right)K_{\lambda\lambda'} \qquad (124)$$

which coincides with Eq. (103) in Ref. [10]. This part must be renormalized in a standard way. Actually, normalizing to $B = 0$ at $T = 0$ it must be simply skipped.

Now, let us calculate the Debye mass squared defined as the limit of the form $m_D^2 = -\Pi_{44}(T, k_4 = 0, \vec{k} \to 0)$ [22]. Within the representation (123), only the last term contributes and we obtain

$$m_D^2 = \frac{1}{4\pi^2}\sum_{N=1}^{+\infty}\frac{N^2}{T^2}\int_0^\infty dq\frac{e^{-(N^2/4qT^2)}}{q^3}. \qquad (125)$$

The integral is simply calculated, $\int_0^\infty dq\frac{e^{-(N^2/4qT^2)}}{q^3} = \frac{16T^4}{N^4}$, and the sum is expressed through Riemann's Zeta function, $\zeta(2) = \frac{\pi^2}{6}$. Thus, we obtain $m_D^2 = \frac{2}{3}T^2$, which is the well-known result [22].

The next important parameter is the "magnetic" mass squared which can be determined as the limit for transversal with respect to the external field direction components, $m^2_{\rm magn} = -\Pi_{12}(T, k_4 = 0, \vec{k} \to 0)$. In this case, the operator K_{12} in Eq. (123) contributes (the component $B_{12} = 0$). To calculate the form factor in the high temperature limit, it is convenient to make an inverse resummation according to Eq. (58). In the static limit, $k_4 = 0$, the parameter $a = 0$ and we have

$$\sum_{N=-\infty}^{+\infty} e^{(-(N^2/4qT^2))} = 2\pi T\left(\frac{q}{\pi}\right)^{1/2}\sum_{N=-\infty}^{+\infty} e^{(-4\pi^2T^2N^2q)}. \qquad (126)$$

For the components Π_{12} we then get

$$\begin{aligned}\Pi_{12}(k,T) = {}& \frac{1}{(4\pi)^2}\sum_{N=-\infty}^{+\infty}\int_0^\infty\frac{dq}{q}\int_0^1 du(2\pi T) \\ & \times\left(\frac{q}{\pi}\right)^{1/2}e^{-k^2qu(1-u)} \\ & \times e^{-N^24\pi^2T^2q}\left(4-2\left(\frac{\xi}{q}\right)^2\right)K_{12}, \end{aligned} \qquad (127)$$

where new variables, $s, t \to s = qu$, $t = q(1-u)$, were introduced. The high temperature limit corresponds to the value of $N = 0$ in the Matsubara sum. In that case the integrals can be calculated easily. First we compute the integral over q which delivers a Gamma function, $\Gamma(\frac{1}{2})/(k^2u(1-u))^{1/2}$. Then the integration over u gives 3π. So, for the form factor we obtain $\Pi(k)^{(1)}_{12} = \frac{3}{8}\frac{T}{k}$. Hence for the Green function,

$$G^0_{12} = \left(-\frac{k_1k_2}{k^2}\right)\frac{1}{k^2-\frac{3}{8}kT} \qquad (128)$$

follows. Here k is the length of the three-momentum vector $\vec{k}$. This expression has the fictitious pole known in gauge theories at finite temperature [22]. In this way we have shown that the known results at a vanishing magnetic field are reproduced in our representation.

VII. THE DEBYE MASS IN THE PRESENCE OF THE BACKGROUND FIELD

In this section we consider the Debye mass squared of the neutral gluon in the background of the magnetic field. It is defined as before by $m_D^2 = -\Pi_{44}(T, B, k_4 = 0, \vec{k} \to 0)$, where we have to use the expression, Eq. (117), from the polarization tensor. It reads

$$m_D^2(B) = \frac{1}{4\pi^2} \int_0^\infty \frac{dq}{q} \sum_{N=1}^{+\infty} \frac{N^2}{qT^2} \frac{B \cosh(2Bq)}{\sinh(Bq)} e^{-(N^2/4qT^2)} \equiv \frac{2}{3} T^2 f\left(\frac{B}{4T^2}\right), \tag{129}$$

where the dimensional parameters are restored. The function $f(s)$ is dimensionless and it depends on the dimensionless variable $s = \frac{B}{4T^2}$. It is chosen to satisfy $f(0) = 1$ such that it describes just the change which comes in from the magnetic field.

The function $f(s)$ can be easily computed numerically. Also, its asymptotic expansion is easy to obtain. We consider small s, i.e., high temperature, and represent $f(s)$ in the form

$$f(s) = \frac{6}{\pi^2} s^2 \int_0^\infty \frac{dq}{q^2} \sum_{N=1}^{+\infty} N^2 \left[\frac{1}{q} + \frac{\cosh(2q)}{\sinh(q)} - \frac{1}{q}\right] e^{-(N^2 s/q)}. \tag{130}$$

Here the first term in the square brackets delivers the zero field limit, $f(0) = 1$. In order to calculate the high temperature behavior, we use Eq. (58) and return to the Matsubara sum. By differentiating Eq. (58) with respect to z with $a = 0$, $z = \frac{s}{q}$, we rewrite $f(s)$ as follows:

$$f(s) = 1 + \frac{3}{2} \frac{1}{\pi^2} \sqrt{\pi s} \int_0^\infty \frac{dq}{\sqrt{q}} \sum_{N=-\infty}^{+\infty} \left[e^q - e^{-q} + \left(\frac{1}{\sinh(q)} - \frac{1}{q}\right)\right] \left(1 - 2\frac{\pi^2 N^2 q}{s}\right) e^{(-(\pi^2 N^2 q/s))}. \tag{131}$$

Here we made a rearrangement of the integrand in the above equation and split it into three parts—the tachyonic one $f_t(s)$ coming from e^q, $f_2(s)$, and $f_3(s)$ coming from the second and third terms in the square brackets. Then we calculate the leading term of the high temperature expansion which is given by $N = 0$ in the temperature sum and the first next-to-leading one resulting from $N \geq 1$.

Now, let us calculate contributions from $N = 0$. For $f_t(s, N = 0)$ we get

$$f_t(s, N = 0) = \frac{3}{2} \frac{1}{\pi^2} \sqrt{\pi s} \int_0^\infty \frac{dq}{\sqrt{q}} e^q. \tag{132}$$

This integral diverges at the upper limit that reflects the tachyonic mode in the spectrum of charged gluons (25). To obtain its value we make the inverse Wick rotation in the q plane, that is, we replace $q \to q e^{-i\pi}$. After this, a simple integration yields

$$f_t(s, N = 0) = -i \frac{3}{2} \frac{1}{\pi^2} \sqrt{\pi s} \sqrt{\pi} = -i \frac{3}{2\pi} \sqrt{s}. \tag{133}$$

The second term is

$$f_2(s, N = 0) = -\frac{3}{2} \frac{1}{\pi^2} \sqrt{\pi s} \int_0^\infty \frac{dq}{\sqrt{q}} e^{-q} \tag{134}$$

and can be easily computed, $f_2(s, N = 0) = -\frac{3}{2\pi} \sqrt{s}$. The third term is

$$f_3(s, N = 0) = \frac{3}{2} \frac{1}{\pi^2} \sqrt{\pi s} \int_0^\infty \frac{dq}{\sqrt{q}} \left(\frac{1}{\sinh(q)} - \frac{1}{q}\right), \tag{135}$$

which can be expressed through a Zeta function, $f_3(s, N = 0) = \frac{3}{2} \frac{1}{\pi^2} \sqrt{\pi s} \sqrt{2\pi} \zeta(\frac{1}{2}, \frac{1}{2})$. Finally we derive

$$f_3(s, N = 0) = \frac{3}{\sqrt{2}\pi} (\sqrt{2} - 1) \zeta\left(\frac{1}{2}\right) \sqrt{s}. \tag{136}$$

Similar simple integrations for the value $N = 1$ result in

$$f(s, N = 1) = \frac{25}{4} \frac{\zeta(3)}{\pi^4} s^2. \tag{137}$$

Thus, for the function $f(s)$ we obtain

$$f(s) = 1 + \left[\frac{3}{\sqrt{2}\pi} (\sqrt{2} - 1) \zeta\left(\frac{1}{2}\right) - \frac{3}{2\pi}\right] \sqrt{s} + \frac{25}{4} \frac{\zeta(3)}{\pi^4} s^2 - i \frac{3}{2\pi} \sqrt{s}. \tag{138}$$

Hence, for the Debye mass we derive

$$m_D^2(B) = \frac{2}{3} T^2 \left[1 - 0.8859 \left(\frac{\sqrt{B}}{2T}\right) + 0.4775 \left(\frac{B^2}{16T^4}\right) - i0.4775 \left(\frac{\sqrt{B}}{2T}\right) + O\left(\frac{B^3}{T^6}\right)\right], \tag{139}$$

where the numeric values of the coefficients in Eq. (138) were substituted.

In this way, from Eq. (130) the Debye mass can be calculated numerically or its asymptotic expansion can be found which appears in terms of the Riemann Zeta function. As expected, it has an imaginary part which results from the tachyonic instability. We note that the Debye mass in the magnetic field is smaller than without the field, confirming a similar observation in [6].

VIII. TRANSVERSAL MODES IN THE PRESENCE OF THE MAGNETIC FIELD

Let us investigate the behavior of the transversal modes in the field at high temperature. We have to calculate the mean value of the polarization tensor in the states given by Eq. (26) for the polarizations $s = 1$ and $s = 2$ in the limit of $k_4 = 0$, $\vec{k} \to 0$. Accounting for Eqs. (26), (34), and (36), we derive for the mean values

$$\langle s=1|\Pi(k)|s=1\rangle = h^2\Pi_2 + l^2(\Pi_3-\Pi_5), \quad (140)$$

$$\langle s=2|\Pi(k)|s=2\rangle = \frac{h^2 l_4^2}{k^2}\Pi_1 + \left(h^2 + \frac{l^2+h^2}{k^2}l_3^2\right)\Pi_3 + \frac{h^4 - l^2 l_3^2}{k^2}\Pi_5, \quad (141)$$

where $l_4 = k_4$, $k^2 = h^2 + l_3^2$, $l^2 = l_3^2 + l_4^2$. To consider the behavior of the static modes in the perpendicular with respect to the field plane, we put $l_3 = 0$ and $k_4 = 0$ and get

$$\begin{aligned} \langle s=1|\Pi(k)|s=1\rangle &= h^2\Pi_2, \\ \langle s=2|\Pi(k)|s=2\rangle &= h^2(\Pi_3+\Pi_5). \end{aligned} \quad (142)$$

We have to calculate the form factors Π_2, Π_3, and Π_5 educed in Eqs. (118) and (119) also for this case.

The procedure of calculations is quite similar to that in the previous sections. We describe its steps by computing the form factor Π_5. To carry out integration over ξ, we change variables, $s = qu$, $t = q(1-u)$, as in Sec. VI. In the limit of interest, $h^2 \to 0$, we restrict ourselves to the leading in this parameter term when the function $\langle\Theta\rangle$ Eq. (47) is substituted. More precisely, we take into account the first term in the expansion,

$$\langle\Theta\rangle = \frac{1}{(4\pi)^2}\frac{1+O(h^2)}{q\sinh(q)}, \quad (143)$$

because, being substituted into Eq. (142), the $\sim O(h^2)$ terms result in a next-to-leading correction. Then we make a resummation of the series in N and take into consideration the term with $N = 0$, which gives the leading high temperature contribution. In this limit we obtain for the form factors

$$\Pi_i^{(N=0)} = \frac{1}{(4\pi)^{3/2}}\frac{T}{\sqrt{B}}\int_0^1 du\int_0^\infty \frac{dq\sqrt{q}}{\sinh(q)}M_i(q,u), \quad (144)$$

where the functions from Eq. (118) should be substituted. Here we restored the overall factor $\frac{T}{\sqrt{B}}$. All other variables are dimensionless.

For the function M_5, integration over u results in the expression

$$M_5(q) = -2 + 2\frac{\cosh(q)\sinh(q)}{q} \quad (145)$$

which after substitution in Eq. (144) leads to the integral

$$I_5 = \int_0^\infty dq\left[-2\frac{\sqrt{q}}{\sinh(q)} + \frac{e^{-q}}{\sqrt{q}} + \frac{e^q}{\sqrt{q}}\right]. \quad (146)$$

The second and the third terms are calculated in Eqs. (132)–(134) in Sec. VII, $I_5^{(2)} = \sqrt{\pi}$, $I_5^{(3)} = -i\sqrt{\pi}$, and for the first one we compute $I_5^{(1)} = \frac{1}{2}(-4+\sqrt{2})\times\sqrt{\pi}\zeta(\frac{3}{2})$. Thus, we obtain in the total

$$\Pi_5 = \frac{1}{(4\pi)^{3/2}}\frac{T}{\sqrt{B}}[-4.214\,05 - 1.772\,45i], \quad (147)$$

where numeric values of the integrals are substituted.

Similar calculations carried out for the form factors Π_2 and Π_3 yield

$$\begin{aligned} \Pi_2 &= \frac{1}{(4\pi)^{3/2}}\frac{T}{\sqrt{B}}[-5.798\,94 - 7.089\,82i], \\ \Pi_3 &= \frac{1}{(4\pi)^{3/2}}\frac{T}{\sqrt{B}}[1.044\,27 - 8.862\,27i]. \end{aligned} \quad (148)$$

The above expressions have to be used in Eq. (142) to obtain the final result. The sum of $\Pi_3 + \Pi_5$ equals $\Pi_3 + \Pi_5 = \frac{T}{\sqrt{B}}[-3.169\,78 - 10.6347i]$. As it is seen, the imaginary part and hence the instability are large.

In this way we have seen that, from the general formula, numbers and asymptotic expansions can be quite easily obtained, demonstrating the effectiveness of the representation (118). We would like to mention that the magnetic field prevents the fictitious pole from appearing. This can be seen from the Schwinger-Dyson equation

$$D^{-1}(k^2) = k^2 - \Pi(k) \quad (149)$$

with the matrix elements (142) inserted,

$$\begin{aligned} \langle s=1|D^{-1}(h^2)|s=1\rangle &= h^2 - \mathrm{Re}(\Pi_2)h^2 \\ &= h^2\left(1 + 5.798\,94\frac{T}{\sqrt{B}}\right) \end{aligned} \quad (150)$$

and

$$\begin{aligned} \langle s=2|D^{-1}(h^2)|s=2\rangle &= h^2 - \mathrm{Re}(\Pi_3+\Pi_5)h^2 \\ &= h^2\left(1 + 3.16978\frac{T}{\sqrt{B}}\right). \end{aligned} \quad (151)$$

We observe that no mass gap shows up, which is an expected feature, and that the "wrong sign" disappeared. This means that the real part of the form factors gives at a finite magnetic field a positive contribution unlike the case $B = 0$ in Eq. (127). It is clear that this can happen because the limits $B \to 0$ and the vanishing external momentum of the polarization tensor do not commute.

IX. DISCUSSION

In the foregoing sections we calculated the one-loop neutral gluon polarization tensor in a magnetic background field at finite temperature in covariant gauge. We represented the polarization tensor in terms of form factors and operator structures. The latter are 10 in number allowed by the weak transversality condition whereby one of them ($T^{(10)}$) goes in the given one-loop approximation with a vanishing form factor. The form factors are represented as double-parametric integrals and the temperature sum. This representation, and especially the explicit functions (118) for the integrands, is the main result of this paper. The functions (118) are quite simple expressions and we believe that this is the most convenient representation.

We checked the obtained expressions to deliver the known results for a vanishing magnetic field or temperature. Also, from our expressions it is easy to obtain the high temperature asymptotics; see Secs. VII and VIII. As an interesting feature we note that in the magnetic mass the fictitious pole known at the zero magnetic field disappears even in this one-loop calculation as soon as the magnetic field is switched on. As a technical side note, we mention that we integrated by parts in the parametric integrals in much the same way as this is known from the corresponding simple graphs in QED. Thereby the tadpole graphs canceled all surface contributions up to one contribution which is just the Debye mass term.

The Debye mass is one of the basic quantities for resummations. In the magnetic field it has an imaginary part, which is, however, for high temperature in the first next-to-leading order, so that one can hope the resummations in the magnetic field will be possible too. This is opposite to the imaginary part in the transverse modes which appears already in the leading order. Finally we mention that the disappearance of the fictitious pole which is some kind of infrared singularity can be understood in the sense that the magnetic field improves the infrared behavior which is not, in general, an unexpected behavior.

A generalization of the techniques demonstrated here to SU(3) or to a massive field (having in mind the W boson) seems to be straightforward. We mention, however, that the treatment of the charged gluon by this method is more complicated since it is unknown at the moment how to integrate by parts in that case.

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APPENDIX

In this appendix we collect formulas which are used to identify the contributions to the form factors, i.e., the contributions which go with the tensor structures $T^{(i)}_{\mu\nu}$.

In the course of the calculations, from the graph (Fig. 2) contributions appear which have up to a constant the following structure. First, from Eq. (70),

$$\mathcal{F}^{1}_{\lambda\lambda'} = P_\lambda P^{\top}_{\lambda'} + a\delta^{\parallel}_{\lambda\lambda'} + b\delta^{\perp}_{\lambda\lambda'} + icF_{\lambda\lambda'}, \tag{A1}$$

where P_λ is given in terms of the vectors (33),

$$P_\lambda = rl_\lambda + \alpha i d_\lambda + \beta h_\lambda, \tag{A2}$$

and r, α, and β are some functions of the variables s and t. The transposition in $P^{\top}_{\lambda'}$ changes the sign of $d_{\lambda'}$, $P^{\top}_{\lambda'} = rl_\lambda - \alpha i d_\lambda + \beta h_\lambda$.

It can be seen that the expression in Eq. (38) fulfills (32) if $(rl^2 + \beta h^2)^2 + al^2 + bh^2 = 0$ holds. In that case it can be represented in terms of form factors according to

$$\mathcal{F}^{1}_{\lambda\lambda'} = -r^2 T^{(1)}_{\lambda\lambda'} + (\alpha^2 - \beta^2)T^{(2)}_{\lambda\lambda'} - r\beta T^{(3)}_{\lambda\lambda'} - r\alpha T^{(4)}_{\lambda\lambda'} + \frac{r(rl^2 + \beta h^2) + a}{h^2} T^{(5)}_{\lambda\lambda'} + (ral^2 + \alpha\beta h^2 + c)T^{(6)}_{\lambda\lambda'} \tag{A3}$$

which can be checked by inserting the explicit expressions (104).

From Eq. (72), a second type of expression appears which has a slightly more complicated form,

$$\mathcal{F}^{2}_{\lambda\lambda'} = P_\lambda Q_{\lambda'} + Q^{\top}_\lambda P^{\top}_{\lambda'} + a\delta^{\parallel}_{\lambda\lambda'} + b\delta^{\perp}_{\lambda\lambda'} + icF_{\lambda\lambda'} \tag{A4}$$

with P_λ from (71) and

$$Q_\lambda = sl_\lambda + \gamma i d_\lambda + \delta h_\lambda, \tag{A5}$$

and s, γ, and δ are also some functions of the variables s and t. In parallel to the above case, if for (A4) the condition (32) is fulfilled, then $(a + 2rsl^2)l^2 + (b + 2\beta\delta h^2)h^2 + (r\delta + s\beta)2l^2h^2 = 0$ must hold. In that case the representation in terms of form factors is

$$\mathcal{F}^{2}_{\lambda\lambda'} = -2rsT^{(1)}_{\lambda\lambda'} - 2(\beta\delta + \alpha\gamma)T^{(2)}_{\lambda\lambda'} - (r\delta + s\beta)T^{(3)}_{\lambda\lambda'} + (r\gamma - s\alpha)T^{(4)}_{\lambda\lambda'} + \left((a + 2rsl^2)\frac{1}{h^2} + r\delta + s\beta\right)T^{(5)}_{\lambda\lambda'} + (c - (r\gamma - s\alpha)l^2 + (\alpha\delta - \beta\gamma)h^2)T^{(6)}_{\lambda\lambda'}. \tag{A6}$$

Formulas (A3) and (A6) are used in Sec. V for the calculation of the form factors.

[1] P. Cea and L. Cosmai, Phys. Rev. D **60**, 094506 (1999).
[2] P. Cea and L. Cosmai, J. High Energy Phys. 02 (2003) 031; 08 (2005) 079.
[3] N. O. Agasian, Phys. Lett. B **562**, 257 (2003).
[4] V. I. Demchik and V. V. Skalozub, arXiv:hep-lat/0601035.
[5] A. O. Starinets, A. V. Vshivtsev, and V. Ch. Zukovskii, Phys. Lett. B **322**, 403 (1994).
[6] V. V. Skalozub and M. Bordag, Nucl. Phys. **B576**, 430

(2000).
[7] V. V. Skalozub and A. V. Strelchenko, Eur. Phys. J. C **33**, 105 (2004).
[8] K. Enqvist and P. Olesen, Phys. Lett. B **329**, 195 (1994).
[9] M. Giovannini, Int. J. Mod. Phys. D **13**, 391 (2004).
[10] M. Bordag and V. V. Skalozub, Eur. Phys. J. C **45**, 159 (2006) (regrettably the paper contains a number of misprints which are corrected for in arXiv:hep-th/0507141v2).
[11] M. Bordag, Yu. O. Grebenyuk, and V. V. Skalozub, Theor. Math. Phys. (Engl. Transl.) **148(1)**, 910 (2006).
[12] J. S. Schwinger, Phys. Rev. D **7**, 1696 (1973).
[13] N. K. Nielsen and P. Olesen, Nucl. Phys. **B144**, 376 (1978).
[14] V. V. Skalozub, Sov. J. Nucl. Phys. **28**, 113 (1978).
[15] J. Ambjorn, N. K. Nielsen, P. Olesen, Nucl. Phys. **B152**, 75 (1979); H. B. Nielsen and P. Olesen, Nucl. Phys. **B160**, 380 (1979); J. Ambjorn and P. Olesen, Nucl. Phys. **B170**, 60 (1980).
[16] Y. M. Cho and D. G. Pak, Phys. Rev. D **65**, 074027 (2002).
[17] V. V. Skalozub, Sov. J. Nucl. Phys. **45**, 1058 (1987).
[18] J. Ambjorn and P. Olesen, Nucl. Phys. **B315**, 606 (1989); **B330**, 193 (1990).
[19] S. MacDowell and O. Tornquist, Phys. Rev. D **45**, 3833 (1992).
[20] I. A. Batalin and A. E. Shabad, Sov. Phys. JETP **33**, 483 (1971).
[21] O. K. Kalashnikov, Mod. Phys. Lett. A **11**, 1825 (1996).
[22] O. K. Kalashnikov, Fortschr. Phys. **32**, 525 (1984).