Note on the first law with p-form potentials

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The conserved charges for p-form gauge fields coupled to gravity are defined using Lagrangian methods. Our expression for the surface charges is compared with an earlier expression derived using covariant phase space methods. Additional properties of the surfaces charges are discussed. The proof of the first law for gauge fields that are regular when pulled back on the future horizon is detailed and is shown to be valid on the bifurcation surface as well. The formalism is applied to black rings with dipole charges and is also used to provide a definition of energy in plane wave backgrounds.

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Remarkably, the first law of black hole mechanics has been demonstrated for arbitrary perturbations around a stationary black hole with bifurcation Killing horizon in any diffeomorphism invariant theory of gravity [1]. Also, this law has been shown to hold when gravity is coupled to Maxwell or Yang-Mills fields as a consequence of conservation laws and of geometric properties of the horizon [2,3].

Recently, black rings with gauge charge along the ring, the so-called dipole charge, have been found in five-dimensional supergravity [4]. As is shown in [5], the black ring solutions with dipole charge have a potential which diverges at the bifurcation surface. This implies that the computations of [1,2] are not directly applicable to that case.

Hamiltonian methods were applied to gravity coupled to a p form and a scalar field in order to explain the occurrence of dipole charges in the first law [5]. Quasilocal formalism [6] as well as covariant phase space methods [7,8] have also been developed. The first aim of this paper is to improve the covariant analysis [7,8] by deriving an expression for the conserved charges taking better care of the form factors. Following the Lagrangian methods based on cohomological results [9,10], our expression for the surface charges will moreover get round the usual ambiguities of covariant phase methods. Several properties of these surface charges will be discussed.

It was observed in [3,8] that a consistent thermodynamics can be done on the future event horizon with gauge potentials that may be irregular on the bifurcation surface if, nevertheless, the potential is regular when pulled back on the future horizon. We will extend the analysis of [7,8] by detailing how this regularity hypothesis allows for proving the first law in that context. We point out that the proof of the first law is valid on the bifurcation surface as well. We will then show that the potential for the black rings [4] admits a regular pullback on the future event horizon and can thus be treated by this method. Note that this analysis covers only electric-type charges and not magnetic charges where the potential is necessarily singular on the future event horizon.

Conservation laws have been defined in asymptotically flat and anti-de Sitter backgrounds, see e.g. the seminal works [11–13]. A natural question, raised in [14–16], is how mass can be defined in asymptotic plane wave geometries. We show in the last section that the conserved charges defined in this paper can be used in this context and lead to the correct first law.

In what follows, we will consider the action

$$S[g, \mathbf{A}, \phi] = \frac{1}{16\pi G}\int\Big[\star 1 R - \star 1\frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}e^{-\alpha\chi}\mathbf{H}\wedge\star\mathbf{H}\Big], \quad (1)$$

where χ is a dilaton and $\mathbf{H} = d\mathbf{A}$ is the field strength of a p form $\mathbf{A}$, $p \geq 1$ [17]. The fields of the theory are collectively denoted by $\phi^i \equiv (g_{\mu\nu}, \mathbf{A}, \chi)$. We will set $16\pi G = 1$ for convenience.

I. CONSERVATION LAWS

A very convenient mathematical setting to handle with $n-1$ or $n-2$-form conservation laws or more generally $(n-q)$-form conservation laws $(0 \leq q < n)$ is the study of local cohomology in field theories [18,19], see also [20] for an introduction. A conservation law consists of the existence of a $(n-q)$ form $k^{(n-q)}$ which is conserved on shell $dk^{(n-q)} \approx 0$ and which is nontrivial, i.e. not the differential of another form on shell, $k^{(n-q)} \not\approx d(\cdot)$.

In Minkowski spacetime $g_{\mu\nu} = \eta_{\mu\nu}$, $\chi = 0$ and for a trivial bundle $\mathbf{A}$, all these lower degree conserved forms are classified by the characteristic cohomology of p-form gauge theories [21]. These laws are generated in the exterior product by the forms $\star\mathbf{H}$ dual to the field strength [22]. More precisely, for odd $n-p-1$, one can construct the conserved $n-p-1$ form $\star\mathbf{H}$. For even $n-p-1$, factors $\star\mathbf{H}$ mutually commute and one may construct the conserved forms $l(n-p-1)$ $\underbrace{\star\mathbf{H}\wedge\cdots\wedge\star\mathbf{H}}_{l}$ for any integer l such that $l(n-p-1)<n-1$.

When gravity and the scalar field are present, the charges

$$\mathbf{Q}^{(n-p-1)} = e^{-\alpha\chi}\star\mathbf{H}, \qquad n-p-1 \text{ odd} \tag{2}$$

$$\mathbf{Q}^{l(n-p-1)} = e^{-l\alpha\chi}\underbrace{\star\mathbf{H}\wedge\cdots\wedge\star\mathbf{H}}_{l}, \qquad n-p-1 \text{ even} \tag{3}$$

still enumerate the nontrivial conservation laws [21,23,24].

In order to investigate the first law of thermodynamics, where variations around a solution are involved, we now extend the analysis to the linearized theory.

In linearized gravity, only $(n-2)$-form conservation laws are allowed [25]. The classification of nontrivial conserved $(n-2)$ forms was described in [9] and is straightforward to specialize in our case. The equivalence classes of conserved $(n-2)$ forms of the linearized theory for the variables $\delta\phi^i$ around a fixed reference solution ϕ^i are in correspondence with equivalence classes of gauge parameters $\xi^\mu(x)$, $\mathbf{\Lambda}(x)$ satisfying the reducibility equations $\delta_{\xi,\Lambda}\phi^i = 0$ [26], i.e.

$$\begin{cases} \mathcal{L}_\xi g_{\mu\nu} = 0, \\ \mathcal{L}_\xi \mathbf{A} + d\mathbf{\Lambda} = 0, \\ \mathcal{L}_\xi \chi = 0. \end{cases} \tag{4}$$

In this paper, we construct a $(n-2)$ form $\mathbf{k}_{\xi,\Lambda}$ enjoying the following properties. First, for each generalized Killing vector $(\xi, \mathbf{\Lambda})$ satisfying the reducibility equations (4), the surface form $\mathbf{k}_{\xi,\Lambda}$ will be closed on shell. As a result, the infinitesimal charge difference between solutions ϕ^i and $\phi^i + \delta\phi^i$ associated with any parameter $(\xi, \mathbf{\Lambda})$ satisfying (4),

$$\delta\mathcal{Q}_{\xi,\Lambda} \hat{=} \oint_S \mathbf{k}_{\xi,\Lambda}[\delta\phi;\phi], \tag{5}$$

will only depend on the homology class of S. Second, since the $(n-2)$ form will be built from the weakly vanishing Noether current, the usual ambiguities that should be treated with care in covariant phase space methods [1] will be avoided here [27]. For additional properties of these surface charges, as the representation theorem of the Lie algebra of reducibility parameters, the reader is referred to the original work [9,10].

II. SURFACE FORMS

Following the lines of [9,10], one can construct the weakly vanishing Noether currents associated with the couple $(\xi, \mathbf{\Lambda})$ by integrating by parts the expression $\delta_{\xi,\Lambda}\phi^i \frac{\delta \mathbf{L}}{\delta\phi^i}$ and using the Noether identities. We obtain

$$\begin{aligned}\mathbf{S}_{\xi,\Lambda} = \star\Big((&-2G_\mu{}^\nu + T_{\mathbf{A}}{}^\nu{}_\mu + T_\chi{}^\nu{}_\mu)\xi_\nu dx^\mu \\ &- \frac{1}{(p-1)!}D_\beta(e^{-\alpha\chi}H_\mu{}^{\beta\mu^1\cdots\mu^{p-1}}) \\ &\times(\xi^\rho A_{\rho\mu^1\cdots\mu^{p-1}} + \Lambda_{\mu^1\cdots\mu^{p-1}})dx^\mu\Big),\end{aligned} \tag{6}$$

where the stress tensors are given by

$$T^{\mu\nu}_{\mathbf{A}} = e^{-\alpha\chi}\Big(\frac{1}{p!}H^\mu{}_{\mu^1\cdots\mu^p}H^{\nu\mu^1\cdots\mu^p} - \frac{1}{2(p+1)!}g^{\mu\nu}H^2\Big), \tag{7}$$

$$T^{\mu\nu}_\chi = (\partial^\mu\chi\partial^\nu\chi - \tfrac{1}{2}g^{\mu\nu}\partial^\alpha\chi\partial_\alpha\chi). \tag{8}$$

The surface form $\mathbf{k}_{\xi,\Lambda}[\delta\phi;\phi] = k^{[\mu\nu]}_{\xi,\mathbf{\Lambda}}(d^{n-2}x)_{\mu\nu}$ can be obtained as a result of a contracting homotopy $\mathbf{I}^{n-1}_{\delta\phi}$ acting on the current $\mathbf{S}_{\xi,\Lambda}$, see e.g. [10,18]. Using the following property of the homotopy operators,

$$d\mathbf{I}^{q-1}_{\delta\phi}\omega^{(q-1)} + \mathbf{I}^q_{\delta\phi}d\omega^{(q-1)} = \delta\omega^{(q-1)}, \ \ \forall\ \omega^{(q-1)}, \ \ q\leq n, \tag{9}$$

one has

$$d\mathbf{k}_{\xi,\Lambda} = \delta\mathbf{S}_{\xi,\Lambda} - \mathbf{I}^{n-2}_{\delta\phi}\Big(\delta_{\xi,\Lambda}\phi^i\frac{\delta\mathbf{L}}{\delta\phi^i}\Big). \tag{10}$$

The closure $d\mathbf{k}_{\xi,\Lambda}[\delta\phi;\phi]\approx 0$ then holds whenever ϕ^i satisfies the equations of motion, $\delta\phi^i$ the linearized equations of motion, and $(\xi, \mathbf{\Lambda})$ the system (4).

Let us now split the current into different contributions, $\mathbf{S}_{\xi,\Lambda} = \mathbf{S}^g_\xi + \mathbf{S}^\chi_\xi + \mathbf{S}^{\mathbf{A}}_{\xi,\Lambda}$ with

$$\mathbf{S}^g_\xi = \star(-2G_\mu{}^\nu\xi_\nu dx^\mu), \tag{11}$$

$$\mathbf{S}^\chi_\xi = \star(T_{\chi\mu}{}^\nu\xi_\nu dx^\mu), \tag{12}$$

and $\mathbf{S}^{\mathbf{A}}_{\xi,\Lambda}$ being the remaining expression. Since the homotopy $\mathbf{I}^{n-1}_{\delta\phi}$ is linear in its argument, the surface form can be decomposed as $\mathbf{k}_{\xi,\Lambda} = \mathbf{k}^g_\xi + \mathbf{k}^\chi_\xi + \mathbf{k}^{\mathbf{A}}_{\xi,\Lambda}$.

The gravitational contribution $\mathbf{k}^g_\xi$, which depends only on the metric and its deviations, coincides with the Abbott-Deser expression [12] and, for Killing vectors, with the expression derived in the Hamiltonian approach of Regge-Teitelboim [13]. It can be written as

$$\mathbf{k}^g_\xi[\delta g; g] = -\delta\mathbf{Q}^g_\xi + \mathbf{Q}^g_{\delta\xi} - i_\xi\mathbf{\Theta}[\delta g] - \mathbf{E}_{\mathcal{L}}[\mathcal{L}_\xi g, \delta g], \tag{13}$$

where

$$\mathbf{Q}^g_\xi = \star(\tfrac{1}{2}(D_\mu\xi_\nu - D_\nu\xi_\mu)dx^\mu\wedge dx^\nu), \tag{14}$$

is the Komar $n-2$ form and

$$\mathbf{\Theta}[\delta g] = \star((D^\sigma\delta g_{\mu\sigma} - g^{\alpha\beta}D_\mu\delta g_{\alpha\beta})dx^\mu), \tag{15}$$

$$\mathbf{E}_{\mathcal{L}}[\mathcal{L}_\xi g, \delta g] = \star\Big(\frac{1}{2}\delta g_{\mu\alpha}(D^\alpha \xi_\nu + D_\nu \xi^\alpha) dx^\mu \wedge dx^\nu\Big). \tag{16}$$

The supplementary term, $E_{\mathcal{L}}$, with respect to the Iyer-Wald form [1] vanishes for Killing vectors.

The scalar contribution is easily found to be $\mathbf{k}^\chi_\xi[\delta g, \delta\chi; g, \chi] = i_\xi \mathbf{\Theta}_\chi$ [28] with

$$\mathbf{\Theta}_\chi = \star(d\chi\delta\chi). \tag{17}$$

Let us now compute the contribution $\mathbf{k}^{\mathbf{A}}_{\xi,\mathbf{\Lambda}}$ from the p form. After some algebra, one can rewrite the current $\mathbf{S}^{\mathbf{A}}_{\xi,\mathbf{\Lambda}}$ as

$$\mathbf{S}^{\mathbf{A}}_{\xi,\mathbf{\Lambda}} = -d\mathbf{Q}^{\mathbf{A}}_{\xi,\mathbf{\Lambda}} + e^{-\alpha\chi}(\mathcal{L}_\xi \mathbf{A} + d\mathbf{\Lambda}) \wedge \star\mathbf{H} - \tfrac{1}{2}e^{-\alpha\chi} i_\xi(\mathbf{H} \wedge \star\mathbf{H}) \tag{18}$$

with

$$\mathbf{Q}^{\mathbf{A}}_{\xi,\mathbf{\Lambda}} = e^{-\alpha\chi}(i_\xi \mathbf{A} + \mathbf{\Lambda}) \wedge \star\mathbf{H}. \tag{19}$$

Using the property (9), the surface form $\mathbf{k}^{\mathbf{A}}_{\xi,\mathbf{\Lambda}}$ reduces to

$$\mathbf{k}^{\mathbf{A}}_{\xi,\mathbf{\Lambda}} = -\delta\mathbf{Q}^{\mathbf{A}}_{\xi,\mathbf{\Lambda}} + \mathbf{Q}^{\mathbf{A}}_{\delta\xi,\delta\mathbf{\Lambda}} + d\mathbf{I}^{n-2}_{\delta\phi}\mathbf{Q}^{\mathbf{A}}_{\xi,\mathbf{\Lambda}} + \mathbf{I}^{n-1}_{\delta\phi}(e^{-\alpha\chi}(\mathcal{L}_\xi \mathbf{A} + d\mathbf{\Lambda}) \wedge \star\mathbf{H} - \tfrac{1}{2}e^{-\alpha\chi} i_\xi(\mathbf{H} \wedge \star\mathbf{H})), \tag{20}$$

where the exact term $d\mathbf{I}^{n-2}_{\delta\phi}\mathbf{Q}^{\mathbf{A}}_{\xi,\mathbf{\Lambda}}$ is trivial and can be dropped. The last term can then be computed easily since it admits only first derivatives of the gauge potential. The homotopy thus reduces in that case to $I^{n-1}_{\delta\mathbf{A}} = \frac{1}{2}\delta\mathbf{A}\frac{\partial}{\partial\mathbf{H}}$. We eventually get

$$\mathbf{k}^{\mathbf{A}}_{\xi,\mathbf{\Lambda}}[\delta g, \delta\mathbf{A}, \delta\chi; g, \mathbf{A}, \chi] = -\delta\mathbf{Q}^{\mathbf{A}}_{\xi,\mathbf{\Lambda}} + \mathbf{Q}^{\mathbf{A}}_{\delta\xi,\delta\mathbf{\Lambda}} + i_\xi\mathbf{\Theta}_{\mathbf{A}} - \mathbf{E}^{\mathbf{A}}_{\mathcal{L}}[\mathcal{L}_\xi\mathbf{A} + d\mathbf{\Lambda}, \delta\mathbf{A}] \tag{21}$$

with

$$\mathbf{\Theta}^{\mathbf{A}} = e^{-\alpha\chi}\delta\mathbf{A} \wedge \star\mathbf{H}, \tag{22}$$

$$\mathbf{E}^{\mathbf{A}}_{\mathcal{L}}[\mathcal{L}_\xi\mathbf{A} + d\mathbf{\Lambda}, \delta\mathbf{A}] = e^{-\alpha\chi}\star\Big(\frac{1}{2}\frac{1}{(p-1)!}\delta\mathbf{A}_{\mu\alpha_1\cdots\alpha_{p-1}} \times(\mathcal{L}_\xi\mathbf{A} + d\mathbf{\Lambda})_\nu^{\ \alpha_1\cdots\alpha_{p-1}} dx^\mu \wedge dx^\nu\Big) \tag{23}$$

which has a very similar structure as the gravitational field contribution (13). For reducibility parameters (4), the term involving $\mathcal{L}_\xi\mathbf{A} + d\mathbf{\Lambda}$ vanishes. The form (19) will be referred to as a Komar term, in analogy with the gravitational Komar term (14).

For $p = 1$ and reducibility parameters, the surface form (21) reduces to the well-known expression for electromagnetism, see e.g. [29]. Expression (21) and the one derived in [7,8] have a similar structure but differ in two respects. First, our surface form contains the additional term $\mathbf{E}^{\mathbf{A}}_{\mathcal{L}}[\mathcal{L}_\xi\mathbf{A} + d\mathbf{\Lambda}, \delta\mathbf{A}]$. Nevertheless, since this term vanishes for reducibility parameters, it will not be relevant for exact conservation laws. Second, the form factors in the Komar term $\mathbf{Q}^{\mathbf{A}}_{\xi,\mathbf{\Lambda}}$ differ from [7,8]. The results of [7,8] agree with ours when the right-hand side of Eq. (10) of [7] and Eq. (4) of [8] are multiplied by $-\frac{p+1}{2}$.

Let us assume that (4) holds for a field configuration $(g, \mathbf{A}, \chi)$. As a consistency check, note that the surface form (21) satisfies the equality on shell $\mathbf{k}^{\mathbf{A}}_{\xi,\mathbf{\Lambda}}[\delta g = 0, \delta\mathbf{A} = d\omega^{(p-1)}, \delta\chi = 0; g, \mathbf{A}, \chi] \approx d(\cdot)$. The charge difference (5) between two configurations differing by a gauge transformation $\delta\mathbf{A} = d\omega^{p-1}$, is thus zero on shell.

Besides generalized Killing vectors $(\xi, \mathbf{\Lambda})$ which are also symmetries of the gauge field and of the scalar χ, there may be charges associated with nontrivial gauge parameters ($\xi = 0$, $\mathbf{\Lambda} \neq d(\cdot)$). For $p = 1$, in electromagnetism, $\mathbf{\Lambda} = \text{constant} \neq 0$ is such a parameter and the associated charge is the electric charge (2). For $p > 1$, nonexact forms $\mathbf{\Lambda}$ may exist if the topology of the manifold is nontrivial. The charges with a nontrivial closed form $\mathbf{\Lambda}$ which does not vary along solutions is given by

$$\mathcal{Q}_{0,-\mathbf{\Lambda}} = \oint_S e^{-\alpha\chi}\mathbf{\Lambda} \wedge \star\mathbf{H} = \oint_T e^{-\alpha\chi} \star \mathbf{H}, \tag{24}$$

where S is a $n-2$ surface enclosing the nontrivial cycle T dual to the form $\mathbf{\Lambda}$. It is simply the integral of (2) on the nontrivial cycle. The charges (24) are thus the generalization for p forms of electric charges.

The properties of the surface form (21) under transformations of the potential $\mathbf{A}$ are worth mentioning. The transformation $\mathbf{A} \to \mathbf{A} + d\epsilon$ preserves the reducibility equations (4) if $d\mathcal{L}_\xi\epsilon = 0$. In that case, $\mathcal{L}_\xi\epsilon$ can be written as the sum of an exact form and a harmonic form that we denote as $f(\epsilon, \xi)\mathbf{\Lambda}'$ with $\mathbf{\Lambda}'$ not varying along solutions, $\delta\mathbf{\Lambda}' = 0$ and $f(\epsilon, \xi)$ constant. In Einstein-Maxwell theory, one has $\mathbf{\Lambda}' = 1$ and $f(\epsilon, \xi) = \mathcal{L}_\xi\epsilon$. Under the transformation $\mathbf{A} \to \mathbf{A} + d\epsilon$, the surface form (21) changes according to

$$\mathbf{k}^{\mathbf{A}}_{\xi,\mathbf{\Lambda}} \to \mathbf{k}^{\mathbf{A}}_{\xi,\mathbf{\Lambda}} - f(\epsilon, \xi)\delta(\mathbf{\Lambda}' \wedge e^{-\alpha\chi} \star \mathbf{H}) + d(\cdot) + \mathbf{t}_\xi, \quad \mathbf{t}_\xi \approx 0. \tag{25}$$

Defining the charge associated to $\mathbf{\Lambda}'$ as (24), one sees that the infinitesimal charge (5) varies on shell as

$$\delta\mathcal{Q}_{\xi,\mathbf{\Lambda}} \to \delta\mathcal{Q}_{\xi,\mathbf{\Lambda}} - f(\epsilon, \xi)\delta\mathcal{Q}_{0,-\mathbf{\Lambda}'}. \tag{26}$$

As a consequence, a transformation $\mathbf{A} \to \mathbf{A} + d\epsilon$ admitting a nonvanishing function $f(\epsilon, \xi)$ cannot be considered as a gauge transformation because such a transformation does not leave the conserved charges of the solution invariant.

III. FIRST LAW

We now assume that ϕ^i and $\phi^i + \delta\phi^i$ are stationary black hole solutions with Killing horizon. The generator of the Killing horizon of ϕ^i, $\xi = \partial_t + \Omega^a \partial_{\varphi_a}$ is a combination of the Killing vectors ∂_t and ∂_{φ_a}, $a = 1 \ldots \lfloor (n-1)/2 \rfloor$. The variation of energy $\delta\mathcal{E}$ and angular momenta $\delta\mathcal{J}_a$ are defined as the charges associated with the Killing vectors ∂_t and $-\partial_{\varphi_a}$, respectively [30]. We remark that this definition of energy is more natural than the one used in [7,8], where a factor $\alpha = \frac{n-3}{n-2}$ was artificially added in Eq. (16) of [7] and in Eq. (8) of [8].

We assume that ξ is a solution of (4) with $\mathbf{\Lambda} = 0$. We also require that $\xi + \delta\xi$ is a symmetry of the perturbed black hole $\phi^i + \delta\phi^i$.

The first law is then a consequence of the equality [31]

$$\oint_{S^\infty} \mathbf{k}_{\xi,0}[\delta\phi;\phi] = \oint_H \mathbf{k}_{\xi,0}[\delta\phi;\phi], \tag{27}$$

where S^∞ is a $(n-2)$-sphere at infinity and H is any cross section of the Killing horizon.

Using the linearity of $\mathbf{k}_{\xi,0}$ with respect to ξ, the left-hand side is simply given by $\delta\mathcal{E} - \Omega^a \delta\mathcal{J}_a$. Splitting the right-hand side, we get

$$\delta\mathcal{E} - \Omega^a \delta\mathcal{J}_a = \oint_H \mathbf{k}^g_{\xi,0}[\delta\phi;\phi] + \oint_H \mathbf{k}^\chi_{\xi,0}[\delta\phi;\phi] + \oint_H \mathbf{k}^{\mathbf{A}}_{\xi,0}[\delta\phi;\phi]. \tag{28}$$

The geometric properties of the Killing horizon then allow one to express the pure gravitational contribution into the form [1,32–34]

$$\oint_H \mathbf{k}^g_{\xi,0}[\delta\phi;\phi] = \frac{\kappa}{8\pi G}\delta\mathcal{A}, \tag{29}$$

where κ is the surface gravity and $\mathcal{A}$ the area of the black hole and where G factors have been restored. Here, the cross section of the horizon could be chosen to lie on the future horizon or, when it exists, to be the bifurcation surface H_B. See also [35] for a derivation of the first law (29) for stationary perturbations on the future event horizon without assumption on the way to perform the variation.

It is now convenient for the rest of the computation to choose a cross -section lying on the future horizon. The integration measure for the $(n-2)$ forms then becomes

$$\sqrt{|g|}(d^{n-2}x)_{\mu\nu} = \tfrac{1}{2}(\xi_\mu n_\nu - n_\mu \xi_\nu) d\mathcal{A}, \tag{30}$$

where $d\mathcal{A}$ is the angular measure and n^μ is an arbitrary null vector transverse to the horizon normalized with $n^\mu \xi_\mu = -1$, see e.g. Eqs. (6.14) and (6.70) of [36] for details.

Using (17), the scalar contribution can be written as

$$\oint_H \mathbf{k}^\chi_{\xi,0}[\delta\phi;\phi] = -\oint_H d\mathcal{A}\,\delta\chi(\mathcal{L}_\xi \chi + \xi^2 \mathcal{L}_n \chi) = 0, \tag{31}$$

which vanishes thanks to the reducibility equations (4), assuming the regularity of the scalar field on the horizon. By continuity, this result is also valid on the bifurcation surface H_B.

The contribution of the p form can be computed using the arguments of [5,37]. The Raychaudhuri equation gives $R_{\mu\nu}\xi^\mu\xi^\nu = 0$ on the horizon. It follows by Einstein's equations and by the identity $\mathcal{L}_\xi \phi = 0$ that $i_\xi \mathbf{H}$ has vanishing norm on the horizon. But as $i_\xi(i_\xi \mathbf{H}) = 0$, $i_\xi \mathbf{H}$ is tangent to the horizon. $i_\xi \mathbf{H}$ has thus the form $\xi \wedge \cdots \wedge \xi$ by antisymmetry of $\mathbf{H}$ and its pullback to the horizon vanishes. The equation $\mathcal{L}_\xi \mathbf{A} = 0$ can be written as $d i_\xi \mathbf{A} = -i_\xi \mathbf{H}$. Therefore, the pullback of $i_\xi \mathbf{A}$ on the horizon is a closed form.

For $p = 1$, $-i_\xi \mathbf{A} = \Phi$ is simply the scalar electric potential at the horizon. When $p > 1$, the quantity $-i_\xi \mathbf{A}$ pulled back on the horizon is the sum of an exact form $d\mathbf{e}$ and a harmonic form $\mathbf{h}$. If the horizon has nontrivial $n-p-1$ cycles T_a, one can define the harmonic forms dual to T_a by duality between homology and cohomology as

$$\int_{T_a} \sigma = \int_H \mathbf{\Omega}_a \wedge \sigma, \quad \forall\ \sigma. \tag{32}$$

The harmonic form $\mathbf{h}$ is then a sum of terms $\mathbf{h} = \Phi^a \mathbf{\Omega}_a$ with Φ^a constant over the nontrivial cycles.

The contribution from the potential contains three terms (21). The Komar term (19) can be written as

$$\oint_H \mathbf{Q}^{\mathbf{A}}_{\xi,0} = -\Phi^a \oint_{T_a} e^{-\alpha\chi} \star \mathbf{H}, \tag{33}$$

where the exact form $d\mathbf{e}$ does not contribute on shell. We recognize on the right-hand side the conserved form written in (24). Let us denote by $\mathcal{Q}_a$ the integral $\oint_{T_a} e^{-\alpha\chi} \star \mathbf{H}$.

Using (30), the contribution $\oint_H i_\xi \mathbf{\Theta_A}[\delta\phi, \phi]$ reads as

$$\oint_H i_\xi \mathbf{\Theta_A}[\delta\phi,\phi] = \oint_H e^{-\alpha\chi}(i_\xi \delta\mathbf{A}) \wedge \star\mathbf{H} - \oint_H d\mathcal{A}\,\xi^2 \star (\delta\mathbf{A} \wedge \star(i_n \mathbf{H})). \tag{34}$$

The first term of (34) nicely combines with the second term of (21) into $-\oint_{T_a} \delta\Phi^a e^{-\alpha\chi} \star \mathbf{H} = -\delta\Phi^a \mathcal{Q}_a$ because $\delta\Phi^a$ is constant as a consequence of the hypotheses on the variation. In the second term of (34), one can replace $\delta\mathbf{A}$ by its pullback $\phi_* \delta\mathbf{A}$ on the future horizon. Indeed, decomposing $\delta\mathbf{A} = \mathbf{n} \wedge \omega^{(1)} + \phi_* \delta\mathbf{A}$, one sees that the term involving $\mathbf{n}$ does not contribute because of the antisymmetry of $\mathbf{H}$. Therefore, the second term in (34) will vanish if $\mathbf{H}$ is regular and if the pullback $\phi_* \delta \mathbf{A}$ on the future horizon is regular.

Finally, the contribution from the potential on the horizon reduces to

$$\oint_H \mathbf{k}^{\mathbf{A}}_{\xi,0}[\delta\phi;\phi] = \Phi^a \delta \mathcal{Q}_a, \tag{35}$$

as it should to give the first law

$$\delta\mathcal{E} - \Omega^a \delta\mathcal{J}_a = \frac{\kappa}{8\pi G}\delta\mathcal{A} + \Phi^a\delta\mathcal{Q}_a. \tag{36}$$

Since the computation can be done entirely on the future horizon, this first law is valid in the extremal case, with $\kappa = 0$. The relations (29) and (31) hold on any cross section of the horizon. Since the surface charges (5) only depend on the homology class of the surface S, the third term in the right-hand side of (28) has to be equal to (35) for any cross section of the horizon as well. Therefore, when the bifurcation surface exists and when the regularity hypotheses are fulfilled, the first law (36) also holds there.

IV. APPLICATION TO BLACK RINGS

Let us consider the black ring with dipole charge described in [4]. This black ring is a solution to the action (1) in five dimensions for a two-form $\mathbf{A}$. The solution admits three independent parameters: the mass, the angular momentum, and a dipole charge $\oint_{S^2} e^{-\alpha\chi} \star \mathbf{H}$ where S^2 is a two-sphere section of the black ring whose topology is $S^2 \times S^1$.

The thermodynamics of this solution was worked out in the original paper [4]. The role of dipole charges in the formalism of Sudarsky and Wald [2] was elucidated in [5]. The metric, the scalar field, and the gauge potential are written in Eqs. (3.2)–(3.4) of [5]. There, the gauge potential

$$\mathbf{A} = B_{t\psi} dt \wedge d\psi, \tag{37}$$

was shown to be singular on the bifurcation surface in order to avoid a delta function in the field strength on the black ring axis. Here, we point out that this singularity in the potential does not prevent one from studying thermodynamics on the future event horizon along the lines above since the pullback of the potential is regular there.

Indeed, following [38], one can introduce ingoing Eddington-Finkelstein coordinates near the horizon of the black ring as

$$d\psi = d\psi' + \frac{dy}{G(y)}\sqrt{-F(y)H^N(y)}, \tag{38}$$

$$dt = dv - CDR\frac{(1+y)\sqrt{-F(y)H^N(y)}}{F(y)G(y)}dy. \tag{39}$$

The metric is regular in these coordinates and the gauge potential can be written as

$$\mathbf{A} = B_{t\psi} dv \wedge d\psi' + dy \wedge \omega^{(1)}, \tag{40}$$

for some $\omega^{(1)}$. The pullback of the gauge potential to the future horizon $y = -1/\nu$ is explicitly regular because $B_{t\psi}$ is finite and v and ψ' are good coordinates.

The first law for black rings may then be seen as a consequence of (36).

V. APPLICATION TO BLACK STRINGS IN PLANE WAVES

We now turn to the definition of mass in asymptotic plane wave geometries. Here, we show that the integration of the surface form $\mathbf{k}_{\partial_t,0}[\delta\phi, \phi]$ along a path γ in solution space [10,39],

$$\mathcal{E} = \int_\gamma \oint_{S^\infty} \mathbf{k}_{\partial_t,0}[\delta\phi,\phi] \tag{41}$$

provides a natural definition of mass, satisfying the first law of thermodynamics.

The action of the Neveu-Schwarz (NS)-NS sector of bosonic supergravity in n dimensions in string frame reads

$$S[G,B,\phi_s] = \frac{1}{16\pi G}\int d^n x\sqrt{-G}e^{-2\phi_s} \times \left[R_G + 4\partial_\mu\phi_s\partial^\mu\phi_s - \frac{1}{12}H^2\right],$$

when all fields in the $D-n$ compactified dimensions vanish. In the Einstein frame, $g_{\mu\nu} = e^{-4\tilde{\phi}/(n-2)}G_{\mu\nu}$, $\phi = \alpha\phi_s$, the action can be written as (1) with $\alpha = \sqrt{8/(n-2)}$ and $\mathbf{A} = B$.

Neutral black string in the n-dimensional maximally symmetric plane wave background $\mathcal{P}_n$, with $n > 4$, are given by [14–16]

$$\begin{aligned} ds_s^2 &= -\frac{f_n(r)(1+\beta^2r^2)}{k_n(r)}dt^2 - \frac{2\beta^2r^2f_n(r)}{k_n(r)}dtdy \\ &\quad + r^2 d\Omega^2_{n-3} + \left(1 - \frac{\beta^2r^2}{k_n(r)}\right)dy^2 + \frac{dr^2}{f_n(r)} \\ &\quad - \frac{r^4\beta^2(1-f_n(r))}{4k_n(r)}\sigma_n^2, \\ e^{\phi_s} &= \frac{1}{\sqrt{k_n(r)}}, \qquad B = \frac{\beta r^2}{2k_n(r)}(f_n(r)dt + dy)\wedge\sigma_n, \end{aligned} \tag{42}$$

where

$$f_n(r) = 1 - \frac{M}{r^{n-4}}, \qquad k_n(r) = 1 + \frac{\beta^2 M}{r^{n-6}}. \tag{43}$$

The black strings have horizon area per unit length given by $\mathcal{A} = M^{(n-3)/(n-4)}A_{n-3}$ where

$$A_{n-3} = \frac{2\pi^{(n-2)/2}}{\Gamma(\frac{n-2}{2})}, \tag{44}$$

is the area of the $n-3$ sphere. Choosing the normalization of the horizon generator as $\xi = \partial_t$, the surface gravity is given by $\kappa = \sqrt{-1/2(D_\mu \xi_\mu D^\mu \xi^\nu)} = \frac{n-4}{2} M^{-(1/(n-4))}$.

Using the surface forms defined above, the charge difference associated with $\frac{\partial}{\partial t}$ between two infinitesimally close black string solutions ϕ, $\phi + \delta\phi$ is given by

$$\delta \mathcal{Q}_{\partial_t} = \oint k_{\partial_t,0}[\delta\phi, \phi] = \frac{n-3}{16\pi G} A_{n-3}\delta M, \tag{45}$$

which reproduces the expectations of [14–16]. This quantity is integrable and allows one to define $\mathcal{Q}_{\partial_t} = \frac{n-3}{16\pi G} A_{n-3} M$ where the normalization of the background has been set to zero. It is easy to check that the first law is satisfied.

Note that one freely can choose a different normalization for the generator $\xi' = N\partial_t$. In that case, the surface gravity changes according to $\kappa' = N\kappa$, the charge associated to ξ' becomes $\delta \mathcal{Q}_{\xi'} = \frac{n-3}{16\pi G} A_{n-3} N \delta M$, and the first law is also satisfied. However, N cannot be a function of β. Otherwise, the charge $\mathcal{Q}_{\xi'}$ would not be defined.

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