

***N*-dimensional Vaidya metric with a cosmological constant in double-null coordinates**

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A recently proposed approach to the construction of the Vaidya metric in double-null coordinates for generic mass functions is extended to the  $n$ -dimensional ( $n > 2$ ) case and to allow the inclusion of a cosmological constant. The approach is based on a qualitative study of the null geodesics, allowing the description of light cones and revealing many features of the underlying causal structure. Possible applications are illustrated by explicit examples. Some new exact solutions are also presented and discussed. The results presented here can simplify considerably the study of spherically symmetric gravitational collapse and mass accretion in arbitrary dimensions.

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**I. INTRODUCTION**

The Vaidya metric [1] is a solution of Einstein's equations for a spherically symmetric body with a unidirectional radial null fluid. It has been used in the analysis of spherically symmetric collapse and the formation of naked singularities for many years (for references, see the extensive list of [2] and also [3]). It is also known that the Vaidya metric can be obtained from the Tolman metric by taking appropriate limits in the self-similar case [4]. This result has shed some light on the nature of the so-called shell-focusing singularities [5], as discussed in detail in [2,6–8]. The Vaidya metric has also proved to be useful in the study of Hawking radiation and the process of black-hole evaporation [9–12], in the stochastic gravity program [13], and, more recently, in the quasinormal modes analysis of varying mass black holes [14,15].

The  $n$ -dimensional Vaidya metric is required in many physically relevant situations. The study of the gravitational collapse in higher dimensional spacetimes [16], for instance, has contributed to the elucidation of the formation, nature, and eventual visibility of singularities. This last topic belongs to the realm of Penrose's celebrated cosmic censorship conjecture, see, e.g., [3] for references. Higher dimensional varying mass black holes are also central protagonists in the new phenomenological models with extra dimensions [17]. These objects might be obtained in the LHC at CERN [18] and, once produced, they are expected to decay driven by the emission of Hawking radiation. The analysis of their quasinormal modes can help the understanding of the dynamics of the evaporation process, and could even lead to some observational signs [15].

The inclusion of a cosmological constant  $\Lambda$  in the  $n$ -dimensional Vaidya metric is mainly motivated by the recent intensive activities in the AdS/CFT and dS/CFT conjectures; see [19] for some references. A cosmological constant is also necessary to allow the discussion of the

$n = 3$  case in the same framework of the higher dimensional cases.

The  $n$ -dimensional Vaidya metric was first discussed in [20]. It can be easily cast in  $n$ -dimensional ( $n > 3$ , the three-dimensional case will be discussed later) radiation coordinates  $(w, r, \theta_1, \dots, \theta_{n-2})$  [16]:

$$ds^2 = -\left(1 - \frac{2m(v)}{(n-3)r^{n-3}}\right)dv^2 + 2cdrdv + r^2d\Omega_{n-2}^2, \quad (1)$$

where  $c = \pm 1$  and  $d\Omega_{n-2}^2$  stands for the metric of the unity  $(n-2)$ -dimensional sphere, assumed here to be spanned by the angular coordinates  $(\theta_1, \theta_2, \dots, \theta_{n-2})$ ,

$$d\Omega_{n-2}^2 = \sum_{i=1}^{n-2} \left( \prod_{j=1}^{i-1} \sin^2 \theta_j \right) d\theta_i^2. \quad (2)$$

For the case of an ingoing radial flow,  $c = 1$  and  $m(v)$  is a monotone increasing mass function in the advanced time  $v$ , while  $c = -1$  corresponds to an outgoing radial flow, with  $m(v)$  being, in this case, a monotone decreasing mass function in the retarded time  $v$ . The four-dimensional Vaidya metric with a cosmological constant in the radiation coordinates has been considered previously in [21].

It is well known that the radiation coordinates are defective at the horizon [22], implying that the Vaidya metric (1) is not geodesically complete in any dimension. (See [23] for a discussion about possible analytical extensions.) Besides, the cross term  $drdv$  can introduce unnecessary oddities in the hyperbolic equations governing the evolution of physical fields on spacetimes with the metric (1). Typically, the double-null coordinates are far more convenient. This was the main motivation of Waugh and Lake's work [24], where the problem of casting the four-dimensional Vaidya metric with  $\Lambda = 0$  in double-null coordinates is considered. As all previous attempts to construct a general transformation from radiation to double-null coordinates have failed, they followed Synge [25] and considered Einstein's equation with spherical symmetry in double-null coordinates *ab initio*. The resulting equations,

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however, are not analytical solvable for generic mass functions. Waugh and Lake's work was recently revisited in [26], where a semianalytical approach allowing for generic mass functions was proposed. Such an approach consists in a qualitative study of the null geodesics, allowing the description of light cones and revealing many features of the underlying causal structure. It can be used also for more quantitative analyses; indeed, it has already enhanced considerably the accuracy of the quasinormal modes analysis of four-dimensional varying mass black holes [14,15], and it can also be applied to the study of gravitational collapse [26]. We notice that another method to construct conformal diagrams based on a systematic study of the null geodesics was also recently proposed in [27].

Here, we extend the approach proposed in [26] to the  $n$ -dimensional ( $n > 2$ ) case with the cosmological constant  $\Lambda$ . Some new exact solutions are also presented. For  $n = 3$ , notably, a crucial nonlinearity can be circumvented and the problem can be reduced to the solution of a second order linear ordinary differential equation, opening the possibility for analytical study of large classes of mass functions. We generalize one of Waugh and Lake's exact solutions by introducing the case corresponding to  $m(v) \propto v^{n-3}$ ,  $n > 3$ , describing a naked or shell-focusing singularity in an  $n$ -dimensional spacetime. We also present some explicit examples of the use of the semianalytical approach to the study of the causal structure of some particular solutions.

In the next section, the main equations are derived and the exact solutions are presented. Section III is devoted to the introduction of the semianalytical approach. Some explicit examples are presented, particularly the two apparent horizons case of an evaporating black hole in a de Sitter spacetime and the BTZ [28] black hole undergoing a finite time interval mass accretion process. The last section is left to some concluding remarks. This work also has two appendixes. The first one presents explicitly the  $n$ -dimensional geometrical quantities necessary to reproduce the equations of Sec. II, and the last contains the polynomial manipulations involved in the calculations of the apparent horizons of Secs. II and III.

## II. THE METRIC

The  $n$ -dimensional spherically symmetric line element in double-null coordinates  $(u, v, \theta_1, \dots, \theta_{n-2})$  is given by

$$ds^2 = -2f(u, v)dudv + r^2(u, v)d\Omega_{n-2}^2, \quad (3)$$

where  $f(u, v)$  and  $r(u, v)$  are nonvanishing smooth functions. We adopt here the same conventions of [24,25]. In particular, the indices 1 and 2 stand for the differentiation with respect to  $u$  and  $v$ , respectively.

The energy-momentum tensor of a unidirectional radial null fluid in the eikonal approximation is given by

$$T_{ab} = \frac{1}{8\pi}h(u, v)k_a k_b, \quad (4)$$

where  $k_a$  is a radial null vector. We will consider here, without loss of generality, the case of a flow along the  $v$  direction. The case of simultaneous ingoing and outgoing flows in four dimensions has already been considered in [29]. Einstein's equations are less constrained in such a case, allowing the construction of some exact similarity solutions, which, incidentally, can also be generalized to the  $n$ -dimensional case in the light of the present work.

Einstein's equations with a cosmological constant  $\Lambda$ ,

$$R_{ab} - \frac{1}{2}g_{ab}R = -\Lambda g_{ab} + 8\pi T_{ab}, \quad (5)$$

imply that, for the energy-momentum tensor (4),

$$R = \frac{2n}{n-2}\Lambda. \quad (6)$$

Using (6) and (A6), Einstein's equations for the metric (3) and the energy-momentum tensor (4) read

$$\frac{f_1}{f} - \frac{r_{11}}{r_1} = 0, \quad (7)$$

$$\frac{f_2}{f} - \frac{r_{22}}{r_2} = \frac{h}{n-2} \frac{r}{r_2}, \quad (8)$$

$$\frac{f_1 f_2}{f^2} - \frac{f_{12}}{f} - (n-2) \frac{r_{12}}{r} = -\frac{2\Lambda}{n-2} f, \quad (9)$$

$$2(rr_{12} + (n-3)r_1 r_2) + (n-3)f = \frac{2\Lambda}{n-2} f r^2. \quad (10)$$

For  $n \neq 3$ , differentiating Eq. (10) with respect to  $u$  and then inserting Eq. (7) leads to

$$\frac{r^{n-2} r_{12}}{f} - \frac{2\Lambda}{(n-2)(n-1)} r^{n-1} = -A, \quad (11)$$

where  $A(v)$  is an arbitrary integration function. The  $n = 3$  case will be considered below. Now, differentiating Eq. (10) with respect to  $v$  and using (8) and (11) gives

$$h = -\left(\frac{n-2}{n-3}\right) \frac{f A_2}{r^{n-2} r_1}. \quad (12)$$

Equation (7) is ready to be integrated,

$$f = 2B r_1, \quad (13)$$

where  $B(v)$  is another arbitrary integration (and nonvanishing) function. From (12) and (13), one has

$$h = -2\left(\frac{n-2}{n-3}\right) B \frac{A_2}{r^{n-2}}. \quad (14)$$

Finally, by using (10) and (13), Eq. (11) can be written as

$$r_2 = -B \left( 1 - \frac{2A}{(n-3)r^{n-3}} - \frac{2\Lambda}{(n-2)(n-1)} r^2 \right). \quad (15)$$

Note that (9) follows from Eqs. (13) and (15). Einstein's equations are, therefore, equivalent to Eqs. (13)–(15), generalizing the results of [24,26].

In order to interpret physically the arbitrary integration functions  $A(v)$  and  $B(v)$ , let us transform from the double-null coordinates back to the radiation coordinates by means of the coordinate change  $(u, v) \rightarrow (r(u, v), v)$ . Such a transformation casts (3) in the form

$$ds^2 = 4Br_2dv^2 - 4Bdrdv + r^2d\Omega_{n-2}^2, \quad (16)$$

where (13) was explicitly used. Comparing (1) and (16) and taking into account (15), it is clear that with the choice

$$B = -\frac{1}{2} \frac{A_2}{|A_2|}, \quad (17)$$

for  $A_2 \neq 0$ , the function  $A(v)$  should represent the mass of the  $n$ -dimensional solution, suggesting the following  $n$ -dimensional generalization of the mass definition based on the angular components of the Riemann tensor proposed for  $\Lambda = 0$  in [30]:

$$m = \frac{n-3}{2} r^{n-3} R_{\theta_1\theta_k\theta_1\theta_k}, \quad (18)$$

$k > 1$ ; see (A5) and (15). Note that, for constant  $A$ , the choice of the function  $B(v)$  is irrelevant since it can be absorbed by a redefinition of  $v$ . Note also that the weak energy condition applied for (4) requires, from (14), that  $BA_2 \leq 0$ . Since  $B$  must be nonvanishing from (13),  $A(v)$  must be a monotone function.

The Kretschmann scalar  $K = R_{abcd}R^{abcd}$  could also be invoked to interpret the integration function  $A$ . Taking into account Eqs. (13) and (15), we have from (A8)

$$K = 4 \frac{(n-1)(n-2)^2}{(n-3)} \frac{A^2}{r^{2(n-1)}} + \frac{8n}{(n-1)(n-2)^2} \Lambda^2. \quad (19)$$

It is clear from (19) that  $r = 0$  is a singularity for  $A \neq 0$  and that  $A$  acts as a gravitational source placed at  $r = 0$ .

The problem now may be stated in the same way as the four-dimensional  $\Lambda = 0$  case [26]: given the mass function  $A(v)$  and the constant  $B$ , one needs to solve Eq. (15), giving rise to the function  $r(u, v)$ . Then,  $f(u, v)$  and  $h(u, v)$  are calculated from (13) and (14). The arbitrary function of  $u$  appearing in the integration of (15) must be chosen properly [24] in order to have a nonvanishing  $f(u, v)$  function from (13). Unfortunately, as stressed earlier by Waugh and Lake [24], such a procedure is not analytically solvable in general. They, nevertheless, were able to find some regular solutions for Eqs. (13)–(15) for  $n = 4$  and  $\Lambda = 0$ , namely, the linear [ $A(v) = \lambda cv$ ] and a certain exponential [ $A(v) = \frac{1}{\beta}(\alpha \exp(\beta cv/2) + 1)$ ] mass functions ( $\lambda$ ,  $\alpha$ , and  $\beta$  are positive constants,  $c = \pm 1$ , corresponding to ingoing/outgoing flow, respectively). The four-dimensional linear mass case was also considered in [8] in great detail and

in a more general situation (the case of a charged radial null fluid). In [26], another four-dimensional exact solution corresponding to  $A(v) = \kappa/v$  and  $\Lambda = 0$  was also presented and discussed. These are the only varying mass analytical solutions obtained in double-null coordinates so far. We notice, however, that Kuroda was able to construct a transformation from radiation to double-null coordinates for some other particular mass functions in four dimensions [10].

In the following section, we will present a semianalytical procedure to attack the problem of solving Eqs. (13)–(15) for general mass functions obeying the weak energy condition, generalizing in this way the results of [26] obtained for  $n = 4$  and  $\Lambda = 0$ . The approach allows us to construct qualitatively conformal diagrams, identifying horizons and singularities, and also to evaluate specific geometric quantities. First, however, we notice that Waugh and Lake's solution corresponding to  $A(v) = \lambda cv$  can also be generalized for  $n$  dimensions. Let us consider the mass function

$$A(v) = \lambda v^{n-3}, \quad (20)$$

$v \geq 0$ ,  $\lambda > 0$ . For this mass function and  $\Lambda = 0$ , with the choice (17), Eq. (15) reads

$$r_2 = \frac{1}{2} - \frac{\lambda}{n-3} \left(\frac{v}{r}\right)^{n-3}, \quad (21)$$

which can be integrated as

$$\ln v + \int^{(r/v)} \left( s - \frac{1}{2} + \frac{\lambda}{n-3} s^{-(n-3)} \right)^{-1} ds = D(u), \quad (22)$$

where  $D(u)$  is an arbitrary integration function, implying, from Eq. (13), that

$$f = -\frac{v^{n-2} D_1}{r^{n-3}} \left[ \left(\frac{r}{v}\right)^{n-2} - \frac{1}{2} \left(\frac{r}{v}\right)^{n-3} + \frac{\lambda}{n-3} \right]. \quad (23)$$

The integration function  $D$  must be chosen in order to have a regular  $f$ . As in the four-dimensional case originally considered by Waugh and Lake, we have three qualitatively distinct cases:  $0 < \lambda < \lambda_n^c$ ,  $\lambda = \lambda_n^c$ , and  $\lambda > \lambda_n^c$ , where

$$\lambda_n^c = \left( \frac{n-3}{2(n-2)} \right)^{n-2}. \quad (24)$$

For  $\lambda > \lambda_n^c$ , for instance, the quantity inside the square brackets in (23) does not vanish; see Appendix B. The integral in (22) can be (numerically) evaluated and, for instance with Waugh and Lake's choice  $D(u) = -u$ ,  $r(u, v)$  can also be (numerically) determined. The resulting causal structures are the same ones as Waugh and Lake's four-dimensional cases. We will construct the relevant conformal diagrams for the three qualitatively distinct cases as an application of the semianalytical approach.

### The three-dimensional case

In lower dimensional ( $n < 4$ ) spacetimes, the Riemann tensor is completely determined by its traces, namely, the Ricci tensor and the scalar curvature, implying some qualitative distinct behavior for the solutions of Einstein's equations in these situations [31]. For  $n = 3$ , Eq. (10) reads

$$\frac{rr_{12}}{f} + \Lambda r^2 = 0, \quad (25)$$

It is interesting to compare this last equation with (11). Equation (13) is still valid and, by using it, Eq. (25) can be easily integrated,

$$r_2 + \Lambda B r^2 = C, \quad (26)$$

where  $C$  is an arbitrary integration function which we call, by convenience,  $C(v) = -B(v)A(v)$ , leading to

$$r_2 = -B(-A - \Lambda r^2), \quad (27)$$

which corresponds to the three-dimensional counterpart of Eq. (15). Equation (9) for  $n = 3$  also follows from (13) and (27). As for the  $n$ -dimensional case, by using (13) and (27), one gets from (8) the last equation,

$$h = -B \frac{A_2}{r}. \quad (28)$$

One can show that the integration functions  $A$  and  $B$  have the same physical interpretation as the higher dimensional cases by introducing the radiation coordinates. We have for the three-dimensional case

$$ds^2 = -(2B)^2(-A - \Lambda r^2)dv^2 - 4Bdrdv + r^2d\theta^2. \quad (29)$$

It is clear that, with the choice (17) for  $B$ , the function  $A$  plays the role of the BTZ black-hole mass [28] ( $\Lambda < 0$  in this case). Note, however, that for  $n = 3$  there is no purely angular components of the Riemann tensor and, therefore, there is no equivalent of the mass definition (18). The Kretschmann scalar in this case reads simply

$$K = 12\Lambda^2. \quad (30)$$

In fact, if terms involving distributions are taken into account properly, curvature invariants such as  $K$  reveal that the origin for the BTZ black hole is a conical singularity (see, for a recent rigorous analysis, [32]).

In contrast with the  $n$ -dimensional case, the nonlinearity present in (27) can be easily circumvented. Let us consider  $A_2 > 0$ ,  $B = -1/2$ , and  $\Lambda = -1/\ell^2$  (other cases follow straightforwardly). By introducing the linear second order ordinary differential equation

$$w''(v) - \frac{A(v)}{4\ell^2} w(v) = 0, \quad (31)$$

it is easy to show that

$$r(u, v) = -2\ell^2 \frac{P(u)w'_a(v) + w'_b(v)}{P(u)w_a(v) + w_b(v)} \quad (32)$$

is the solution of (27), where  $w_a(v)$  and  $w_b(v)$  are the two linearly independent solutions of (31), and  $P(u)$  is an arbitrary function. The second order linear equation (31) has an analytical solution in closed form for many functions  $A(v)$ . For instance, for  $A(v) \propto v^\alpha$  [or  $A(v) \propto \exp \alpha v$ ],  $\alpha \in \mathbb{R}$ , Eq. (31) is equivalent to the Bessel equation, after an appropriate redefinition of  $v$  ( $v \rightarrow v^{(\alpha/2+1)}$  or  $v \rightarrow e^{\alpha v/2}$ ). Even for the case where no solution in closed form can be obtained, the main analytical properties of  $w(v)$  can be inferred easily from (31) due to its linearity.

### III. SEMIANALYTICAL APPROACH

We review here the semianalytical approach proposed in [26] for the solution of Eq. (15) [or, analogously, (27) for the three-dimensional case]. First, notice that in double-null coordinates the light cones correspond to the hypersurfaces with constant  $u$  or constant  $v$ . One can, in this case, deduce the causal structure of a given spacetime by considering the set of null geodesics corresponding to  $u = \text{constant}$  or  $v = \text{constant}$ . The function  $r(u, v)$ , obtained as the solution of (15) with a careful choice of initial conditions [24,26], also has a clear geometrical interpretation: it is the radius of the  $(n-2)$ -dimensional sphere defined by the intersection between the hypersurfaces  $u = \text{constant}$  and  $v = \text{constant}$ . Once  $r(u, v)$  is obtained from (15),  $f(u, v)$  and  $h(u, v)$  can be calculated directly from Eqs. (13) and (14), giving all the information about the spacetime in question. Incidentally, Eq. (15) along constant  $u$ , and, consequently, along a portion of the light cone, is a first order ordinary differential equation in  $v$ . One can evaluate the function  $r(u, v)$  in any spacetime point by solving the  $v$ -initial value problem starting with  $r(u, 0)$ . For instance, the usual constant curvature empty spacetimes ( $A = 0$ ) can be obtained by choosing  $r(u, 0) = u/2$ , leading to

$$\int_{r_0}^r \left(1 - \frac{2\Lambda}{(n-2)(n-1)} s^2\right)^{-1} ds = \frac{1}{2}(v+u). \quad (33)$$

As is expected, for the  $n$ -dimensional de Sitter case ( $\Lambda > 0$ ),  $r = \sqrt{(n-2)(n-1)}/2\Lambda$  corresponds to a cosmological horizon. The condition  $r_1(u, 0) \neq 0$  is sufficient to assure that  $f(u, v) \neq 0$  everywhere [26].

Let us focus now on the solutions  $r(v) = r(\bar{u}, v)$  of (15), considered as a first order ordinary differential equation in  $v$  for constant  $u = \bar{u}$ , keeping in mind that they indeed describe how  $r(\bar{u}, v)$  varies along the  $u = \bar{u}$  portion of the light cone and, thus, that they are closely related to the causal structure of the underlying spacetime. This situation is shown schematically in Fig. 1.

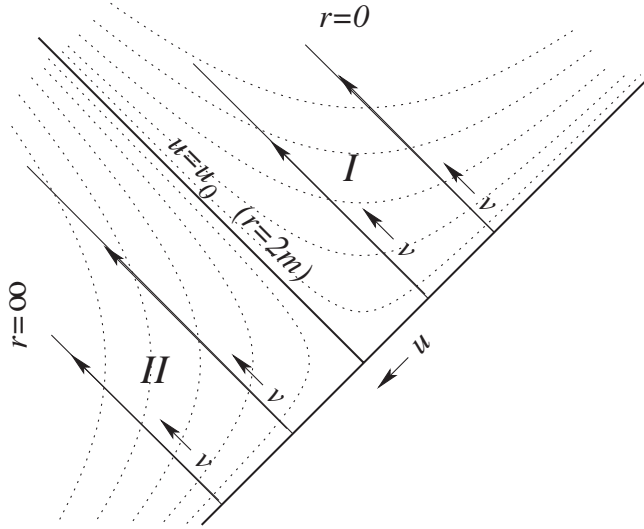


FIG. 1. The solutions  $r(u, v)$  of (15) for the Schwarzschild case [ $A(v) = m$ ; see [24]]. The dotted lines correspond to the lines of constant  $r$ . In region I (the interior region), the values of  $r$  decrease monotonically upward, varying from  $r = 2m$  (the horizon corresponding to the degenerated hyperbole  $u = u_0$  and  $v = v_0$ ) to the singularity  $r = 0$ . In the exterior region (II), on the other hand, the values of  $r$  increase monotonically leftward, varying from  $r = 2m$  to the spatial infinity  $r = \infty$ . Some slices of constant  $u = \bar{u}$  are depicted by the lines with arrows. The values of  $r(\bar{u}, v)$  read along such lines correspond to the solutions of (15) with a certain initial condition  $r(0) = r_{\bar{u}}$ . Since the lines of constant  $u$  are portions of the light cone, the function  $r(\bar{u}, v)$  describes how the light cones cross the  $(n - 2)$ -dimensional spheres of radius  $r$ . In the interior region, for instance, all null geodesics eventually reach the singularity, whereas in the exterior the null geodesics with constant  $u$  escape to the null infinity. Between such regions, there is an event horizon at  $u = u_0$ . We notice that all this information is coded in the qualitative behavior of the solutions  $r(\bar{u}, v)$  for the first order ordinary differential equation corresponding to (15) considered along constant  $u = \bar{u}$ . This is the basic idea of our semianalytical approach.

For  $n > 3$ , the curves defined by

$$-\frac{2\Lambda}{(n-2)(n-1)}r^{n-1} + r^{n-3} = \frac{2A(v)}{n-3} \quad (34)$$

correspond (if they indeed exist) to the frontiers of certain regions of the  $(v, r)$  plane where the solutions  $r(v)$  of (15) have qualitative distinct behaviors. For the de Sitter case, (34) can define two nonintersecting curves, whereas for the anti-de Sitter case ( $\Lambda < 0$ ) there is only one curve. (See Appendix B.) Let us suppose, for instance, a de Sitter case with  $A_2(v) > 0$  and  $B = -1/2$ . (Outgoing radiation flows, three-dimensional spacetimes, or the anti-de Sitter case follow in a straightforward manner.) Let us call  $r_{EH}(v)$  and  $r_{CH}(v)$  the curves defined by (34); see Fig. 2. Because of (15), for all points of the plane  $(v, r)$  below  $r_{EH}(v)$ ,  $r_2 < 0$ . Hence, any solution  $r(v)$  entering in this region will, un-

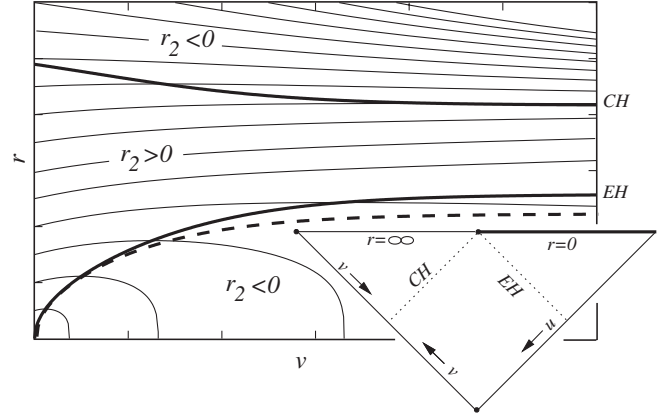


FIG. 2. In the region below the curve  $r_{EH}(v)$  (the apparent event horizon),  $r_2 < 0$ . Any solution  $r(v)$  of (15) that enters into this region will reach the singularity at  $r = 0$  with finite  $v$ . Solutions confined to the  $r_2 > 0$  region always escape from the singularity and tend asymptotically to  $r_{CH}(v)$  (the apparent cosmological horizon). The region above  $r_{CH}(v)$  corresponds to the cosmological exterior; any solution there also tends asymptotically to  $r_{CH}(v)$ . The dashed line corresponds to the apparent event horizon for the case of a vanishing  $\Lambda$  (the unique solution of (34) for  $\Lambda = 0$ ). The conformal diagram is inserted. [The case depicted here corresponds to  $n = 5$ ,  $\Lambda = 1$ , and  $A(v) = (6/5) \times \tanh v$ ,  $v \geq 0$ . In this work, we do not consider maximal extensions. For a discussion of the maximal conformal diagram for de Sitter and anti-de Sitter black holes, see [39], for instance.]

avoidably, reach the singularity at  $r = 0$ , with finite  $v$ . Suppose a given solution  $r_i(v)$  with the initial condition  $r_i(0) = r_i$  enters into the region below  $r_{EH}(v)$ . As for smooth  $A$ , the uniqueness of solutions for (15) is guaranteed for any point with  $r > 0$ ; any solution starting at  $r(0) < r_i$  is confined to the region below  $r_i(v)$  and will also reach the singularity at  $r = 0$ , with finite  $v$ . On the other hand, suppose that a given solution  $r_e(v)$  with the initial condition  $r_e(0) = r_e$  never enters into the region below the curve  $r_{EH}(v)$ . Solutions starting at  $r(0) > r_e$  are, therefore, confined to the region above  $r_e(v)$  and will escape from the singularity at  $r = 0$ . Hence, one has two qualitatively distinct behaviors for the light cones; see Fig. 2. For  $r(0) < r_i$ , the future direction points toward the singularity and all null geodesics eventually reach  $r = 0$ . This is the typical situation in the interior region of a black hole. For  $r(0) > r_e$ , the  $u = \text{constant}$  portion of the light cones escapes from the singularity. The curve  $r_{EH}(v)$  plays the role of an apparent event horizon. For a given  $v$ , all solutions  $r(v)$  in Fig. 2 such that  $r(v) \leq r_{EH}(v)$  will be captured by the singularity. Solutions for which  $r(v) > r_{EH}(v)$  are temporally free, but they may find themselves trapped later if  $r_{EH}(v)$  increases. The event horizon corresponds to the last of these solutions trapped by the singularity and is located somewhere between  $r_i(v)$  and  $r_e(v)$ .

In the region between  $r_{EH}(v)$  and  $r_{CH}(v)$ , the solutions  $r(v)$  have  $r_2 > 0$ . They escape from  $r_{EH}(v)$  and tend

asymptotically to  $r_{\text{CH}}(v)$ , which plays the role of an apparent cosmological horizon. The region above  $r_{\text{CH}}(v)$  corresponds to the cosmological exterior. [See, for instance, [33]. For  $A = 0$ , the cosmological exterior region corresponds to the choice  $r_0 = \infty$  in (33).] There,  $r_2 < 0$  and the solutions also tend asymptotically to  $r_{\text{CH}}(v)$ . The case where  $r_{\text{EH}}(v)$  and  $r_{\text{CH}}(v)$  converge to the same function gives rise to the interesting  $n$ -dimensional Nariai solution [34].

Summarizing, since the partial differential equation (15) for  $r(u, v)$  does not involve  $u$  explicitly, it is always possible to discover how the light cones cross the  $(n-2)$ -dimensional spheres of radius  $r$  by exploring, in the plane  $(v, r)$ , the solutions  $r(\bar{u}, v)$  of the first order differential equation corresponding to (15) considered on the constant  $u$  null geodesics. From this information, one can reconstruct the causal structure of the underlying spacetime. For typical cases, we will have some spacetime regions where the light cones necessarily end in the singularity at  $r = 0$ , whereas for some other regions it is always possible to escape to the null infinity. The frontier of these regions must correspond to event horizons. Let us illustrate the approach with some explicit examples.

### A. Waugh and Lake generalized solutions

As the first application of the semianalytical approach, let us consider the generalized Waugh and Lake solutions corresponding to the choice (20) for the mass function. From the discussion of Sec. II, we expect three qualitatively distinct cases according to the value of  $\lambda$ . In fact, these solutions have the same causal structure for any  $n > 3$ . The frontier of the region in the  $(v, r)$  plane where all the solutions of (21) reach the singularity at  $r = 0$  corresponds to the straight line

$$r_0(v) = \left(\frac{2\lambda}{n-3}\right)^{1/(n-3)} v. \quad (35)$$

Taking the  $v$  derivative of (21), one gets

$$r_{22} = \lambda \frac{v^{n-2}}{r^{n-3}} \left(-1 + \frac{1}{2} \frac{v}{r} - \frac{\lambda}{n-3} \left(\frac{v}{r}\right)^{n-3}\right). \quad (36)$$

The regions in the plane  $(v, r)$  where the solutions obey  $r_{22} = 0$  are the straight lines defined by

$$\left(\frac{r}{v}\right)^{n-2} - \frac{1}{2} \left(\frac{r}{v}\right)^{n-3} + \frac{\lambda}{n-3} = 0. \quad (37)$$

One can now repeat the analysis done for the  $n = 4$  case by considering the three qualitatively different cases according to the value of  $\lambda$  and the possible solutions of (37). For this purpose, we will consider the solutions of (21) with the initial condition  $r(u, 0) \propto u$ .

For  $\lambda > \lambda_n^c$ , with  $\lambda_n^c$  given by (24), Eq. (37) has no solution and  $r_{22} < 0$  for all points with  $v > 0$ . The only relevant frontier in the plane  $(v, r)$  is the  $r_0(v)$  straight line. All solutions of (21) are concave functions and cross the

$r_2 = 0$  line, reaching the singularity with a finite  $v$  which increases monotonically with  $u$ . The causal structure of the corresponding spacetime is very simple. There is no horizon, and all future cones end in the singularity at  $r = 0$ . The associated conformal diagram is depicted in Fig. 3.

For  $\lambda = \lambda_n^c$ ,  $r_{22} = 0$  only on the line

$$r_h(v) = \frac{n-3}{2(n-2)} v. \quad (38)$$

This straight line itself is a solution of (21). All other solutions are concave functions. Note that, in this case,  $r_h(v) > r_0(v)$  for  $v > 0$ . We have two distinct qualitative behaviors for the null trajectories along the  $u$  constant. All solutions starting at  $r(0) > 0$  are confined to the region above  $r_h(v)$ . They never reach the singularity; all trajectories reach  $I^+$ . However, in the region below  $r_h(v)$ , we have infinitely many concave trajectories starting and ending in the (shell-focusing) singularity. They start at  $r(0) = 0$ , increase in the region between  $r_h(v)$  and  $r_0(v)$ , cross the last line, and reach  $r = 0$  again, with finite  $v$ . The trajectory  $r_h(v)$  plays the role of an event horizon, separating two regions with distinct qualitative behavior: one where constant  $u$  null trajectories reach  $I^+$  and another where they start and end in the singularity. This behavior is only possible, of course, because the solutions of (21) fail to be unique at  $r = 0$ . The relevant conformal diagram is also shown in Fig. 3.

For  $0 < \lambda < \lambda_n^c$ , we have three distinct regions according to the concavity of the solutions. They are limited by the two straight lines  $r_-(v)$  and  $r_+(v)$  defined by (37). We have  $r_0(v) < r_-(v) < r_+(v)$  for  $v > 0$ . Both straight lines

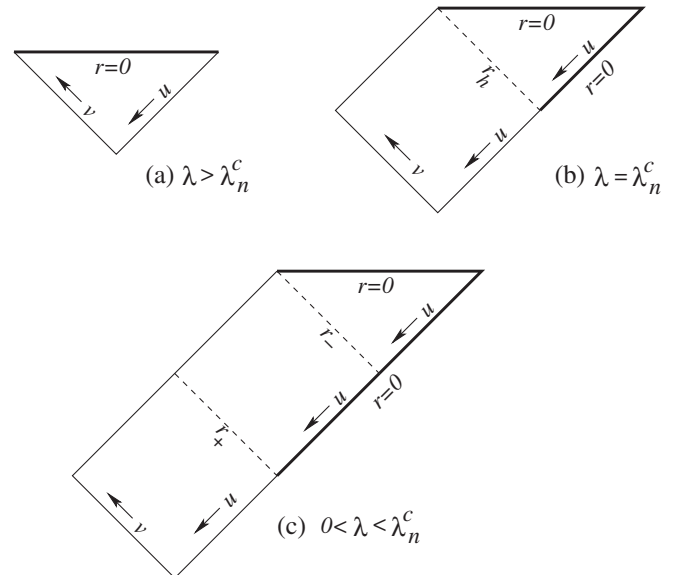


FIG. 3. Conformal diagrams corresponding to the mass function (20) and  $\Lambda = 0$ . Cases (b) and (c) exhibit naked and shell-focusing singularities. For all cases, the curves in  $(v, r)$  are similar to the ones presented in [26], corresponding to the  $n = 4$  case.

$r_-(v)$  and  $r_+(v)$  are also solutions of (21). Between them, solutions are convex. Above  $r_+$  and below  $r_-$ , solutions are concave. The last line is the inner horizon. Inside the horizon, the null trajectories with constant  $u$  start and end in the singularity  $r = 0$ . The line  $r_+$  is an outer horizon. Between them, the constant  $u$  null trajectories start in the naked singularity and reach  $I^+$ . Beyond the outer horizon, all constant  $u$  null trajectories escape the singularity; see Fig. 3.

**B. Black-hole evaporation in a de Sitter spacetime**

As an example of using the semianalytical approach for  $\Lambda \neq 0$ , let us consider the case of an evaporating black hole due to Hawking radiation in a de Sitter  $n$ -dimensional ( $n > 3$ ) spacetime. The mass decreasing rate for these  $n$ -dimensional black holes can be obtained from the  $n$ -dimensional Stefan-Boltzmann law (see, for instance, [35]), leading to

$$\frac{dm}{du} = -am^{-2/(n-3)} \tag{39}$$

where  $a > 0$  is the effective  $n$ -dimensional Stefan-Boltzmann constant. Equation (39) can be immediately integrated, leading to

$$m(u) = m_0 \left(1 - \frac{u}{u_0}\right)^{(n-3)/(n-1)}, \tag{40}$$

where  $u_0$  is the lifetime of a black hole with initial mass  $m_0$ ,

$$au_0 = \frac{n-3}{n-1} m_0^{(n-1)/(n-3)}. \tag{41}$$

We recall that (39) is not expected to be valid in the very final stages of the black-hole evaporation, where the appearance of new emission channels for Hawking radiation [36] can induce changes in the value of the constant  $a$ , and maybe even the usual adiabatic derivation of Hawking radiation would not be valid anymore. We do not address these points here. We assume that the black hole evaporates completely (40) for all  $0 \leq u \leq u_0$  and that  $m = 0$  for  $u > u_0$ . Notice that, since  $m' \rightarrow \infty$  for  $u \rightarrow u_0$ , a naked singularity will be formed after the vanishing of the black hole [10].

We can consider now an  $n$ -dimensional Vaidya metric with a mass term of the form given by (40). This is the first case of an outgoing radial flux we consider, and according to (17),  $B = 1/2$ . However, in this case,  $f(u, v)$  has different sign and the temporal directions for the  $B = -1/2$  case are now spacelike. The transformation  $(u, v) \rightarrow (v, -u)$  restores the temporal and spatial directions [26]. One needs to keep in mind that, for  $B < 0$  (ingoing radiation fields),  $v$  corresponds to the advanced time, whereas for  $B > 0$  (outgoing radiation fields),  $u$  should correspond to the retarded time. In this case, the equivalent of Eq. (15) for the mass function (40) reads

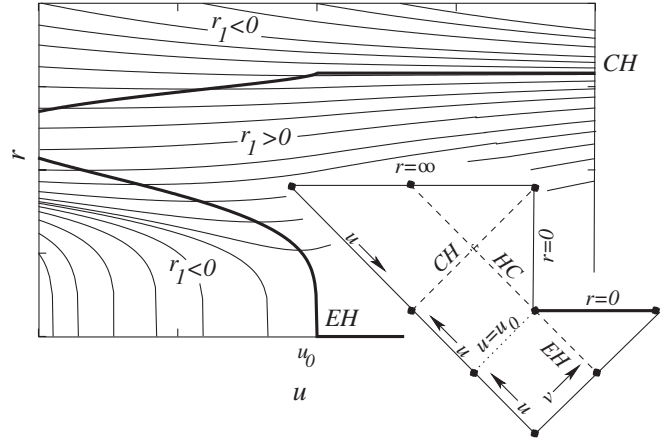


FIG. 4. The  $(u, r)$  plane for an evaporating black hole in an  $n$ -dimensional de Sitter spacetime. The apparent horizons  $r_{EH}(u)$  and  $r_{CH}(u)$  delimit the regions where the solutions of (42) have qualitatively distinct behavior. Since the black hole vanishes for  $u = u_0$ , for  $u > u_0$  the only remaining horizon corresponds to  $r_{CH}(u)$ . The conformal diagram is inserted. For  $u < u_0$ , it represents a usual black hole. For  $u > u_0$ , we have the empty space left after the vanishing of the black hole. The dashed line HC (extension of the black-hole horizon EH) is a Cauchy horizon. Beyond it, we have a breakdown of predictability due to the naked singularity left by the black hole. Part of the cosmological exterior region (the region above CH and below HC) is completely insensitive to the singularity. (The case depicted here corresponds to  $n = 6$ ,  $\Lambda = 5/2$ , and  $m_0 = u_0 = 1, u \geq 0$ .)

$$r_1 = \frac{1}{2} \left( 1 - \frac{2m_0}{n-3} \frac{(1 - u/u_0)^{(n-3)/(n-1)}}{r^{n-3}} - \frac{2\Lambda}{(n-1)(n-2)} r^2 \right), \tag{42}$$

$0 \leq u \leq 0$ . For  $u > u_0$ , we have an empty  $n$ -dimensional de Sitter spacetime. We can now apply the same procedure we have used for the previous cases. We call  $r_{EH}(u)$  and  $r_{CH}(u)$  the curves on the  $(u, r)$  plane delimiting the regions where the solutions of (42) have qualitative distinct behavior. (See Fig. 4.) In this case, since  $A_1 < 0$ , the curve  $r_{EH}(v)$  decreases and reaches  $r = 0$  for  $u = u_0$ , corresponding to the total vanishing of the black hole. After this point,  $r_{CH}(v)$  is the only horizon (a cosmological one) of this spacetime. Before  $u = u_0$ , however, some solutions of (42) that crossed  $r_{EH}(v)$  could reach the singularity at  $r = 0$ . The exterior cosmological region [region of the  $(u, r)$  plane above  $r_{CH}(v)$ ] is insensitive to the singularity. The resulting conformal diagram is shown in Fig. 4. It represents a spacetime with a cosmological horizon and, in its interior, a black hole that evaporates completely leaving behind a naked singularity.

**C. Mass accretion by BTZ black holes**

As an explicit example of the  $n = 3$  case, let us consider a BTZ black hole [28] undergoing a mass accretion process

during a finite time interval. The  $C^1$ -class mass function

$$A(v) = \begin{cases} m_i, & v < -v_0, \\ av(3v_0^2 - v^2) + b, & -v_0 \leq v \leq v_0, \\ m_f, & v > v_0 \end{cases} \quad (43)$$

does correspond to such a situation. Here,  $m_i$  and  $m_f$  stand for, respectively, the initial and final mass, and the constants  $a = (m_f - m_i)/4v_0^3$  and  $b = (m_f + m_i)/2$  are chosen in order to have smooth matches at  $v = \pm v_0$ . For the BTZ case,  $\Lambda = -1/\ell^2$  and the apparent horizon corresponding to the region of the  $(r, v)$  plane where  $r_2 = 0$  is given simply by  $r_{\text{EH}}(v) = \ell\sqrt{A(v)}$ ; see (27). Since  $A_2 > 0$ , we adopt  $B = -1/2$ . The solutions of (27) below the curve  $r_{\text{EH}}(v)$  have  $r_2 < 0$ , whereas the ones above have  $r_2 > 0$ ; see Fig. 5. For  $v < -v_0$ , and analogously for  $v > v_0$ , Eq. (27) can be easily integrated,

$$\frac{r(u, v) - \ell\sqrt{m_i}}{r(u, v) + \ell\sqrt{m_i}} = D_i(u)e^{\sqrt{m_i}v/\ell}, \quad (44)$$

where  $D_i(u)$  is a monotone arbitrary integration function. The region  $r > \ell\sqrt{m_i}$  is obtained by choosing  $D_i(u) > 0$ , whereas  $r < \ell\sqrt{m_i}$  corresponds to  $D_i(u) < 0$ . In the interior region of a BTZ black hole, we see from (44) that the constant  $u$  null geodesics reach the conical singularity at  $r = 0$  in a finite time given by  $(1/\ell)\ln(-1/D_i(u))$ . In contrast with the  $n > 3$  cases, the solutions  $r(v)$  here do not reach  $r = 0$  perpendicularly (see Fig. 5). For the exterior region, the null geodesics can reach the spatial infinity  $r = \infty$  also in a finite time given by  $(1/\ell) \times \ln(1/D_i(u))$ .

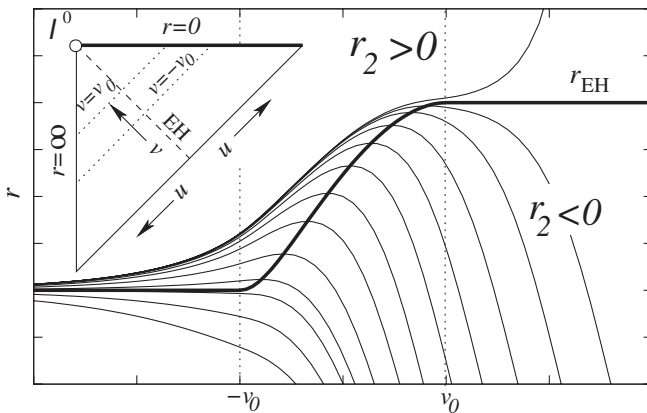


FIG. 5. The  $(v, r)$  plane for a BTZ black hole accreting mass according to (43). In the region below the curve  $r_{\text{EH}}(v)$  (the apparent event horizon),  $r_2 < 0$ . The constant  $u = \bar{u}$  null geodesics associated with the solutions  $r(\bar{u}, v)$  of (27) entering into this region reach the conical singularity at  $r = 0$  with finite  $v$ . The null geodesics confined to the  $r_2 > 0$  region can escape from the singularity, reaching the null and spatial infinity. The conformal diagram is inserted. (The case depicted here corresponds to  $\Lambda = -1$ ,  $B = -1/2$ , and  $m_i = m_f/9 = 1$ .)

The solutions  $r(v)$  for the mass function (43) are obtained by matching, at  $v = \pm v_0$ , the BTZ black hole (44) with the accretion process corresponding to  $-v_0 < v < v_0$ . This situation is depicted in Fig. 5.

#### IV. CONCLUSION

We considered here the problem of constructing the  $n$ -dimensional Vaidya metric with a cosmological constant  $\Lambda$  in double-null coordinates, generalizing the work of [26], where a semianalytical approach is proposed to attack the equations derived previously by Waugh and Lake [24] for  $n = 4$  and  $\Lambda = 0$ . Some new exact solutions are also presented. For  $n = 3$ , in particular, the problem reduces to the solution of the second order linear ordinary differential equation (31), allowing the analytical study of large classes of mass functions  $A(v)$ . As an example, for  $\Lambda < 0$  and  $A(v) = m_0 + m_1 \tanh v$ ,  $m_0 > m_1$ , Eq. (31) is equivalent to the hypergeometric equation, after a proper redefinition of  $v$ . This situation corresponds to a BTZ black hole with mass  $m_0 - m_1$  that accretes some amount of mass smoothly and ends with mass  $m_0 + m_1$ . This kind of smooth exact solution, with two ‘‘asymptotic BTZ’’ regions ( $v \rightarrow \pm\infty$ ), is very interesting to the study of creation of particles in nonstationary spacetimes [37].

The generalized semianalytical approach is also useful to the study of quasinormal modes of varying mass  $n$ -dimensional black holes. In particular, an interesting point is the study of the stationary regime for quasinormal modes, described in [14], in higher dimensional space and in the presence of a cosmological constant. The semianalytical approach can help the understanding of the relation anticipated by Konoplya [38] for the quasinormal modes of  $n$ -dimensional black holes with  $\Lambda = 0$ :  $\omega_R \propto nr_0^{-1}$ , where  $\omega_R$  and  $r_0$  stand for the oscillation frequency and the black hole radius, respectively. Some preliminary results [15] suggest the existence of a stationary regime such that, for slowly varying mass  $n$ -dimensional black holes,  $\omega_R(v) \propto nr_0^{-1}(v)$ . As another application for  $\Lambda \neq 0$ , we have the dynamical formation of a Nariai solution, which could be accomplished by choosing  $A(v)$  such that  $r_{\text{EH}}(v) \rightarrow r_{\text{CH}}(v)$  for large  $v$ . These points are still under investigation.

#### ACKNOWLEDGMENTS

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#### APPENDIX A: SOME GEOMETRICAL QUANTITIES

We present here some geometrical quantities necessary to reproduce the equations of Sec. II. Such quantities are the  $n$ -dimensional ( $n > 2$ ) generalization of the ones calculated previously by Synge [25] and Waugh and Lake [24] for  $n = 4$ . Only nonvanishing terms are given.



Recall that our  $n$ -dimensional line element is given by

$$ds^2 = -2f(u, v)dudv + r^2(u, v)d\Omega_{n-2}^2, \quad (\text{A1})$$

where  $d\Omega_{n-2}^2$  is the unity  $(n - 2)$  sphere, assumed here to be spanned by the angular coordinates  $\theta_i, i = 1, \dots, (n - 2)$ ,

$$d\Omega_{n-2}^2 = \sum_{i=1}^{n-2} \left( \prod_{j=1}^{i-1} \sin^2 \theta_j \right) d\theta_i^2. \quad (\text{A2})$$

The Christoffel symbols are

$$\begin{aligned} \Gamma_{uu}^u &= \frac{f_1}{f}, & \Gamma_{vv}^v &= \frac{f_2}{f}, \\ \Gamma_{u\theta_k}^{\theta_k} &= \frac{r_1}{r}, & \Gamma_{v\theta_k}^{\theta_k} &= \frac{r_2}{r}, \\ \Gamma_{\theta_k\theta_k}^u &= r \left( \prod_{i=1}^{k-1} \sin^2 \theta_i \right) \frac{r_2}{f}, \\ \Gamma_{\theta_k\theta_k}^v &= r \left( \prod_{i=1}^{k-1} \sin^2 \theta_i \right) \frac{r_1}{f}, \\ \Gamma_{\theta_k\theta_k}^{\theta_j} &= -\frac{1}{2} \left( \prod_{i=j+1}^{k-1} \sin^2 \theta_i \right) \sin 2\theta_j, & (k > j) \\ \Gamma_{\theta_j\theta_k}^{\theta_k} &= \cot \theta_j, & (k \neq j). \end{aligned} \quad (\text{A3})$$

The totally covariant Riemann tensor  $R_{abcd}$  is given by

$$\begin{aligned} R_{uvuv} &= \frac{f_1 f_2}{f} - f_{12}, \\ R_{u\theta_k u \theta_k} &= r \left( \prod_{i=1}^{k-1} \sin^2 \theta_i \right) \left( \frac{f_1 r_1}{f} - r_{11} \right), \\ R_{v\theta_k v \theta_k} &= r \left( \prod_{i=1}^{k-1} \sin^2 \theta_i \right) \left( \frac{f_2 r_2}{f} - r_{22} \right), \\ R_{u\theta_k \theta_k v} &= r \left( \prod_{i=1}^{k-1} \sin^2 \theta_i \right) r_{12}, \\ R_{\theta_j \theta_k \theta_j \theta_k} &= r^2 \left( \prod_{l=1}^{j-1} \sin^2 \theta_l \right) \left( \prod_{i=j+1}^{k-1} \sin^2 \theta_i \right) \left( 1 + 2 \frac{r_1 r_2}{f} \right) \\ & \quad (k > j). \end{aligned} \quad (\text{A4})$$

Using (13), we get from the purely angular components of the Riemann tensor

$$R_{\theta_1 \theta_k \theta_1 \theta_k}^{\theta_k} = 1 + \frac{r_2}{B}, \quad (\text{A5})$$

for any  $k > 1$ . As for the  $n = 4$  case [30], such components are invariant under transformations involving solely the double-null coordinates  $(u, v) \rightarrow (U(u, v), V(u, v))$  and are candidates for defining the mass of the solution for  $\Lambda = 0$ .

The Ricci tensor, obtained by the contraction  $R_{ab} = R_{cab}{}^c$ , is given by

$$\begin{aligned} R_{uu} &= (n - 2) \left( \frac{f_1 r_1}{f r} - \frac{r_{11}}{r} \right), \\ R_{uv} &= \frac{f_1 f_2}{f^2} - \frac{f_{12}}{f} - (n - 2) \frac{r_{12}}{r}, \\ R_{vv} &= (n - 2) \left( \frac{f_2 r_2}{f r} - \frac{r_{22}}{r} \right), \\ R_{\theta_k \theta_k} &= \left( \prod_{i=1}^{k-1} \sin^2 \theta_i \right) \left[ \frac{2}{f} (r r_{12} + (n - 3) r_1 r_2) + (n - 3) \right]. \end{aligned} \quad (\text{A6})$$

Two scalar quantities are relevant for our purposes: the scalar curvature  $R = g^{ab} R_{ab}$  and the Kretschmann scalar  $K = R_{abcd} R^{abcd}$ . They are given by

$$R = \frac{2}{f} \left( f_{12} - \frac{f_1 f_2}{f} \right) + \frac{n - 2}{r^2} \left( 4 \frac{r_1 r_2}{f} + (n - 3) \left( 1 + \frac{r r_{12}}{f} \right) \right) \quad (\text{A7})$$

and

$$\begin{aligned} K &= 2 \frac{(n - 1)(n - 3)}{r^2} \left[ \frac{4}{f^2} \left( r_{12}^2 + r_{11} r_{22} + \frac{r_1 r_2 f_1 f_2}{f^2} \right. \right. \\ & \quad \left. \left. - \frac{r_{22} r_1 f_1}{f} - \frac{r_{11} r_2 f_2}{f} \right) + \frac{1}{r^2} \left( 1 + 4 \frac{r_1^2 r_2^2}{f^2} + \frac{r_1 r_2}{f} \right) \right] \\ & \quad + \frac{4}{f^2} \left( \frac{f_1^2 f_2^2}{f^2} + f_{12}^2 - 2 \frac{f_{12} f_1 f_2}{f} \right). \end{aligned} \quad (\text{A8})$$

### APPENDIX B: SOME POLYNOMIAL RESULTS

The evaluation of the zeros of  $f$  given by (23) involves finding the positive roots of the polynomial

$$p_n(x) = x^{n-3} \left( x - \frac{1}{2} \right) = -\frac{\lambda}{n - 3}, \quad (\text{B1})$$

for  $\lambda > 0$  and  $n > 3$ . The only roots of  $p_n(x)$  are  $x = 1/2$  and  $x = 0$ , the latter with multiplicity  $n - 3$ . For positive  $x, p_n(x) < 0$  only in the interval  $x \in (0, 1/2)$ . The minimal value of  $p_n(x)$  in such an interval corresponds to

$$p_n(x_{\max}^p) = -\frac{1}{n - 3} \lambda_n^c = -\frac{1}{n - 3} \left( \frac{n - 3}{2(n - 2)} \right)^{n-2}, \quad (\text{B2})$$

where

$$x_{\max}^p = \frac{n - 3}{2(n - 2)}, \quad (\text{B3})$$

the unique root of  $p'_n(x)$  in the interval. Thus, for  $\lambda > \lambda_n^c$ , the polynomial (B1) has no roots. For  $\lambda = \lambda_n^c$  there is only one root ( $x = x_{\max}^p$ ), while for  $0 < \lambda < \lambda_n^c$  there are two roots,  $x_1$  and  $x_2$ , with  $0 < x_1 < x_{\max}^p < x_2 < 1/2$ .

The determination of the curves (34) involves the evaluation of the positive roots of the polynomial

$$q_n(x) = x^{n-3} \left( 1 - \frac{2\Lambda}{(n - 1)(n - 2)} x^2 \right) = \frac{2A}{n - 3}, \quad (\text{B4})$$

with  $A > 0$  and  $n > 3$ . For the anti-de Sitter case,  $\Lambda < 0$ , the only root of  $q_n(x)$  is  $x = 0$ ;  $q_n(x)$  and  $q'_n(x)$  are both positive for all  $x > 0$ , and, therefore, (B4) has only one positive root. Equation (34), in this case, defines only one curve.

On the other hand, for the de Sitter case,  $\Lambda > 0$ ,  $q_n(x)$  always has three roots:  $x = \pm\sqrt{(n-1)(n-2)/2\Lambda}$  and  $x = 0$ , the last with multiplicity  $n-3$ . Moreover, for positive  $x$ ,  $q_n(x) \geq 0$  only for  $x \in [0, \sqrt{(n-1)(n-2)/2\Lambda}]$ , and the maximum value of  $q_n(x)$  in such an interval is  $q_n(x_{\max}^q)$ , where

$$x_{\max}^q = \sqrt{\frac{(n-2)(n-3)}{2\Lambda}}, \quad (\text{B5})$$

the only point of the interval where  $q'_n(x) = 0$ . For  $0 < 2A/(n-3) < q_n(x_{\max}^q)$ , (B4) always has two roots,  $x_1$  and

$x_2$ ,  $x_1 < x_{\max}^q < x_2$ . For  $2A/(n-3) = q_n(x_{\max}^q)$ , there is only one root,  $x = x_{\max}^q$ , and for  $2A/(n-3) > q_n(x_{\max}^q)$  there is no root at all. In this case, provided that  $0 < 2A/(n-3) < q_n(x_{\max}^q)$ , Eq. (34) defines two curves,  $r_{\text{EH}}(v)$  and  $r_{\text{CH}}(v)$  (see Fig. 2), such that  $r_{\text{EH}}(v) < x_{\max}^q < r_{\text{CH}}(v)$ .

If  $r(v)$  is a curve defined by (34), its derivative is given by

$$r'(v) = \frac{2}{n-3} \frac{A'(v)}{r^{n-4}} \left( (n-3) - \frac{2\Lambda}{n-2} r^2 \right)^{-1}. \quad (\text{B6})$$

If  $\Lambda < 0$ , it is clear that  $r'(v)$  has the same sign of  $A'(v)$ . For  $\Lambda > 0$ , the curves  $r_{\text{EH}}(v)$  and  $r_{\text{CH}}(v)$  have qualitatively distinct behavior. Since  $r_{\text{EH}}(v) < x_{\max}^q$ ,  $r'_{\text{EH}}(v)$  has the same sign as  $A'(v)$ . The curve  $r_{\text{CH}}(v)$  has the converse behavior, since  $r_{\text{CH}}(v) > x_{\max}^q$ .

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