Can Gravity Probe B usefully constrain torsion gravity theories?

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In most theories of gravity involving torsion, the source for torsion is the intrinsic spin of matter. Since the spins of fermions are normally randomly oriented in macroscopic bodies, the amount of torsion generated by macroscopic bodies is normally negligible. However, in a recent paper, Mao *et al.* (arXiv:gr-qc/0608121) point out that there is a class of theories, including the Hayashi-Shirafuji (1979) theory, in which the angular momentum of macroscopic spinning bodies generates a significant amount of torsion. They further argue that, by the principle of action equals reaction, one would expect the angular momentum of test bodies to couple to a background torsion field, and therefore the precession of the Gravity Probe B gyroscopes should be affected in these theories by the torsion generated by the Earth. We show that in fact the principle of action equals reaction does not apply to these theories, essentially because the torsion is not an independent dynamical degree of freedom. We examine in detail a generalization of the Hayashi-Shirafuji theory suggested by Mao *et al.* called Einstein-Hayashi-Shirafuji theory. There are a variety of different versions of this theory, depending on the precise form of the coupling to matter chosen for the torsion. We show that, for any coupling to matter that is compatible with the spin transport equation postulated by Mao *et al.*, the theory has either ghosts or an ill-posed initial-value formulation. These theoretical problems can be avoided by specializing the parameters of the theory and in addition choosing the standard minimal coupling to matter of the torsion tensor. This yields a consistent theory, but one in which the action equals reaction principle is violated, and in which the angular momentum of the gyroscopes does not couple to the Earth's torsion field. Thus, the Einstein-Hayashi-Shirafuji theory does not predict a detectable torsion signal for Gravity Probe B. There may be other torsion theories which do.

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I. INTRODUCTION AND SUMMARY

A. Theories of gravity with torsion

General relativity (GR) is in good agreement with all current experimental data from laboratory tests, the solar system,[1] and binary pulsars [2]. However there is good motivation to consider modifications and extensions of GR: low energy limits of string theory and higher dimensional models usually involve extra, long range universal forces mediated by scalar fields, and in addition the observed acceleration of the Universe may be due to a modification of GR at large distances. One may hope that new and highly accurate experiments, such as Gravity Probe B (GPB) [3], will enable one to test for deviations from GR.

One natural framework in which to generalize GR is to allow the connection $\Gamma^{\mu}{}_{\nu\lambda}$ to be a nonsymmetric independent dynamical variable instead of being determined by the metric. The covariant derivative of a vector v^{μ} is defined in the usual way as

$$\nabla_{\mu} v^{\nu} = \partial_{\mu} v^{\nu} + \Gamma^{\nu}{}_{\mu\lambda} v^{\lambda}. \tag{1.1}$$

If one retains the assumption that the connection is metric compatible, $\nabla_{\mu} g_{\nu\lambda} = 0$, then it can be shown that the connection is determined uniquely by the metric and the torsion tensor

$$S_{\mu\nu}{}^{\lambda} \equiv \Gamma^{\lambda}{}_{[\mu\nu]}. \tag{1.2}$$

One obtains

$$\Gamma^{\lambda}{}_{\mu\nu} = \{^{\lambda}_{\mu\nu}\} - K_{\mu\nu}{}^{\lambda} \tag{1.3}$$

where the first term is the Levi-Civita connection determined by the metric and

$$K_{\mu\nu}{}^{\lambda} = -S_{\mu\nu}{}^{\lambda} - S^{\lambda}{}_{\nu\mu} - S^{\lambda}{}_{\mu\nu} \tag{1.4}$$

is called the contorsion tensor. A spacetime equipped with a metric $g_{\mu\nu}$ and a torsion tensor is called a Riemann-Cartan spacetime. The Riemann tensor $R^{\mu}{}_{\nu\lambda\rho}$ of the full connection (1.3) is related to the usual Riemann tensor $\tilde{R}^{\mu}{}_{\nu\lambda\rho}$ of the Levi-Civita connection by

$$R^{\mu}{}_{\nu\lambda\rho} = \tilde{R}^{\mu}{}_{\nu\lambda\rho} + \tilde{\nabla}_{\rho} K_{\lambda\nu}{}^{\mu} - \tilde{\nabla}_{\lambda} K_{\rho\nu}{}^{\mu} + K_{\lambda\sigma}{}^{\mu} K_{\rho\nu}{}^{\sigma} - K_{\rho\sigma}{}^{\mu} K_{\lambda\nu}{}^{\sigma}, \tag{1.5}$$

where $\tilde{\nabla}_{\mu}$ is the Levi-Civita derivative operator, and our convention for the Riemann tensor is given by Eq. (B5). The action for theories of gravity in this framework has the generic form

$$S[g_{\mu\nu}, \Gamma^{\lambda}{}_{\mu\nu}, \Psi] = S_G[g_{\mu\nu}, \Gamma^{\lambda}{}_{\mu\nu}] + S_{\rm matter}[g_{\mu\nu}, \Gamma^{\lambda}{}_{\mu\nu}, \Psi], \tag{1.6}$$

where S_G and $S_{\rm matter}$ are the gravitational and matter actions and Ψ collectively denotes the matter fields. There is

[1] An exception is the Pioneer anomaly [1], which remains controversial.

extensive literature on theories of gravity of this type; see the review articles [4–7].

It is often useful to reexpress these theories using the tetrad formalism [8]. In this formalism the independent variables are taken to be a tetrad of four linearly independent vector fields $e_a{}^\mu(x)$, and a tetrad connection $\omega_\mu{}^{ab} = -\omega_\mu{}^{ba}$ defined by

$$\vec{\nabla}_{\vec{e}_a}\vec{e}_b = e_a{}^\mu \omega_\mu{}^c{}_b \vec{e}_c. \quad (1.7)$$

Here tetrad indices a run from 0 to 3 and are raised and lowered using the Minkowski metric $\eta_{ab} \equiv \mathrm{diag}(-1, 1, 1, 1)$. The two sets of variables, $g_{\mu\nu}, \Gamma^\lambda{}_{\mu\nu}$ and $e_a{}^\mu$, $\omega_\mu{}^{ab}$, are related by[2]

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu, \quad (1.8a)$$
$$\Gamma^\lambda{}_{\mu\nu} = e_a{}^\lambda [e^a{}_{\nu,\mu} + \omega_\mu{}^a{}_b e^b{}_\nu], \quad (1.8b)$$

where the dual basis of one-forms $e^a{}_\mu$ is defined by

$$e_a{}^\mu e^b{}_\mu = \delta^b_a. \quad (1.9)$$

The action of the theory in terms of the tetrad variables is

$$S[e_a{}^\mu, \omega_\mu{}^{ab}, \Psi] = S_G[e_a{}^\mu, \omega_\mu{}^{ab}] + S_{\mathrm{matter}}[e_a{}^\mu, \omega_\mu{}^{ab}, \Psi]. \quad (1.10)$$

Normally the theory is invariant under local Lorentz transformations $\Lambda_a{}^b = \Lambda_a{}^b(x)$ of the tetrad

$$e_a{}^\mu \to \Lambda_a{}^b e_b{}^\mu, \quad (1.11a)$$
$$\omega_\mu{}^a{}_b \to \Lambda^a{}_c \Lambda_b{}^d \omega_\mu{}^c{}_d - \Lambda^a{}_{c,\mu}\Lambda_b{}^c, \quad (1.11b)$$

together with the corresponding transformations of any fermionic matter fields.

There are three different categories of theories involving torsion:

(i) Theories in which torsion is an independent dynamical variable and the field equations for torsion are algebraic, for example, the Einstein-Cartan theory [4] in which the gravitational action is proportional to the Ricci scalar. In these theories the torsion vanishes in vacuum.

(ii) Theories in which the torsion tensor is an independent dynamical variable and a propagating degree of freedom, for example, Refs. [4,6,7,9–12]. The source for torsion is the tensor

$$\sigma_{ab}{}^\mu \equiv \frac{1}{\sqrt{-g}} \frac{\delta S_{\mathrm{matter}}}{\delta \omega_\mu{}^{ab}}.$$

For the standard, minimal coupling of torsion to matter, this tensor is a measure of density of fundamental or intrinsic spin, and thus is very small when averaged over macroscopic distances in unpolarized matter [4,6]. (For nonstandard couplings it is conceivable that this tensor could be non-negligible.)

(iii) Theories in which the torsion is not an independent dynamical variable, but is specified in terms of some other degrees of freedom in the theory, for example, a scalar potential [13,14] or a rank 2 tensor potential [7,15,16].

A particular special case of the third category are the so-called *teleparallel theories* [17–32]. In these theories the only dynamical variable is the tetrad $e_a{}^\mu$; the tetrad connection $\omega_\mu{}^{ab}$ is not an independent variable. In addition, the local Lorentz transformations (1.11a) are *not* a symmetry of the theory. The tetrad therefore contains 6 extra physical degrees of freedom which are normally gauged away by the local Lorentz symmetry. In linear perturbation theory about flat spacetime these extra degrees of freedom act like an antisymmetric, rank 2 tensor potential for the torsion; see Sec. II below for more details. In teleparallel theories the torsion is defined to be

$$S_{\mu\nu}{}^\lambda = \tfrac{1}{2} e_a{}^\lambda (e^a{}_{\nu,\mu} - e^a{}_{\mu,\nu}). \quad (1.12)$$

The form of this equation is invariant under coordinate transformations but not under the local Lorentz transformations (1.11a). It follows from this definition and from Eq. (1.8b) that the tetrad connection $\omega_\mu{}^{ab}$ vanishes, and it follows that the Riemann tensor

$$R^{ab}{}_{\mu\nu} \equiv e^a{}_\lambda e^b{}_\sigma R^{\lambda\sigma}{}_{\mu\nu} = \omega_\nu{}^{ab}{}_{,\mu} - \omega_\mu{}^{ab}{}_{,\nu} + \omega_\mu{}^a{}_c \omega_\nu{}^{cb} - \omega_\nu{}^a{}_c \omega_\mu{}^{cb} \quad (1.13)$$

also vanishes. Thus in teleparallel theories the curvature (1.5) of the full connection vanishes, and so on the right-hand side of Eq. (1.5) the contorsion terms must cancel the curvature term. Hence, if the spacetime metric is close to that predicted by general relativity, so that $\tilde{R} \sim M/r^3$ at a distance r from a mass M, then the torsion must be of order $S \sim K \sim M/r^2$. Therefore, teleparallel theories generically predict a non-negligible torsion for macroscopic, unpolarized bodies, unlike conventional torsion theories.

B. Constraining torsion with Gravity Probe B

The prevailing lore about torsion theories has been that they are very difficult to distinguish from general relativity, since the torsion generated by macroscopic bodies is normally negligibly small for the reasons discussed above [7,33]. However, a recent paper by Mao, Tegmark, Guth, and Cabi (MTGC) [34] points out that teleparallel theories are an exception in this regard. They suggest that GPB might be an ideal tool to probe such torsion theories. In particular, they argue that, since the angular momentum of macroscopic bodies generates torsion, one would expect that the angular momentum of test bodies such as the GPB gyroscopes would couple to the Earth's torsion field, by the principle of "action equals reaction."

[2]The connection (1.8b) is automatically metric compatible by virtue of the antisymmetry of $\omega_\mu{}^{ab}$ on a and b.

MTGC also review the literature on the equations of motion and spin precession of test bodies in torsion theories [14,16,35–44]. They argue that, because there is some disagreement in this literature, and because the precise form of the coupling of torsion to matter is not known, it is reasonable to assume that test bodies fall along geodesics of the full connection (called autoparallels), and to assume that the spin of a gyroscope is parallel transported with respect to the full connection. They introduce a theory of gravity called the Einstein-Hayashi-Shirafuji (EHS) theory, a generalization of an earlier teleparallel theory of Hayashi and Shirafuji [18], and compute the constraints that GPB will be able to place on this theory for their assumed equations of motion and spin transport.

In this paper we reexamine the utility of GPB as a probe of torsion gravity theories. We agree with the general philosophy expressed by MTGC that the precise form of the coupling of torsion to matter is something that should be tested experimentally rather than assumed *a priori*. However, while it is conceivable that there could exist couplings which would predict a detectable torsion signal for GPB, we show that teleparallel theories and the EHS theory, in particular, do not.

We start by discussing the action equals reaction principle. This appears to be a robust and very generic argument, indicating that the angular momentum of a test body should couple to torsion in theories where spinning bodies generate torsion. In fact, there is a loophole in the argument, and, in particular, it does not apply to the EHS theory, as we show in detail in Sec. IV below. The nature of the loophole can be understood using a simple model. Consider in Minkowski spacetime the following theory of two scalar fields Φ_1 and Φ_2 and a particle of mass m,

$$S = -\frac{1}{2}\int d^4x[(\nabla\Phi_1)^2 + (\nabla\Phi_2)^2] - \int d\lambda(m + q\Phi_1)\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}. \tag{1.14}$$

Here λ is a parameter along the worldline and q is a scalar charge. In this theory the particle generates a Φ_1 field but not a Φ_2 field, and correspondingly it feels a force from the Φ_1 field but not the Φ_2 field, in accordance with the action equals reaction idea. However, consider now the theory in noncanonical variables $\tilde\Phi_1 = \Phi_1$, $\tilde\Phi_2 = \Phi_1 + \Phi_2$. In terms of these variables the particle generates both a $\tilde\Phi_1$ field and a $\tilde\Phi_2$ field, but feels a force only from the $\tilde\Phi_1$ field, in violation of action equals reaction. Thus, we see that the action equals reaction principle can only be applied to the independent dynamical variables in the theory, which diagonalize the kinetic energy term in the action. In the EHS theory, the torsion and metric are not independent dynamical variables; see Sec. IV below.

C. The Einstein-Hayashi-Shirafuji theory

In the remainder of this paper we examine in detail the EHS theory suggested by MTGC. In this theory the defining relation (1.12) between torsion and tetrad for teleparallel theories is replaced by

$$S_{\mu\nu}{}^{\lambda} = \frac{\sigma}{2}e_a{}^{\lambda}(e^a{}_{\nu,\mu} - e^a{}_{\mu,\nu}), \tag{1.15}$$

where parameter σ lies in the range $0 \le \sigma \le 1$. For $\sigma = 1$ this reduces to the teleparallel case, while for $\sigma = 0$ the torsion tensor vanishes. Thus, the EHS theory interpolates between GR at $\sigma = 0$, in which the torsion vanishes but the Riemann tensor is, in general, different from zero, and the Hayashi-Shirafuji teleparallel theory [18] at $\sigma = 1$, in which the Riemann tensor vanishes but the torsion tensor is, in general, different from zero.

The only dynamical variable in the gravitational sector of this theory is the tetrad $e_a{}^\mu$, and, as for the teleparallel theories, the theory is generally covariant but not invariant under the local Lorentz transformations (1.11a) of the tetrad. The action for the theory can be written as

$$S[e_a{}^\mu, \Psi] = S_G[e_a{}^\mu] + S_{\rm matter}[e_a{}^\mu, \omega_\mu{}^{ab}, \Psi], \tag{1.16}$$

where Ψ denotes the matter fields and, in the second term, $\omega_\mu{}^{ab}$ denotes the tetrad connection obtained from the torsion tensor (1.15) via Eqs. (1.3), (1.4), and (1.8b). The matter action $S_{\rm matter}$ is not specified by MTGC, so there are different versions of the EHS theory depending on the form of the coupling to torsion chosen in this matter action. The gravitational action is given by

$$S_G[e_a{}^\mu] = \int d^4x\sqrt{-g}[a_1 t_{\mu\nu\lambda}t^{\mu\nu\lambda} + a_2 v_\nu v^\nu + a_3 a_\nu a^\nu]. \tag{1.17}$$

Here a_1, a_2, and a_3 are free parameters with dimensions of mass squared (we use units with $\hbar = c = 1$), and the tensors $t_{\mu\nu\lambda}$, v^ν, and a^ν are defined to be the irreducible pieces of the torsion tensor (1.15), but with the factor of σ removed:

$$v_\mu = \sigma^{-1}S_{\mu\lambda}{}^{\lambda}, \tag{1.18a}$$

$$a_\mu = \tfrac{1}{6}\sigma^{-1}\epsilon_{\mu\nu\rho\sigma}S^{\sigma\rho\nu}, \tag{1.18b}$$

$$t_{\lambda\mu\nu} = \sigma^{-1}S_{\nu(\mu\lambda)} + \tfrac{1}{6}(g_{\nu\lambda}v_\mu + g_{\nu\mu}v_\lambda) - \tfrac{1}{3}g_{\lambda\mu}v_\nu. \tag{1.18c}$$

Also in Eq. (1.17) g denotes the metric determinant, where $g_{\mu\nu}$ is given in terms of the tetrad by Eq. (1.8a). For fixed a_1, a_2, and a_3, this gravitational action is independent of the parameter σ; this parameter enters the theory only through the dependence of the matter action on the torsion tensor (1.15).[3]

In Appendix B we show that the gravitational action can be rewritten in terms of the Ricci scalar $R(\{\})$ of the Levi-Civita connection as

$$S_G[e_a{}^\mu] = \int d^4x\sqrt{-g}[d_1 R(\{\}) + 4d_2 v_\nu v^\nu + 9d_3 a_\nu a^\nu], \tag{1.19}$$

where $d_1 = -3a_1/8$, $d_2 = (a_1 + a_2)/4$, $d_3 = a_3/9 - a_1/4$. We shall refer to the three-dimensional space parametrized by (d_1, d_2, d_3) as the gravitational-action parameter space.

As mentioned above, there are different versions of the EHS theory, depending on the form of the matter action $S_{\rm matter}$ chosen; MTGC do not specify a matter action. Consider now what is required in order to predict the signal seen by GPB. The experiment consists of an Earth-orbiting satellite carrying four very stable gyroscopes, and the measured quantity is the time dependence of the angles between the spins of the gyroscopes and the direction to a fixed guide star. To compute this quantity in an arbitrary Riemann-Cartan spacetime, it is sufficient to know the equations of motion and of spin transport for a spinning point particle.[4] These equations can be computed, in principle, for any matter action. MTGC assume that the matter action is such that the trajectory of the spinning point particle is either an *autoparallel* (a geodesic of the full connection) or an *extremal* (a geodesic of the metric), and that its spin is parallel transported with respect to the full connection.

D. Requirements necessary to ensure physical viability of the theory

In this paper we constrain the parameter space of the EHS theory by imposing a set of physical requirements. To simplify the analysis we first linearize the EHS theory with respect to a flat torsion-free spacetime, and then impose physical requirements on the linearized theory. The linearized theory is completely characterized by two tensor fields: a symmetric field $h_{\mu\nu}$, and an antisymmetric field $a_{\mu\nu}$. In terms of these fields the torsion tensor is given by $S_{\mu\nu}{}^\lambda = \sigma(h^\lambda{}_{[\nu,\mu]} - 2a^\lambda{}_{[\nu,\mu]})/2$, and the metric is given by $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

We impose three types of requirements. First, we require that the theory have no ghosts, i.e., that the Hamiltonian of the theory be bounded from below. This requirement rules out most of the three-dimensional parameter space of the gravitational action S_G. The remaining viable subdomain of the parameter space consists of two intersecting two-dimensional planes.

Second, we further require that the theory have a well-posed initial-value formulation. This means that, if the physical degrees of freedom are specified on an initial spacelike hypersurface, the future evolution of these degrees of freedom is uniquely determined. Now, for many theories some of the degrees of freedom are nonphysical and are associated with a gauge symmetry. For example, in classical electrodynamics the field equations for the vector potential A_μ are invariant under the gauge transformation $A_\mu \to A_\mu + \varphi_{,\mu}$. These field equations therefore do not predict a unique evolution for the vector potential, and correspondingly consist of a set of underdetermined partial differential equations. Nevertheless, classical electrodynamics has a well-posed initial-value formulation, because the degrees of freedom whose evolution cannot be predicted are pure gauge.

A similar situation arises in linearized EHS theory. There, the gravitational action is invariant under certain symmetries of the dynamical variables (not diffeomorphisms), and correspondingly the field equations form a set of an underdetermined partial differential equations. Therefore the theory can have a well-posed initial-value formulation only if the undetermined degrees of freedom are pure gauge. This can be the case only if the matter action is also invariant under the symmetries. However, the equations of motion and spin precession postulated by MTGC do not respect these symmetries, and they should inherit such a property from the matter action. We conclude therefore that the initial-value formulation is ill-posed. This argument applies in most of the remaining portion of the parameter space of the theory. The subdomain that is not excluded by this argument and by the requirement of no ghosts consists of a single line in the three-dimensional space.

Third, in this remaining subdomain, the linearized gravitational action reduces to that of GR; it depends on the tetrad only through the metric $g_{\mu\nu}$. In particular, this means that the torsion tensor is completely undetermined in vacuum, and so for a generic matter action, the motion of test bodies cannot be predicted. The corresponding inconsistency of the Hayashi-Shirafuji theory in this limit has been previously discussed in Refs. [18,23,25–27,29–31]. This inconsistency is avoided if one constructs a special matter action in which the unpredictability of the torsion tensor is associated with a gauge symmetry of the theory, in this case the predications of the EHS theory coincide with the predictions of GR.

Finally, our argument that the initial-value formulation is ill posed can be evaded by modifying the coupling of the torsion tensor to matter in the theory. Rather than postulat-

[3]The parameters defining the theory are therefore (a_1, a_2, a_3, σ). MTGC use a different set of parameters $(c_1, c_2, c_3, \kappa, \sigma)$ which are not all independent. The relation between the two sets of parameters can be derived using the identity of Appendix B and is $a_1 = \sigma^2 c_1 - 4/(3\kappa)$, $a_2 = \sigma^2 c_2 + 4/(3\kappa)$, $a_3 = \sigma^2 c_3 - 3/\kappa$.

[4]One also needs to know the trajectories of photons, but these are determined by gauge invariance to be just the null geodesics of the metric, as in GR [4].

TABLE I. A summary of the status of the Einstein-Hayashi-Shirafuji theory [34] in different sectors of its parameter space. The rows of the table represent these different sectors; the parameters d_2 and d_3, which appear in the gravitational part of the action, are defined in Eq. (1.19) in the text. There are different versions of the Einstein-Hayashi-Shirafuji theory depending on the precise form chosen of the coupling of the torsion tensor to matter fields. These different versions are the columns of the table. ''Autoparallel'' means that it is assumed that the matter coupling is such that freely falling bodies move on geodesics of the full connection, while ''Extremal'' means they move on geodesics of the Levi-Civita connection determined by the metric. These were the two cases considered by Mao *et al.* [34]. ''Standard matter coupling'' means the standard, minimal coupling of torsion to matter fields [4,6], which, in general, gives rise to motions of test bodies that are neither autoparallel nor extremal. The meanings of the various entries in the table are as follows. ''Ghosts'' means that some of the degrees of freedom in the theory are ghostlike at short distances, signaling an instability that rules out the theory. ''I.V.F.'' means that the theory does not have a well-posed initial-value formulation, and so is ruled out. ''Inconsistent'' means that the theory does not predict the value of the torsion tensor, so the motion of test bodies cannot be predicted, while ''GR'' means that the theory reduces to general relativity. Finally ''no GPB torsion signal'' means that there is no torsion-induced coupling between the Earth's angular momentum and that of the Gravity Probe B gyroscopes; there is only a coupling between the fundamental spins of the Earth's fermions and of those in the gyroscopes, which gives a negligible signal as those spins are randomly oriented.

	Autoparallel	Extremal	Standard matter coupling
Sector of parameter space			
$\mathcal{D}_0 = \{d_2 \neq 0, d_3 \neq 0\}$	Ghosts	Ghosts	Ghosts
$\mathcal{D}_1 = \{d_2 \neq 0, d_3 = 0\}$	I.V.F.	I.V.F.	I.V.F.
$\mathcal{D}_2 = \{d_2 = 0, d_3 \neq 0\}$	I.V.F.	I.V.F.	Consistent but no GPB torsion signal (action = reaction violated)
$\mathcal{D}_3 = \{d_2 = 0, d_3 = 0\}$	Inconsistent	Inconsistent/GR	Inconsistent

ing the equations of motion and spin precession used by MTGC, we instead assume that the coupling of torsion to matter is the standard, minimal coupling described in Refs. [4,6]. For this coupling, the matter action *is* invariant under the symmetries of the gravitational action discussed above, in a portion of the parameter space, and so the theory has a well-posed initial-value formulation in which the undetermined degrees of freedom are interpreted as gauge degrees of freedom. This is the interpretation suggested in the original paper by Hayashi and Shirafuji [18]. For this case, we again examine the linearized theory for the fields $h_{\mu\nu}$ and $a_{\mu\nu}$. We find that $h_{\mu\nu}$ satisfies the same equation as the metric perturbation in GR, while $a_{\mu\nu}$ satisfies a wave equation (with a suitable choice of gauge) whose source is obtained from the intrinsic spin density of matter. As discussed earlier, this implies that, for a macroscopic object for which the spins of the elementary particles are not correlated over macroscopic scales, $a_{\mu\nu}$ will be negligible. Hence the spacetime of the linearized theory is completely characterized by the metric alone, and so its predictions coincide with those of GR[5] and there will be no extra signal in GPB.

A summary of the status of the EHS theory in various different cases discussed above is given in Table I.

To summarize, there are no cases in which the EHS theory gives a detectable torsion signal in GPB. However, it is nevertheless possible that other torsion theories in the other categories discussed in Sec. I A, with a suitable choice of matter coupling, could predict a detectable signal. Various possibilities for nonminimal couplings are discussed by Shapiro [6]. It would be interesting to find a torsion theory that predicts a detectable torsion signal for GPB; such a theory would be an example to which the theory-independent framework developed by MTGC (a generalization of the parametrized post-Newtonian framework to include torsion) could be applied.

E. Organization of this paper

This paper is organized as follows. In Sec. II we derive the dynamical variables and the action of the linearized EHS theory. In Sec. II B we study the action for the antisymmetric field $a_{\mu\nu}$, temporarily setting the symmetric field $h_{\mu\nu}$ to zero. We show that this theory has ghosts on a subdomain of the parameter space. Section II E and Appendix A extend this result to the complete linearized theory, including the symmetric field $h_{\mu\nu}$, thereby ruling out a subdomain of the parameter space. We then focus on the complementary subdomain and show that it is invariant under certain symmetries. Section III reviews the necessary requirements for a well-posed initial-value formulation. In the subsection of Sec. III we use these requirements to rule out a subdomain of the parameter space that has an ill-posed initial-value formulation. The remaining portion of the parameter space is discussed in Sec. II F

Finally, Sec. IV considers the EHS theory with the standard matter-torsion coupling. In a certain portion of parameter space this theory has a well-posed initial-value formulation and no ghosts, but we show that the deviations

[5]This is despite the fact that the torsion tensor is generically nonzero. The torsion tensor is not an independent degree of freedom in this limit; it is given in terms of the metric by the first term in Eq. (2.3b).

of its predictions from those of GR are negligible for unpolarized macroscopic bodies. Final conclusions are given in Sec. V.

II. LINEARIZATION ABOUT FLAT, TORSION-FREE SPACETIME OF THE EINSTEIN-HAYASHI-SHIRAFUJI THEORY

A. Action and variables of linearized theory

To linearize the EHS theory we first decompose the tetrads $e_a{}^\mu$ and dual one-forms $e^a{}_\mu$ into background tetrads and one-forms and perturbations:

$$e_a{}^\mu = b_a{}^\mu + \delta c_a{}^\mu, \tag{2.1a}$$

$$e^a{}_\mu = b^a{}_\mu + \delta e^a{}_\mu. \tag{2.1b}$$

We assume that the background tetrads $b_a{}^\mu$ are constants for which the metric (1.8a) is the Minkowski metric, $g_{\alpha\beta} = \eta_{\alpha\beta}$, and for which the torsion (1.15) is vanishing, $S_{\mu\nu}{}^\lambda = 0$. Thus, to zeroth order, the spacetime is flat and torsion-free. Throughout this paper we will work to leading order in the tetrad perturbations. Hereafter, unless we explicitly state otherwise, Greek indices are raised and lowered with $\eta_{\mu\nu}$ and Latin indices with η_{ab}.

From the definition (1.9) of the dual basis applied to both the full tetrads and the background tetrads, we find that $\delta e^a{}_\mu b_a{}^\nu = -\delta c_a{}^\nu b^a{}_\mu$. Thus we can take $\delta e^a{}_\mu$ to be the fundamental variable of the theory. We next convert this quantity into a spacetime rank 2 tensor using the background tetrad, and take the independent symmetric and antisymmetric pieces. This yields the definitions

$$h_{\mu\nu} \equiv 2\delta e^b{}_{(\mu} b^a{}_{\nu)} \eta_{ab}, \tag{2.2a}$$

$$a_{\mu\nu} \equiv \delta e^b{}_{[\mu} b^a{}_{\nu]} \eta_{ab}. \tag{2.2b}$$

The formulas (1.8a) and (1.15) for the metric and torsion now yield

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{2.3a}$$

$$S_{\mu\nu}{}^\lambda = \frac{\sigma}{2}(h^\lambda{}_{[\nu,\mu]} - 2a^\lambda{}_{[\nu,\mu]}). \tag{2.3b}$$

The linearized gravitational action $S_G^{\rm linear}$ can now be obtained by substituting the expressions (2.3) for the metric and torsion into the action (1.19) and expanding to quadratic order in $h_{\mu\nu}$ and $a_{\mu\nu}$. The resulting action can be written schematically in the form

$$S_G^{\rm linear} = S_S[h_{\mu\nu}] + S_C[h_{\mu\nu}, a_{\alpha\beta}] + S_A[a_{\mu\nu}], \tag{2.4}$$

where S_S is quadratic in the symmetric tensor $h_{\mu\nu}$, S_A is quadratic in the antisymmetric tensor $a_{\mu\nu}$, and S_C contains the cross terms.

B. The antisymmetric term in the action

We now focus on the antisymmetric term S_A in the action, ignoring for the moment the other two terms. For this theory we derive two kinds of results. First, we constrain the parameter space (d_1, d_2, d_3) by imposing the requirement of no ghosts, and second we derive symmetry transformations under which the action S_A (with specific parameters) is invariant. Later in Sec. II E we will extend some of these results to the complete linearized action $S_G^{\rm linear}$.

The antisymmetric action $S_A[a_{\mu\nu}]$ is constructed from the antisymmetric field $a_{\mu\nu}$ in Minkowski spacetime, and consists of terms that are products of derivatives of $a_{\mu\nu}$, of the form $(\partial_\lambda a_{\mu\nu})^2$. Actions of this type also arise in gravitational theories with a nonsymmetric metric and have been extensively studied. See Refs. [45,46] for a discussion of the existence of ghosts in theories of this type.

There are only three linearly independent terms of the form $(\partial_\lambda a_{\mu\nu})^2$, namely,

$$a_{\mu\lambda}{}^{,\lambda} a^\mu{}_\sigma{}^{,\sigma}, \qquad a_{\mu\nu,\lambda} a^{\mu\nu,\lambda}, \qquad a_{\mu\nu,\lambda} a^{\mu\lambda,\nu}.$$

From these terms one can construct only two functionally independent actions, since the identity

$$a^{\mu\lambda}{}_{,\lambda} a_{\mu\sigma}{}^{,\sigma} - a^{\mu\lambda,\sigma} a_{\mu\sigma,\lambda} = \partial_\lambda (a^{\mu\lambda} a_{\mu\sigma}{}^{,\sigma} - a^{\mu\lambda,\sigma} a_{\mu\sigma}) \tag{2.5}$$

shows that a linear combination of the terms is a divergence which can be converted to a surface term upon integrating and thereby discarded. Therefore, the most general action of this type can be written as

$$S_A = \int d^4x [d_2 a_{\mu\lambda}{}^{,\lambda} a^{\mu\sigma}{}_{,\sigma} + d_3 a^*_{\mu\lambda} a^{*\mu\sigma}{}_{,\sigma}], \tag{2.6}$$

where $a^{*\mu\nu} \equiv -\epsilon^{\mu\nu\rho\sigma} a_{\rho\sigma}/2$ (here $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita tensor of a flat spacetime), and d_2, d_3 are free parameters. Indeed an explicit calculation using Eqs. (1.18), (1.19), and (2.3) shows that S_A is given by the expression (2.6), where the parameters d_2 and d_3 are those defined after Eq. (1.19). Next, we specialize to a particular Lorentz frame, and rewrite the action in terms of the vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ defined by $E_i = a_{0i}$ and $B_i = \frac{1}{2}\epsilon_{ijk} a_{jk}$, where i, j, k run from 1 to 3. This gives

$$a_{\mu\lambda}{}^{,\lambda} a^{\mu\sigma}{}_{,\sigma} = (\dot{\mathbf{E}} + \nabla \times \mathbf{B})^2 - (\nabla \cdot \mathbf{E})^2, \tag{2.7}$$

where an overdot denotes differentiation with respect to time. The corresponding expression for $a^*_{\mu\lambda}{}^{,\lambda} a^{*\mu\sigma}{}_{,\sigma}$ has the same form, except for the substitutions $\mathbf{E} \rightarrow \mathbf{B}$ and $\mathbf{B} \rightarrow -\mathbf{E}$.

We now consider the case where both d_2 and d_3 are nonzero. In this case Hamiltonian density corresponding to the action (2.6) takes the form

$$\begin{aligned} H_A = {}& \frac{1}{4d_3}(\boldsymbol{\pi}_{\mathbf{B}} + 2d_3 \nabla \times \mathbf{E})^2 + d_3(\nabla \cdot \mathbf{B})^2 \\ & - d_3(\nabla \times \mathbf{E})^2 + \frac{1}{4d_2}(\boldsymbol{\pi}_{\mathbf{E}} - 2d_2 \nabla \times \mathbf{B})^2 \\ & + d_2(\nabla \cdot \mathbf{E})^2 - d_2(\nabla \times \mathbf{B})^2, \end{aligned} \tag{2.8}$$

where $\boldsymbol{\pi}_{\mathbf{E}}$ and $\boldsymbol{\pi}_{\mathbf{B}}$ denote the momenta conjugate to $\mathbf{E}$ and $\mathbf{B}$.

C. Ghosts

We now constrain the values of the coefficients d_2 and d_3 by demanding that $\int H_A d^3x$ be bounded from below. Let us start by considering the coefficient d_3, and suppose first that $d_3 < 0$. At a given point in space, keeping the values of $\mathbf{E}$, $\mathbf{B}$, and $\boldsymbol{\pi}_{\mathbf{E}}$ fixed, we can make H_A arbitrarily negative by choosing $\boldsymbol{\pi}_{\mathbf{B}}$ to be arbitrarily large. The same is clearly true for the integrated Hamiltonian $\int H_A d^3x$. Next suppose that $d_3 > 0$. By fixing $\mathbf{B}$, $\nabla \cdot \mathbf{E}$, $\boldsymbol{\pi}_{\mathbf{E}}$ and adjusting the value of $\boldsymbol{\pi}_{\mathbf{B}}$ to keep the value of $\boldsymbol{\pi}_{\mathbf{B}} + 2d_3 \nabla \times \mathbf{E}$ fixed, we can make H_A arbitrarily negative, this time by choosing an arbitrarily large value for $\nabla \times \mathbf{E}$. By applying analogous considerations to d_2 we reach the conclusion that the theory defined by H_A has ghosts on the domain

$$\mathcal{D}_0 = \{(d_1, d_2, d_3) | d_2 \neq 0, d_3 \neq 0\} \tag{2.9}$$

in parameter space. Similar analyses can be found in Refs. [45,46] for nonsymmetric gravity theories, and in Ref. [47] for teleparallel gravity theories.

D. Symmetries

We now consider the domain in parameter space not excluded by the above analysis, which consists of the two 2D regions

$$\mathcal{D}_1 = \{(d_1, d_2, d_3) | d_2 \neq 0, d_3 = 0\}, \tag{2.10a}$$

$$\mathcal{D}_2 = \{(d_1, d_2, d_3) | d_2 = 0, d_3 \neq 0\}, \tag{2.10b}$$

together with the line

$$\mathcal{D}_3 = \{(d_1, d_2, d_3) | d_2 = 0, d_3 = 0\}. \tag{2.10c}$$

The antisymmetric action S_A vanishes identically on $\mathcal{D}_3$, so in this subsection we will not consider $\mathcal{D}_3$ any further.

We are interested in the symmetries of the antisymmetric action S_A on the domains $\mathcal{D}_1$ and $\mathcal{D}_2$. From the formula (2.6) for the action we see that these two domains are isomorphic to one another under the duality transformation

$$a_{\mu\nu} \rightarrow a^*_{\mu\nu} \quad \text{or} \quad \boldsymbol{E} \rightarrow \boldsymbol{B}, \qquad \boldsymbol{B} \rightarrow -\boldsymbol{E}. \tag{2.11}$$

Therefore it is sufficient to focus on one of the domains, say $\mathcal{D}_1$. From Eqs. (2.6) and (2.7), the antisymmetric action on this domain is

$$S_{A|\mathcal{D}_1} = \int d_2[(\dot{\mathbf{E}} + \nabla \times \mathbf{B})^2 - (\nabla \cdot \mathbf{E})^2] d^4x. \tag{2.12}$$

Consider now the initial-value problem for $\mathbf{E}$ and $\mathbf{B}$. Suppose that we are given sufficient initial data on some constant time hypersurface, and that we wish to determine the time evolution of $\mathbf{E}(t)$ and $\mathbf{B}(t)$ using the action (2.12). Note that this action is independent of the longitudinal part of $\mathbf{B}$, which means that the evolution of this longitudinal part can be prescribed arbitrarily, independent of the initial data. Therefore, the field equations for $\mathbf{E}$ and $\mathbf{B}$ must form a set of an underdetermined partial differential equations. As discussed in the Introduction, this can only be consistent if the undetermined degrees of freedom can be interpreted as being gauge degrees of freedom.

The action (2.12) is invariant under the symmetry

$$a_{\mu\nu} \rightarrow a_{\mu\nu} + \epsilon_{\mu\nu\alpha\beta}\chi^{\alpha,\beta}, \tag{2.13}$$

where $\chi^\alpha(x)$ is an arbitrary vector field. This can be seen from the fact that the action (2.12) is given by the first term in Eq. (2.6), which depends on $a_{\mu\nu}$ only through its divergence $a_{\mu\nu}{}^{,\nu}$. Similarly on the domain $\mathcal{D}_2$ the antisymmetric action (2.6) is invariant under the symmetry

$$a_{\mu\nu} \rightarrow a_{\mu\nu} + \chi_{[\mu,\nu]}. \tag{2.14}$$

These are the symmetries that are responsible for the indeterminacy in the evolution equations. We will study in later sections the conditions under which these symmetries can be interpreted as gauge symmetries, thus allowing the theory to have a well-posed initial-value formulation. As discussed in the Introduction, the gauge symmetry interpretation requires the matter action to be invariant[6] under the symmetries (2.13) and (2.14).

E. The complete linearized action

Up to now we have ignored the pieces S_S and S_C of the complete linearized action (2.4), and have studied a reduced theory depending only on the antisymmetric field $a_{\mu\nu}$ described by the action S_A alone. We showed that this reduced theory has ghosts if both d_2 and d_3 are nonzero, i.e., on the domain $\mathcal{D}_0$. In Appendix A this result is generalized to the complete linearized theory, including the symmetric field $h_{\mu\nu}$, showing that the complete theory also has ghosts in the domain $\mathcal{D}_0$. Essentially we show that, whenever the Hamiltonian $\int H_A d^3x$ is unbounded from below, then the corresponding Hamiltonian of the full linearized theory is also unbounded from below.

The symmetries (2.13) and (2.14) of the reduced theory on the domains $\mathcal{D}_1$ and $\mathcal{D}_2$ also generalize to the complete linearized theory. This can be seen as follows. Since the symmetries only involve the antisymmetric field $a_{\mu\nu}$, the only additional term in the complete action (2.4) whose invariance needs to be checked is the cross term $S_C[h_{\mu\nu}, a_{\rho\sigma}]$. This cross term can be written as

$$S_C = \int h_{\alpha\beta,\gamma} a_{\mu\nu,\rho} P^{\alpha\beta\gamma\mu\nu\rho} d^4x,$$

where $P^{\alpha\beta\gamma\mu\nu\rho}$ is a tensor constructed from the Minkowski metric. Integrating by parts and discarding a surface term yields

[6] A general discussion of the problems that arise when the gravitational action is invariant under a symmetry not shared by the matter action can be found in Leclerc [31].

$$S_C = -\int h_{\alpha\beta} a_{\mu\nu,\rho\gamma} P^{\alpha\beta\gamma\mu\nu\rho} d^4x.$$

At least two of the indices on $a_{\mu\nu,\rho\gamma}$ must be contracted with one another, and since $a_{\mu\nu}$ is antisymmetric it follows that only divergence terms of the form $a_{\mu\sigma,}{}^{\sigma}{}_{\gamma}$ can appear. These divergence terms are invariant under the symmetry (2.13).

For the symmetry (2.14), we compute explicitly the cross term S_C specialized to the domain $\mathcal{D}_2$. Since $d_2 = 0$, the only term in the general action (1.19) that can contribute to this cross term is the $a_\nu a^\nu$ term involving the square of the axial piece of the torsion; the Ricci scalar term depends only on $h_{\mu\nu}$. Using the definition (1.18b) of this axial piece together with the formula (2.3b) for the torsion in terms of $h_{\mu\nu}$ and $a_{\mu\nu}$ gives

$$a_\mu = \tfrac{1}{6}\eta_{\mu\lambda}\epsilon^{\lambda\nu\rho\sigma} a_{\nu\sigma,\rho}. \tag{2.15}$$

Since this depends only on the antisymmetric field $a_{\mu\nu}$ it does not generate any cross terms, and we conclude that the cross term S_C vanishes identically on the domain $\mathcal{D}_2$. Therefore the complete action is invariant under the symmetry (2.14) on $\mathcal{D}_2$.

F. The general relativity limit of the gravitational action

So far we have considered the domains $\mathcal{D}_0$, $\mathcal{D}_1$, and $\mathcal{D}_2$ of the gravitational-action parameter space. We now focus on the remaining domain $\mathcal{D}_3$. From the definitions (1.19) and (2.10c) we find that in this domain the gravitational action reduces to that of general relativity, so it is invariant under local Lorentz transformations of the tetrad $e_a{}^\mu \to \Lambda_a{}^b(x) e_b{}^\mu$. This invariance guarantees that the left-hand side of the Euler-Lagrange equation

$$-\frac{\delta S_G}{\delta e^a{}_\rho} e^a{}_\mu g_{\rho\nu} = \frac{\delta S_{\rm matter}}{\delta e^a{}_\rho} e^a{}_\mu g_{\rho\nu}$$

is a symmetric tensor [8]. Consistency now requires the right-hand side to be symmetric. However, the torsion tensor is not invariant under local Lorentz transformations. Therefore, a generic matter action that couples between the torsion tensor and matter fields would break the local Lorentz symmetry. This matter action produces a nonsymmetric right-hand side for the Euler-Lagrange equation, thereby rendering the theory inconsistent. The corresponding inconsistency of the Hayashi-Shirafuji theory in this limit has been previously discussed in Refs. [18,23,25–27,29–31]. The inconsistency is avoided if the matter action is invariant under local Lorentz transformations. In this case the torsion tensor is undetermined by the EHS theory, and the theory reduces to GR.

III. INITIAL-VALUE FORMULATION OF THE THEORY

In this section we focus on the domains $\mathcal{D}_1$ and $\mathcal{D}_2$ that are not ruled out by the existence of ghosts, and examine in more detail the conditions under which the theory on these domains has a well-posed initial-value formulation.

Suppose that we specify as initial data $\{a_{\mu\nu}, h_{\alpha\beta}\}$ and $\{\dot{a}_{\mu\nu}, \dot{h}_{\alpha\beta}\}$ on some initial constant time hypersurface, and ask whether the evolution of the fields for all subsequent times is uniquely determined. Now the action (2.6) is invariant under certain symmetries which allow us to generate new solutions that correspond to the same initial data. These symmetries consist of, first, diffeomorphisms $x^\mu \to x^\mu - \xi^\mu(x)$ under which the fields transform as

$$h_{\mu\nu} \to h_{\mu\nu} + 2\xi_{(\mu,\nu)}, \tag{3.1a}$$

$$a_{\mu\nu} \to a_{\mu\nu} + \xi_{[\nu,\mu]}, \tag{3.1b}$$

and second, the symmetries (2.13) on $\mathcal{D}_1$ and (2.14) on $\mathcal{D}_2$. By invoking one of these transformations in a spacetime region to the future of the initial data hypersurface, we can generate new solutions for the field equations that correspond to the same initial data. Therefore the evolution of the fields $h_{\mu\nu}$ and $a_{\mu\nu}$ cannot be uniquely predicted. For the theory to have a well-posed initial-value formulation it is necessary that *all* of these transformations correspond to gauge symmetries, which means that all observables should remain invariant under these transformations. Then the failure of the theory to uniquely predict $h_{\mu\nu}$ and $a_{\mu\nu}$ is merely associated with the unpredictability of nonphysical degrees of freedom.

We now focus on the observables that will be measured by the GPB experiment. If we use the equations of motion and spin transport for test bodies postulated by MTGC, then these observables are not invariant under the symmetries (2.13) and (2.14), as we now show. Thus the initial-value formulation is ill posed for these postulated equations of motion.

Observations with Gravity Probe B

We focus on one of the four GPB gyroscopes, and represent it as a particle with trajectory $z^\alpha(\tau)$ where τ is proper time, and with spin $s^\alpha(\tau)$. We let the 4-momentum of the photons from the distant fixed guide star be k^α. Let θ be the angle, as measured in the frame of the gyroscope, between its spin and the direction to the guide star. Then we have

$$\cos\theta = \frac{\vec{s}\cdot\vec{k}}{(\vec{u}\cdot\vec{k})\sqrt{\vec{s}^2}}, \tag{3.2}$$

where $\vec{s}\cdot\vec{k} = g_{\mu\nu}s^\mu k^\nu$. This can be seen from the formulas for these vectors in the rest frame of the gyroscope: $\vec{u} = (1, \mathbf{0})$, $\vec{k} = \omega(1, -\mathbf{n})$, and $\vec{s} = (0, \mathbf{s})$, where $\mathbf{n}$ is a unit vector in the direction of the star.

The equations of motion and spin precession postulated by MTGC are

$$\frac{Du^\mu}{D\tau} = 0, \qquad \frac{Ds^\mu}{D\tau} = 0, \tag{3.3}$$

where $u^\mu = dz^\mu/d\tau$ is the 4-velocity and

$$\frac{D}{D\tau} \equiv u^\mu \nabla_\mu \tag{3.4}$$

is the covariant derivative operator along the worldline with respect to the full connection ∇_μ. In other words, the particle travels on a geodesic of the full connection (an autoparallel curve). MTGC also consider the possibility that the equation of motion is

$$\frac{\tilde{D}u^\mu}{\tilde{D}\tau} \equiv u^\nu \tilde{\nabla}_\nu u^\mu = 0, \tag{3.5}$$

where $\tilde{\nabla}_\nu$ is the Levi-Civita connection determined by the metric, so that the particle travels along a geodesic of the metric (an extremal curve). As mentioned by MTGC this possibility is theoretically inconsistent since by Eqs. (3.3) the orthogonality of u^μ and s_μ is not maintained during the evolution. Nevertheless, we shall also consider this possibility below. Finally, as mentioned above, photons follow null geodesics of the metric:

$$k^\mu \tilde{\nabla}_\mu k^\nu = 0. \tag{3.6}$$

We now apply the operator $D/D\tau$ to the formula (3.2) for the angle θ, and use the autoparallel equations of motion (3.3) together with $\nabla_\mu g_{\mu\lambda} = 0$. This gives

$$\frac{d[\cos\theta]}{d\tau} = \frac{1}{(\vec{u}\cdot\vec{k})\sqrt{\vec{s}^2}}\left[\left(\vec{s} - \frac{\vec{s}\cdot\vec{k}}{\vec{u}\cdot\vec{k}}\vec{u}\right)\cdot\frac{D\vec{k}}{D\tau}\right]. \tag{3.7}$$

The measurable accumulated change in $\cos(\theta)$ in the interval $\tau_1 \rightarrow \tau_2$ is therefore

$$\Delta[\cos(\theta)] = \int_{\tau_1}^{\tau_2} d\tau \frac{1}{(\vec{u}\cdot\vec{k})\sqrt{\vec{s}^2}}\left[\left(\vec{s} - \frac{\vec{s}\cdot\vec{k}}{\vec{u}\cdot\vec{k}}\vec{u}\right)\cdot\frac{D\vec{k}}{D\tau}\right]. \tag{3.8}$$

Next, we examine how the change in angle (3.8) transforms under the symmetry transformations (2.13) and (2.14). For this purpose it is sufficient to consider the motion of the gyroscope in a flat, torsion-free spacetime, since we are working to linear order. We use Lorentzian coordinates where

$$\left\{ {\mu \atop \alpha\beta} \right\} = \Gamma^\mu{}_{\alpha\beta} = 0,$$

which implies that for an initially static gyroscope u^μ, s^μ, and k^μ are all constants, so that $\Delta[\cos(\theta)] = 0$.

Consider now the effect of the transformations (2.13) or (2.14). Denoting the transformed quantities with primes we find that

$$\left\{ {\mu \atop \alpha\beta} \right\}' = \left\{ {\mu \atop \alpha\beta} \right\} = 0,$$

$\Gamma'^\alpha{}_{\mu\nu} = 0 + \delta\Gamma^\alpha{}_{\mu\nu}$. From Eqs. (3.3) we find that $z'^\mu(\tau') = z^\mu(\tau) + \delta z^\mu(\tau)$ and $s'^\mu(\tau') = s^\mu(\tau) + \delta s^\mu(\tau)$, where both $\delta z(\tau)$ and δs^μ are $O(\chi)$. It turns out that the precise expressions for these quantities are not required for our calculation. Notice that for a fixed distant star the field $k^\mu(x)$ (before the transformation) is approximately constant in a neighborhood of the gyroscope, in the sense that $k^\mu{}_{,\nu} = 0$. Now, Eq. (3.6) implies that $k^\mu(x)$ remains invariant under the transformations. Therefore the derivative of $k'^\mu(x)$ along $z'(\tau')$ is given by

$$\frac{Dk'^\mu}{D\tau'} = \delta\Gamma^\mu{}_{\alpha\beta} u'^\alpha k'^\beta. \tag{3.9}$$

Substituting Eq. (3.9) and $s'_\mu(\tau')$ into Eq. (3.7), and retaining only terms which are $O(\chi)$ (so we can drop the distinction between τ and τ') we obtain

$$\frac{d[\cos(\theta')]}{d\tau} = \frac{1}{(\vec{u}\cdot\vec{k})\sqrt{\vec{s}^2}}\left[s_\mu \delta\Gamma^\mu{}_{\alpha\beta} u^\alpha k^\beta - \frac{\vec{s}\cdot\vec{k}}{\vec{u}\cdot\vec{k}} \delta\Gamma^\mu{}_{\alpha\beta} u_\mu u^\alpha k^\beta \right]. \tag{3.10}$$

From Eqs. (1.3), (1.4), and (2.3) the change $\delta\Gamma^\mu{}_{\alpha\beta}$ in the connection coefficients is

$$\delta\Gamma^\mu{}_{\alpha\beta} = \sigma \eta_{\beta\lambda} \epsilon^{\mu\lambda\rho\sigma} \chi_{\sigma,\rho\alpha}, \tag{3.11}$$

for the symmetry (2.13) and

$$\delta\Gamma^\mu{}_{\alpha\beta} = (\sigma/2)(\chi_{\beta,\alpha}{}^\mu - \chi^\mu{}_{,\alpha\beta}) \tag{3.12}$$

for the symmetry (2.14).

We substitute these transformation rules into Eq. (3.10) and then substitute the result into Eq. (3.8). Recalling that χ_α are arbitrary functions of the coordinates, we find that by invoking the transformations (2.13) or (2.14) we can set $\Delta[\cos(\theta)]$ to have an arbitrary value. Thus, the observable angle is not invariant under the symmetry transformations, and hence they cannot be interpreted to be gauge transformations. [In particular, this implies that the matter action must be noninvariant]. Consequently the initial-value formulation of the theory is ill posed.

We now repeat this analysis for extremal worldlines satisfying Eq. (3.5). Equation (3.10) acquires an additional term

$$-\frac{\vec{s}\cdot\vec{k}}{(\vec{u}\cdot\vec{k})^2\sqrt{\vec{s}^2}}\vec{k}\cdot\frac{D\vec{u}}{D\tau} = -\frac{\vec{s}\cdot\vec{k}}{(\vec{u}\cdot\vec{k})^2\sqrt{\vec{s}^2}}(\delta\Gamma^\mu{}_{\alpha\beta}k_\mu u^\alpha u^\beta)$$

on the right-hand side. In addition, the change to extremal worldlines alters the quantity δu^μ, but this does not appear in the final formula (3.10). As before we find that the initial-value formulation is ill posed.

Finally, MTGC also consider the possibility of an extremal worldline together with the following equation for the spin precession:

$$\frac{Ds_{\alpha\beta}}{D\tau} = 0. \tag{3.13}$$

Here the antisymmetric tensor $s_{\alpha\beta}$ is related to the particle spin through $s^\mu = \epsilon^{\mu\nu\rho\sigma} u_\nu s_{\rho\sigma}$. This relation guarantees that the orthogonality condition $s^\mu u_\mu = 0$ is satisfied throughout the motion of the particle.[7] By examining the transformation of $\Delta[\cos(\theta)]$ under the symmetries (2.13) and (2.14) for an extremal worldline for which the law for the spin precession is given by (3.13), we find as before that the initial-value formulation is ill posed.

IV. EINSTEIN-HAYASHI-SHIRAFUJI THEORY WITH STANDARD MATTER COUPLING

The analysis so far suggests that, in order to obtain a consistent theory, one needs to choose a matter action which is invariant under the symmetries (2.13) or (2.14) of the gravitational action. This would allow those symmetries to be interpreted as gauge symmetries and allow the theory to have a well-posed initial-value formulation. In this section we show that the standard, minimal coupling of matter to torsion [4,6] does respect the symmetry (2.14), and so one does obtain a consistent linearized theory on the domain $\mathcal{D}_2$ by choosing this coupling.[8] For the special case of $\sigma = 1$, this matter action is the one suggested in the original Hayashi-Shirafuji paper [18]. Following the analysis of Hayashi and Shirafuji, we show that the predictions of this linearized theory coincide with those of linearized GR for macroscopic sources with negligible net intrinsic spin.

The standard Dirac action in an Einstein-Cartan spacetime is $S_D = \int d^4x\sqrt{-g}L_D$, where

$$L_D = \frac{i}{2} e_a{}^\mu(\bar{\psi}\gamma^a D_\mu \psi - \overline{D_\mu\psi}\gamma^a\psi) - m\bar{\psi}\psi, \tag{4.1a}$$

$$D_\mu = \partial_\mu - \frac{i}{4}\sigma^{bc} g_{\rho\nu} e_b{}^\nu \nabla_\mu e_c{}^\rho, \tag{4.1b}$$

$$\nabla_\mu e_c{}^\rho = \partial_\mu e_c{}^\rho + \Gamma^\rho{}_{\mu\nu} e_c{}^\nu, \tag{4.1c}$$

$$\sigma^{bc} = \frac{i}{2}[\gamma^b, \gamma^c], \tag{4.1d}$$

and γ^b are Dirac matrices with the representation given in Ref. [48] that satisfy $\gamma^a\gamma^b + \gamma^b\gamma^a = -2\eta^{ab}$. Also $\bar{\psi}$ denotes the adjoint spinor defined by $\bar{\psi} = \psi^\dagger\gamma^0$, where $\dagger$ denotes Hermitian conjugation. The torsion tensor enters this action only through the covariant derivative in Eq. (4.1b). As is well known, this action can be written as the usual torsion-free action together with a coupling of the fermion to the axial piece (1.18b) of the torsion tensor [6]. Using the definitions (1.4) and (1.18) we obtain

$$L_D = \frac{i}{2} e_a{}^\mu(\bar{\psi}\gamma^a \tilde{D}_\mu \psi - \overline{\tilde{D}_\mu\psi}\gamma^a\psi) - m\bar{\psi}\psi + \frac{3}{2}\sigma e_b{}^\mu a_\mu \bar{\psi}\gamma^5\gamma^b\psi. \tag{4.2}$$

Here $\tilde{D}_\mu = \partial_\mu - i\sigma^{bc} g_{\rho\nu} e_b{}^\nu \tilde{\nabla}_\mu e_c{}^\rho/4$,

$$\tilde{\nabla}_\mu e_c{}^\rho = \partial_\mu e_c{}^\rho + \left\{ {\rho \atop \mu\nu} \right\} e_c{}^\nu,$$

and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, where we use the convention $\epsilon_{0123} = \sqrt{-g}$.

Now the first line in the Dirac action (4.2) depends only on the metric, and, in particular, it is independent of $a_{\mu\nu}$, so it is trivially invariant under the transformation (2.14). The second line depends on $a_{\mu\nu}$ only through the axial piece a_μ of the torsion, which is given by Eq. (2.15), and which is also invariant under (2.14). Therefore the entire action (4.2) is invariant under the symmetry.

We conclude that on the domain $\mathcal{D}_2$ we have a consistent theory with a well-posed initial-value formulation, in which the symmetry (2.14) is a gauge symmetry. The self-consistency of this theory for $\sigma = 1$ was previously discussed by Leclerc [32]. From Eqs. (1.19) and (2.10b), the complete action for the theory is

$$S[e^a{}_\mu, \Psi] = \int d^4x\sqrt{-g}\left[\frac{1}{2\hat{\kappa}}R(\{\}) + 9d_3 a_\mu a^\mu\right] + S_D[e^a{}_\mu, \psi, \bar{\psi}] + \ldots, \tag{4.3}$$

where $\hat{\kappa} = 1/(2d_1)$. Here the ellipsis denotes additional terms in the standard model of particle physics that are not coupled to the torsion; the only term that couples to the torsion is the Dirac action for the fermions (the "minimal coupling" scheme of Refs. [4,6]).

The linearized equation of motion for $e^a{}_\mu$ obtained from this action gives equations for $h_{\mu\nu}$ and $a_{\mu\nu}$:

$$-\tfrac{1}{2}\Box\bar{h}_{\mu\nu} + \bar{h}_{\rho(\mu,\nu)}{}^{\rho} - \tfrac{1}{2}\eta_{\mu\nu}\bar{h}_{\rho\lambda}{}^{,\rho\lambda} = \hat{\kappa}T_{(\mu\nu)}, \tag{4.4a}$$

$$\Box a_{\mu\nu} + 2a_{\rho[\mu,\nu]}{}^{\rho} = \frac{1}{d_3}T_{[\mu\nu]}. \tag{4.4b}$$

Here $\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$, $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \eta_{\mu\nu}h_\rho{}^\rho$, $\mathcal{T}_{\mu\nu}$ is the nonsymmetric energy-momentum tensor defined by

$$\mathcal{T}_\mu{}^\nu \equiv \frac{1}{\sqrt{-g}}\frac{\delta S_{\rm matter}}{\delta e^a{}_\nu} e^a{}_\mu, \tag{4.5}$$

and $T_{\mu\nu}$ is its leading order term in a perturbative expansion (i.e. $T_{\mu\nu}$ is independent of $h_{\mu\nu}$ and $a_{\mu\nu}$). The source terms in Eqs. (4.4) obey the conservation laws $T_{[\mu\nu]}{}^{,\nu} = 0$ and $T_{(\mu\nu)}{}^{,\nu} = 0$, by virtue of the invariance of the matter action under the transformations (2.14) and (3.1). By using these transformations we can impose the gauge conditions $\bar{h}_{\mu\nu}{}^{,\nu} = 0$ and $a_{\mu\nu}{}^{,\nu} = 0$, thereby reducing the field equa-

[7] As noted by MTGC, normally one also demands a "transversality" condition $s_{\alpha\beta}u^\beta = 0$. However, by Eqs. (3.5) and (3.13) this condition cannot be maintained along an extremal worldline. This also implies that the norm of s^μ is not constant along the evolution.

[8] Note that this implies, in particular, that the standard matter coupling does not predict either extremal curves or autoparallel curves for the motions of test bodies, since those cases are not invariant under the symmetry (2.14).

tions to wave equations:

$$\Box \bar{h}_{\mu\nu} = -2\hat{\kappa}T_{(\mu\nu)}, \tag{4.6a}$$

$$\Box a_{\mu\nu} = \frac{1}{d_3}T_{[\mu\nu]}. \tag{4.6b}$$

The first of these is the usual linearized Einstein equation.

Next we examine the antisymmetric piece of the stress energy tensor which acts as a source for $a_{\mu\nu}$ in Eq. (4.6b). One can show [18] that this antisymmetric piece is related to the divergence of the spin density tensor by

$$T_{[\beta\alpha]} = \sigma b^a{}_\alpha b^b{}_\beta [\sigma_{ab}{}^\mu{}_{,\mu}]_{b_c{}^\nu}, \tag{4.7}$$

where the subscript $b_c{}^\nu$ indicates evaluation at the background values of the tetrad. Recall that the spin density is defined by

$$\sigma_{ab}{}^\mu \equiv \frac{1}{\sqrt{-g}} \frac{\delta S_{\text{matter}}[e^f{}_\alpha, \omega_\nu{}^{cd}, \Psi]}{\delta \omega_\mu{}^{ab}},$$

where in this definition the matter action is considered to be a functional of the *independent* variables $e^f{}_\alpha$, $\omega_\nu{}^{cd}$, and Ψ. The matter action can be brought to this desired form by substituting $D_\mu = \partial_\mu - \frac{i}{4}\sigma^{bc}g_{\rho\mu}\omega^\rho{}_{bc}$ into the expression for L_D in Eq. (4.1a). Relation (1.7) guarantees that this expression for D_μ is equal to our original expression in Eq. (4.1b).

Equations (4.6b) and (4.7) imply that $a_{\mu\nu}$ is sourced only by the intrinsic spin density of matter. As we have discussed, integrating Eq. (4.6b) for a macroscopic object for which the spins of the elementary particles are not aligned over a macroscopic scale gives a negligible $a_{\mu\nu}$ [18], and consequently the predictions of the linearized theory coincide with those of GR. Thus there is no extra torsion-related signal predicted for GPB for this theory.

The lack of an experimental signature of torsion may seem strange, since the torsion tensor is nonvanishing even in the limit where one can neglect intrinsic spin. As discussed in the Introduction, the explanation is that the torsion is not an independent dynamical degree of freedom. More specifically, to linear order, macroscopic bodies give rise to a metric perturbation $h_{\mu\nu}$ in the same way as in GR, and then the torsion is simply defined to be

$$S_{\mu\nu}{}^\lambda = \frac{\sigma}{2} h^\lambda{}_{[\nu,\mu]} \tag{4.8}$$

[c.f. the first term in Eq. (2.3b)]. This definition has no dynamical consequence, since only the axial piece (1.18b) of torsion couples to matter, and the expression (4.8) has no axial piece.

V. CONCLUSIONS

Preliminary results from Gravity Probe B will be announced in April 2007. The primary scientific goals of the experiment are to verify the predictions of general relativity for geodetic precession and for dragging of inertial frames [3]. However the mission is also potentially useful as a probe of modifications of general relativity.

One class of theories of gravity that GPB could potentially usefully constrain consists of theories involving a dynamical torsion tensor. Mao *et al.* suggested a particular class of torsion theories that they argued would predict a measurable torsion signal for GPB [34]. We have shown that this particular class of theories is internally consistent in only a small region of its parameter space, and in that consistent region it does not predict any signal for GPB. There may exist other torsion theories which could be usefully constrained by GPB. It would be interesting to find such theories.

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APPENDIX A: GHOSTS IN THE LINEARIZED THEORY

In this appendix we show that the Hamiltonian of the complete linearized EHS theory (2.4) is unbounded below whenever the corresponding Hamiltonian of the antisymmetric term (2.6) is unbounded from below. This allows us to deduce the existence of ghosts in the complete theory from their existence in the reduced theory (2.6).

It is sufficient to show this property for the case of finite dimensional, quadratic dynamical systems. We consider a system with N degrees of freedom whose Lagrangian and Hamiltonian are given by

$$L = \tfrac{1}{2}\dot{q}_m\dot{q}_n K_{mn} - V(q_m), \tag{A1}$$

$$H = \tfrac{1}{2}p_m p_n (K^{-1})_{mn} + V(q_m). \tag{A2}$$

Here q_m denotes the generalized coordinates and $p_m = K_{mn}\dot{q}_n$ denotes the conjugate momenta, where the indices m, n run from 1 to N, and $K_{mn} = K_{nm}$. Recall that the antisymmetric action S_A was obtained by ignoring some of the dynamical variables of the linearized EHS theory. In a discrete theory this corresponds to writing a reduced Lagrangian L_r that depends only on some of the generalized coordinates q_i, where i runs from 1 to M, $M < N$. The reduced theory has the following Lagrangian and Hamiltonian:

$$L_r = \tfrac{1}{2}\dot{q}_i\dot{q}_j k_{ij} - U(q_i), \tag{A3}$$

$$H_r = \tfrac{1}{2}\tilde{p}_i\tilde{p}_j (k^{-1})_{ij} + U(q_i). \tag{A4}$$

Here $\tilde{p}_i = k_{ij}\dot{q}_j$, where $k_{ij} = K_{ij}$ for $i, j = 1, \ldots, M$ and

$$U(q_1, \ldots, q_M) = V(q_1, \ldots, q_M, 0, \ldots, 0), \tag{A5}$$

i.e., the potential $U(q_i)$ is obtained from $V(q_m)$ by setting $q_m = 0$ for $m = M + 1, \ldots, N$.

Now suppose that the potential term $U(q_i)$ of the reduced Hamiltonian H_r is unbounded from below. It follows immediately from the definition (A5) that $V(q_m)$ is also unbounded from below, and so the complete Hamiltonian (A2) is unbounded from below.

Suppose next that the kinetic term of the reduced Hamiltonian H_r is unbounded from below. By virtue of Eq. (A4) this means that at least one of the eigenvalues of $(k^{-1})_{ij}$ must be negative. Denoting the eigenvalues of the matrix k_{ij} by $\lambda_{(i)}$, $1 \leq i \leq M$, we find that there exists an eigenvalue l for which $\lambda_{(l)}^{-1} < 0$. This implies that there exists an M-dimensional eigenvector $w_i^{(l)}$ for which $w_i^{(l)} w_j^{(l)} k_{ij} = \lambda_{(l)} < 0$. We now construct an N-dimensional vector defined by $\eta_m = (w_1^{(l)}, \ldots, w_M^{(l)}, 0, \ldots, 0)$. By definition, this vector satisfies $\eta_m \eta_n K_{mn} = \lambda_{(l)} < 0$, implying that K_{mn} has a negative eigenvalue. This means that the complete Hamiltonian (A2) is unbounded from below.

APPENDIX B: ALTERNATIVE FORM OF GRAVITATIONAL ACTION

In this appendix we derive the identity

$$\int \sqrt{-g} R(\{\}) d^4x = \int \sqrt{-g} \left[-\frac{8}{3} t_{\mu\nu\lambda} t^{\mu\nu\lambda} + \frac{8}{3} v_\nu v^\nu - 6 a_\nu a^\nu \right], \quad \text{(B1)}$$

where $R(\{\})$ is the Ricci scalar of the Levi-Civita connection, and $t_{\mu\nu\lambda}$, v^μ, and a^μ are the irreducible pieces (1.18) of the torsion tensor with the factor of σ removed. Combining this identity with the formula (1.17) for the gravitational action of the EHS theory yields the alternative form (1.19) of that action.

The idea is to introduce a new torsion tensor

$$\bar{S}_{\mu\nu}{}^{\lambda} \equiv \tfrac{1}{2} e_a{}^{\lambda} (e^a{}_{\nu,\mu} - e^a{}_{\mu,\nu}). \quad \text{(B2)}$$

This is just the torsion tensor (1.15) of the EHS theory but specialized to $\sigma = 1$, i.e., it is the torsion tensor of the Hayashi-Shirafuji teleparallel theory [18]. From Eqs. (1.15) and (1.18) it is related to the fields $t_{\mu\nu\lambda}$, v^μ, and a^ν by

$$\bar{S}_{\nu\mu\lambda} = \tfrac{2}{3}(t_{\lambda\mu\nu} - t_{\lambda\nu\mu}) + \tfrac{1}{3}(g_{\lambda\mu} v_\nu - g_{\lambda\nu} v_\mu) + \epsilon_{\lambda\mu\nu\rho} a^\rho. \quad \text{(B3)}$$

We denote the corresponding metric compatible connection by $\bar{\Gamma}^\alpha{}_{\beta\gamma}$ and the corresponding Riemann tensor by $\bar{R}^\mu{}_{\nu\lambda\sigma}$. This Riemann tensor vanishes identically by virtue of the definition (B2), as we discussed in the Introduction. The Ricci scalar $\bar{R} \equiv g^{\beta\delta} \bar{R}^\alpha{}_{\beta\alpha\delta}$ also vanishes, which implies

$$\int \sqrt{-g} \bar{R} d^4x = 0. \quad \text{(B4)}$$

We now substitute into Eq. (B4) the formula

$$\bar{R}^\alpha{}_{\beta\gamma\delta} = \bar{\Gamma}^\alpha{}_{\delta\beta,\gamma} - \bar{\Gamma}^\alpha{}_{\gamma\beta,\delta} + \bar{\Gamma}^\alpha{}_{\gamma\mu} \bar{\Gamma}^\mu{}_{\delta\beta} - \bar{\Gamma}^\alpha{}_{\delta\mu} \bar{\Gamma}^\mu{}_{\gamma\beta}, \quad \text{(B5)}$$

together with barred versions of Eqs. (1.3) and (1.4). This gives

$$\int \sqrt{-g} [R(\{\}) - 2\tilde{\nabla}_\alpha \bar{K}^\beta{}_\beta{}^\alpha - \bar{K}^\alpha{}_{\alpha\mu} \bar{K}^\delta{}_\delta{}^\mu + \bar{K}^\beta{}_\mu{}^\alpha \bar{K}^\alpha{}_\mu{}^\beta] = 0. \quad \text{(B6)}$$

Here $\tilde{\nabla}_\alpha$ denotes the Levi-Civita derivative operator. Discarding the total derivative term and using the decomposition (B3) together with the barred version of Eq. (1.4) now yields the desired identity (B1).

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