Solar System constraints to general $f(R)$ gravity

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It has been proposed that cosmic acceleration or inflation can be driven by replacing the Einstein-Hilbert action of general relativity with a function $f(R)$ of the Ricci scalar R. Such $f(R)$ gravity theories have been shown to be equivalent to scalar-tensor theories of gravity that are incompatible with Solar System tests of general relativity, as long as the scalar field propagates over Solar System scales. Specifically, the parameterized post-Newtonian (PPN) parameter in the equivalent scalar-tensor theory is $\gamma = 1/2$, which is far outside the range allowed by observations. In response to a flurry of papers that questioned the equivalence of $f(R)$ theory to scalar-tensor theories, it was recently shown explicitly, without resorting to the scalar-tensor equivalence, that the vacuum field equations for $1/R$ gravity around a spherically symmetric mass also yield $\gamma = 1/2$. Here we generalize this analysis to $f(R)$ gravity and enumerate the conditions that, when satisfied by the function $f(R)$, lead to the prediction that $\gamma = 1/2$.

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I. INTRODUCTION

The evidence that the expansion of the Universe is currently accelerating [1,2] suggests that the Universe is dominated by dark energy with a large negative pressure. The predominant hypothesis is that a nonzero vacuum energy drives the acceleration, but this poses two serious theoretical questions: why is the vacuum energy nonzero, and why is it so miniscule? An equally plausible alternative to dark energy is a modification of general relativity that would generate cosmic acceleration [3,4]. Modifying general relativity in this manner eliminates the need for dark energy, but it does not explain why the vacuum energy is zero. Similar modifications of general relativity have also been proposed to drive inflation [5].

A possible modification to general relativity that generates an accelerated expansion is $1/R$ gravity [3], in which a term proportional to $1/R$, where R is the Ricci scalar, is added to the Einstein-Hilbert action so that the $1/R$ term dominates as the Hubble parameter decreases. Soon after the introduction of this theory, it was shown that $1/R$ gravity is dynamically equivalent to a scalar-tensor gravity with no scalar kinetic term [6]. Moreover, the equivalence to scalar-tensor gravity applies to all modified gravity theories that replace the Einstein-Hilbert action with some function of the Ricci scalar [known as $f(R)$ gravity], provided that $f(R)$ has a nonzero second derivative with respect to R. When the scalar field is light, this theory makes predictions that are incompatible with Solar System tests of general relativity [7–9]. Consequently, Ref. [6] concluded that a broad class of $f(R)$ gravity theories, including $1/R$ gravity, are ruled out by Solar System tests.

Since then, however, the results in Ref. [6] were criticized by a number of papers [10–14] and some even claim that Solar System experiments do not rule out any form of $f(R)$ gravity. The essence of the criticism is that $f(R)$ gravity admits the Schwarzschild–de Sitter solution

and hence the vacuum spacetime in the Solar System is not different from that in general relativity, although there were also broader objections to the equivalence between $f(R)$ and scalar-tensor gravity [12]. Working directly with the field equations, a recent paper [15] found that even though the Schwarzschild–de Sitter metric is a vacuum solution in $1/R$ gravity, it does not correspond to the solution around a spherically symmetric massive body.[1] They found that the solution for the Solar System is identical to the spacetime derived using the corresponding scalar-tensor theory.

In this paper, we generalize the analysis of Ref. [15] to a broad class of $f(R)$ gravities, namely, those theories that admit a Taylor expansion of $f(R)$ around the background value of the Ricci scalar. We work in the metric formalism, where the field equations are obtained by varying the action with respect to the metric and treating the Ricci scalar as a function of the metric. The Palatini formalism, which treats the Ricci scalar as a function of the connection and varies the action with respect to the connection and the metric independently, yields different field equations for $f(R)$ gravity and has been studied extensively elsewhere (see e.g. Refs. [18–22]).

This paper is organized as follows: In Sec. II, we solve the linearized field equations around a spherical mass and find that the solution in the Solar System is in agreement with the solution obtained using the equivalent scalar-tensor theory. When $f(R)$ satisfies a condition that is analogous to the scalar field being light in the equivalent scalar-tensor theory, the resulting spacetime is incompatible with Solar System tests of general relativity. In Sec. III, we consider how our analysis applies to several $f(R)$ gravity theories, including general relativity. This particular example illustrates the connection between $f(R)$ gravity

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[1]Eddington made a similar mistake in R^2 gravity [16], which was later corrected by [17].

and general relativity and clarifies the requirements for a general relativistic limit of an $f(R)$ theory. We summarize our conclusions in Sec. IV.

II. WEAK-FIELD SOLUTION AROUND A SPHERICAL STAR

We consider gravitational theories with actions of the form

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S_m, \tag{1}$$

where $f(R)$ is a function of the Ricci scalar R and S_m is the matter action. The field equation obtained by varying the action with respect to the metric is

$$f_R R_{\mu\nu} - \tfrac{1}{2} f g_{\mu\nu} - \nabla_\mu \nabla_\nu f_R + \Box f_R g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2}$$

where $f_R \equiv df/dR$. In previous studies, predictions of Solar System dynamics in these theories were analyzed by appealing to an equivalence with scalar-tensor theories [6]. We review this equivalence in the appendix. Since the equivalent scalar-tensor theory is incompatible with Solar System observations if the scalar field propagates on Solar System scales, the authors of Ref. [6] concluded that the corresponding $f(R)$ theories are ruled out. We now show that this conclusion can be made *without* appealing to the equivalence between $f(R)$ and scalar-tensor gravity. Instead, we work directly with the linearized field equations about a spherical mass distribution. Our treatment clarifies and amends a similar analysis presented in Ref. [23], and we extend it to cases where the background value of the Ricci scalar equals zero.

We now find the metric that describes the spacetime around a spherical body in $f(R)$ gravity in the weak-field regime. To do this, we must choose a background spacetime around which to linearize the field equations. The only physically relevant choice is an isotropic and homogeneous background spacetime that solves Eq. (2) for some spatially uniform cosmological stress-energy tensor $T_{\mu\nu}^{\text{cos}}$. The evolution of the time-dependent and spatially homogeneous background scalar curvature $R_0(t)$ is determined by the trace of Eq. (2),

$$f_{R0}(t)R_0(t) - 2f_0(t) + 3\Box f_{R0}(t) = \kappa T^{\text{cos}}(t), \tag{3}$$

where $f_{R0} \equiv df/dR|_{R=R_0}$, $f_0 \equiv f(R_0)$, and $T^{\text{cos}} \equiv g^{\mu\nu} T_{\mu\nu}^{\text{cos}}$.

In order to investigate perturbations away from this background, we express the Ricci scalar as the sum of two components:

$$R(r, t) \equiv R_0(t) + R_1(r), \tag{4}$$

where $R_0(t)$ is the spatially homogenous background curvature that solves Eq. (3) and $R_1(r)$ is a time-independent perturbation to this background curvature. We assume that all derivatives of $f(R)$ are well defined at the present-day

value of R_0 so that we may use a Taylor expansion of $f(R)$ around $R = R_0$ to evaluate $f(R_0 + R_1)$ and $f_R(R_0 + R_1)$. We will terminate the expansion by neglecting terms nonlinear in R_1. Provided that the higher-order terms of the Taylor series do not cancel in some contrived way, neglecting the higher-order terms is only justified if the sum of the zeroth-order and linear terms is greater than all other terms in the Taylor expansion. Specifically, we require that

$$f_0 + f_{R0}R_1 \gg \frac{1}{n!} f^{(n)}(R_0)R_1^n, \tag{5}$$

$$f_{R0} + f_{RR0}R_1 \gg \frac{1}{n!} f^{(n+1)}(R_0)R_1^n, \quad \text{for all } n > 1, \tag{6}$$

where $f_{RR0} \equiv d^2f/dR^2|_{R=R_0}$ and $f^{(n)}(R_0) \equiv d^nf/dR^n|_{R=R_0}$.

Now we consider the trace of Eq. (2) with both a cosmological matter source described by T^{cos} and a finite, time-independent, spherically symmetric matter source, described by T^{s}:

$$f_R R - 2f + 3\Box f_R = \kappa(T^{\text{cos}} + T^{\text{s}}). \tag{7}$$

Using first-order Taylor expansions to evaluate f_R and f and neglecting $\mathcal{O}(R_1^2)$ terms, we obtain a linearized version of Eq. (7):

$$3f_{RR0}\Box R_1(r) - [f_{R0}(t) - f_{RR0}(t)R_0(t) - 3\Box f_{RR0}(t)]R_1$$
$$= \kappa T^{\text{s}}. \tag{8}$$

To obtain this equation, we used the fact that $R_0(t)$ solves Eq. (3) to eliminate terms that are independent of R_1. By dropping $\mathcal{O}(f_{RR0}R_1^2)$ terms from Eq. (8) while keeping the $f_{RR0}R_0R_1$ term, we have implicitly assumed that $R_1 \ll R_0$ if R_0 is nonzero. We will check that this condition is satisfied after the discussion following Eq. (31). If R_0 is zero, then the $\mathcal{O}(f_{RR0}R_1^2)$ is guaranteed to be smaller than the nonzero terms in Eq. (8) by virtue of Eq. (5). Note that if $f_{RR0} = 0$, as in general relativity, this equation becomes simply $f_{R0}R_1 = -\kappa T^{\text{s}}$. If in addition f_{R0} is nonzero then R_1 must vanish outside the star and hence the Schwarzschild–de Sitter solution becomes the solution to the field equation outside the source. However, if $f_{RR0} \neq 0$, this is no longer necessarily the case.

Finally, we take our background metric to be a flat Friedmann-Robertson-Walker (FRW) metric. We then consider a spherically symmetric perturbation to this background so that the linearized perturbed metric takes the form

$$ds^2 = -[1 + 2\Psi(r)]dt^2 + a(t)^2\{[1 + 2\Phi(r)]dr^2 + r^2 d\Omega^2\}, \tag{9}$$

where the present value of $a(t)$ is one. When solving the field equations, we will keep only terms linear in the perturbations Ψ and Φ.

We will now solve Eq. (8) for a nonzero f_{RR0}. Since we confine our analysis to a static perturbation $R_1(r)$, $\Box$ becomes the flat-space Laplacian operator ∇^2. Restricting our analysis to a source with mass density $\rho(r)$ and negligible pressure, we may rewrite Eq. (8) as

$$\nabla^2 R_1 - m^2 R_1 = -\frac{\kappa\rho}{3f_{RR0}}, \qquad (10)$$

where we have defined a mass parameter

$$m^2 \equiv \frac{1}{3}\left(\frac{f_{R0}}{f_{RR0}} - R_0 - 3\frac{\Box f_{RR0}}{f_{RR0}}\right). \qquad (11)$$

Due to the evolution of $R_0(t)$, this mass parameter varies in time. However, the time-scale of variation in the cosmological background spacetime is comparable to the current Hubble time. Since this time-scale is much longer than the time-scale of Solar System dynamics, we may neglect the time variation of the background spacetime when considering the behavior of bodies within the Solar System [24]. Therefore, for the purposes of this calculation, we take m to be time-independent.

The Green's function $G(r)$ for this differential equation depends on the sign of m^2:

$$G(r) = \begin{cases} -\cos(mr)/(4\pi r) & m^2 < 0, \\ -\exp(-mr)/(4\pi r) & m^2 > 0, \end{cases} \qquad (12)$$

where $m \equiv \sqrt{|m^2|}$. If $mr \ll 1$, then both Green's functions are approximately $-1/(4\pi r)$, which is the Green's function for Laplace's equation. In this case, the term proportional to m^2 in Eq. (10) may be neglected and the solution outside the star is given by

$$R_1 = \frac{\kappa}{12\pi f_{RR0}}\frac{M}{r}, \qquad (13)$$

where M is the total mass of the source. We note that when applied to $1/R$ gravity with a static de Sitter background, this result agrees with the result presented in Ref. [15].

We emphasize that in order for this solution for R_1 to be valid, we must have $mr \ll 1$. Only when this condition is satisfied is the trace of the field equation well approximated by Laplace's equation. This restriction was not mentioned in Ref. [23]. The physical interpretation of this constraint is clear when one considers the equivalent scalar-tensor theory. When one switches to a frame where the scalar degree of freedom is canonical, the effective mass of the scalar field evaluated in the Jordan frame is [6]

$$m_\varphi^2 = \frac{f_{R0}}{3}\left(\frac{1}{f_{RR0}} + \frac{R_0}{f_{R0}} - \frac{4f_0}{(f_{R0})^2} - \frac{2\kappa T^{\cos}}{(f_{R0})^2}\right). \qquad (14)$$

Since R_0 is the solution to Eq. (3), this expression may be simplified to

$$m_\varphi^2 = \frac{1}{3}\left(\frac{f_{R0}}{f_{RR0}} - R_0 - 6\frac{\Box f_{R0}}{f_{R0}}\right). \qquad (15)$$

It is clear that both m_φ and m [defined by Eq. (11)] are of

the same order. Therefore, the condition that $mr \ll 1$ is equivalent to demanding that the scalar field be light ($m_\varphi r \ll 1$). See the appendix for more details.

In summary, Eq. (13) is a solution to the trace of the field equation within the Solar System only if *the scalar degree of freedom propagates on Solar System scales.* In terms of $f(R)$, the necessary condition is

$$|m^2|r^2 \equiv \left|\frac{1}{3}\left(\frac{f_{R0}}{f_{RR0}} - R_0 - 3\frac{\Box f_{RR0}}{f_{RR0}}\right)\right|r^2 \ll 1. \qquad (16)$$

The triangle inequality tells us that the mass constraint given by Eq. (16) implies that

$$\left|\frac{f_{R0}}{f_{RR0}}\right|r^2 - \left|R_0 - 3\frac{\Box f_{RR0}}{f_{RR0}}\right|r^2 \ll 1. \qquad (17)$$

Finally, since $\Box f_{RR0}/f_{RR0} \sim H^2$, where $H \equiv \dot{a}/a$ is the current Hubble parameter, and we know that $R_0 r^2 \sim H^2 r \ll 1$ by cosmological constraints, the mass constraint implies that

$$\left|\frac{f_{R0}}{f_{RR0}}\right|r^2 \ll 1. \qquad (18)$$

We will now use the expression for R_1 given by Eq. (13) to solve the field equations for the metric perturbations Ψ and Φ. As we did for the trace of the field equation, we simplify the field equations by replacing $f(R)$ and $f_R(R)$ with first-order Taylor expansions around the background value R_0 to obtain field equations that are linear in R_1. Using Eq. (3) to simplify this expression, we obtain

$$f_{R0}(R^\mu_\nu - [R_0]^\mu_\nu) + f_{RR0}R_1 R^\mu_\nu - \tfrac{1}{2}f_{R0}R_1\delta^\mu_\nu$$
$$- f_{RR0}\nabla^\mu\nabla_\nu R_1 + \delta^\mu_\nu f_{RR0}\Box R_1 = \kappa T^{\text{s}\mu}_\nu, \qquad (19)$$

where $[R_0]^\mu_\nu$ is the unperturbed FRW Ricci tensor and δ^μ_ν is the Kronecker delta. We neglected time derivatives of the background metric when deriving this equation. As previously noted, the time scale of variations in R_0 is much longer than that of Solar System dynamics, making the terms involving time derivatives of R_0 irrelevant to gravitational effects within the Solar System.

We simplify Eq. (19) further by dropping several negligible terms. We continue to ignore terms that depend on the variation of the background spacetime by dropping terms that involve products of Φ, Ψ, and $f_{RR0}R_1$ with H and dH/dt. Since we are working in the weak-field regime, we neglect all terms that are nonlinear functions of the metric perturbations Φ and Ψ. Keeping only terms that are linear in Φ and Ψ allows us to replace the $\Box$ with the flat-space Laplacian operator ∇^2 since the perturbation is assumed to be static. Finally, we know from Eq. (13) that $f_{RR0}R_1 \sim \kappa M/r$, and we expect Ψ and Φ to be proportional to $\kappa M/r$ as well. Therefore, $f_{RR0}R_1\Psi$ and $f_{RR0}R_1\Phi$ are second-order quantities, and we may neglect them. With these simplifications, the tt, rr, $\theta\theta$ components of Eq. (19) are, respectively,

$$f_{R0}\nabla^2\Psi + \tfrac{1}{2}f_{R0}R_1 - f_{RR0}\nabla^2 R_1 = \kappa\rho, \qquad (20)$$

$$f_{R0}\left(-\Psi'' + \frac{2}{r}\Phi'\right) - \frac{1}{2}f_{R0}R_1 + \frac{2}{r}f_{RR0}R_1' = 0, \qquad (21)$$

$$f_{R0}\left(\frac{1}{r}\Phi' - \frac{1}{r}\Psi' + \frac{2}{r^2}\Phi\right) - \frac{1}{2}f_{R0}R_1 + \frac{1}{r}f_{RR0}R_1'$$
$$+ f_{RR0}R_1'' = 0, \qquad (22)$$

where the prime denotes differentiation with respect to r. The $\phi\phi$ component of Eq. (19) is identical to the $\theta\theta$ component given by Eq. (22).

Recalling that R_1 solves Eq. (10) with $m^2 = 0$ so that $\nabla^2 R_1$ is proportional to the density ρ, Eq. (20) may be rewritten

$$f_{R0}\nabla^2\Psi = \tfrac{2}{3}\kappa\rho - \tfrac{1}{2}f_{R0}R_1. \qquad (23)$$

We express Ψ as the sum of two functions: $\Psi = \Psi_0 + \Psi_1$, where

$$f_{R0}\nabla^2\Psi_0 = \tfrac{2}{3}\kappa\rho, \qquad (24)$$

$$f_{R0}\nabla^2\Psi_1 = -\frac{1}{2}f_{R0}R_1. \qquad (25)$$

Provided that $f_{R0} \neq 0$, Eq. (24) may be integrated via Gauss's Law to give

$$\Psi_0'(r) = \frac{\kappa}{6\pi f_{R0}}\frac{m(r)}{r^2}, \qquad (26)$$

where $m(r)$ is the mass enclosed in a sphere of radius r. If we assume that Ψ_0 vanishes as $r \to \infty$, we may integrate Eq. (26) to obtain

$$\Psi_0 = -\frac{\kappa}{6\pi f_{R0}}\frac{M}{r}, \qquad (27)$$

outside the star. Solving Eq. (25) outside the star using Eq. (13) for R_1 yields

$$|\Psi_1| = \frac{1}{48\pi f_{RR0}}\kappa M r \ll \frac{1}{f_{R0}}\frac{\kappa M}{r}, \qquad (28)$$

where the inequality follows from Eq. (18). Since $\Psi_0 \sim \kappa M/(f_{R0}r)$ outside the star we have shown that $|\Psi_1| \ll |\Psi_0|$. Therefore, we may neglect Ψ_1 and conclude that $\Psi = \Psi_0$ as given by Eq. (27). This expression for Ψ is used to define Newton's constant: $G \equiv \kappa/(6\pi f_{R0})$. For $1/R$ gravity with a static vacuum de Sitter background, $f_{R0} = 4/3$, so κ takes its standard value of $8\pi G$ and Eq. (27) matches the corresponding result in Ref. [15].

We now turn our attention to Eq. (21), which we will solve for Φ. First, we note that Eq. (13) implies that $R_1' = -R_1/r$. Therefore, the ratio of the second two terms in Eq. (21) is

$$\left|\frac{(1/2)f_{R0}R_1}{2f_{RR0}R_1'/r}\right| \sim \left|\frac{f_{R0}}{f_{RR0}}\right| r^2 \ll 1, \qquad (29)$$

where the inequality follows from Eq. (18). Consequently, the $f_{R0}R_1$ term is negligible, and we drop it from the equation. Differentiating Eq. (26) to find Ψ'', and using Gauss's Law to obtain R_1' from Eq. (10) (with $m^2 = 0$), we may then rewrite Eq. (21) as

$$\Phi'(r) = \frac{\kappa}{12\pi f_{R0}}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{m(r)}{r}\right). \qquad (30)$$

Assuming that Φ vanishes as $r \to \infty$, this equation may be integrated to obtain

$$\Phi = \frac{\kappa}{12\pi f_{R0}}\frac{M}{r}, \qquad (31)$$

outside the star. It is easy to verify that Eqs. (27) and (31) also satisfy the third field equation, Eq. (22).

We may now check our assumption that $R_1 \ll R_0$ for nonzero R_0. From the expression for R_1 given by Eq. (13) and our definition that $\kappa \equiv 6\pi f_{R0}G$, we see that

$$\frac{R_1}{R_0} \lesssim \frac{1}{R_0}\left(\frac{GM}{R_s}\right)\frac{f_{R0}}{f_{RR0}}, \qquad (32)$$

where R_s is the radius of the star. It is easy to check that this expression holds inside the star as well by integrating Eq. (10) into the interior of the star. Therefore, our assumption that $R_1 \ll R_0$ places an additional condition on the ratio f_{R0}/f_{RR0}:

$$\left|\frac{f_{R0}}{f_{RR0}}\right| \ll R_0\left(\frac{R_s}{GM}\right) \quad \text{for } R_0 \neq 0. \qquad (33)$$

If $f_{R0}/f_{RR0} \sim R_0$, as is the case for many $f(R)$ theories with nonzero R_0, then this condition is always satisfied.

Thus we have shown explicitly that $\Psi = -2\Phi = -GM/r$ for all $f(R)$ theories with nonzero f_{RR0} that satisfy the conditions given by Eqs. (5), (6), (16), and (33). Transforming the metric given by Eq. (9) to isotropic coordinates, taking $a = 1$ today, and keeping only terms that are linear in GM/r gives

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 + \frac{GM}{r}\right)[dr^2 + r^2 d\Omega^2]. \qquad (34)$$

It is clear that this spacetime is equivalent to a parameterized post-Newtonian (PPN) spacetime with PPN parameter $\gamma = 1/2$. This result is in gross violation of observations; Solar System tests require that $\gamma = 1 + (2.1 \pm 2.3) \times 10^{-5}$ [8,9]. We also note that this result is in precise agreement with the results obtained using the equivalent scalar-tensor theory [6] (see also [25]).

III. CASE STUDIES

First, we show how we regain the results of general relativity if we take $f_{RR0} = 0$ and assume that our linearized Taylor expansion is a valid approximation. We note that general relativity $[f(R) = R]$ satisfies both of these conditions.

Taking $f_{RR0} = 0$, Eq. (8) yields

$$f_{R0}R_1 = \kappa\rho. \qquad (35)$$

When $f_{RR0} = 0$, the $f_{R0}R_1$ terms in the field equations [Eqs. (21) and (22)] are no longer negligible compared to the terms proportional to f_{RR0} since these terms vanish. The field equations then become

$$f_{R0}\nabla^2\Psi + \tfrac{1}{2}f_{R0}R_1 = \kappa\rho, \qquad (36)$$

$$f_{R0}\left(-\Psi'' + \frac{2}{r}\Phi'\right) - \frac{1}{2}f_{R0}R_1 = 0, \qquad (37)$$

$$f_{R0}\left(\frac{1}{r}\Phi' - \frac{1}{r}\Psi' + \frac{2}{r^2}\Phi\right) - \frac{1}{2}f_{R0}R_1 = 0. \qquad (38)$$

Using Eq. (35), Eq. (36) becomes

$$f_{R0}\nabla^2\Psi = \frac{\kappa}{2}\rho, \qquad (39)$$

and the solution outside the star is

$$\Psi = -\frac{\kappa}{8\pi f_{R0}}\frac{M}{r}. \qquad (40)$$

From Eqs. (37) and (38), we have

$$\frac{f_{R0}}{r^2}(r\Phi)' = \frac{\kappa}{2}\rho, \qquad (41)$$

and the solution outside the star is

$$\Phi = \frac{\kappa}{8\pi f_{R0}}\frac{M}{r} = -\Psi. \qquad (42)$$

Since $\Psi = -\Phi = -GM/r$, transforming to isotropic coordinates reveals that $\gamma = 1$ as expected.

With this result it is easy to see why the $\mu \to 0$ limit in $1/R^n$ ($n > 0$) gravity *does not* recover general relativity. In $1/R^n$ gravity [3], we have

$$f(R) = R - \frac{\mu^{2+2n}}{R^n}, \qquad n > 0. \qquad (43)$$

The static solution to Eq. (3) with $T^{\rm cos} = 0$ is $R_0 = (n+2)^{1/(n+1)}\mu^2$, and $f_{RR0} \propto \mu^{-2}$. Therefore, f_{RR0} *diverges rather than vanishes* in the limit that $\mu \to 0$, and general relativity is *not* regained. The mass parameter for this theory has the dependence $m^2 \propto \mu^2$ and hence it vanishes in the limit that $\mu \to 0$. Furthermore, a Taylor series of Eq. (43) around R_0 is well behaved and cosmological

constraints tell us that $\mu \sim H$ so that $m^2r^2 \ll 1$ in the Solar System. We conclude that the analysis of general $f(R)$ gravity given in Sec. II applies and $\gamma = 1/2$ for these theories in a static background.

We note, however, that the static solution to Eq. (3) may not describe the current cosmological background in $1/R^n$ gravity. This solution is unstable, and without fine-tuning of the initial conditions, this spacetime will evolve toward a spacetime with $R_0 \ll \mu^2$ [3]. In that case, we note that

$$\frac{(m!)^{-1}f^{(m)}(R_0)R_1^m}{f_0 + f_{R0}R_1} \lesssim \left(\frac{GM}{r}\right)^m \ll 1, \qquad (44)$$

$$\frac{(m!)^{-1}f^{(m+1)}(R_0)R_1^m}{f_{R0} + f_{RR0}R_1} \lesssim \left(\frac{GM}{r}\right)^m \ll 1, \qquad (45)$$

so that Eqs. (5) and (6) are still satisfied. Furthermore, $m^2 \propto R_0$, so, as in the static-background case, the mass is of order the Hubble parameter today. Therefore, the $\gamma = 1/2$ result holds even during the late-time evolution of $1/R^n$ gravity.

Next we consider Starobinsky gravity [5] which has

$$f(R) = R + \frac{R^2}{\alpha^2}. \qquad (46)$$

The static solution to Eq. (3) with $T^{\rm cos} = 0$ is $R_0 = 0$ for this theory. Since $f(R)$ is a second-order polynomial, the first-order Taylor expansion of $f_R(R_0 + R_1)$ is exact. The $\mathcal{O}(R_1^2)$ term in the Taylor expansion of $f(R_0 + R_1)$ is suppressed compared to the linear term by a factor of GM/r and is therefore negligible. The mass parameter for this theory is proportional to α^2, so Eq. (13) is a solution for R_1 if $\alpha^2r^2 \ll 1$. Therefore, $\gamma = 1/2$ in this theory if $\alpha^2r^2 \ll 1$ inside the Solar System. If the mass parameter α is made large (i.e. if $\alpha \simeq 10^{12}$ GeV as proposed in Ref. [5]), then this condition is not satisfied and we cannot use the analysis in Sec. II to calculate γ for this theory.

Next we consider an example of a theory that uses two mass parameters: a hybrid between Starobinsky gravity and $1/R$ gravity. In particular, consider the function

$$f(R) = R + \frac{1}{\alpha^2}R^2 - \frac{\mu^4}{R}. \qquad (47)$$

We then find that, as in the usual $1/R$ case, we have $R_0 = \sqrt{3}\mu^2$ (for a static background in vacuum). However,

$$m^2 = 3\mu^2\left(\frac{\alpha^2}{9\mu^2 - \sqrt{3}\alpha^2}\right). \qquad (48)$$

We can make this quantity as large as we want by letting the denominator tend towards zero, which gives the condition $\alpha \to 3^{3/4}\mu$. Thus, in this model we can violate the conditions listed in Sec. II by fine-tuning the parameters.

Finally, we consider power-law gravitational actions [26]:

$$f(R) = \left(\frac{R}{\alpha}\right)^{1+\delta}. \tag{49}$$

Assuming that $\delta \neq 1$, the static vacuum solution to Eq. (3) is $R_0 = 0$. If δ is not an integer, there will be some derivative that is not defined at $R = 0$, which causes the Taylor expansion to fail around that point. In particular, if it is supposed that $\delta \ll 1$, then at least the second derivative will be undefined so that the Taylor expansion will fail. For $\delta = 1$ the static vacuum background value R_0 is undetermined. However, if we choose $R_0 \neq 0$ then all of the conditions listed in Sec. II are satisfied and we conclude that $\gamma = 1/2$ in agreement with Ref. [27]. If δ is an integer greater than one, then the Taylor expansion around $f(R_0 = 0)$ is well defined, but we cannot drop the terms that are nonlinear in R_1 since the linearized function vanishes. Therefore, this analysis is incapable of determining whether $f(R) = R^{1+\delta}$ gravity with $\delta \neq 1$ conflicts with Solar System tests.

IV. CONCLUSIONS

By analyzing the field equations around a spherically symmetric mass, we have shown that, in agreement with the analysis in Ref. [6], the PPN parameter γ of general $f(R)$ gravity is $\gamma = 1/2$ given the following conditions:

(I) The Taylor expansions of $f(R)$ and df/dR about the current background value $R = R_0$, where R_0 solves Eq. (3), are well defined and dominated by terms that are linear in deviations away from $R = R_0$. If R_0 is nonzero, then the deviations from R_0 are small compared to R_0. This condition may be reexpressed as Eq. (33) and is closely related to the third condition stated below.

(II) The second derivative of $f(R)$ with respect to R is nonzero when evaluated at the background value of $R = R_0$.

(III) The mass parameter given by Eq. (11) respects the condition $mr \ll 1$ within the Solar System.

For theories with one extra mass parameter and nonzero R_0, as in $1/R$ gravity, it is reasonable to assume that $f_{R0}/f_{RR0} \sim R_0$. In that case, the latter part of the first condition is always true and the third condition is satisfied provided that $R_0 r^2 \ll 1$ within the Solar System. However, for theories with multiple mass parameters, such as the Starobinsky-$1/R$ hybrid presented in this paper, it is possible that this condition can be violated.

The second and third conditions listed above correspond to synonymous conditions in the scalar-tensor treatment: $f(R)$ and scalar-tensor gravity are equivalent only if the second derivative of $f(R)$ is nonzero, and $\gamma = 1/2$ only if the scalar field is light enough to propagate through the Solar System. Therefore, we have also verified that, contrary to the claim of some authors [10–14], calculating the Solar System predictions of $f(R)$ gravity using the equivalent scalar-tensor theory is a valid technique.

APPENDIX: REVIEW OF SCALAR-TENSOR EQUIVALENCE

The action for the scalar-tensor theory that is equivalent to $f(R)$ gravity is

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g}[f(\phi) + f_\phi(\phi)(R - \phi)] + S_m, \tag{A1}$$

where $f_\phi(\phi) \equiv df/d\phi$ and S_m is the matter action. The field equation for ϕ is $\phi = R$ if $d^2f/d\phi^2 \neq 0$. Since the relation between ϕ and R is purely algebraic, it can be resubstituted into the action to reproduce the action for $f(R)$ gravity given by Eq. (1). After the conformal transformation $g_{\mu\nu}^E \equiv f_\phi(\phi)g_{\mu\nu}$, the action becomes that of general relativity with a minimally coupled scalar field:

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g_E} \left(R_E - \frac{3}{2f_\phi(\phi)^2} g_E^{\mu\nu}[\nabla_{E\mu}f_\phi(\phi)] \right.$$
$$\left. \times [\nabla_{E\nu}f_\phi(\phi)] - \frac{1}{f_\phi(\phi)^2}[\phi f_\phi(\phi) - f(\phi)] \right) + S_m. \tag{A2}$$

Introducing a canonical scalar field φ such that $f_\phi(\phi) = \exp(\sqrt{2\kappa/3}\varphi)$, Eq. (A2) can be rewritten as

$$S = \int d^4x \sqrt{-g_E} \left(\frac{1}{2\kappa} R_E - \frac{1}{2}(\nabla_E \varphi)^2 - V(\varphi) \right) + S_m, \tag{A3}$$

where the potential is defined by

$$V(\varphi) \equiv \frac{\phi(\varphi)f_\phi[\phi(\varphi)] - f[\phi(\varphi)]}{2\kappa f_\phi[\phi(\varphi)]^2}. \tag{A4}$$

The absence of the kinetic term in Eq. (A1) implies the Brans-Dicke parameter of $f(R)$ gravity theories is $\omega = 0$ [7]. From an analysis of Brans-Dicke gravity, if the scalar degree of freedom can propagate on scales much larger than the Solar System, we can conclude that $\gamma = (1 + \omega)/(2 + \omega) = 1/2$ [7].

In the frame where φ is canonical (the Einstein frame) φ has the equation of motion

$$\Box_E \varphi = \frac{dV}{d\varphi} + \sqrt{\frac{\kappa}{6}} f'(\phi)^{-2} T^{\mathrm{M}}, \qquad (A5)$$

where the prime denotes differentiation with respect to ϕ. When we reexpress Eq. (A5) in terms of $f'(\phi)$ and the usual metric $g_{\mu\nu}$, we recover Eq. (7). Therefore, we stress that this reformulation contains *no new dynamics* compared to the expressions used in this paper. The two formulations are entirely equivalent.

In order to derive the mass m_φ, we let $\varphi = \varphi_0(t) + \varphi_1(r)$ and $T^{\mathrm{M}} = T^{\mathrm{cos}} + T^{\mathrm{s}}$ so that $\varphi_0(t)$ satisfies Eq. (A5) with T^{cos}. We then expand to linear order in the perturbation φ_1, writing Eq. (A5) in terms of the physical metric $g_{\mu\nu}$. We find

$$\Box \varphi_1 = f'(\phi_0) \left(\frac{d^2 V}{d\varphi^2} \Big|_{\varphi_0} - \frac{2}{3} \kappa \frac{T^{\mathrm{cos}}}{[f'(\phi_0)]^2} \right) \varphi_1 + \sqrt{\frac{\kappa}{6}} \frac{T^{\mathrm{s}}}{f'(\phi_0)}, \qquad (A6)$$

where ϕ_0 denotes the background field value for the ϕ field. Using Eq. (A4) to evaluate $d^2 V/d\varphi^2$, we have

$$m_\varphi^2 = \frac{f'(\phi_0)}{3} \left[\frac{1}{f''(\phi_0)} + \frac{\phi_0}{f'(\phi_0)} - \frac{4 f(\phi_0)}{[f'(\phi_0)]^2} - 2\kappa \frac{T^{\mathrm{cos}}}{[f'(\phi_0)]^2} \right]. \qquad (A7)$$

Finally, we may rewrite m_φ^2 as Eq. (14) since $\phi_0 = R_0$. We conclude that if $m_\varphi^2 r^2 \ll 1$ then $\gamma = 1/2$ as discussed in Ref. [6].

[1] S. Perlmutter *et al.* (Supernova Cosmology Project Collaboration), Astrophys. J. **517**, 565 (1999).
[2] A. G. Riess *et al.* (Supernova Search Team Collaboration), Astron. J. **116**, 1009 (1998).
[3] S. M. Carroll, V. Duvvuri, M. Trodden, and M. S. Turner, Phys. Rev. D **70**, 043528 (2004).
[4] S. Capozziello, S. Carloni, and A. Troisi, arXiv:astro-ph/0303041.
[5] A. A. Starobinsky, Phys. Lett. **91B**, 99 (1980).
[6] T. Chiba, Phys. Lett. B **575**, 1 (2003).
[7] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).
[8] B. Bertotti, L. Iess, and P. Tortora, Nature (London) **425**, 374 (2003).
[9] S. S. Shapiro, J. L. Davis, D. E. Lebach, and J. S. Gregory, Phys. Rev. Lett. **92**, 121101 (2004).
[10] A. Rajaraman, arXiv:astro-ph/0311160.
[11] T. Multamaki and I. Vilja, Phys. Rev. D **74**, 064022 (2006).
[12] V. Faraoni, Phys. Rev. D **74**, 023529 (2006).
[13] M. L. Ruggiero and L. Iorio, J. Cosmol. Astropart. Phys. 01 (2007) 010.
[14] G. Allemandi, M. Francaviglia, M. L. Ruggiero, and A. Tartaglia, Gen. Relativ. Gravit. **37**, 1891 (2005).

[15] A. L. Erickcek, T. L. Smith, and M. Kamionkowski, Phys. Rev. D **74**, 121501 (2006).
[16] A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, Cambridge, England, 1922).
[17] E. Pechlaner and R. Sexl, Commun. Math. Phys. **2**, 165 (1966).
[18] D. N. Vollick, Classical Quantum Gravity **21**, 3813 (2004).
[19] E. E. Flanagan, Phys. Rev. Lett. **92**, 071101 (2004).
[20] S. M. Carroll *et al.*, Phys. Rev. D **71**, 063513 (2005).
[21] S. M. Carroll, I. Sawicki, A. Silvestri, and M. Trodden, New J. Phys. **8**, 323 (2006).
[22] K. Kainulainen, V. Reijonen, and D. Sunhede, arXiv:gr-qc/0611132.
[23] X. H. Jin, D. J. Liu, and X. Z. Li, arXiv:astro-ph/0610854.
[24] See C. M. Will, *Theory and Experiment in Gravitational Physics,* revised edition (Cambridge University Press, Cambridge, England, 1993), pp. 91–92.
[25] G. J. Olmo, Phys. Rev. D **72**, 083505 (2005).
[26] T. Clifton and J. D. Barrow, Phys. Rev. D **72**, 103005 (2005).
[27] G. V. Bicknell, J. Phys. A **7**, 1061 (1974).
[28] I. Navarro and K. Van Acoleyen, J. Cosmol. Astropart. Phys. 02 (2007) 022.