Higher-spin currents and thermal flux from Hawking radiation

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Quantum fields near black hole horizons can be described in terms of an infinite set of $d = 2$ conformal fields. In this paper, by investigating transformation properties of general higher-spin currents under a conformal transformation, we reproduce the thermal distribution of Hawking radiation in both cases of bosons and fermions. As a by-product, we obtain a generalization of the Schwarzian derivative for higher-spin currents.

I. INTRODUCTION

Hawking studied quantum effects of matter in the background of a black hole formed in collapse and concluded that the black hole will emit thermal radiation as if it was a blackbody at the Hawking temperature [1,2]. Soon after Unruh [3] realized importance of choice of vacua and showed that eternal black holes emit the same radiation. There are many other derivations and all of them universally give the same result.

Recently Robinson and Wilczek proposed a new partial derivation of Hawking radiation [4], which ties its existence to the cancellation of gravitational anomalies at the horizon. The method was then generalized in [5] to charged black holes by using the gauge anomaly in addition to the gravitational anomaly and further applied to rotating black holes [6,7] and others [8–13]. The essential observation is that quantum fields near the horizon behave as an infinite set of two-dimensional fields and ingoing modes at the horizon can be considered as left moving modes while outgoing modes as right moving modes. Once the ingoing modes fall into the black hole, they never come out classically and cannot affect the physics outside the black hole. Quantum mechanically, however, they cannot be neglected because, without the ingoing modes, the theory becomes chiral at the horizon, which makes the effective theory anomalous under general coordinate or gauge transformations. In this sense, the ingoing modes at the horizon only affect the exterior region through quantum anomalies.

The derivation of the Hawking flux through quantum anomalies indicates universality of the Hawking radiation, but it is so far only partial because the full thermal distribution has not yet been obtained. Schwarzschild black hole emits thermal radiation with the Planck distribution $N^{\pm}(\omega)$. Here $N^{\pm}(\omega)$ is given by

$$N^{\pm}(\omega) = \frac{1}{e^{\beta\omega} \pm 1}, \tag{1}$$

where $1/\beta$ is the Hawking temperature of the black hole and $\pm$ corresponds to fermions and bosons, respectively. The energy flux is given by the following specific moment $F_2^{\pm}$ of $N^{\pm}(\omega)$, but in order to derive the complete thermal radiation it is necessary to obtain all the fluxes with higher moments $F_n^{\pm}$ [14]. Here we define $F_n^{\pm}$ ($n \geq 1$) as an $(n-1)$th moment of the Planck distribution

$$F_n^{\pm} = \int_0^{\infty} \frac{d\omega}{2\pi} \omega^{n-1} N^{\pm}(\omega). \tag{2}$$

The integrals can be performed as

$$F_{2n}^{+} = (1 - 2^{1-2n}) \frac{B_n}{8\pi n} \kappa^{2n}, \qquad F_{2n}^{-} = \frac{1}{8\pi n} B_n \kappa^{2n}, \tag{3}$$

where B_n's are the Bernoulli numbers ($B_1 = 1/6$, $B_2 = 1/30$) and $\kappa = 2\pi/\beta$ is the surface gravity of the black hole.

$F_2^{\pm}$ is the energy flux from the black hole. Similarly $F_{2n}^{\pm}$ can be expected to be given by fluxes of higher-spin currents. Since each partial wave of quantum fields near the horizon is described by $d = 2$ conformal field, there are infinitely many higher-spin conserved currents near the horizon. To determine their fluxes in a similar method adopted for the energy flux [4] or the charge flux [5], we need to calculate quantum anomalies for higher-spin currents to fix the boundary conditions of the currents at the horizon. Covariant calculations of these anomalies are quite involved and we leave them for a future publication [15]. In this paper, we use a much simpler and powerful technique of conformal field theories to derive the complete thermal spectrum of Hawking radiation.

II. ENERGY FLUX

We first review a derivation of the energy flux from Hawking radiation for Schwarzschild black hole by using a conformal field theory technique [16,17]. Since each partial wave of quantum fields in a Schwarzschild black

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hole background behaves as an independent two-dimensional conformal field, we can treat them separately in the two-dimensional (t, r) plane. Covariant energy-momentum tensor $T_{\mu\nu} = (2/\sqrt{-g})\delta S/\delta g^{\mu\nu}$ is classically traceless and conserved but when the matter field is quantized they give rise to a conformal anomaly $T^{\mu}_{\mu} = c/24\pi R$ where c is the central charge of the matter field. In the conformal gauge $g_{\mu\nu} = e^{\varphi}\eta_{\mu\nu}$, the conservation of the energy-momentum tensor can be solved and the (uu) component becomes

$$T_{uu}(u, v) = \frac{c}{24\pi}\left(\partial_u^2\varphi - \frac{1}{2}(\partial_u\varphi)^2\right) + T_{uu}^{(\text{conf})}(u). \quad (4)$$

Here $u = t - x$ and $v = t + x$ are light cone coordinates. $T_{uu}^{(\text{conf})}(u)$ is holomorphic (i.e. independent of v) and independent of the conformal factor φ. We need a boundary condition to determine $T_{uu}^{(\text{conf})}(u)$ in the black hole background. This is essentially the same as the derivation given by Christensen and Fulling [18]. Under a holomorphic coordinate transformation $(u, v) \rightarrow (w(u), y(v))$, since T_{uu} transforms covariantly and the conformal factors are related as $\tilde{\varphi}(w, y) = \varphi(u, v) - \ln\frac{dw}{du}\frac{dy}{dv}$, the holomorphic part transforms as

$$T_{ww}^{(\text{conf})}(w) = \left(\frac{dw}{du}\right)^{-2}\left(T_{uu}^{(\text{conf})}(u) + \frac{c}{24\pi}\{w, u\}\right), \quad (5)$$

where $\{w, u\}$ is the Schwarzian derivative,

$$\{w, u\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2. \quad (6)$$

The prime means a derivative with respect to u.

In the case of Schwarzschild black hole, we are interested in the outgoing energy flux measured by asymptotic inertial observers whose natural coordinates are asymptotic Minkowski variables (u, v) where $u = t - r_*$ ($v = t + r_*$) are outgoing (ingoing) light-cone coordinates, and r_* is the tortoise coordinate. Regular coordinates near the horizon, on the other hand, are given by the Kruskal coordinates (U, V) which are related to (u, v) as $U = -e^{-\kappa u}$ and $V = e^{\kappa v}$. Hence we have a relation

$$T_{UU}^{(\text{conf})}(U) = \left(\frac{1}{\kappa U}\right)^2\left(T_{uu}^{(\text{conf})}(u) + \frac{c}{24\pi}\{U, u\}\right). \quad (7)$$

As in [3] we impose a boundary condition for the outgoing energy flux such that physical quantities should be regular at the future horizon $U = 0$ in the Kruskal coordinate. This requires that $\langle T_{UU}^{(\text{conf})}(U)\rangle$ is finite at the horizon, and hence $\langle T_{uu}^{(\text{conf})}\rangle$ is determined to be $-(c/24\pi)\{U, u\}$. We further assume that there is no ingoing flux at infinity. As a result we obtain the asymptotic flux by the value of the Schwarzian derivative

$$\langle T^r_t\rangle = \langle T_{uu}\rangle - \langle T_{vv}\rangle = -\frac{c}{24\pi}\{U, u\} = \frac{c}{48\pi}\kappa^2. \quad (8)$$

If we put $c = 1$ ($c = 1/2$), this coincides with the energy flux F_2^- (F_2^+) from the thermal radiation of black holes.

The purpose of the present paper is to reproduce all the fluxes $F_{2n}^{\pm}$ from conformal transformation properties of higher-spin currents.

III. FOURTH-RANK CURRENT

In order to reproduce the full spectrum of thermal Hawking radiation, we need to calculate all the moments F_{2n}. We can obtain these higher moments by considering behavior of higher-spin currents in the black hole background. First let us consider a fourth-rank current for a scalar field ϕ [19]. In flat $d = 2$ space-time, there is a conserved traceless current

$$\begin{aligned}J_{\mu\nu\rho\sigma} = (&8\partial_\mu\phi\partial_\nu\partial_\rho\partial_\sigma\phi - 12\partial_\mu\partial_\nu\phi\partial_\rho\partial_\sigma\phi \\ &- 4g_{\mu\nu}\partial^\lambda\phi\partial_\lambda\partial_\rho\partial_\sigma\phi + 8g_{\mu\nu}\partial^\lambda\partial_\rho\phi\partial_\lambda\partial_\sigma\phi \\ &- g_{\mu\nu}g_{\rho\sigma}\partial^\lambda\partial^\tau\phi\partial_\lambda\partial_\tau\phi) + \text{symm.} \quad (9)\end{aligned}$$

The "symm." means symmetrization under $(\mu\nu\rho\sigma)$. Its holomorphic component is given by a sum of two terms $J_{uuuu} \propto \partial_u\phi\partial_u^3\phi - 3/2\partial_u^2\phi\partial_u^2\phi$. In curved space the current will acquire trace anomaly and the holomorphy will be lost, but it is plausible to think that we can separate the nonholomorphic part as we did in the case of energy-momentum tensor and define a holomorphic fourth-rank current, whose conformal transformation laws contain a generalization of the Schwarzian derivative. From the OPE between J_{uuuu} and $T_{uu}^{(\text{conf})}$, we can obtain a transformation law of J_{uuuu} under an infinitesimal conformal transformation. In order to obtain its finite transformation law, we follow the derivation of the ordinary Schwarzian derivative in [20]. Two operators, $: -\partial_u\phi\partial_u^3\phi:$ and $:\partial_u^2\phi\partial_u^2\phi:$, have different transformation properties but we can show that they give the same value of the Schwarzian derivatives for our specific transformation from u to the Kruskal U. Therefore we will restrict ourselves to considering only the first type $: -\partial\phi\partial^3\phi(u):$. The normal ordering of the operator is defined by a point splitting method as

$$:\partial\phi\partial^3\phi(u): = \lim_{\epsilon\rightarrow 0}\left(\partial\phi(u + \epsilon/2)\partial^3\phi(u - \epsilon/2) + \frac{3}{2\pi\epsilon^4}\right). \quad (10)$$

Here $\phi(u_1, v_1)\phi(u_2, v_2) \sim -(1/4\pi)\ln(u_1 - u_2) \times (v_1 - v_2)$. Conformal transformations of the right-hand side under $u \rightarrow w(u)$ can be easily calculated. Taking the limit $\epsilon \rightarrow 0$, we obtain

$$\begin{aligned}:\partial\phi\partial^3\phi(u): = \ &w'w''':\partial\phi^{(w)}\partial\phi^{(w)}(w): \\ &+ 3(w')^2w'':\partial\phi^{(w)}\partial^2\phi^{(w)}(w): \\ &+ (w')^4:\partial\phi^{(w)}\partial^3\phi^{(w)}(w): \\ &- \frac{1}{480\pi}\{w, u\}_{(1,3)}, \quad (11)\end{aligned}$$

where

$$\{w, u\}_{(1,3)} = 6\frac{w'''''}{w'} - 20\left(\frac{w'''}{w'}\right)^2 - 45\left(\frac{w''}{w'}\right)^4$$
$$+ 90\frac{(w'')^2 w'''}{(w')^3} - 30\frac{w''''w''}{(w')^2} \quad (12)$$

is a generalized Schwarzian derivative for the operator $:\partial\phi\partial^3\phi(u):$.

We apply this transformation law to the conformal transformation from u to the Kruskal coordinate $w = U = -e^{-\kappa u}$. By imposing the regularity condition at the future horizon in the Kruskal coordinate, it is necessary to require that all the operators in the Kruskal coordinate must be finite at the horizon. Hence the outgoing flux of the fourth-rank current from the black hole in the (u, v) coordinate is given by

$$\langle : -\partial\phi\partial^3\phi(u):\rangle = \frac{1}{480\pi}\{U, u\}_{(1,3)} = \frac{\kappa^4}{480\pi}. \quad (13)$$

This precisely agrees with the 3rd moment F_4^- for bosons.

IV. HIGHER-SPIN CURRENTS FOR BOSONS

The derivation of F_4^- can be generalized to the higher moment F_{2n}^- whose current is given by a linear combination of $\partial_{\mu_1}\ldots\partial_{\mu_m}\phi\partial_{\mu_{m+1}}\ldots\partial_{\mu_{2n}}\phi$. The holomorphic component is a linear combination of $:(-1)^{n+m}\partial_u^m\phi\partial_u^{2n-m}\phi:$. Since we can show that all these terms give the same value of the Schwarzian derivative for the conformal transformation $u \to U$, we consider the $m = 1$ case. A conformal transformation for $:(-1)^{n-1}\partial\phi\partial^{2n-1}\phi:$ can be more easily obtained by introducing its generating function

$$:\partial\phi(u)\partial\phi(u + a): \equiv \sum_{n=0}^{\infty}\frac{a^n}{n!}:\partial\phi(u)\partial^{n+1}\phi(u): \quad (14)$$

than calculating each term separately [21]. Its conformal transformation under $u \to w(u)$ can be calculated as

$$:\partial\phi(u)\partial\phi(u + a): = \partial_u w(u)\partial_u w(u + a):$$
$$\times \partial_w\phi^{(w)}(w(u))\partial_w\phi^{(w)}(w(u + a)):$$
$$+ A_b(w, u), \quad (15)$$

where $A_b(w, u)$ is a generating function of the generalized Schwarzian derivatives for bosons $\{w, u\}_{(1,n)}$ and given by

$$A_b(w, u) = -\frac{1}{4\pi}\frac{\partial_u w(u)\partial_u w(u + a)}{(w(u) - w(u + a))^2} + \frac{1}{4\pi a^2}. \quad (16)$$

In our case, $w = U = -e^{-\kappa u}$, the conformal transformation becomes

$$:\partial_U\phi^{(U)}(U(u))\partial_U\phi^{(U)}(U(u + a)):$$
$$= e^{\kappa a}\left(\frac{1}{\kappa U}\right)^2[:\partial\phi(u)\partial\phi(u + a): - A_b(U, u)], \quad (17)$$

and

$$A_b(U, u) = -\frac{\kappa^2}{16\pi}\frac{1}{\sinh^2\frac{\kappa a}{2}} + \frac{1}{4\pi a^2}. \quad (18)$$

Regularity at the future horizon in the Kruskal coordinate determines the expectation value of the generating function of the higher-rank currents as

$$\langle :\partial\phi(u)\partial\phi(u + a):\rangle = A_b(U, u). \quad (19)$$

$A_b(U, u)$ can be expanded as the following power series of a,

$$A_b(U, u) = \sum_{n=0}^{\infty}(-1)^n\frac{B_{n+1}\kappa^{2(n+1)}}{8\pi(n + 1)}\frac{a^{2n}}{(2n)!}. \quad (20)$$

Hence the outgoing flux of the $2n$th rank current for bosons is given by

$$\langle :(-1)^{n-1}\partial\phi\partial^{2n-1}\phi(u):\rangle = \frac{1}{8\pi n}B_n\kappa^{2n}. \quad (21)$$

This reproduces the flux from thermal Hawking radiation F_{2n}^-. The relation (21) is a higher-spin generalization of (8) and (13). We now conclude that the conformal field theory technique gives the full thermal spectrum of Hawking radiation for the bosonic case.

Two comments are in order. First, by using the same technique of the generating function, we can show that the Schwarzian derivative for the operator $:(-1)^{n+m}\partial^m\phi\partial^{2n-m}\phi:$ gives the same value $(B_n/8\pi n)\kappa^{2n}$ for its expectation value in the black hole background. Or it is equivalent to state that the Schwarzian derivative vanishes for a total derivative operator $\partial(\partial^k\phi\partial^l\phi)$ under the conformal transformation $u \to U$, and we do not need to worry which linear combination of them is a holomorphic component of the conserved current.

The second comment is the meaning of $A_b(U, u)$. It is a generating function of the Schwarzian derivatives of higher-spin currents under a conformal transformation $u \to U = -e^{\kappa u}$. It can be written in an integral form as

$$A_b(U, u) = \int_0^{\infty}\frac{d\omega}{2\pi}\omega N^-(\omega)\cos(a\omega). \quad (22)$$

This is the temperature-dependent part of a finite temperature Green function for $\langle T\partial\phi(x)\partial\phi(x + a)\rangle_\beta$ as can be seen from

$$\langle T\phi(x)\phi(y)\rangle_\beta = \int\frac{d^2k}{(2\pi)^2}\left(\frac{i}{k^2 + i\epsilon}\right.$$
$$\left. + 2\pi N^-(|\omega|)\delta(k^2)\right)e^{-ik(x-y)}. \quad (23)$$

It is the reason why we can reproduce the fluxes of all the higher-spin currents. Appearance of the finite temperature Green function [22] is natural because the conformal transformation from u to the Kruskal coordinate $U = -e^{-\kappa u}$ is

nothing but a conformal transformation from zero temperature to finite temperature.

V. HIGHER-SPIN CURRENTS FOR FERMIONS

The calculation can be similarly applied to a fermionic case. A difference here is that spinor has a conformal weight $1/2$ and transforms as $\psi(z) = (\partial_z w(z))^{1/2} \psi^{(w)}(w(z))$. In order to calculate the conformal transformation of operators like $:\!\psi \partial^n \psi(z)\!:$, we again consider its generating function;

$$:\psi(z)\psi(z+a): \equiv \sum_{n=0}^{\infty} \frac{a^n}{n!} :\psi \partial^n \psi(z):. \qquad (24)$$

Under a conformal transformation, we have

$$\begin{aligned} :\psi(u)\psi(u+a): &= (\partial_u w(u)\partial_u w(u+a))^{1/2}: \\ &\quad \times \psi^{(w)}(w(u))\psi^{(w)}(w(u+a)): \\ &\quad + A_f(w, u), \end{aligned} \qquad (25)$$

where

$$A_f(w, u) = \frac{i}{2\pi} \frac{(\partial_u w(u)\partial_u w(u+a))^{1/2}}{w(u+a) - w(u)} - \frac{i}{2\pi a}. \qquad (26)$$

Here we have used $\psi(u_1)\psi(u_2) \sim -i/2\pi(u_1 - u_2)$. In the case, $w = U = -e^{-\kappa u}$, it becomes

$$A_f(U, u) = \frac{i}{2\pi a}\left(\frac{\kappa a}{2} \frac{1}{\sinh\frac{\kappa a}{2}} - 1\right). \qquad (27)$$

This is the temperature-dependent part of a finite temperature Green function for fermions. Expanding this as a power series of a, we find

$$A_f(U, u) = \sum_{n=1}^{\infty} i^{2n+1} \frac{(1 - 2^{1-2n})B_n \kappa^{2n}}{4\pi n} \frac{a^{2n-1}}{(2n-1)!}. \qquad (28)$$

Hence the flux of the $2n$th rank current from a black hole is given by the value of its Schwarzian derivative as

$$\left\langle :\frac{i^{2n-1}}{2}\psi \partial^{2n-1}\psi(u): \right\rangle = \frac{(1 - 2^{1-2n})B_n}{8\pi n}\kappa^{2n}. \qquad (29)$$

This again precisely gives the thermal flux F_{2n}^+ for fermions. The difference in the Planck distribution between scalars and spinors comes from the difference of conformal weights. It is amusing to consider fields with another conformal weight to derive possibly different thermal spectrum from $N^{\pm}(\omega)$.

VI. DISCUSSION

In this paper, we have applied a conformal field theory technique to obtain the full thermal spectrum from Hawking radiation in both cases of bosons and fermions. We have shown that, by investigating transformation properties of higher-spin currents under a conformal transformation which maps the null coordinate u to the Kruskal coordinate $U = -e^{-\kappa u}$, the expectation value of each higher-spin current in the Unruh vacuum exactly coincides with a corresponding specific moment of the Planck distribution. The full thermal distribution of the Hawking radiation can be reproduced from these expectation values for the higher-spin currents. In the previous analyses using anomalies in [4–13,18], only the zeroth and first moments of the Planck distribution, corresponding to fluxes of charge (or angular momentum) and energy, could be derived. In this sense, our present calculation has made the previous partial results complete. But it is more desirable if we can reinforce the present conformal field theory calculation by a covariant calculation of higher-spin currents. It is known that even in the gravitational and gauge field background, two-dimensional massless field theories can be described by conformal field theories, and the present calculation of using holomorphic (or antiholomorphic) higher-spin currents is naturally justified. In a covariant formulation, these higher-spin currents are given by specific components of covariant tensors. To clarify them, it is necessary first to construct covariant traceless higher-spin tensors and then to calculate the trace anomalies of these tensors. By solving these equations, we should be able to extract the (anti)holomorphic components, corresponding to those we have used in our paper. It is also interesting to see what kind of transformations these higher-spin currents will generate as the conformal energy-momentum tensor generates conformal transformations which are combinations of coordinate transformations and Weyl transformations. We will come back to these problems in the near future [15].

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