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Surface geometry of 5D black holes and black rings

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We discuss geometrical properties of the horizon surface of five-dimensional rotating black holes and black rings. Geometrical invariants characterizing these 3D geometries are calculated. We obtain a global embedding of the 5D rotating black horizon surface into a flat space. We also describe the Kaluza-Klein reduction of the black ring solution (along the direction of its rotation) which, though it is nakedly singular, relates this solution to the 4D metric of a static black hole distorted by the presence of external scalar (dilaton) and vector (''electromagnetic'') fields. The properties of the reduced black hole horizon and its embedding in $\mathbb{E}^3$ are briefly discussed.

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I. INTRODUCTION

Black objects (holes, strings, rings, etc.) in higher dimensional spacetimes have attracted a lot of attention recently. The existence of higher than 4 dimensions of the spacetime is a natural consequence of the consistency requirement in string theory. Models with large extra dimensions, originally proposed to solve such long-standing fundamental ''puzzles'' as the hierarchy and cosmological constant problems, became very popular recently. In these models mini black holes and other black objects play a special role serving as natural probes of extra dimensions. This is one of the reasons why the questions about what kind of black objects can exist in higher dimensions and what are their properties are now discussed so intensively.

Higher dimensional generalizations of the Kerr metric for a rotating black hole were obtained quite a long time ago by Myers and Perry (MP) [1]. In a D-dimensional spacetime the MP metrics, besides the mass M, also contain $[(D-1)/2]$ parameters connected with the independent components of the angular momentum of the black hole. (Here $[A]$ means the integer part of A.) The event horizon of the MP black holes has the spherical topology S^{D-2}. This makes them in many aspects similar to the 4D Kerr black hole. According to the Hawking theorem [2] any stationary black hole in a 4D spacetime obeying the dominant energy condition has the topology of the horizon S^2. Black hole surface topologies distinct from S^2 are possible if the dominant energy condition is violated [3]. Moreover, a vacuum, stationary black hole is uniquely specified by its mass and angular momentum. Recent discovery of different black ring solutions [4–12] (which includes both one parameter and two parameters of rotation and also black saturn) demonstrated that both the restriction on the topology of the horizon and the uniqueness property of black holes are violated in the 5D spacetime.

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In this paper we discuss the geometry of the horizon surfaces of 5D black rings and 5D black holes with one rotation parameter. A similar problem for the 4D rotating black holes was studied in detail by Smarr [13]. We generalize his approach to the 5D case. After a brief summary of known properties of 3D round spheres and tori in the flat 4D space (Sec. II), we consider a geometry of 3D space which admits two orthogonal commuting Killing vectors (Sec. III). In particular, we calculate its Gauss curvature. In Sec. IV we apply these results to the horizon surface of 5D rotating black holes with one rotation parameter. The embedding of this 3D surface into the flat spacetime is considered in Sec. V. The horizon surface geometry for a 5D rotating black ring is discussed in Sec. VI. This section also considers a Kaluza-Klein reduction of the black ring metric along the direction of its rotation which maps this solution onto a black hole solution of 4D Einstein equations with the dilaton and ''electromagnetic'' fields. The geometry and embedding of the horizon in the $\mathbb{E}^3$ for this metric are obtained. Section VII contains the discussion of the results.

II. SPHERE S^3 AND TORUS $S^2 \times S^1$ IN $\mathbb{E}^4$

A. Sphere S^3

In this section we briefly remind the reader about some known properties of a 3D sphere and a torus in a flat 4D space.

Consider four-dimensional Euclidean space $\mathbb{E}^4$ and denote by X_i ($i = 1, \dots, 4$) the Cartesian coordinates in it. A 3-sphere consists of all points equidistant from a single point $X_i = 0$ in $\mathbb{R}^4$. A unit round sphere S^3 is a surface defined by the equation $\sum_{i=1}^4 X_i^2 = 1$. Using complex coordinates $z_1 = X_1 + iX_2$ and $z_2 = X_3 + iX_4$, one can also equivalently define the unit 3-sphere as a subset of $\mathbb{C}^2$,

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1|^2 + |z_2|^2 = 1\}. \tag{1}$$

We use the embedding of S^3 in $\mathbb{C}^2$ to introduce the *Hopf* coordinates (θ, ϕ, ψ) as

$$z_1 = \sin(\theta)e^{i\phi}; \qquad z_2 = \cos(\theta)e^{i\psi}. \tag{2}$$

Here θ runs over the range $[0, \pi/2]$, and ϕ and ψ can take any values between 0 and 2π. In these coordinates the metric on the 3-sphere is

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2. \tag{3}$$

The volume of the unit 3-sphere is $2\pi^2$. Coordinate lines of ϕ and ψ are circles. The lengths of these circles take the maximum value 2π at $\theta = \pi/2$ for the ϕ line and at $\theta = 0$ for the ψ line, respectively. These largest circles are geodesics. Similarly, the coordinate lines of the θ coordinate are geodesics. For the fixed values of $\phi = \phi_0$ and $\psi = \psi_0$ and $\theta \in [0, \pi/2]$, this line is a segment of the length $\pi/2$ connecting the fixed points of the Killing vectors ∂_ϕ and ∂_ψ. Four such segments, $\phi = \phi_0, \phi_0 + \pi, \psi = \psi_0, \psi_0 + \pi$, form the largest circle of length 2π.

The surfaces of constant θ are flat *tori* T^2. For instance, $\theta = \theta_0$ can be cut apart to give a rectangle with horizontal edge length $\cos\theta_0$ and vertical edge length $\sin\theta_0$. These tori are called *Hopf tori* and they are pairwise linked. The fixed points of the vectors ∂_ϕ and ∂_ψ ($\theta = 0$ for ∂_ϕ and $\theta = \pi/2$ for ∂_ψ) form a pair of linked great circles. Every other Hopf torus passes between these circles. The equatorial Hopf torus is the one which can be made from a square. The others are all rectangular. Also we can easily see that the surfaces of constant ϕ or constant ψ are *half* 2-spheres or topologically disks.

B. Torus $\mathcal{T}^3 = S^2 \times S^1$

The equation of a torus $\mathcal{T}^3 = S^2 \times S^1$ in $\mathbb{E}^4$ is

$$X_1^2 + X_2^2 + \left(\sqrt{X_3^2 + X_4^2} - a\right)^2 = b^2. \tag{4}$$

The surface $\mathcal{T}^3$ is obtained by the rotation of a sphere S^2 of the radius b around a circle S^1 of the radius a ($a > b$). In this paper we call $\mathcal{T}^3$ a *flat space torus*. Let us emphasize that the word *torus* is commonly used for the topological product of circles. Such a space can be obtained, for example, from a unit cube in $\mathbb{E}^3$ through proper identification of the *opposite* boundary planes. This space has the topology $S^1 \times S^1 \times S^1$ and a flat metric. However, the geometry of the *flat space torus* ($S^2 \times S^1$) is determined by its embedding in $\mathbb{E}^4$ and it is not flat. Let us define toroidal coordinates as

$$\begin{aligned} X_1 &= \frac{\alpha \sin\hat{\theta}}{B}\cos\phi, \qquad X_2 = \frac{\alpha \sin\hat{\theta}}{B}\sin\phi, \\ X_3 &= \frac{\alpha \sinh\eta}{B}\cos\psi, \qquad X_4 = \frac{\alpha \sinh\eta}{B}\sin\psi, \end{aligned} \tag{5}$$

where $B = \cosh\eta - \cos\hat{\theta}$. The toroidal coordinates $(\eta, \hat{\theta}, \phi, \psi)$ change in the following intervals:

$$0 < \eta < \infty, \qquad 0 \le \hat{\theta} \le \pi, \qquad 0 \le \phi, \qquad \psi \le 2\pi. \tag{6}$$

The flat metric in these coordinates takes the form

$$ds^2 = \frac{\alpha^2}{B^2}(d\eta^2 + \sinh^2\eta d\psi^2 + d\hat{\theta}^2 + \sin^2\hat{\theta} d\phi^2). \tag{7}$$

In these coordinates the surface of constant $\eta = \eta_0$ is a torus $\mathcal{T}^3$ and one has

$$\alpha = \sqrt{a^2 - b^2}, \qquad \cosh\eta_0 = a/b. \tag{8}$$

Introducing new coordinates $y = \cosh\eta$ and $x = \cos\hat{\theta}$, one can also write the metric (7) in the form [7]

$$ds^2 = \frac{\alpha^2}{(y-x)^2}\left[\frac{dy^2}{y^2 - 1} + (y^2 - 1)d\psi^2 + \frac{dx^2}{1 - x^2} + (1 - x^2)d\phi^2\right]. \tag{9}$$

The points with $\eta < \eta_0$ lie in the exterior of $\mathcal{T}^3$. The induced geometry on the 3-surface $\eta = \eta_0$ is

$$ds^2 = \frac{a^2 - b^2}{(a - b\cos\hat{\theta})^2}[(a^2 - b^2)d\psi^2 + b^2(d\hat{\theta}^2 + \sin^2\hat{\theta} d\phi^2)]. \tag{10}$$

This metric has two Killing vectors, ∂_ϕ and ∂_ψ. The first one has two sets of fixed points, $\theta = 0$ and $\theta = \pi$, which are circles S^1. The second Killing vector, ∂_ψ, does not have fixed points. The 3-volume of the torus $\mathcal{T}^3$ is $8\pi^2 ab^2$.

Since the sections $\psi =$ const are round spheres, instead of $\hat{\theta}$ it is convenient to use another coordinate, $\theta \in [0, \pi]$,

$$\sin\theta = \frac{\sqrt{a^2 - b^2}\sin\hat{\theta}}{a - b\cos\hat{\theta}}. \tag{11}$$

Using this coordinate one can rewrite the metric (10) in the form

$$ds^2 = (a + b\cos\theta)^2 d\psi^2 + b^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{12}$$

Once again we can easily see that the surfaces of constant θ are flat *tori* T^2 except for $\theta = 0$ or $\theta = \pi$, which are circles. The surfaces of constant ψ are 2-spheres whereas the surfaces of constant ϕ are 2-tori.

Sometimes it is convenient to consider special foliations of $\mathcal{T}^3$ [14]. This foliation is a kind of "clothing" worn on a manifold, cut from a stripy fabric. These stripes are called plaques of the foliation. On each sufficiently small piece of the manifold, these stripes give the manifold a local product structure. This product structure does not have to be consistent outside local patches; a stripe followed around long enough might return to a different, nearby stripe. As an example of foliations let us consider the manifold $\mathbb{R}^3$. The foliations are generated by two-dimensional leaves or plaques with one coordinate as a constant. That is, the surfaces $z =$ constant would be the plaques of the folia-

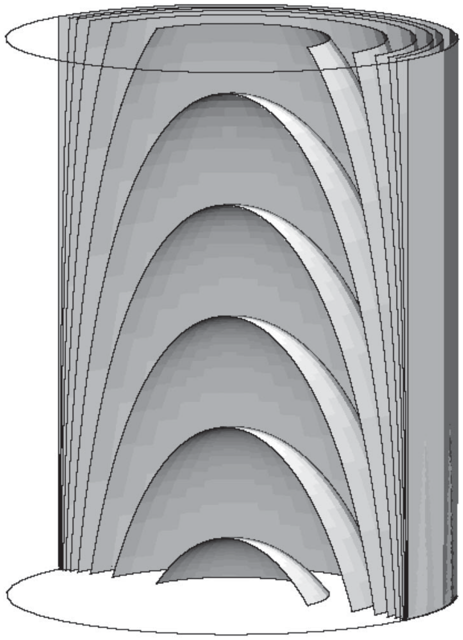

FIG. 1. This picture shows the transverse Reeb foliations of the cylindrical section of $S^2 \times S^1$. The two-dimensional spherical shaped stripes or "plaques" are stacked, giving rise to a cylindrical section of the 3-torus. (courtesy: http://kyokan.ms.u-tokyo.ac.jp)

tions, and in this case there is a global product structure. Similarly, one can consider the foliations of $S^2 \times S^1$. Figure 1 shows the transverse *Reeb* foliations of the cylindrical section of $S^2 \times S^1$ [14]. We can see the stacking of spherically shaped plaques giving rise to the cylindrical section.

III. GEOMETRY OF THREE-DIMENSIONAL SPACE WITH TWO ORTHOGONAL COMMUTING KILLING VECTORS

As we shall see, both metrics of the horizon surface of a 5D black ring and a black hole with one rotation parameter can be written in the form

$$ds_H^2 = f(\zeta)d\zeta^2 + g(\zeta)d\phi^2 + h(\zeta)d\psi^2. \tag{13}$$

Here f, g, and h are non-negative functions of the coordinate ζ. One can use an ambiguity in the choice of the coordinate ζ to put $f = 1$. For this choice ζ has the meaning of the proper distance along the ζ-coordinate line. We call such a parametrization canonical. The coordinates ϕ and ψ have a period of 2π and $\zeta \in [\zeta_0, \zeta_1]$. ∂_ϕ and ∂_ψ are two mutually orthogonal Killing vectors. If $g(\zeta)$ $[h(\zeta)]$ vanishes at some point, then the Killing vector ∂_ϕ (∂_ψ) has a fixed point at this point. The metric (13) does not have a conelike singularity at a fixed point of ∂_ψ if at this point the following condition is satisfied:

$$\frac{1}{2\sqrt{hf}}\frac{dh}{d\zeta} = 1. \tag{14}$$

A condition of regularity of a fixed point of ∂_ϕ can be obtained from (14) by changing h to g.

By comparing the metric (13) with the metric for the 3-sphere (3), one can conclude that (13) describes the geometric of a distorted 3D sphere if g and h are positive inside some interval (ζ_1, ζ_2), while g vanishes at one of its end points (say, ζ_1) and h vanishes at the other (say, ζ_2). Similarly, by comparing (13) with (12) one concludes that, if, for example, g is positive in the interval (ζ_1, ζ_2) and vanishes at its ends, while h is positive everywhere on this interval, including its ends, the metric (13) describes a topological torus.

For the metric (13), the nonvanishing components of the curvature tensor are

$$R_{\zeta\phi\zeta\phi} = \frac{g'(fg)'}{4fg} - \frac{1}{2}g'', \tag{15}$$

$$R_{\zeta\psi\zeta\psi} = \frac{h'(fh)'}{4fh} - \frac{1}{2}h'', \tag{16}$$

$$R_{\phi\psi\phi\psi} = -\frac{g'h'}{4f}. \tag{17}$$

Here $(')$ denotes the differentiation with respect to the coordinate ζ.

Denote by e_a^i $(i, a = 1, 2, 3)$ 3 orthonormal vectors and introduce the Gauss curvature tensor as follows:

$$K_{ab} = -R_{ijkl}e_a^i e_b^j e_a^k e_b^l. \tag{18}$$

The component K_{ab} of this tensor coincides with the curvature in the 2D direction for the 2D plane spanned by e_a^i and e_b^j. One has

$$\sum_{b=1}^{3} K_{ab} = R_{ij}e_a^i e_a^j, \qquad \sum_{a=1}^{3}\sum_{b=1}^{3} K_{ab} = R. \tag{19}$$

For the metric (13) the directions of the coordinate lines θ, ϕ, and ψ are eigenvectors of K_{ab} and the corresponding eigenvalues are K_a

$$K_\psi = \frac{R_{\zeta\phi\zeta\phi}}{fg}, \qquad K_\phi = \frac{R_{\zeta\psi\zeta\psi}}{fh}, \qquad K_\zeta = \frac{R_{\phi\psi\phi\psi}}{gh}. \tag{20}$$

These quantities are the curvatures of the 2D sections orthogonal to the ψ, ϕ, and ζ lines, respectively. For brevity, we call these 2D surfaces ψ, ϕ, and ζ sections.

For the unit sphere S^3, from (3) one can easily see that

$$K_\psi = K_\phi = K_\theta = 1. \tag{21}$$

However, for the torus $S^2 \times S^1$, from (12) we have

$$K_\psi = \frac{1}{b^2}; \qquad K_\phi = K_\theta = \frac{\cos\theta}{b(a + b\cos\theta)}. \tag{22}$$

Thus we see that K_ψ always remains positive, while K_ϕ and K_θ are positive in the interval $(0 \le \theta < \pi/2)$. Thus the *equatorial plane* $(\theta = \pi/2)$ divides the torus in two halves, one in which all the sectional curvatures are positive and the other in which two of the sectional curvatures are negative. In fact, the surface $\theta = \pi/2$ is topologically $S^1 \times S^1$ with the metric

$$ds^2 = a^2 d\psi^2 + b^2 d\phi^2. \tag{23}$$

Equations (19) imply

$$R^\zeta_\zeta = K_\phi + K_\psi, \qquad R^\phi_\phi = K_\psi + K_\zeta, \qquad R^\psi_\psi = K_\zeta + K_\phi. \tag{24}$$

$$R = 2(K_\zeta + K_\phi + K_\psi). \tag{25}$$

From the above expression it is clear that $K_\psi < 0$ if g' and $\ln[fg/(g')^2]'$ have opposite signs. Similarly, $K_\phi < 0$ implies that h' and $\ln[fh/(h')^2]'$ have opposite signs. For $K_\zeta < 0$, g' and h' must have the same sign.

Let us consider now *Euler characteristics* of the two-dimensional sections of the horizon surface. We denote by χ_a the Euler characteristic for the 2-surface $x^a = \text{const}$. By using the *Gauss Bonnet* theorem we have

$$2\pi\chi_a = \iint_{\mathcal{M}} K_a dA + \int_{\partial\mathcal{M}} k_g ds. \tag{26}$$

Here dA is the element of area on the surface and k_g is the geodesic curvature on the boundary. If the surface has no boundary or the boundary line is a geodesic, then the last term vanishes. For the metric (13) simple calculations give

$$2\pi\chi_\psi = -\pi\left[\frac{g'}{\sqrt{fg}}\right]_{\zeta_0}^{\zeta_1} + \int_{\partial\mathcal{M}} k_g ds, \tag{27}$$

$$2\pi\chi_\phi = -\pi\left[\frac{h'}{\sqrt{fh}}\right]_{\zeta_0}^{\zeta_1} + \int_{\partial\mathcal{M}} k_g ds, \tag{28}$$

$$\chi_\zeta = 0. \tag{29}$$

Thus we see that the Gaussian curvatures of sections completely describe the topology and geometry of the 3-horizons.

IV. A 5D ROTATING BLACK HOLE WITH ONE ROTATION PARAMETER

A. Volume and shape of the horizon surface

For the five-dimensional MP black hole with a single parameter of rotation, the induced metric on the horizon is [1]

$$ds^2 = r_0^2 ds_H^2, \tag{30}$$

$$ds_H^2 = f(\theta)d\theta^2 + \frac{\sin^2\theta}{f(\theta)}d\phi^2 + (1 - \alpha^2)\cos^2\theta d\psi^2. \tag{31}$$

Here $f(\theta) = (1 - \alpha^2\sin^2\theta)$ and r_0 is the length parameter related to the mass M of the black hole as

$$r_0^2 = \frac{8\sqrt{\pi}GM}{3}. \tag{32}$$

The metric (31) is in the *Hopf coordinates* and hence the coordinate θ varies from 0 to $\pi/2$. The rotation is along the ϕ direction. The quantity $\alpha = |a|/r_0$ characterizes the rapidity of the rotation. It vanishes for a nonrotating black hole and takes the maximal value $\alpha = 1$ for an extremely rotating one. In what follows we put $r_0 = 1$, so that α coincides with the rotation parameter. Different quantities (such as lengths and curvature components) can be easily obtained from the corresponding dimensionless expressions by using their scaling properties.

For $\alpha = 0$ the horizon is a round sphere S^3 of the unit radius. In the presence of rotation this sphere is distorted. Its 3-volume is $V_3 = 2\pi^2\sqrt{1-\alpha^2}$. In the limiting case of an extremely rotating black hole, $\alpha = 1$, V_3 vanishes.

The coordinate lines of ϕ and ψ on this distorted sphere remain closed circles. The length of the circle corresponding to the ϕ coordinate changes from 0 (at $\theta = 0$) to its largest value (at $\theta = \pi/2$)

$$l_\phi = \frac{2\pi}{\sqrt{1-\alpha^2}}. \tag{33}$$

Similarly, the length of the circles connected with the ψ coordinate changes from its maximal value (at $\theta = 0$)

$$l_\psi = 2\pi\sqrt{1-\alpha^2} \tag{34}$$

to 0 at $\theta = \pi/2$. A line $\phi, \psi = \text{const}$ on the distorted sphere is again a geodesic:

$$l_\theta = 4\mathbf{E}(\alpha), \tag{35}$$

where $\mathbf{E}$ is the complete elliptic integral of the second kind.

The lengths l_ψ, l_θ, and l_ϕ as the functions of the rotation parameter α are shown at Fig. 2 by lines 1, 2, and 3, respectively. All these lines start at the same point $(0, 2\pi)$. In the limit of the extremely rotating black hole $(\alpha = 1)$, the horizon volume vanishes, $l_\psi = 0$, $l_\theta = 4$, and l_ϕ infinitely grows.

B. Gaussian curvature

Calculations of the eigenvalues K_a of the Gaussian curvatures give

$$K_\psi = \frac{[1 - \alpha^2(1 + 3\cos^2\theta)]}{f(\theta)^3}, \qquad K_\phi = K_\theta = \frac{1}{f(\theta)^2}. \tag{36}$$

From these relations it follows that the quantity K_ψ is

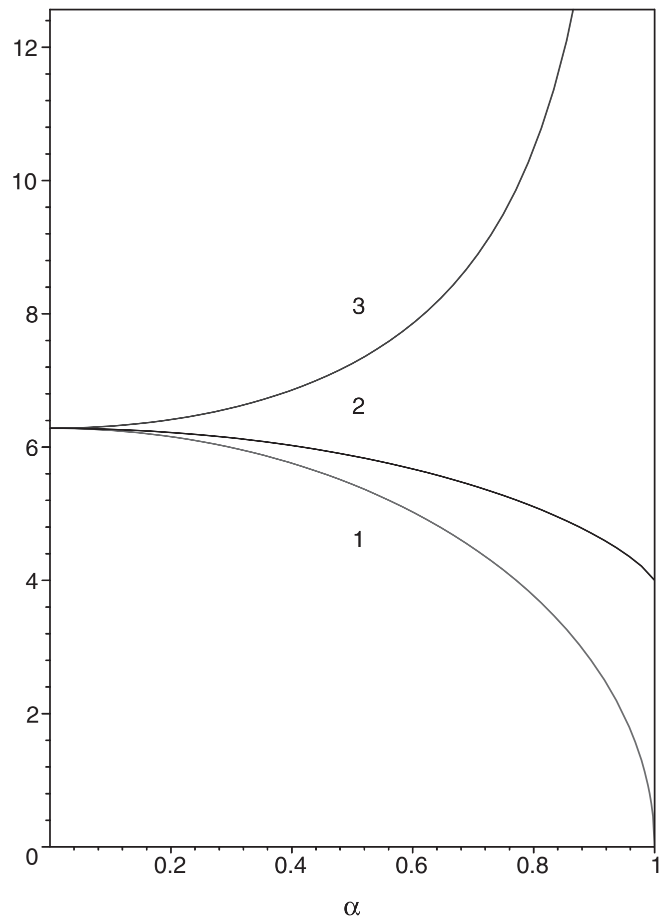

FIG. 2. Lengths l_ψ (1), l_θ (2), and l_ϕ (3) as the functions of the rotation parameter α.

negative in the vicinity of the "pole" $\theta = 0$ for $1/2 < \alpha < 1$, while the other two quantities, K_θ and K_ϕ, are always positive. This is similar to the 4D Kerr black hole where the Gaussian curvature of the two-dimensional horizon becomes negative near the pole for $\alpha > 1/2$. This is not surprising since the 2D section $\psi = \text{const}$ of the metric is isometric to the geometry of the horizon surface of the Kerr black hole.

The Ricci tensor and Ricci scalar for the metric on the surface of the horizon of the 5D black hole are

$$R_\theta^\theta = R_\phi^\phi = \frac{2[1 - \alpha^2(1 + \cos^2\theta)]}{f(\theta)^3}, \tag{37}$$

$$R_\psi^\psi = \frac{2}{f(\theta)^2}, \qquad R = \frac{2[3 - \alpha^2(3 + \cos^2\theta)]}{f(\theta)^3}. \tag{38}$$

The components of the Ricci tensors R_θ^θ and R_ϕ^ϕ become negative for certain values near the pole $\theta = 0$, when $\alpha > 1/\sqrt{2}$, while the Ricci scalar is negative when $\alpha > \sqrt{3/4}$.

It is interesting to note that the surfaces of constant ϕ or constant ψ are topologically disks with Euler characteristics equal to unity. The boundaries of these disks are on $\theta = \pi/2$. It is easy to check from Eqs. (27) and (28) that boundary terms of the *Gauss Bonnet* equation vanish on this boundary. This shows that the boundary, which is the equatorial line on the deformed hemisphere, is a geodesic of the induced metric. Another important point is, while approaching the naked singularity limit ($\alpha = 1$), the Gaussian curvatures of all the three sections, as well as the negative Ricci scalar, blow up along the "equator" ($\theta = \pi/2$). This shows the extreme flattening of the horizon along the equatorial plane, before the horizon shrinks to zero volume.

V. EMBEDDING

A. Embedding of the horizon in 5D pseudo-Euclidean space

Let us discuss now the problem of the embedding of the horizon surface of a rotating 5D black hole into a flat space. We start by reminding the reader that a similar problem for a 4D (Kerr) black hole was considered a long time ago by Smarr [13]. He showed that, if the rotation parameter of the Kerr metric $\alpha < 1/2$, then the 2D surface of the horizon can be globally embedded in $\mathbb{E}^3$ as a rotation surface. For $\alpha > 1/2$ such an embedding is possible if the signature of the 3D flat space is $(-, +, +)$. The reason why a rapidly rotating black hole horizon cannot be embedded in $\mathbb{E}^3$ is as follows. Let us consider a 2D geometry which has a symmetry generated by a Killing vector field ξ. Suppose ξ has a fixed point "P" where it vanishes. It is then possible to show that if a vicinity of such a fixed point is to be embedded in $\mathbb{E}^3$ then the Gaussian curvature at P must be non-negative (see for example [15]). The Gaussian curvature at the pole of the horizon of a rotating black hole becomes negative for $\alpha > 1/2$. In a recent paper [15] a global embedding of the horizon of a rapidly rotating black hole into $\mathbb{E}^4$ was constructed.

It should be emphasized here that embeddings into higher dimensional Euclidean or pseudo-Euclidean spaces are often used as models to understand the global properties of the lower dimensional geometry. A well-known example in which the embedding is done in a pseudo-Euclidean space is a Lorentz (or hyperboloid) model of a 2D hyperboloid of revolution. The necessity of the choice of the Minkowskian signature in the embedding space is directly connected to the property that the Gaussian curvature of a 2D hyperboloid of revolution is negative. In four-dimensional geometry such an example is a Euclidean anti-de Sitter space which is embedded in the 5D flat space with a Minkowskian signature. In this section we consider such embedding models for the horizon of a 5D rotating black hole.

Since the 3D surface of a rotating 5D black hole has two commuting orthogonal Killing vectors, it is natural to consider its embedding into the flat space which has at least two independent orthogonal 2-planes of the rotation. In this case the minimal number of dimensions of the space of the embedding is 5. We write the metric in the form

$$dS^2 = \varepsilon dz^2 + dx_1^2 + dx_2^2 + dy_1^2 + dy_2^2, \tag{39}$$

where $\varepsilon = \pm 1$. By introducing polar coordinates (ρ, ϕ) and (r, ψ) in the 2-planes (x_1, x_2) and (y_1, y_2), respectively, we obtain

$$dS^2 = \varepsilon dz^2 + d\rho^2 + \rho^2 d\phi^2 + dr^2 + r^2 d\psi^2. \tag{40}$$

Using $\mu = \cos\theta$ as a new coordinate, one can rewrite the metric on the horizon (31) in the form

$$ds^2 = f d\mu^2 + \rho^2 d\phi^2 + r^2 d\psi^2, \tag{41}$$

$$f = \frac{1 - \alpha^2\mu^2}{1 - \mu^2}, \qquad \rho = \frac{\mu}{\sqrt{1 - \alpha^2\mu^2}}, \tag{42}$$

$$r = \sqrt{(1 - \alpha^2)(1 - \mu^2)}. \tag{43}$$

Assuming that z is a function of μ, and identifying ρ and r in (40) with (43), one obtains the metric (41) provided the function $z(\mu)$ obeys the equation

$$\left(\frac{dz}{d\mu}\right)^2 = \varepsilon\left[f - \left(\frac{d\rho}{d\mu}\right)^2 - \left(\frac{dr}{d\mu}\right)^2\right]. \tag{44}$$

By substituting (43) into (44) one obtains

$$\left(\frac{dz}{d\mu}\right)^2 = \varepsilon\frac{\alpha^2\mu^2(3\alpha^2\mu^2 - \alpha^4\mu^4 - 3)}{(1 - \alpha^2\mu^2)^3}. \tag{45}$$

It is easy to check that for $|\alpha| \leq 1$ and $0 \leq \mu \leq 1$ the expression on the right-hand side of (45) always has the sign opposite to the sign of ε. Thus one must choose $\varepsilon = -1$ and one has

$$z = \frac{1}{2\alpha}\int_{\sqrt{1-\alpha^2\mu^2}}^{1} \frac{dy}{y^{3/2}}\sqrt{1 + y + y^2}. \tag{46}$$

Let us emphasize that this result is valid for both the slowly and the rapidly rotating black holes.

B. Global embedding into $\mathbb{E}^6$

1. Construction of an embedding

It is possible, however, to find a global isometric embedding of the 3-horizon of a rotating black hole in a flat space with positive signature, if the number of dimensions is 6. This embedding is analogous to the one discussed in [15] for the rapidly rotating Kerr black hole.

Let us denote by X_i $(i = 1, \ldots, 6)$ the Cartesian coordinates in $\mathbb{E}^6$. We write the embedding equations in the form

$$X_i = \frac{\eta(\theta)}{\rho_0} n^i(\tilde{\phi}) \qquad (i = 1, 2, 3), \tag{47}$$

$$X_4 = \nu(\theta)\cos\psi, \qquad X_5 = \nu(\theta)\sin\psi, \qquad X_6 = \chi(\theta). \tag{48}$$

Here the functions n^i obey the condition

$$\sum_{i=1}^{3}(n^i(\tilde{\phi}))^2 = 1. \tag{49}$$

In other words, the 3D vector n^i as a function of $\tilde{\phi}$ describes a line on the unit round sphere S^2. We require this line to be a smooth closed loop $[\mathbf{n}(0) = \mathbf{n}(2\pi)]$ without self-interactions. We denote

$$\rho(\tilde{\phi}) = \left[\sum_{i=1}^{3}(n^i_{,\tilde{\phi}})^2\right]^{1/2}. \tag{50}$$

Then $dl = \rho(\tilde{\phi})d\tilde{\phi}$ is the line element along the loop. The total length of the loop is

$$l_0 = 2\pi\rho_0 = \int_0^{2\pi}\rho(\tilde{\phi})d\tilde{\phi}. \tag{51}$$

We define a new coordinate ϕ as

$$\phi = \frac{1}{\rho_0}\int_0^{\tilde{\phi}}\rho(\tilde{\phi})d\tilde{\phi}. \tag{52}$$

It is a monotonic function of $\tilde{\phi}$ and has the same period 2π as $\tilde{\phi}$. The induced metric for the embedded 3D surface defined by (47) and (48) becomes

$$ds^2 = \left[\frac{\eta_{,\theta}^2}{\rho_0^2} + \nu_{,\theta}^2 + \chi_{,\theta}^2\right]d\theta^2 + \eta^2 d\phi^2 + \nu^2 d\psi^2. \tag{53}$$

Now comparing Eqs. (31) and (53) we get

$$\eta(\theta) = \frac{\sin\theta}{\sqrt{1 - \alpha^2\sin^2\theta}}, \qquad \nu(\theta) = \sqrt{1 - \alpha^2}\cos\theta, \tag{54}$$

$$\chi(\theta) = \int_0^{\theta}\cos\theta\sqrt{\left[1 - \frac{1}{\rho_0^2(1 - \alpha^2\sin^2\theta)^3}\right]}d\theta. \tag{55}$$

We choose the functions n^i in such a way that

$$\rho_0^2 \geq \frac{1}{(1 - \alpha^2)^3}, \tag{56}$$

so that the function $\chi(\theta)$ remains real valued for all θ; hence we can globally embed the horizon in $\mathbb{E}^6$.

2. A special example

To give an explicit example of the above-described embedding let us put

$$n^1 = \frac{\cos\tilde{\phi}}{F}, \qquad n^2 = \frac{\sin\tilde{\phi}}{F}, \qquad n^2 = \frac{a\sin(N\tilde{\phi})}{F}, \tag{57}$$

$$F = \sqrt{1 + a^2\sin^2(N\tilde{\phi})}. \tag{58}$$

Here $N \geq 1$ is a positive integer. For this choice the value of ρ_0 is

$$\rho_0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{\sqrt{a^2\cos^2(\tilde{\phi})(N^2-1)+a^2+1}}{1+a^2\sin^2(\tilde{\phi})} d\tilde{\phi}. \quad (59)$$

For $N = 1$, $\rho_0 = 1$. For $N > 1$ the above integral can be exactly evaluated to give

$$\rho_0 = \frac{2}{\pi} k_1 [N^2 \Pi(a^2 k_1^2, ik_2) - (N^2-1)K(ik_2)], \quad (60)$$

$$k_1 = \frac{1}{\sqrt{1+a^2}}, \qquad k_2 = a\sqrt{\frac{N^2-1}{1+a^2}}. \quad (61)$$

Here K and Π are elliptic integrals of the first and third kind, respectively. For a fixed value of N, ρ_0 is a monotonically growing function of a (see Fig. 3). This figure shows the length of the closed loop $(n^1(\tilde{\phi}), n^2(\tilde{\phi}), n^3(\tilde{\phi}))$ on a unit sphere [as described by Eqs. (57) and (58)] as a function of the parameters N and a. It can be easily seen from Eq. (55) that the length of this loop has to be greater than 2π (i.e. the equatorial circumference) for the embedding in $\mathbb{E}^6$ to exist. In principle, we can have an arbitrarily long closed loop on a unit sphere. As a special example we consider a "*transverse wave*" loop around the equator of the sphere with N as the frequency of the wave and a as the amplitude. From Fig. 3 it is clear that we can have an arbitrarily long loop by making the frequency arbitrarily large. For a fixed value of a the value of ρ_0 increases monotonically with N. The asymptotic form of ρ_0 for large values of a can be easily obtained as follows. Notice that for large a the denominator in the integral (59) is large unless $\tilde{\phi}$ is close to 0, π, or 2π. Near these points $\cos\tilde{\phi}$ can be approximated by 1, and the expression for ρ_0 takes the form

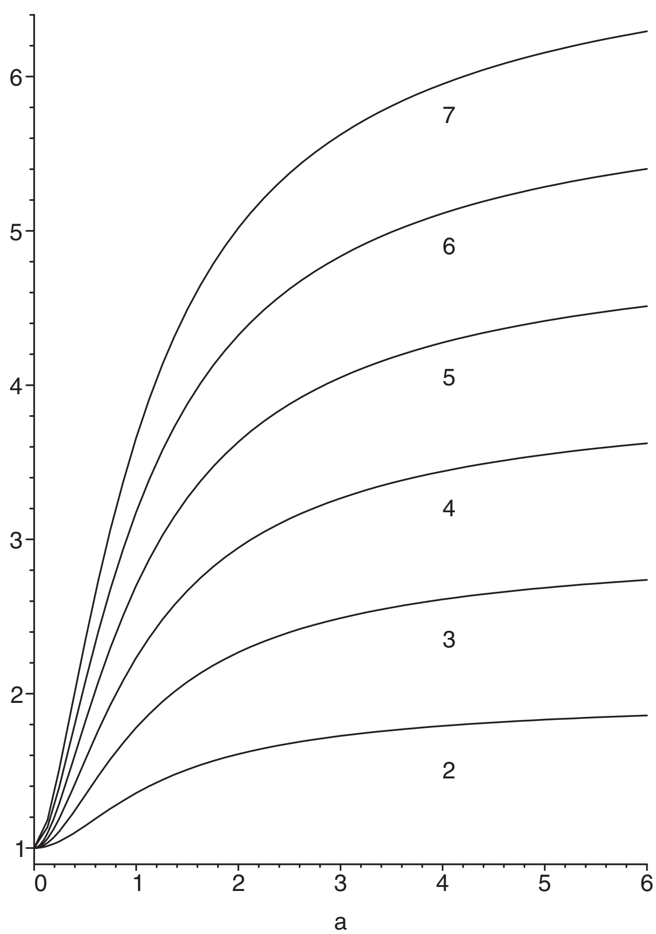

FIG. 3. ρ_0 as a function of a for the values of N from 2 (line 2) to 7 (line 7).

$$\rho_0 \approx \frac{aN}{2\pi} \int \frac{d\tilde{\phi}}{1+a^2\sin^2\tilde{\phi}} = \frac{aN}{\sqrt{1+a^2}}. \quad (62)$$

Using these properties of ρ_0, one can show that for large enough values of N and a the quantity ρ_0 can be made arbitrarily large, so that the condition (56) is satisfied and we have the global embedding of the horizon surface for any $\alpha < 1$.

VI. A 5D ROTATING BLACK RING

A. Horizon surface of a black ring

Now we consider properties of horizon surfaces of stationary black strings in an asymptotically 5D flat spacetime [4]. In this paper we would only consider the *balanced* black ring in the sense that there is no angular deficit or angular excess causing a conical singularity. The ring rotates along S^1 and this balances the gravitational self-attraction. The geometry of a balanced rotating black ring has been studied in [10]. We focus here mainly on the properties of the geometrical invariants in light of the discussions in Sec. III. The metric of the rotating black ring is [5,6]

$$ds^2 = -[F(x)/F(y)]\left[dt + r_0 \frac{\sqrt{2}\nu}{\sqrt{1+\nu^2}}(1+y)d\tilde{\psi}\right]^2 + \frac{r_0^2}{(x-y)^2}[-F(x)(G(y)d\tilde{\psi}^2 + [F(y)/G(y)]dy^2) + F(y)^2([dx^2/G(x)] + [G(x)/F(x)]d\phi^2)], \quad (63)$$

where

$$F(\zeta) = 1 - \frac{2\nu}{1+\nu^2}\zeta, \qquad G(\zeta) = (1-\zeta^2)(1-\nu\zeta). \quad (64)$$

The quantity r_0 is the radius scale of the ring. The parameter $\nu \in [0, 1]$ determines the shape of the ring. The coordinate x changes in the interval $-1 \leq x \leq 1$, while $y^{-1} \in [-1, (2\nu)/(1+\nu^2)]$. The black ring is rotating in the $\tilde{\psi}$ direction. The positive "y" region is the ergosphere of the rotating black ring while the negative "y" region lies outside the ergosphere with spatial infinity at $x = y = -1$.

The metric (63) has a coordinate singularity at $y = 1/\nu$. However, after the transformation

$$d\psi = d\tilde{\psi} + J(y)dy,$$
$$dv = dt - r_0 \frac{\sqrt{2}\nu}{\sqrt{1+\nu^2}}(1+y)J(y)dy, \tag{65}$$

with $J(y) = \sqrt{-F(y)}/G(y)$, the metric is regular at $y = 1/\nu$. In these regular coordinates one can show that the surface $y = 1/\nu$ is the horizon.

The induced metric on the horizon of a rotating black ring is given by

$$ds^2 = r_0^2 ds_H^2, \tag{66}$$

$$ds_H^2 = \frac{p}{k(\theta)}\left[\frac{d\theta^2}{k(\theta)^2} + \frac{\sin^2\theta d\phi^2}{l(\theta)}\right] + ql(\theta)d\psi^2, \tag{67}$$

$$k(\theta) = 1 + \nu\cos\theta, \qquad l(\theta) = 1 + \nu^2 + 2\nu\cos\theta, \tag{68}$$

$$p = \frac{\nu^2(1-\nu^2)^2}{1+\nu^2}, \qquad q = 2\frac{1+\nu}{(1-\nu)1+\nu^2}. \tag{69}$$

In this metric the coordinates ϕ and ψ have a period of 2π and $\theta \in [0, \pi]$. $\theta = 0$ is the axis pointing outwards (i.e. increasing S^1 radius), while $\theta = \pi$ points inwards. The volume of the horizon surface for the metric (67) is

$$V = 8\sqrt{2}\pi^2\nu^2\sqrt{1-\nu}\left[\frac{\sqrt{1+\nu}}{\sqrt{1+\nu^2}}\right]^3. \tag{70}$$

B. Gaussian curvature

The metric (67) is of the form (13), so that one can apply to it the results of Sec. III. For example, its Gaussian curvatures have the following eigenvalues ($i = \psi, \phi, \theta$) ,

$$K_i = \frac{k(\theta)^2 F_i(\cos\theta)}{2\nu(1-\nu^2)^2(1+\nu^2)l(\theta)^2}, \tag{71}$$

where the functions $F_i = F_i(\zeta)$ are defined as follows:

$$F_\psi = \nu(3+\nu^2)\zeta^2 + 2(\nu^4+\nu^2+2)\zeta + 2\nu^{-1} - \nu + 3\nu^3,$$
$$F_\phi = 8\nu^2\zeta^3 + \nu(5\nu^2+7)\zeta^2 + 2(1-\nu^2)\zeta - \nu(3\nu^2+1),$$
$$F_\theta = \nu^3\zeta^2 + (6\nu^2+3\nu+2)\zeta + \nu(3+\nu^2). \tag{72}$$

From the above equations it is clear that for any value of ν and θ the Gaussian curvature of the ψ sections (i.e. K_ψ) always remains positive. This is absolutely similar to the flat space torus case as described by (22). The sign of the Gaussian curvatures for the other two sections, K_ϕ and K_θ, depends on the values of ν and θ. For example, for $\theta = 0$, both these curvatures are positive for all values of ν. But as we increase θ, K_ϕ becomes negative for higher values of ν, and ultimately when $\theta = \pi/2$ it becomes negative for all ν and continues to be negative until $\theta = \pi$. On the other hand, K_θ remains positive and grows with ν for all ν until $\theta = \pi/2$, then starts becoming negative for higher values of ν and ultimately becomes negative for all ν at $\theta = \pi$.

Let us emphasize that, because of the distortions due to the rotation, K_ϕ and K_θ do not become negative at the same value of θ as it was for the flat space torus case. To get an invariant measure of distortion produced due to rotation, let us define two invariant lengths in the following way. We know that an $S^2 \times S^1$ surface can be divided in two halves. In one, all the sectional curvatures are positive, and in the other, at least one of the sectional curvatures is negative. The length of a $\phi = \psi =$ const line in these two halves is invariant (coordinate independent) and gives a measure of the distortion. Let $\theta = \theta_i$ $(i = \theta, \phi)$ be the point where $F_i(\cos\theta)$ vanishes. Then the two invariant lengths are

$$\lambda_{i1} = 2\sqrt{p}\int_0^{\theta_i}\frac{d\theta}{\sqrt{k(\theta)^3}}, \qquad \lambda_{i2} = 2\sqrt{p}\int_{\theta_i}^{\pi}\frac{d\theta}{\sqrt{k(\theta)^3}}. \tag{73}$$

It is easy to check from (12) that in the case of a flat space torus we have $\lambda_{i1} = \lambda_{i2}$. However, for the rotating black rings they are different functions of the parameter ν. Figures 4 and 5 show the invariant lengths for $(i = \theta, \phi)$ as function of ν, respectively. We see that for $i = \theta$, $\lambda_{i2} < \lambda_{i1}$ for small ν. However, as we increase ν the difference between them reduces, and ultimately at $\nu \approx 0.615$, λ_{i2} overtakes λ_{i1}, whereas for $i = \phi$, λ_{i2} is always greater than λ_{i1}.

It is evident that both the ψ and ϕ sections are closed and do not have a boundary. Calculating the *Euler numbers* for these surface we get

$$\chi_\psi = 2, \qquad \chi_\phi = 0. \tag{74}$$

This shows that the ψ section is a deformed 2-sphere with positive Gaussian curvature. Its rotation in the ψ direction generates the horizon surface of the rotating black ring.

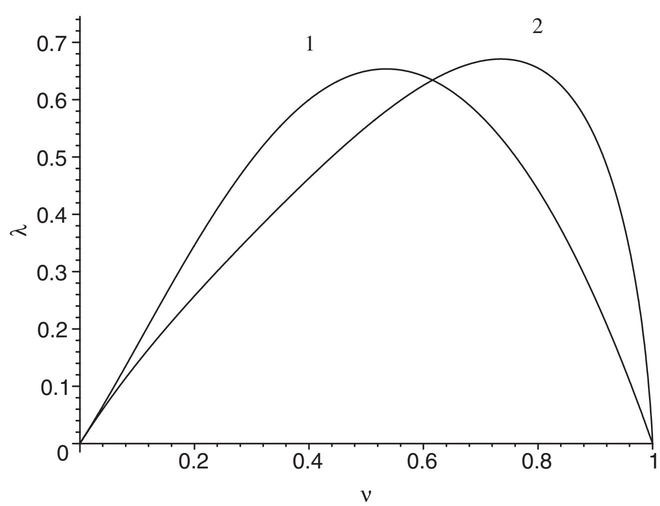

FIG. 4. $\lambda_{\theta 1}$ and $\lambda_{\theta 2}$ as a function of ν.

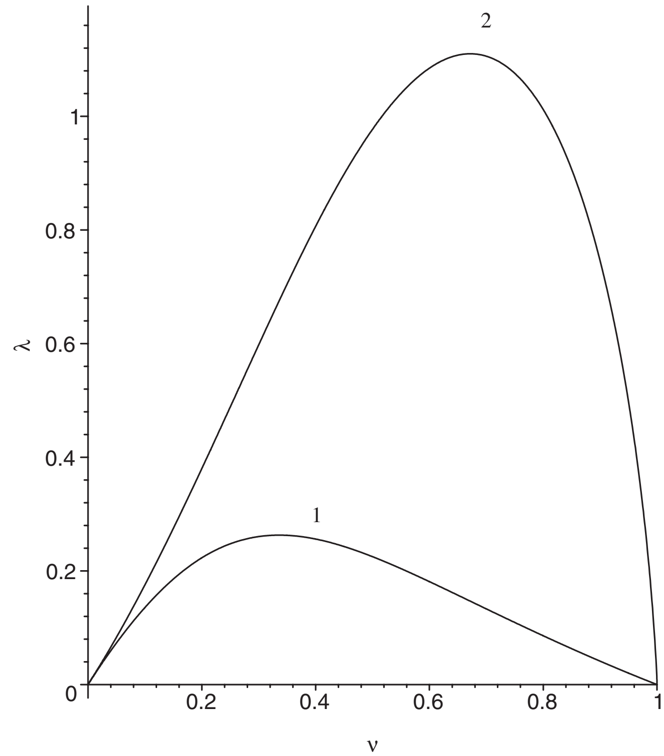

FIG. 5. $\lambda_{\phi 1}$ and $\lambda_{\phi 2}$ as a function of ν.

C. Kaluza-Klein reduction of the rotating black ring

The absence of the conelike singularities in the black ring solution (63) is a consequence of the exact balance between the gravitational attraction and the centrifugal forces generated by the ring's rotation. We discuss the effects connected with the ring rotation from a slightly different point of view. Let us write the metric (63) in the following Kaluza-Klein form (see e.g. [16–18]):

$$ds_5^2 = \Phi^{-(1/3)}[h_{\alpha\beta}dx^\alpha dx^\beta + \Phi(A_t dt + d\phi)^2]. \quad (75)$$

The 4D reduced *Pauli* metric in this space is ($a, b = 0, \dots, 3$)

$$ds_4^2 = h_{ab}dx^a dx^a = \Phi^{1/3}[g_{ab}dx^a dx^b - \Phi A_t^2 dt^2]. \quad (76)$$

Here g_{ab} is the four-dimensional metric on the $\tilde{\psi}$ section of the 5D black ring. By comparison of (75) and (76) one has

$$\Phi^{2/3} = \xi_{\tilde{\psi}}^2 = -\frac{F(x)}{F(y)}L(x, y), \quad (77)$$

$$A_t = \frac{(\xi_t \cdot \xi_{\tilde{\psi}})}{\xi_{\tilde{\psi}}^2} = \frac{\sqrt{2}\nu}{\sqrt{1+\nu^2}}\frac{(1+y)}{L(x, y)}, \quad (78)$$

$$L(x, y) = \left[\frac{2\nu^2}{1+\nu^2}(1+y)^2 + \frac{F(y)G(y)}{(x-y)^2}\right], \quad (79)$$

where $\xi_t = \partial_t$, $\xi_{\tilde{\psi}} = \partial_{\tilde{\psi}}$, and $\xi_\phi = \partial_\phi$ are the Killing vectors of (75). The quantities $\ln(\Phi)$ and A_t can be interpreted as a "dilaton field" and an "electromagnetic potential" in the 4D spacetime.

Since the original space contains points where $\xi_{\tilde{\psi}}^2$ vanishes (or becomes infinite), the 4D metric (76) is singular at these points. As we shall see, this metric describes a distorted 4D black hole. From the 4D point of view this distortion is induced by the dilaton field $\ln(\phi)$ and the electromagnetic field $F_{\mu\nu}$, supported by sources located at the singular points in the black hole exterior. Namely, the presence of these sources and the absence of asymptotic flatness explain the apparent "violation" of *uniqueness theorems*.

The horizon for the 4D metric (76) is defined by the condition

$$h_{tt} = \xi_t^2 - (\xi_{\tilde{\psi}}^2)^{-1}(\xi_t \cdot \xi_{\tilde{\psi}})^2 = 0. \quad (80)$$

It is easy to show that this condition is equivalent to the condition defining the horizon of the 5D metric. Thus both horizons are located at $y = 1/\nu$.

To summarize, the 4D metric (76) obtained after the reduction describes a static 4D black hole in the presence of an external dilaton and "electromagnetic" field. The dilaton field $\ln\Phi$ [as well as the metric (76)] has a singularity at the points where $\xi_{\tilde{\psi}}^2$ either vanishes (at the axis of symmetry, $x = 1$, $y = -1$) or infinitely grows (at the spatial infinity, $x = y = -1$). Outside these regions the dilaton field is regular everywhere including the horizon where it takes the value

$$\Phi_H = \left[\frac{2}{1+\nu^2}\left(\frac{1+\nu}{1-\nu}\right)l(\theta)\right]^{3/2}. \quad (81)$$

The "electromagnetic field strength," which has nonzero components

$$F_{tx} = -A_{t,x}, \qquad F_{ty} = -A_{t,y}, \quad (82)$$

is regular everywhere and vanishes at the spatial infinity. However, the $F^2 = F_{ab}F^{ab}$ invariant is well defined throughout the spacetime but drops (towards negative infinity) at the axis of symmetry.

The metric ds_1^2 on the 2D horizon surface for the reduced metric (76) is conformal to the metric ds_0^2 of the 2D section $\tilde{\psi} = \text{const}$ of the black ring horizon, (67). These metrics are of the form [$k = k(\theta)$, $l = l(\theta)$]

$$ds_\epsilon^2 = \Phi_H^{\epsilon/3}\frac{p}{k}\left[\frac{d\theta^2}{k^2} + \frac{\sin^2\theta d\phi^2}{l}\right], \qquad \epsilon = 0, 1. \quad (83)$$

Both metrics ds_ϵ^2 can be embedded in $\mathbb{E}^3$ as rotation surfaces. The embedding equations are

$$X_1 = m\beta\cos\phi, \qquad X_2 = m\beta\sin\phi, \qquad X_3 = m\gamma, \quad (84)$$

where

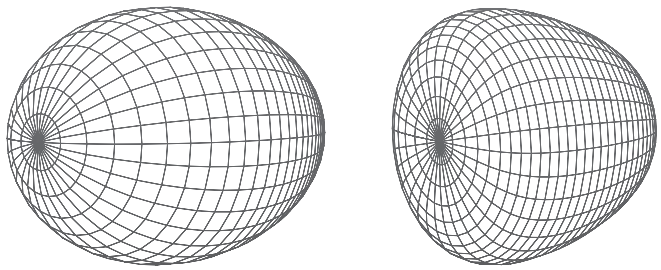

FIG. 6. The embedding diagrams for the metrics ds_0^2 (left) and ds_1^2 (right) for $\nu = 2/3$.

$$\beta = k^{-1/2} l^{-1/2+\epsilon/4} \sin\theta,$$
$$\gamma = \int_0^\theta (k^{-3} l^{\epsilon/2} - \beta_{,\theta}^2)^{1/2} d\theta, \qquad (85)$$
$$m = \sqrt{p}\left[\frac{2}{1+\nu^2}\left(\frac{1+\nu}{1-\nu}\right)\right]^\epsilon.$$

The embedding diagrams for the metrics ds_0^2 and ds_1^2 are shown in Fig. 6 by the left and right plots, respectively. Both rotation surfaces are deformed spheres. The surface with the geometry ds_1^2 is more flattened at poles.

VII. DISCUSSION

In this paper we discussed and analyzed the surface geometry of five-dimensional black holes and black rings with one parameter of rotation. We found that the sectional Gaussian curvature and the Ricci scalar of the horizon surface of the 5D rotating black hole are negative if the rotation parameter is greater than some critical value, similarly to the case of the 4D Kerr black hole. However, there is an important difference between the embeddings of the horizon surfaces of 5D and 4D black holes in the flat space. As was shown in [13], a rotating 2-horizon can be embedded as a surface of rotation in a three-dimensional Euclidean space only when the rotation parameter is less than the critical value. For other examples of embeddings of rotating horizons see [19–22]. For ''supercritical'' rotation the global embedding is possible either in 3D flat space with the signature of the metric $(-, +, +)$ [13] or in $\mathbb{E}^4$ with the positive signature [15]. For the 5D black hole for any value of its rotation parameter the horizon surface cannot be embedded in 5D Euclidean space as a surface of rotation. Such an embedding requires that the signature of the flat 5D space is $(-, +, +, +, +)$. However, we found a global embedding of this surface in 6D Euclidean space.

We calculated the surface invariants for the rotating black ring and analyzed the effect of rotation on these invariants. Finally, we considered the Kaluza-Klein reduction of the rotating black ring which maps its metric onto the metric of the 4D black hole in the presence of external dilaton and electromagnetic fields. Under this map, the horizon of the 5D black ring transforms into the horizon of the 4D black hole. The ''reduced'' black hole is static and axisymmetric. Distorted black holes in the Einstein-Maxwell-dilaton gravity were discussed in [23]. This paper generalizes the well-known results of [24] for vacuum distorted black holes. It would be interesting to compare the reduced distorted black hole discussed in this paper with solutions presented in [23].

ACKNOWLEDGMENTS

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