Radiation equation of state and loop quantum gravity corrections

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(Received 16 January 2007; published 28 June 2007)

The equation of state for radiation is derived in a canonical formulation of the electromagnetic field. This allows one to include correction terms expected from canonical quantum gravity and to infer implications to the universe evolution in radiation dominated epochs. Corrections implied by quantum geometry can be interpreted in physically appealing ways, relating to the conformal invariance of the classical equations.

DOI: 10.1103/PhysRevD.75.123521 PACS numbers: 98.80.Jk, 04.60.Pp

I. INTRODUCTION

In theoretical cosmology, many insights can already be gained from spatially isotropic Friedmann-Robertson-Walker (FRW) models

$$ds^2 = -d\tau^2 + a(\tau)^2\left(\frac{dr^2}{1-kr^2} + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\right) \tag{1}$$

with $k = 0$ or ± 1. The matter content in such a highly symmetric space-time can only be of the form of a perfect fluid with stress-energy tensor $T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b)$ where ρ is the energy density of the fluid, P its pressure and u^a the 4-velocity vector field of isotropic comoving observers. Once an equation of state $P = P(\rho)$ is specified to characterize the matter ingredients, the continuity equation $\dot{\rho} + 3H(\rho + P) = 0$ with the Hubble parameter $H = \dot{a}/a$ allows one to determine the behavior of $\rho(a)$ in which energy density changes during the expansion or contraction of the universe. This function, in turn, enters the Friedmann equation $H^2 + k/a^2 = 8\pi G\rho/3$ and allows one to derive solutions for $a(\tau)$.

In general, one would expect the equation of state $P = P(\rho)$ to be nonlinear which would make an explicit solution of the continuity and Friedmann equations difficult. It is thus quite fortunate that in many cases linear equations of state $P = w\rho$ with w constant are sufficient to describe the main matter contributions encountered in cosmology at least phenomenologically. The influence of compact objects on cosmological scales is, for instance, described well by the simple dust equation of state $P(\rho) = 0$. Relativistic matter, mainly electromagnetic radiation, satisfies the linear equation of state $P = \frac{1}{3}\rho$. The latter example is an exact equation describing the Maxwell field, rather than an approximation for large scale cosmology. It is thus, at first sight, rather surprising that the dynamics of electromagnetic waves in a universe can be summarized in such a simple equation of state irrespective of details of the field configuration. The result follows in the standard way from the trace-freedom of the electromagnetic stress-energy tensor and is thus related to the conformal symmetry of Maxwell's equations. That the availability of such a simple equation of state is very special for a matter field can be seen by taking the example of a scalar field ϕ with potential $V(\phi)$. In this case, we have an energy density $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ and pressure $P = \frac{1}{2}\dot{\phi}^2 - V(\phi)$. Unless the scalar is free and massless, $V(\phi) = 0$ for which we have a stiff fluid $P = \rho$, there is no simple relation between pressure and energy density independently of a specific solution.

Any conformal symmetry such as that of elecromagnetism might be broken by quantum effects especially when quantum gravity with its new scale provided by the Planck length is taken into account. The coupling of the electromagnetic field to geometry will then change, and exact conformal symmetries can easily be violated. Accordingly, one expects corrections from quantum gravity to the radiation equation of state and corresponding effects in the universe evolution during radiation dominated epochs. In loop quantum cosmology [1] equations of state of matter fields are in general modified by perturbative corrections at large scales and nonperturbative ones on small scales [2]. This has mainly been studied so far for a scalar field for which quantum modifications can be so strong that negative pressure results independently of the chosen potential [3]. The main reason is the fact that the isotropic scalar field Hamiltonian $H_\phi = \frac{1}{2}a^{-3}p_\phi^2 + a^3V(\phi)$, where p_ϕ is the momentum of ϕ, contains an inverse power of the scale factor a. For quantum gravity, this factor has to be quantized, too. Using the methods of Ref. [4], it turns out that inverse powers receive strong loop quantum corrections at small length scales [2]. Accordingly, such modifications play a role for effective equations describing the universe after the big bang (or even during the quantum transition through the big bang singularity). During later stages, modifications are expected to decrease in size, but they might still be relevant due to sometimes tight constraints on evolution parameters.

An extension to the usual matter ingredients of cosmology with linear equations of state is, however, difficult since the modification is based on quantizations of the fundamental field Hamiltonians. Equations of state are obtained from fundamental Hamiltonians after an analysis of the matter field equations, which can be difficult in general especially when quantum effects are taken into account. The only exception is the dust case since it implies a constant Hamiltonian (the total mass of dust) which is straightforwardly quantized without any corrections. Thus, although the dust energy density is proportional to a^{-3} and metric dependent in a way which involves the inverse, it does not receive any modification since the Hamiltonian, i.e. total energy $a^3\rho$, is the essential object to be quantized. For radiation with $\rho \propto a^{-4}$ the expectation is not clear since the total energy does behave like an inverse power of a, but this follows only after an indirect analysis of the field dynamics. It is not the solution $\rho(a) \propto a^{-4}$ of the continuity equation which is quantized but the original field Hamiltonian from which the equation of state has to be derived first. One thus has to go back to the fundamental Maxwell Hamiltonian, derive energy density and pressure and see how quantum effects change the equation of state. If this is completed, one may attempt to solve the continuity equation to obtain corrections to $\rho(a)$.

We will derive such corrections in this article, using the canonical quantization given by loop quantum gravity [5–7]. Candidates for Hamiltonian operators of the Maxwell field have been proposed [4] which show several sources of correction terms. To derive corrections to the equation of state, however, we need to perform the usual calculation in a Hamiltonian formulation. Thus, we first present the canonical formulation for the free classical Maxwell field to rederive the standard result for the equation of state parameter w without reference to an action or the stress-energy tensor. Appropriate modifications to the matter Hamiltonian H_M are then made to derive possible loop quantum gravity corrections to the equation of state w. We will show that one case of corrections results again in a linear equation of state, albeit in a corrected way which depends on the basic discreteness scale of quantum gravity. In this case we are able to express, as in the classical case, the full field dynamics in terms of a simple modified w, and to solve explicitly for $\rho(a)$. Our derivation takes into account inhomogeneous field configurations and presents the first modified equation of state obtained for a realistic matter source in loop quantum gravity.

II. CANONICAL FORMULATION

In a canonical formulation, the Hamiltonian H_M rather than the action is used to determine equations of motion of any function f on the phase space by means of Poisson brackets, $\dot{f} = \{f, H_M\}$. The Poisson structure defines the kinematical arena which follows from the field variables and momenta. The basic configuration variable in a Lagrangian formulation of Maxwell's field theory is the vector potential A_a which determines the field strength tensor

$$F_{ab} = \nabla_a A_b - \nabla_b A_a, \tag{2}$$

where ∇_a is the covariant derivative operator. Notice that ∇_a can be replaced by the partial derivative operator ∂_a even on a curved space-time since the field strength tensor F_{ab} is antisymmetric. The action for the free Maxwell field in an arbitrary background space is given by

$$\begin{aligned} S_M &= -\frac{1}{16\pi}\int d^4x\sqrt{-g}F_{ab}F^{ab} \\ &= -\frac{1}{16\pi}\int d^4x\sqrt{-g}F_{ab}F_{cd}g^{ac}g^{bd}, \end{aligned} \tag{3}$$

where g is the determinant of the Lorentzian space-time metric g_{ab}. From the action one obtains Maxwell's equations as the Euler-Lagrange equations extremizing S_M.

A canonical formalism (Hamiltonian framework) is achieved by performing a Legendre transform of this action S_M, replacing time derivatives of configuration variables by momenta. This, as always, requires one to treat space and time differently and is the reason why the canonical formulation is not manifestly covariant. We introduce a foliation of the space-time (M, g_{ab}) by a family of spacelike hypersurfaces Σ_t:$t = \text{const}$ in terms of a time function t on M. Canonical variables will depend on which time function one chooses, but the resulting dynamics of observable quantities will remain covariant. Furthermore, let t^a be a timelike vector field whose integral curves intersect each leaf Σ_t of the foliation precisely once and which is normalized such that $t^a\nabla_a t = 1$. This t^a is the "evolution vector field" along whose orbits different

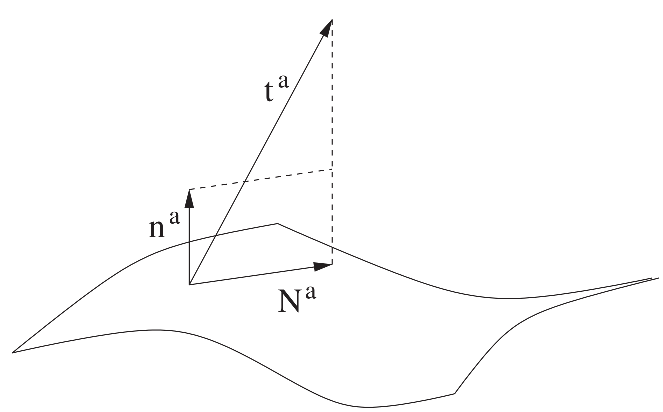

FIG. 1. Decomposition of the evolution vector field t^a in terms of the normal n^a to spatial slices and a spacelike part N^a.

points on all $\Sigma_t \equiv \Sigma$ can be identified. This allows us to write all space-time fields in terms of t-dependent components defined on a spatial manifold Σ. Lie derivatives of space-time fields along t^a are identified with "time derivatives" of the spatial fields.

A. Hamiltonian

Let us, as illustrated in Fig. 1, decompose t^a into normal and tangential parts with respect to Σ_t by defining the lapse function N and the shift vector N^a as $t^a = Nn^a + N^a$ with $N^a n_a = 0$, where n^a is the unit normal vector field to the hypersurfaces Σ_t. The space-time metric g_{ab} induces a spatial metric q_{ab} by the formula $g_{ab} = q_{ab} - n_a n_b$. Now using $n^a = N^{-1}(t^a - N^a)$ and $q^{ab} = g^{ab} + n^a n^b$ to project fields normal and tangential to Σ_t, we can decompose the field strength tensor F_{ab} and the action S_M as follows:

$$F_{ab}n^a = \frac{1}{N}(F_{ab}t^a - N^a F_{ab}) = \frac{1}{N}(\dot{A}_b - \partial_b(A_a t^a) - N^a F_{ab}), \tag{4}$$

$$\begin{aligned} F_{ab}F^{ab} &= F_{ab}F_{cd}g^{ac}g^{bd} \\ &= F_{ab}F_{cd}(q^{ac} - n^a n^c)(q^{bd} - n^b n^d) \\ &= F_{ab}F_{cd}q^{ac}q^{bd} - 2F_{ab}F_{cd}n^a n^c q^{bd} \\ &= F_{ab}F_{cd}q^{ac}q^{bd} - \frac{2}{N^2}(\dot{A}_b - \partial_b(A_a t^a) \\ &\quad - N^a F_{ab})(\dot{A}_d - \partial_d(A_a t^a) - N^c F_{cd})q^{bd}, \end{aligned} \tag{5}$$

where $\dot{A}_b = \mathcal{L}_t A_b = t^a \partial_a A_b + A_a \partial_b t^a$, and the action takes the form

$$\begin{aligned} S_M &= -\frac{1}{16\pi}\int d^4x\sqrt{-g}F_{ab}F^{ab} = -\frac{1}{16\pi}\int dt \int_{\Sigma_t} d^3x N\sqrt{q}F_{ab}F^{ab} \\ &= -\frac{1}{16\pi}\int dt \int_{\Sigma_t} d^3x N\sqrt{q}\Big(-\frac{2}{N^2}(\dot{A}_b - \partial_b(A_a t^a) - N^a F_{ab})(\dot{A}_d - \partial_d(A_a t^a) - N^c F_{cd})q^{bd} + F_{ab}F_{cd}q^{ac}q^{bd}\Big) \\ &= \int dt \int_{\Sigma_t} d^3x\Big(\frac{\sqrt{q}}{8\pi N}(\dot{A}_b - \partial_b(A_a t^a) - N^a F_{ab})(\dot{A}_d - \partial_d(A_a t^a) - N^c F_{cd})q^{bd} - \frac{N\sqrt{q}}{16\pi}F_{ab}F_{cd}q^{ac}q^{bd}\Big). \end{aligned} \tag{6}$$

It follows that the conjugate momentum π^a to the configuration variable A_a is given by

$$\pi^e = \frac{\delta S_M}{\delta \dot{A}_e} = \frac{\sqrt{q}}{4\pi N}(\dot{A}_d - \partial_d(A_a t^a) - N^c F_{cd})q^{ed}, \tag{7}$$

which is a densitized vector field because of the presence of $\sqrt{q}$. Its physical interpretation is as the electric field measured by an observer with 4-velocity n^a. Now the action can be expressed in terms of the canonical variables A_a and π^a,

$$S_M(A_a, \pi^a) = \int dt \int_{\Sigma_t} d^3x\Big(\frac{2\pi N}{\sqrt{q}}\pi^a\pi^b q_{ab} - \frac{N\sqrt{q}}{16\pi}F_{ab}F_{cd}q^{ac}q^{bd}\Big). \tag{8}$$

We can cast the action in Eq. (8) into the desired form $S_M = \int dt[\int_{\Sigma_t} d^3x \pi^a \dot{A}_a - H_M]$ by writing the integrand in the following manner:

$$\begin{aligned} S_M(A_a, \pi^a) &= \int dt \int_{\Sigma_t} d^3x\Big[\frac{4\pi N}{\sqrt{q}}\pi^a\pi^b q_{ab} - N\Big(\frac{2\pi}{\sqrt{q}}\pi^a\pi^b q_{ab} + \frac{\sqrt{q}}{16\pi}F_{ab}F_{cd}q^{ac}q^{bd}\Big)\Big] \\ &= \int dt \int_{\Sigma_t} d^3x\Big[\pi^a(\dot{A}_a - \partial_a(A_d t^d) - N^c F_{ca}) - N\Big(\frac{2\pi}{\sqrt{q}}\pi^a\pi^b q_{ab} + \frac{\sqrt{q}}{16\pi}F_{ab}F_{cd}q^{ac}q^{bd}\Big)\Big] \\ &= \int dt \int_{\Sigma_t} d^3x\Big[\pi^a\dot{A}_a + (A_d t^d)\partial_a\pi^a - N^c\pi^a F_{ca} - N\Big(\frac{2\pi}{\sqrt{q}}\pi^a\pi^b q_{ab} + \frac{\sqrt{q}}{16\pi}F_{ab}F_{cd}q^{ac}q^{bd}\Big)\Big] \end{aligned} \tag{9}$$

having integrated by parts in the second term. This completes the Legendre transform and we can read off the equations of motion from Eq. (9). First, since the momentum conjugate to the time component of A_a is absent, extremization of the action with respect to $A_a t^a$ results in

$$G = \partial_a \pi^a = 0 \tag{10}$$

as the usual Gauss constraint. The total Hamiltonian of the Maxwell field then is

$$\begin{aligned} H_M = \int_{\Sigma_t} d^3x\Big[&-(A_d t^d)\partial_a\pi^a + N^c\pi^a F_{ca} \\ &+ N\Big(\frac{2\pi}{\sqrt{q}}\pi^a\pi^b q_{ab} + \frac{\sqrt{q}}{16\pi}F_{ab}F_{cd}q^{ac}q^{bd}\Big)\Big] \end{aligned} \tag{11}$$

with two contributions

$$D_c = \pi^a F_{ca} \tag{12}$$

and

$$\mathcal{H} = \frac{2\pi}{\sqrt{q}} \pi^a \pi^b q_{ab} + \frac{\sqrt{q}}{16\pi} F_{ab} F_{cd} q^{ac} q^{bd}, \tag{13}$$

which, when added to the gravitational Hamiltonian, give matter contributions to the diffeomorphism and Hamiltonian constraint, respectively. From Eq. (13) we obtain the usual expression $\int d^3x \mathcal{H}$ for the energy of the electromagnetic field.

B. Equation of state

The evolution equations can be obtained by evaluating the Poisson brackets of A_a and π^a with the Hamiltonian. Although we will not need the explicit form of these equations, we present them in Appendix A for the sake of completeness. Here we rather determine energy and pressure from our canonical expressions (see also Ref. [8]) in order to formulate the equation of state. The matter Hamiltonian is directly related to energy density [9] by

$$\rho = \frac{1}{\sqrt{q}} \frac{\delta H_M}{\delta N}, \tag{14}$$

and thus, from Eq. (11), it is

$$\rho = \frac{2\pi}{q} \pi^a \pi^b q_{ab} + \frac{1}{16\pi} F_{ab} F_{cd} q^{ac} q^{bd}. \tag{15}$$

The canonical formula for pressure is given by

$$P = -\frac{2}{3N\sqrt{q}} q_{ab} \frac{\delta H_M}{\delta q_{ab}} = \frac{2}{3N\sqrt{q}} q^{ab} \frac{\delta H_M}{\delta q^{ab}} \tag{16}$$

as shown in Appendix B. This gives

$$\begin{aligned} P &= \frac{2}{3N\sqrt{q}} q^{ef} \bigg(\frac{\pi N}{\sqrt{q}} \pi^a \pi^b (q_{ab} q_{ef} - 2 q_{ae} q_{bf}) \\ &\quad + \frac{\sqrt{q} N}{8\pi} q^{ac} F_{ae} F_{cf} - \frac{\sqrt{q} N}{32\pi} F_{ab} F^{ab} q_{ef} \bigg) \\ &= \frac{2}{3N\sqrt{q}} \bigg(\frac{\pi N}{\sqrt{q}} \pi^a \pi^b q_{ab} + \frac{\sqrt{q} N}{32\pi} F_{ab} F^{ab} \bigg) \\ &= \frac{1}{3} \bigg[\frac{2\pi}{q} \pi^a \pi^b q_{ab} + \frac{1}{16\pi} F_{ab} F^{ab} \bigg]. \end{aligned} \tag{17}$$

Finally, the equation of state can easily be obtained from Eqs. (15) and (17):

$$w = \frac{P}{\rho} = \frac{1}{3}, \tag{18}$$

which is the standard result.

III. QUANTIZATION

Being interested in effects from quantum gravity, we have to quantize metric components in the matter Hamiltonian (11), not just the matter field itself. Metric factors are thus not treated as a given classical background but become operators. This requires a quantum representation, which can only be found if we also use canonical variables for the geometry. We thus need to use momenta of q_{ab} even though they do not appear in the matter Hamiltonian.

In loop quantum gravity, the basic objects appropriate for a canonical quantization are constructed from a densitized triad E^a_i and the SU(2)-connection $A^i_a = \Gamma^i_a - \gamma K^i_a$ where Γ^i_a is the spin connection compatible with the triad, K^i_a the extrinsic curvature and γ is the Barbero-Immirzi parameter [10–12]. Instead of the spatial metric q_{ab} we thus use the densitized vector fields E^a_i which are related to the metric by $E^a_i E^b_i = q^{ab} \det q_{cd}$.

These fields cannot be quantized directly but must be integrated suitably to remove local divergences in delta functions. The basic ingredient of a loop quantization is to use holonomies $h_e(A) = \mathcal{P}\exp\int_e A^i_a \dot{e}^a \tau_i dt \in SU(2)$ for all curves $e \subset \Sigma$ and fluxes $F_S(E) = \int_S E^a_i n_a \tau^i d^2y$ for all surfaces $S \subset \Sigma$ where τ_i are Pauli matrices, $\dot{e}^a$ is the tangent vector to the edge e and n_a the conormal to the surface S. Thus the canonical quantization is performed by using holonomies and fluxes as operators, turning their Poisson brackets into commutators [13,14]. A quantum representation is easily constructed by using states which are functionals of connections. Since holonomies are our basic connection dependent operators, they serve to generate all states from a basic one which is just a constant on the space of connections. All states are then functionals depending on connections through holonomies, and they can be associated with graphs collecting the edges of holonomies used in the generation process. An orthonormal basis can be determined explicitly in terms of spin network states [15].

An immediate consequence of this quantization is that fluxes and spatial geometrical operators such as area and volume [16–18] have discrete spectra containing zero. Hence, their inverses do not exist as densely defined operators. However, a quantization of the matter Hamiltonian such as (11) demands the quantization of such inverse expressions since, e.g., $q^{-(1/2)}$ or the metric q_{ab} which can only be obtained by inverting the densitized triad, appear in the matter Hamiltonian. Therefore, the quantization of the matter Hamiltonian seems, at first, to be seriously problematic. However, a well-defined quantization is possible after noticing that the Poisson bracket of the volume with connection components,

$$\begin{aligned} \left\{ A^i_a, \int \sqrt{|\det E|} d^3x \right\} &= 2\pi\gamma G \epsilon^{ijk} \epsilon_{abc} \frac{E^b_j E^c_k}{\sqrt{|\det E|}} \\ &= 4\pi\gamma G e^i_a, \end{aligned} \tag{19}$$

amounts to an inverse of densitized triad components [19]. This is written here in terms of the cotriad e^i_a from which we can directly obtain the metric $q_{ab} = e^i_a e^i_b$. Similar expressions allow one to include the inverse determinant of the metric as we need it in the Maxwell Hamiltonian. The left hand side of Eq. (19) does not refer to inverse densitized triad components and can be quantized: we can express the connection component through holonomies, use the volume operator and turn the Poisson bracket into a commutator. This observation enables us to quantize inverse powers of the densitized triad. Leading to well-defined operators, this quantization process implies characteristic modifications of the classical expressions such as Eq. (11) on small scales, where densitized triad components are small. Moreover, since there are many different but classically equivalent ways to rewrite expressions like Eq. (19) for which the quantization would give different results, there are quantization ambiguities. However, several characteristic effects occur for any quantization choice such that they can be studied reliably with phenomenological applications in mind.

A. Effective Maxwell Hamiltonian

Hamiltonian operators of a quantum theory can, in semiclassical regimes, be approximated by effective expressions which amend the classical ones by quantum correction terms. The general procedure, detailed in Refs. [20,21], requires one to evaluate expectation values of the Hamiltonian in suitable semiclassical states. A crucial ingredient in loop quantum gravity is the discrete, nonlocal nature of states written in terms of holonomies as basic objects. Although Hamiltonian operators on such discrete lattice states are quite complicated, expectation values can often be evaluated explicitly in perturbative regimes where one assumes the geometry to be close to a symmetric one. This is certainly allowed in our applications to derive the effective equation of state of radiation in a flat FRW universe. The background symmetry implies the existence of three approximate spatial Killing vector fields X^a_I generating transitive isometries. We will only make use of this translational symmetry, not of the additional rotations in the construction of states. These vector fields can be used as a tangent space basis, thus denoting tensor indices for components in this basis by capital letters $I, J, \ldots$.

The background symmetry also has implications for the selection of states of the quantum theory. A general quantization has to consider arbitrary states, but for effective equations one computes expectation values only in states suitable for a semiclassical regime. For perturbative inhomogeneities, one can restrict lattices as they occur in general graphs to regular cubic ones and thus simplify geometrical operators. This has been developed recently in Ref. [22] for metric perturbations as well as for a scalar field, and we can directly apply the same techniques to the Maxwell Hamiltonian. We refer the reader to this paper for more details.

B. Gravitational variables and lattice states

In a perturbative regime around a spatially flat isotropic solution, one can choose the canonical variables to be given by functions $(\tilde{p}^I(x), \tilde{k}_J(x))$ which determine a densitized triad by $E^a_i = \tilde{p}^{(i)}(x)\delta^a_i$ and extrinsic curvature by $K^i_a = \tilde{k}_{(i)}(x)\delta^i_a$. Thus, one can diagonalize the canonical variables compared to the general situation where all matrix elements of E^a_i and K^i_a would be independent. As seen in many symmetric models, this simplifies the calculations considerably: it allows one to replace involved SU(2) calculations by much simpler U(1) calculations [23,24]. SU(2) matrices arise because loop quantum gravity is based on holonomies $h_e = \mathcal{P}\exp(\int_e dt\dot{e}^a A^i_a \tau_i)$ of a connection A^i_a related to extrinsic curvature. For unrestricted connections, holonomies can take any SU(2) value, but a diagonalization implies that all quantities can be reduced to a maximal Abelian subgroup U(1). Matrix elements of Hamiltonians and other operators can then be computed in explicit form.

Using properties of the general loop representation mentioned before, basic variables of the quantum theory are, for a chosen lattice, U(1) elements $\eta_{v,I}$ attached to a lattice link $e_{v,I}$ starting at a vertex v and pointing in direction X^a_I, and their conjugate fluxes $F_{v,I}$. The U(1) elements $\eta_{v,I}$ appear as matrix elements in SU(2) holonomies $h_{v,I} = \mathrm{Re}\,\eta_{v,I} + 2\tau_I \,\mathrm{Im}\,\eta_{v,I}$ along edges $e_{v,I}$. Following the construction of the Hilbert space using holonomies as "creation" operators by acting on a state which is constant on the space of connections, a general state is a functional $|\ldots, \mu_{v,I}, \ldots\rangle = \prod_{v,I} \eta_{v,I}^{\mu_{v,I}}$. Allowing all possible values of assignments of integers $\mu_{v,I} \in \mathbb{Z}$ to the lattice edges $e_{v,I}$, this defines an orthonormal basis of the Hilbert space. Basic operators are represented as holonomies

$$\hat{\eta}_{v,I}|\ldots, \mu_{v',J}, \ldots\rangle = |\ldots, \mu_{v,I} + 1, \ldots\rangle \tag{20}$$

for each pair (v, I) where all labels other than $\mu_{v,I}$ remain unchanged, and fluxes

$$\hat{\mathcal{F}}_{v,I}|\ldots, \mu_{v',J}, \ldots\rangle = 2\pi\gamma\ell_P^2(\mu_{v,I} + \mu_{v,-I})|\ldots, \mu_{v',J}, \ldots\rangle, \tag{21}$$

where $\ell_P = \sqrt{\hbar G}$ is the Planck length and a subscript $-I$ means that the edge preceding the vertex v in the chosen orientation is taken. These and the following constructions are explained in more detail in Ref. [22].

Effective equations are obtained by taking expectation values of the Hamiltonian operator and computing a continuum approximation of the result (similar to a derivative expansion in low energy effective actions). The result is a local field theory which includes quantum corrections. This is done by relating holonomies

$$\eta_{v,I} = \exp(i\int_{e_{v,I}} \mathrm{d}t\gamma\tilde{k}_I/2) \approx \exp(i\ell_0\gamma\tilde{k}_I(v+I/2)/2) \tag{22}$$

to continuum fields $\tilde{k}_I$ through midpoint evaluation on the edges $e_{v,I}$ (denoted by an argument $v + I/2$ of the fields), and similarly for fluxes

$$F_{v,I} = \int_{S_{v,I}} \tilde{p}^I(y)\mathrm{d}^2y \approx \ell_0^2\tilde{p}^I(v+I/2). \tag{23}$$

Although the nonlocal basic objects do not allow us to define continuum fields at all spatial points, in a slowly varying field approximation the midpoint evaluations are sufficient to define the continuum fields by interpolation. Here, ℓ_0 is the coordinate length of lattice links. It does not appear in the quantum theory which only refers to states and their labels $\mu_{v,I}$. This is independent of coordinates and only makes use of an abstract, labeled graph. The parameter ℓ_0 only enters in the continuum approximation since it is classical fields which are integrated and related to holonomies and fluxes. These continuum fields, or tensor components $\tilde{p}^I$ and $\tilde{k}_I$, must depend on which coordinates are chosen to represent them. For the situation given here, the combinations $p^I: = \ell_0^2\tilde{p}^I$ and $k_I: = \ell_0\tilde{k}_I$, as they appear in holonomies and fluxes evaluated for slowly varying fields, are coordinate independent.

A further operator we can immediately define is the volume operator. Using the classical expression $V = \int \mathrm{d}^3x\sqrt{|\tilde{p}^1\tilde{p}^2\tilde{p}^3|} \approx \sum_v \ell_0^3\sqrt{|\tilde{p}^1\tilde{p}^2\tilde{p}^3|} = \sum_v\sqrt{|p^1p^2p^3|}$ we introduce the volume operator $\hat{V} = \sum_v\prod_{I=1}^3\sqrt{|\hat{\mathcal{F}}_{v,I}|}$ which, using Eq. (21), has eigenvalues

$$V(\{\mu_{v,I}\}) = (2\pi\gamma\ell_P^2)^{3/2}\sum_v\prod_{I=1}^3\sqrt{|\mu_{v,I}+\mu_{v,-I}|}. \tag{24}$$

This operator is not only interesting for geometrical purposes, but also for making use of the identity (19) or, more generally

$$\{A_a^i, V_v^r\} = 4\pi\gamma G r V_v^{r-1}e_a^i, \tag{25}$$

which gives inverse powers of the densitized triad for any $0 < r < 2$ often appearing in matter Hamiltonians. When quantizing this expression using holonomies, the volume operator and a commutator for the Poisson bracket, we obtain

$$\begin{aligned}\widehat{V_v^{r-1}}e_I^i &= \frac{-2}{8\pi i r\gamma\ell_P^2\ell_0}\sum_{\sigma\in\{\pm1\}}\sigma\,\mathrm{tr}(\tau^i h_{v,\sigma I}[h_{v,\sigma I}^{-1}, \hat{V}_v^r]) \\ &= \frac{1}{2\ell_0}(\hat{B}_{v,I}^{(r)} - \hat{B}_{v,-I}^{(r)})\delta_{(I)}^i =: \frac{1}{\ell_0}\hat{C}_{v,I}^{(r)}.\end{aligned} \tag{26}$$

For symmetry, we use both edges $e_{v,I}$ and $e_{v,-I}$ touching the vertex v along direction X_I^a. The operator $\hat{B}_{v,I}^{(r)}$ is obtained by taking the trace in Eq. (26) and using $h_{v,I} = \mathrm{Re}\eta_{v,I} + 2\tau_I\,\mathrm{Im}\eta_{v,I}$,

$$\hat{B}_{v,I}^{(r)} := \frac{1}{4\pi i\gamma G\hbar r}(s_{v,I}\hat{V}_v^r c_{v,I} - c_{v,I}\hat{V}_v^r s_{v,I}) \tag{27}$$

with

$$c_{v,I} = \frac{1}{2}(\eta_{v,I} + \eta_{v,I}^*) \quad \text{and} \quad s_{v,I} = \frac{1}{2i}(\eta_{v,I} - \eta_{v,I}^*).$$

Such expressions can be used for the electric field part of Eq. (11) where the metric factor to be quantized is

$$\frac{q_{ab}}{\ell_0\sqrt{q}} = \frac{e_a^ie_b^i}{\ell_0\sqrt{q}} \approx \frac{\ell_0^2e_a^ie_b^i}{V_v}$$

in terms of the volume $V_v \approx \ell_0^3\sqrt{q(v)}$ of a lattice site. This can then be quantized, using Eq. (26) with $r = 1/2$, to

$$\widehat{\frac{q_{IJ}}{\ell_0\sqrt{q}}} = (\ell_0\widehat{V_v^{-1/2}e_I^i})(\ell_0\widehat{V_v^{-1/2}e_J^i}) = \hat{C}_{v,I}^{(1/2)}\hat{C}_{v,J}^{(1/2)}. \tag{28}$$

Noticing that the momentum π^a of the electromagnetic field is quantized, just as the densitized triad, by a flux operator $\Pi_{v,I} := \int_{S_{v,I}}\mathrm{d}^2yn_a\pi^a \approx \ell_0^2\pi^I(v)$, the whole electric field term can be written as

$$\begin{aligned}H_\pi &= 2\pi\int \mathrm{d}^3xN(x)\frac{q_{ab}}{\sqrt{q}}\pi^a\pi^b \approx 2\pi\sum_v N(v)\ell_0^3\frac{q_{ab}}{\sqrt{q}}\pi^a\pi^b \\ &= 2\pi\sum_{v,I,J}N(v)\frac{q_{IJ}}{\ell_0\sqrt{q}}\Pi_{v,I}\Pi_{v,J},\end{aligned}$$

which is then quantized to

$$\hat{H}_\pi = 2\pi\sum_v N(v)\hat{C}_{v,I}^{(1/2)}\hat{C}_{v,J}^{(1/2)}\hat{\Pi}_{v,I}\hat{\Pi}_{v,J}. \tag{29}$$

For the magnetic field term in Eq. (11), at first sight, a different metric expression arises: $\sqrt{q}q^{ac}q^{bd}$ which also involves inverse components when expressed in terms of the densitized triad. The term appears different from the electric field term and could thus be quantized differently. However, noting

$$\begin{aligned}F_{ab}F_{cd}q^{ac}q^{bd} &= B^eB^f\epsilon_{eab}\epsilon_{fcd}q^{ac}q^{bd} \\ &= \epsilon_{eab}B^eB^fq_{fd}\epsilon^{abd}q^{-1} = 2q^{-1}q_{ab}B^aB^b\end{aligned}$$

in terms of the magnetic field $B^a = \epsilon^{abc}F_{bc}$ shows that the metric dependence is the same as in the electric part. We thus expect the same metric operator and correspondingly the same quantum gravity corrections in both terms, although different ones are mathematically possible owing to quantization ambiguities. The magnetic contribution to the Maxwell Hamiltonian then is

$$H_B = \frac{1}{8\pi}\int d^3x N(x)\frac{q_{ab}}{\sqrt{q}}B^aB^b \approx \frac{1}{8\pi}\sum_v N(v)\ell_0^3\frac{q_{ab}}{\sqrt{q}}B^aB^b = \frac{1}{8\pi}\sum_{v,I,J}N(v)\frac{q_{IJ}}{\ell_0\sqrt{q}}B_{v,I}B_{v,J}$$

with the magnetic flux $B_{v,I} := \int_{S_{v,I}} d^2y n_a B^a \approx \ell_0^2 B^I(v)$. Magnetic flux components $B_{v,I}$ are quantized using U(1) holonomies of the electromagnetic vector potential along closed loops transversal to the direction I:

$$\hat{B}_{v,I} = \frac{1}{4}\sum_{J,K}\sum_{\sigma_J,\sigma_K\in\{\pm 1\}}\sigma_J\sigma_K\epsilon^{IJK}\lambda_{v,\sigma_J J,\sigma_K K}.$$

We use the symbol λ to distinguish an electromagnetic holonomy λ from a gravitational one, η. The loop holonomy $\lambda_{v,\pm J,\pm K}$ is then computed around an elementary lattice loop starting in v in direction $\pm X_J^a$ and returning to v along $\pm X_K^a$. Summing over J, K and the two sign factors σ_J and σ_K accounts for all four loops starting in v transversally to $e_{v,I}$. The resulting quantized magnetic part of the Hamiltonian is

$$\hat{H}_\pi = \frac{1}{8\pi}\sum_v N(v)\hat{C}^{(1/2)}_{v,I}\hat{C}^{(1/2)}_{v,J}\hat{B}_{v,I}\hat{B}_{v,J} \tag{30}$$

with the same gravitational operator $\hat{C}^{(1/2)}_{v,I}\hat{C}^{(1/2)}_{v,J}$ as in the electric term. It is thus natural to use the same quantum operators and corresponding corrections in both terms, even though mathematically it is possible to quantize them differently. This aspect will be used in the following calculations.

C. Effective Hamiltonian and equation of state

As in Ref. [22] we can include effects of the quantization of metric coefficients by inserting correction functions in the classical Hamiltonian which follow, e.g., from the eigenvalues [22]

$$C^{(1/2)}_{v,I}(\{\mu_{v',I'}\}) = 2(2\pi\gamma\ell_P^2)^{-1/4}|\mu_{v,J}+\mu_{v,-J}|^{1/4} \times|\mu_{v,K}+\mu_{v,-K}|^{1/4} \times(|\mu_{v,K}+\mu_{v,-K}+1|^{1/4} - |\mu_{v,K}+\mu_{v,-K}-1|^{1/4}) \tag{31}$$

(where indices J and K are defined such that $\epsilon_{IJK}\neq 0$) of operators $\hat{C}^{(1/2)}_{v,I}$. Although for large $\mu_{v,I}$ these eigenvalues approach the function

$$C^{(1/2)}_{v,I}(\{\mu_{v',I'}\})C^{(1/2)}_{v,J}(\{\mu_{v',I'}\}) \sim (2\pi\gamma\ell_P^2)^{-1/2}\frac{\prod_{K=1}^3\sqrt{|\mu_{v,K}+\mu_{v,-K}|}}{|\mu_{v,I}+\mu_{v,-I}||\mu_{v,J}+\mu_{v,-J}|}$$

expected classically for $q_{IJ}/\sqrt{q} = \sqrt{|p^1p^2p^3|}/p^Ip^J$ with a densitized triad $E^a_i = p^{(i)}\delta^a_i$ and using the relation (21) between labels and flux components, they differ for values of $\mu_{v,I}$ closer to one. This deviation can, for an isotropic background, be captured in a single correction function

$$\alpha_{v,K} = \frac{1}{3}\sum_I C^{(1/2)}_{v,I}(\{\mu_{v',I'}\})^2\cdot\frac{\sqrt{2\pi\gamma\ell_P^2}(\mu_{v,I}+\mu_{v,-I})^2}{\prod_{J=1}^3\sqrt{|\mu_{v,J}+\mu_{v,-J}|}}, \tag{32}$$

which would equal one in the absence of quantum corrections. This is indeed approached in the limit where all $\mu_{v,I}\gg 1$, but for any finite values there are corrections. If all $\mu_{v,I} > 1$ one can directly check that corrections are positive, i.e. $\alpha_{v,K} > 1$ in this regime. Expressing the labels in terms of the densitized triad through fluxes (21) results in functionals

$$\alpha[p^I(v)] = \alpha_{v,K}(4\pi\gamma\ell_P^2\mu_{v,I}), \tag{33}$$

which enter effective Hamiltonians. The general expression one can expect is thus

$$H_{\rm eff} = \int_\Sigma d^3x N\left[\alpha[q_{cd}]\frac{2\pi}{\sqrt{q}}\pi^a\pi^b q_{ab} + \beta[q_{cd}] \times\frac{\sqrt{q}}{16\pi}F_{ab}F_{cd}q^{ac}q^{bd}\right] \tag{34}$$

with two possibly different correction functions α and β depending on the lattice values $\mu_{v,I}$. As shown before, the case $\alpha=\beta$ is preferred, and we will see soon that this has implications for the effective equation of state. (In Ref. [4] a Hamiltonian operator was introduced which did not use the same quantizations for metric coefficients in the electric and magnetic parts, thus giving $\alpha\neq\beta$. A quantization as described here, using the same quantization in both parts, was formulated in Ref. [25]. Phenomenological implications of a quantization of the latter type, concerning Lorentz invariance, are discussed in Ref. [26].) There are other possible sources for corrections, such as higher order powers and higher derivatives of the electric and magnetic fields. But these terms would not be metric dependent and are thus not crucial for the following arguments.

Now using Eq. (34), we get the modified expression

$$\frac{1}{N}q^{ab}\frac{\delta H_M}{\delta q^{ab}} = -\frac{q_{ab}}{N}\frac{\delta H_M}{\delta q_{ab}} = \frac{\pi}{\sqrt{q}}\pi^c\pi^d q_{cd}(\alpha + 2q^{ab}\delta\alpha/\delta q^{ab}) + \frac{\sqrt{q}}{32\pi}F_{cd}F^{cd}(\beta + 2q^{ab}\delta\beta/\delta q^{ab}), \tag{35}$$

depending on α and β. For a nearly isotropic background geometry, for instance, α only depends on the determinant q of the spatial metric and, from Appendix B, $q^{ab}\delta\alpha/\delta q^{ab} = -3q\,d\alpha/dq$, which we assume in what follows.

The modified energy density and pressure then are

$$\rho_{\rm eff} = \frac{2\pi}{q}\pi^a\pi^b q_{ab}\alpha + \frac{1}{16\pi}F_{ab}F_{cd}q^{ac}q^{bd}\beta, \qquad (36)$$

$$\begin{aligned} 3P_{\rm eff} &= \frac{2\pi}{q}\pi^a\pi^b q_{ab}(\alpha - 6q\mathrm{d}\alpha/\mathrm{d}q) \\ &\quad + \frac{1}{16\pi}F_{ab}F^{ab}(\beta - 6q\mathrm{d}\beta/\mathrm{d}q) \\ &= \frac{2\pi}{q}\pi^a\pi^b q_{ab}\alpha\left(1 - 6\frac{\mathrm{d}\log\alpha}{\mathrm{d}\log q}\right) \\ &\quad + \frac{1}{16\pi}F_{ab}F^{ab}\beta\left(1 - 6\frac{\mathrm{d}\log\beta}{\mathrm{d}\log q}\right). \end{aligned} \qquad (37)$$

It follows easily from Eqs. (35)–(37) that the classical behavior is reproduced for $\alpha = \beta = 1$. Interestingly, for $\alpha = \beta$, the equation of state w can easily be computed and is modified as

$$w_{\rm eff} = \frac{1}{3} - 2\frac{\mathrm{d}\log\alpha}{\mathrm{d}\log q}. \qquad (38)$$

This modification is independent of the specific matter dynamics as in the classical case, and it results in an equation of state which is linear in ρ, but depends on the geometrical scales (and the Planck length) through α.

IV. COSMOLOGICAL APPLICATIONS

In an isotropic and homogeneous universe (FRW), it follows from the FRW metric and Einstein's equation that the evolution of the energy density is given by the continuity equation, i.e.,

$$\dot\rho + 3\frac{\dot a}{a}(\rho + P) = 0, \qquad (39)$$

where a is the scale factor and the dot indicates a proper time derivative. Using the definition of the equation of state and eliminating the time derivative, this equation can be cast into the following useful form:

$$\frac{\mathrm{d}\log\rho(a)}{\mathrm{d}\log a} = -3(1 + w(a)). \qquad (40)$$

Here we have shown the dependence of the equation of state on the scale factor explicitly. It can easily be shown that the solution to the above equation is

$$\rho(a) = \rho_0 \exp\left[-3\int(1 + w(a))\mathrm{d}\log a\right], \qquad (41)$$

where ρ_0 is the integration constant. Now by inserting the modified equation of state in the radiation era, Eq. (38) with $q = a^6$, we obtain

$$\rho(a) = \rho_0\alpha(a)a^{-4}. \qquad (42)$$

Again, for $\alpha = 1$, we retrieve the classical result $\rho(a) \propto a^{-4}$. Therefore, loop quantum gravity corrections induced by discreteness of the flux operator are reflected even in the evolution of the FRW universe.

Although one can write α as a function of the scale factor for perturbations around a flat isotropic model, it is important to note that corrections are well defined even though the scale factor can be rescaled arbitrarily. One can express α as a function of a only after coordinates have been specified, such that there is no ambiguity in relating the scale appearing in α (such as the Planck length) to a. More precisely, we first obtained $\alpha[p^I(v)]$ in Eq. (36) with lattice values for the fluxes $p^I(v) = \ell_0^2\tilde p^I(v)$ which are coordinate independent while $\tilde p^I$ would be rescaled just as the scale factor. The quantum state, through its lattice building blocks, unambiguously determines the magnitude of the elementary variables as they appear in corrections. Under rescalings or other coordinate changes, both the classical field $\tilde p^I$ and the coordinate form of the lattice change in such a way that elementary fluxes remain unchanged. In a nearly isotropic context, for instance, one has $p^I(v) \approx p = \ell_0^2 a^2$ spatially constant which can be related to the Hubble scale by $Np^{3/2} = H^{-3}$. Here, we use the number N of lattice sites of elementary area p in the Hubble volume H^{-3} as a measure of how fine the lattice is. Inserting all this in correction functions yields $\alpha(p) = \alpha(N^{-2/3}H^{-2})$ expressed purely in terms of coordinate independent quantities. The function N enters as an additional ingredient to describe the microstructure of the underlying quantum state. In a given solution including the time dependence $H(t)$ of the Hubble scale as well as a function $N(t)$ describing the quantum state one could relate all this, in a secondary step, to the scale factor $a(t)$. But since the scale factor is not the primary argument of correction functions, there is no problem with rescalings. See also Ref. [27] for further clarifications of this issue which was not clear in all the literature on purely homogeneous models.

V. DISCUSSIONS

We have derived here the equation of state of the Maxwell field in a canonical form, including corrections expected from loop quantum gravity. In the canonical derivation, the reason for a linear equation of state, which is trace-freedom in the Lagrangean derivation, is the fact that the same metric dependent factor $q_{ab}/\sqrt{q}$ multiplies both terms in the Hamiltonian. The Maxwell Hamiltonian is thus simply rescaled if the metric is conformally transformed, which explains the conformal invariance of Maxwell's equations. This is special for the Maxwell field and different from, e.g., a scalar field with a nonvanishing potential.

The same fact allows one to quantize the Hamiltonian in a way which affects both the electric and magnetic term in the same way, at least as far as the metric dependence is concerned. One then obtains a single correction function $\alpha = \beta$ which only corrects the metric dependence of the total scale of the Hamiltonian. In this sense, conformal invariance is preserved even after quantization. (But this would not be the case if a quantization is used which results in $\alpha \neq \beta$.)

This preservation of the form of the Hamiltonian explains why we are still able to derive an equation of state independently of the specific field dynamics and that it remains linear. However, the classical value $w = \frac{1}{3}$ is corrected due to quantum effects in the space-time structure. This modification is also understandable from a Lagrangean perspective, together with basic information from the loop quantization. Employing trace-freedom of the stress-energy tensor to derive the equation of state, we have to use the inverse metric in $g^{ab}T_{ab}$. But from loop quantum gravity we know that, when quantized, not all components of the inverse metric agree with inverse operators of the quantization. For the scale factor of an isotropic metric, for instance, we have $\widehat{a^{-1}} \neq$ "$\hat{a}^{-1}$" since the right hand side is not even defined [2]. While the left hand side is defined through identities such as Eq. (19), it satisfies $\widehat{a^{-1}}\,\hat{a} \neq 1$ and thus shows deviations from the classical expectation $a^{-1}a = 1$ on small scales which were captured here in correction functions. As derived in detail, this implies scale dependent modifications to the equation of state parameter $w_{\rm eff}$.

The result can also be interpreted in more physical terms. The classical behavior $\rho(a) \propto a^{-4}$ can be understood as a combination of a dilution factor a^{-3} and an additional redshift factor a^{-1} for radiation in an expanding universe. As we have seen, this is corrected to $\alpha(a)a^{-4}$ where $\alpha(a)$ corrects the metric factor $q_{ab}/\sqrt{q} \sim a^{-1}\delta_{ab}$. Since this is only a single inverse power of a for an isotropic solution, we can interpret the result as saying that only redshift receives corrections due to quantum effects on electromagnetic propagation. The dilution factor due to expansion is unmodified, except that the background evolution $a(t)$ itself receives corrections. This agrees with the result for dust, which is only diluted and has an unmodified equation of state even after quantization [28]. Unlike dust, for radiation one has to refer to the inhomogeneous field and its quantum Hamiltonian to derive a reliable equation of state, as presented here.

ACKNOWLEDGMENTS

M. B. was supported by NSF Grant No. PHY0554771.

APPENDIX A: EQUATIONS OF MOTION

It is straightforward to derive the equations of motion for the canonical variables A_a and π^a from the Poisson brackets of each of these variables with the matter Hamiltonian H_M. Then

$$\dot{A}_a = \{A_a, H_M\} = \frac{\delta H_M}{\delta \pi^a} = \partial_a(A_c t^c) + N^c F_{ca} + \frac{4\pi N}{\sqrt{q}} \pi^c q_{ca}, \quad \text{(A1)}$$

and

$$\dot{\pi}^a = \{\pi^a, H_M\} = -\frac{\delta H_M}{\delta A_a} = \partial_c(N^c \pi^a) - \partial_d(N^a \pi^d) - 4\partial_c(N\sqrt{q}F_{ef}q^{ec}q^{fa}). \quad \text{(A2)}$$

The modified Hamiltonian gives rise to the following new set of equations of motion:

$$\dot{A}_a = \{A_a, H_{\rm eff}\} = \frac{\delta H_{\rm eff}}{\delta \pi^a} = \partial_a(A_c t^c) + N^c F_{ca} + \frac{4\pi N}{\sqrt{q}} \alpha(q)\pi^c q_{ca}, \quad \text{(A3)}$$

and

$$\dot{\pi}^a = \{\pi^a, H_{\rm eff}\} = -\frac{\delta H_{\rm eff}}{\delta A_a} = \partial_c(N^c \pi^a) - \partial_d(N^a \pi^d) - 4\partial_c(N\beta(q)\sqrt{q}F_{ef}q^{ec}q^{fa}), \quad \text{(A4)}$$

where $H_{\rm eff}$ is the effective Hamiltonian of the Maxwell's field (H_M with α and β inserted).

APPENDIX B: PRESSURE

The general, thermodynamical definition of pressure is the negative change of energy by volume, which we can write as

$$P = -\frac{1}{N}\frac{\delta H}{\delta\sqrt{q}} \quad \text{(B1)}$$

whenever the Hamiltonian $H = \int d^3x N(x)\mathcal{H}(x)$ is depends isotropically on the metric. Otherwise, one has to use all components of the stress tensor $\delta H/\delta q^{ab}$ which is not proportional to the identity. The derivative by the determinant of the metric can be expressed in terms of metric components by using a suitable change of variables which includes q as an independent one. We thus introduce $q_{ab} =: q^{1/3}\bar{q}_{ab}$ with $\det\bar{q}_{ab} = 1$ such that $\partial q_{ab}/\partial q = \frac{1}{3}q^{-1}q_{ab}$ where all components of $\bar{q}_{ab}$ are kept fixed in the partial derivative. This is exactly what we need to compute pressure since only the volume but not the shape of the fluid is varied. This change of variables implies

$$\frac{\delta}{\delta\sqrt{q}} = 2\sqrt{q}\frac{\delta}{\delta q} = 2\sqrt{q}\sum_{ab}\frac{\partial q_{ab}}{\partial q}\frac{\delta}{\delta q_{ab}} = \frac{2}{3\sqrt{q}}\sum_{ab} q_{ab}\frac{\delta}{\delta q_{ab}}$$

and thus

$$P = -\frac{2}{3N\sqrt{q}} q_{ab} \frac{\delta H}{\delta q_{ab}}. \quad \text{(B2)}$$

We can also verify this by comparing the dynamical effects of H on the metric with the Raychaudhuri equation expressed in terms of the canonical variables which for simplicity we do for homogeneous metrics. Using the following definitions for the extrinsic curvature tensor $K_{ab} = \nabla_a n_b$ (which turns out to be automatically spatial and symmetric without projection if homogeneity is used), the expansion parameter $\theta = K_{ab}q^{ab}$ and the shear $\sigma_{ab} = K_{(ab)} - \frac{1}{3}\theta q_{ab}$, the canonical momentum conjugate to q_{ab} derived from the gravitational Lagrangian is

$$\pi^{ab} = \frac{\sqrt{q}}{16\pi G}(K^{ab} - K^c_c q^{ab}) = \frac{\sqrt{q}}{16\pi G}\left(\sigma^{ab} - \frac{2}{3}\theta q^{ab}\right),$$

where G is the gravitational constant. Then the Raychaudhuri equation in terms of the canonical variables takes the following form:

$$\dot{\theta} = -8\pi G \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\pi^{ab} q_{ab}}{\sqrt{q}}\right). \quad \text{(B3)}$$

The canonical equations of motion, in the presence of a matter Hamiltonian H added to the gravitational Hamiltonian to form H_{Total}, become

$$\dot{q}_{ab} = \frac{\delta H_{\mathrm{Total}}}{\delta \pi^{ab}} = \frac{16\pi G N}{\sqrt{q}}(2\pi_{ab} - q_{ab}\pi^c_c) + 2D_{(a}N_{b)} \quad \text{(B4)}$$

and

$$\begin{aligned} \dot{\pi}^{ab} &= -\frac{\delta H_{\mathrm{Total}}}{\delta q_{ab}} \\ &= -\frac{N\sqrt{q}}{16\pi G}\left({}^{(3)}R^{ab} - \frac{1}{2}{}^{(3)}R q^{ab}\right) \\ &\quad + \frac{8\pi G N}{\sqrt{q}} q^{ab}\left(\pi_{cd}\pi^{cd} - \frac{1}{2}\pi^2\right) \\ &\quad - \frac{32\pi G N}{\sqrt{q}} q^{ab}\left(\pi^{ac}\pi^b_c - \frac{1}{2}\pi\pi^{ab}\right) - \frac{\delta H}{\delta q_{ab}} \\ &\quad + \frac{\sqrt{q}}{16\pi G}(D^a D^b N - q^{ab} D^c D_c N) \\ &\quad + \sqrt{q} D_c\left(\frac{N^c \pi^{ab}}{\sqrt{q}}\right) - 2\pi^{c(a} D_c N^{b)}, \end{aligned} \quad \text{(B5)}$$

where D_a is the derivative operator compatible with q_{ab}. Variation of the total action with respect to the lapse function N yields the Hamiltonian constraint equation

$$-\frac{\sqrt{q}}{16\pi G}{}^{(3)}R + \frac{16\pi G}{\sqrt{q}}\left(\pi^{ab}\pi_{ab} - \frac{1}{2}\pi^2\right) + H = 0. \quad \text{(B6)}$$

Upon inserting Eqs. (B4)–(B6) into Eq. (B3), the Raychaudhuri equation becomes

$$\begin{aligned} \frac{\dot{\theta}}{N} &= -\frac{1}{3}\theta^2 - \sigma^{ab}\sigma_{ab} - \frac{4\pi G}{N\sqrt{q}} H + \frac{8\pi G}{N\sqrt{q}} q_{ab} \frac{\delta H}{\delta q_{ab}} \\ &\quad + D^a D_a N - 8\pi G D_c\left(\frac{N^c \pi^a_a}{\sqrt{q}}\right) + \frac{16\pi G}{\sqrt{q}} \pi^{ca} D_c N_a, \end{aligned} \quad \text{(B7)}$$

which, for a homogeneous universe, reduces to

$$\frac{\dot{\theta}}{N} = -\frac{1}{3}\theta^2 - \sigma^{ab}\sigma_{ab} - \frac{4\pi G}{N\sqrt{q}} H + \frac{8\pi G}{N\sqrt{q}} q_{ab} \frac{\delta H}{\delta q_{ab}}. \quad \text{(B8)}$$

On the other hand, for a perfect fluid distribution, the Raychaudhuri equation is found to be

$$\frac{\dot{\theta}}{N} = -\frac{1}{3}\theta^2 - \sigma^{ab}\sigma_{ab} - 4\pi G(\rho + 3P). \quad \text{(B9)}$$

Now comparing Eq. (B8) with Eq. (B9), we verify the canonical formula for the average pressure for a perfect fluid distribution in an anisotropic geometry

$$P = -\frac{2}{3N\sqrt{q}} q_{ab} \frac{\delta H}{\delta q_{ab}} = \frac{2}{3N\sqrt{q}} q^{ab} \frac{\delta H}{\delta q^{ab}}. \quad \text{(B10)}$$

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