Dynamical coherent states and physical solutions of quantum cosmological bounces

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A new model is studied which describes the quantum behavior of transitions through an isotropic quantum cosmological bounce in loop quantum cosmology sourced by a free and massless scalar field. As an exactly solvable model even at the quantum level, it illustrates properties of dynamical coherent states and provides the basis for a systematic perturbation theory of loop quantum gravity. The detailed analysis is remarkably different from what is known for harmonic oscillator coherent states. Results are evaluated with regard to their implications in cosmology, including a demonstration that in general quantum fluctuations before and after the bounce are unrelated. Thus, even within this solvable model the condition of classicality at late times does not imply classicality at early times before the bounce without further assumptions. Nevertheless, the quantum state does evolve deterministically through the bounce.

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I. INTRODUCTION

An understanding of high curvature regimes of a universe is likely to require a quantization of gravity which is nonperturbative and background independent. Background independence means that one does not base the theory on preexisting causal or geometrical structures because they are to be provided by the quantized gravitational field itself. Not surprisingly, many difficulties in this new setting have to be overcome even when the aim is "only" to verify that a proposed theory will have the correct semiclassical limit. One first has to determine appropriate semiclassical states of the *interacting* quantum theory of gravity. Thus, already in the definition of states in which the classical limit is to be probed one has to face quantum dynamics. Unlike perturbative quantizations on a background, no exactly known free vacuum state is available which one could use to determine properties of an interacting vacuum state perturbatively. Vacuum or coherent states of general interacting theories then have to be constructed anew, which can show properties quite different from the well-known (Gaussian) states of free theories or the harmonic oscillator. Although it is often assumed, Gaussian states may not capture the right semiclassical properties in any given system. They may be assumed as "prepared" initial states, but crucial deviations from Gaussian form can occur especially in systems with long evolution times, for which cosmology is the example *par excellence.*

Quantum gravity is not only an interacting quantum field theory whose interacting semiclassical states are to be determined, it also, in general, has no close relation to a free quantum field theory as it is often exploited in effective field theories of particle physics. For correct predictions it is, first of all, necessary to determine precise states which capture semiclassical properties. In this paper, a model, introduced in [1], is studied which is exactly solvable and includes characteristic effects of loop quantum gravity, one candidate for a background independent quantization [2–4]. The model itself is based on loop quantum cosmology [5]. With new techniques [6,7], coherent state properties can be determined explicitly. In this sense, the model is analogous to the harmonic oscillator in quantum mechanics and it has indeed the same solvability properties as explained in more detail below. This will allow us to perform a complete dynamical coherent state analysis, demonstrating how properties can differ considerably for distinct systems even when one considers only solvable models. The model we study is not only illuminating in this regard, but it also is of direct physical interest since it describes nonsingular cosmological bounce models.

Bouncing solutions of cosmological models have recently received much attention. Although they are generally very special, they can indicate how transitions through the classical big bang singularity may be possible. Many different examples exist by now, which have most systematically been developed in loop quantum cosmology. Most arguments are based on "effective" equations which import some quantum effects into classical equations, and which sometimes allow exact analytical solutions (such as in [1,8,9]) or can at least be studied numerically; see e.g. [10–14]. Also in this context the above question of what a semiclassical state of an interacting quantum theory looks like is relevant, although it is often overlooked. It enters in the derivation or justification of those effective equations which are supposed to capture properties of semiclassical states. If the correct type of semiclassical states is not known, one cannot be sure to have included all relevant corrections to the classical equations in the right way. As a by-product, our solvable model presents the first case of a complete set of effective equations in quantum cosmology.

Our model, used to illustrate semiclassical state issues, is solvable exactly at the quantum level [1]. This is much

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stronger than having analytical solutions of effective equations since full states, including not only expectation values but also fluctuations and higher moments, are under full control. In this sense, the system is comparable to a harmonic oscillator. A complete analysis becomes possible, including e.g. the evolution of coherent states and their long-term dynamical properties. This is especially relevant in the light of recent numerical analyses of related models where initial Gaussian states without squeezing were evolved [15]. We will see that the coherent state structure of the model is much richer than that of unsqueezed Gaussian states, with squeezing influencing the general behavior significantly. This is an instructive example for the importance of a dedicated coherent state analysis, rather than taking over harmonic oscillator properties to a new model.

The model is paradigmatic for background independent quantum gravity obtained from a loop quantization where the usual free field theory basis is not available. Since the bounce model is exactly solvable, it can provide a perturbative basis for quantum gravity including all possible interactions and degrees of freedom. Thus, the form of coherent states determined here is relevant not only for the model itself but for quantum gravity in general. At this stage, perturbative inhomogeneities are not included explicitly and thus the question of how they evolve through a bounce is not addressed in this paper. We rather show and emphasize that even the unperturbed isotropic situation poses several important questions for how quantum fluctuations of the isotropic mode evolve through a bounce. We follow a general method, summarized in Sec. II. The solvable models relevant for cosmology are introduced and analyzed in Sec. III, and discussed more broadly in Sec. IV.

II. THE METHOD

In what follows, we will not use a fixed, or any, representation of our quantum system on a specific Hilbert space. Rather, we take an algebraic viewpoint and treat the algebra of basic operators, such as $[\hat{q}, \hat{p}] = i\hbar$ in quantum mechanics, together with the Hamiltonian $\hat{H}$ as primary. The quantities we are most interested in are expectation values $\langle\hat{q}\rangle_\psi = \langle\hat{q}\rangle$, $\langle\hat{p}\rangle_\psi = \langle\hat{p}\rangle$ in a given state ψ, which we often drop as a label if no confusion can arise, and fluctuations and correlations $\Delta q = \sqrt{\langle\hat{q}^2\rangle - \langle\hat{q}\rangle^2}$, $\Delta p = \sqrt{\langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2}$ and $C_{qp} = \frac{1}{2}\langle\hat{q}\hat{p} + \hat{p}\hat{q}\rangle - \langle\hat{q}\rangle\langle\hat{p}\rangle$. Higher moments, involving higher than quadratic powers of basic operators, could be included by the same means although we will not need to do so here. Nevertheless, it is important to note that, would we determine all the moments, we could reconstruct the state ψ provided that the moments satisfy appropriate conditions. The most basic such condition is Heisenberg's uncertainty relation $(\Delta q)^2(\Delta p)^2 - C_{qp}^2 \geq \hbar^2/4$ (also called Schrödinger-Robertson uncertainty relation in this form) which will be used frequently below. As in [6], we call all fluctuations, correlations and higher moments *quantum variables* since, unlike expectation values, they describe typical quantum properties.

Determining the evolution of moments is thus sufficient to find properties of states. One can sidestep the explicit construction of states in a representation by deriving and solving equations of motion for moments directly, such as $\mathrm{d}\langle\hat{p}\rangle/\mathrm{d}t = \langle[\hat{p}, \hat{H}]\rangle/i\hbar$ and $\mathrm{d}(\Delta p)^2/\mathrm{d}t = \langle[\hat{p}^2, \hat{H}]\rangle/i\hbar - 2\langle\hat{p}\rangle\mathrm{d}\langle\hat{p}\rangle/\mathrm{d}t$. In general, this set of equations is highly coupled because, unless $[\hat{p}, \hat{H}]$ is linear in basic operators, $\langle[\hat{p}, \hat{H}]\rangle$ is a function of expectation values and quantum variables. The quantum variables, in turn, will satisfy equations of motion involving moments of higher degree. (See [6,16] for examples.) This coupling describes the backreaction of spreading and deformations on the peak trajectory of a state, which is a crucial quantum effect. It is, for instance, the reason for the usual nonlocality in time of effective actions.

This is the place where solvability properties of a model become important. If $\hat{H}$ is quadratic, for instance, $[\hat{p}, \hat{H}]$ and $[\hat{q}, \hat{H}]$ will be linear in $\hat{q}$ and $\hat{p}$ and $\langle\hat{q}\rangle$ and $\langle\hat{p}\rangle$ will not couple to quantum variables. As is well-known for the harmonic oscillator, the spreading of states then does not influence the peak motion at all and no nontrivial quantum corrections arise in effective equations. More generally, this happens whenever basic operators taken together with the Hamiltonian form a linear commutator algebra. Our solvable bounce model is precisely of this type.

It is then feasible to solve equations of motion for $\langle\hat{q}\rangle$, $\langle\hat{p}\rangle$ and any desired quantum variables directly without taking the detour of first computing a state in a chosen representation. Many representation dependent difficulties can be avoided, such as explicit formulas for inner products and normalizations [17]. Instead, properties of the Hilbert space structure, such as the self-adjointness of operators, can be implemented straightforwardly through reality conditions for the solutions of $\langle\hat{q}\rangle$, $\langle\hat{p}\rangle$ and quantum variables.

III. A SOLVABLE BOUNCE MODEL AND ITS PROPERTIES

A free isotropic scalar field ϕ couples to gravity through the Friedmann equation

$$\frac{3}{8\pi G}c^2\sqrt{p} = \frac{1}{2}p^{-3/2}p_\phi^2. \tag{1}$$

We use canonical variables, explained in more detail in [18], whose relation to the scale factor a, the scalar ϕ and their derivatives in proper time τ is $c = \mathrm{d}a/\mathrm{d}\tau$ (extrinsic curvature), $p^{3/2} = a^3$ (volume) and $p_\phi = p^{3/2}\mathrm{d}\phi/\mathrm{d}\tau$. Geometrically, p is the isotropic component of a densitized triad and can be positive or negative for the two triad representations. In what follows, we assume positive p

without loss of generality and drop absolute values. Moreover, for later convenience we rescale ϕ by $\sqrt{2}$ and drop factors of $8\pi G/3$ such that $\{c, p\} = 1$. Solving (1) for p_ϕ, which is a constant of motion, yields $p_\phi \propto \pm cp$, allowing four possible choices for the signs: c can be positive (expanding universe) or negative (contracting universe), and for each case p_ϕ can take any sign (such that ϕ runs opposite or along coordinate time τ). We can interpret $H = cp$ as the Hamiltonian which generates the flow in the variable ϕ, playing the role of internal time. The deparametrized Hamiltonian constraint then reads $p_\phi + H = 0$.

This Hamiltonian is quadratic, although not of the harmonic oscillator form, and thus solvable as explained above. Unlike for the harmonic oscillator, the Hamiltonian does not have a definite sign. One can easily understand the behavior of solutions, and of the energy spectrum of the quantum theory, by performing a canonical transformation to new canonical variables (π, q) in which $c = \frac{1}{\sqrt{2}}(\pi + q)$, $p = \frac{1}{\sqrt{2}}(\pi - q)$. The Hamiltonian then becomes the upside-down harmonic oscillator $H = \frac{1}{2}\pi^2 - \frac{1}{2}q^2$ which obviously allows positive as well as negative energy solutions. Classical solutions can easily be determined as $\pi(\phi) = A\cosh\phi + B\sinh\phi$, $q(\phi) = B\cosh\phi + A\sinh\phi$. In terms of the integration constants A and B, the Hamiltonian is $H = \frac{1}{2}(A^2 - B^2)$. Corresponding solutions in the original variables are $c(\phi) = \frac{1}{\sqrt{2}}(A + B)e^{\phi}$, $p(\phi) = \frac{1}{\sqrt{2}}(A - B)e^{-\phi}$.

Since we assume positive p, solutions as functions of q can only be incoming from the left of the upside-down potential, where π is positive and q negative. This assumption implies $A - B > 0$. The sign of H then depends on whether we describe an expanding or contracting universe, $c > 0$ implying $H = \frac{1}{2}(A + B)(A - B) > 0$ while $c < 0$ implies $H < 0$. In the first case, $p_\phi < 0$ and ϕ runs opposite to proper time, while it runs along proper time for a collapsing universe. The opposite case would be realized had we chosen the opposite sign for H. The behavior of solutions is illustrated in Fig. 1.

This Hamiltonian is directly quantized, following the rules of quantum mechanics, in the Wheeler-DeWitt approach [19,20]. It is thus helpful to recall the properties of the resulting quantum system as derived in [1] using the method summarized in Sec. II. The quadratic Hamiltonian implies that the system is solvable, which can be exploited to determine explicit solutions $\langle\hat{c}\rangle(\phi) = c_1e^{\phi}$ and $\langle\hat{p}\rangle(\phi) = c_2e^{-\phi}$ for expectation values of the basic operators $\hat{c}$ and $\hat{p}$, in full agreement with the classical solutions. Although the functional expressions for expectation values agree with classical solutions, a difference to the classical case is that states may be superpositions of positive and negative eigenfunctions of $\hat{H}$ even though the expectation value of $\hat{p}$ is either contracting or expanding [21]. We will later discuss the effect of possible admixtures of negative to positive energy states in solutions.

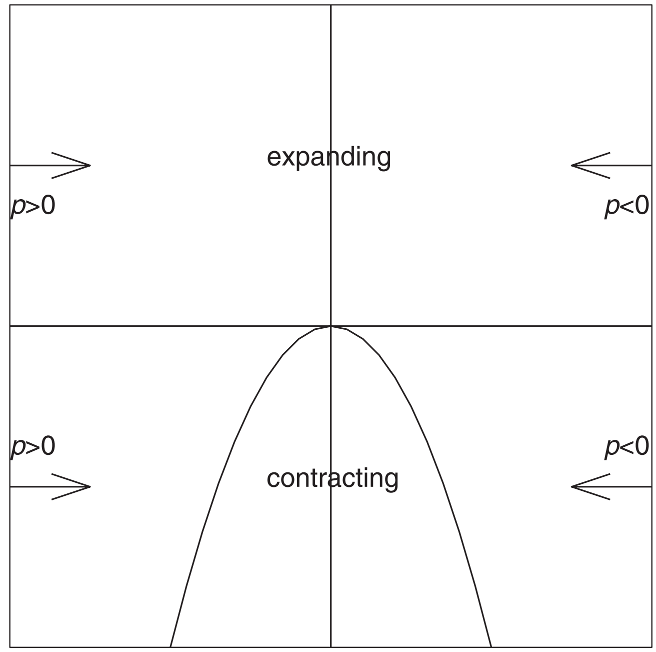

FIG. 1. Illustration of positive and negative energy solutions in an upside-down harmonic oscillator potential $V(q) = -\frac{1}{2}q^2$. The sign of the corresponding triad variable p is indicated.

Moreover, fluctuations can be solved explicitly by the same methods. They are $(\Delta c)(\phi)^2 = c_3e^{2\phi}$, $(\Delta p)(\phi)^2 = c_5e^{-2\phi}$ and $C_{qp}(\phi) = c_4$ for the correlation. While fluctuations are not constant, they satisfy $(\Delta c)/\langle\hat{c}\rangle = \text{const}$ and $(\Delta p)/\langle\hat{p}\rangle = \text{const}$. Uncertainty relations require $c_3c_5 - c_4^2 \geq \frac{1}{4}\hbar^2$. Typically, Δc and Δp are thus of the order $\sqrt{\hbar}$, although the precise value requires more specific information about the state. For dimensional reasons, this must involve the only available scale H such as in the form $\Delta p = \sqrt{\hbar H}$ and $\Delta c = \sqrt{\hbar/H}$. In contrast to the harmonic oscillator where the ground state provides specific values for fluctuations, no distinguished state is available for the Hamiltonian encountered here. Later on we will see that the loop quantization does provide more information which one can use to estimate fluctuations. Still, the information will not be as complete as it is for the harmonic oscillator ground state.

The key feature of a loop quantization is that its representation does not provide a c-operator but only operators for almost periodic functions of c [22] (i.e. countable superpositions of $\exp(i\mu c)$ with $\mu \in \mathbb{R}$). This is sufficient for constructing a Hamiltonian operator which is well-defined and agrees with the classical one, cp, in low curvature regimes where $c \ll 1$. A loop quantization can thus be understood as implying that c in the classical Hamiltonian is replaced by $\sin c$ when it is quantized. Instead of $\sin c$ one could, of course, choose any other almost periodic function which reduces to c when $c \ll 1$. The freedom can be restricted by relating the model to quantizations of the Hamiltonian constraint as it can be defined in the full theory [23]. This distinguishes the use of $\sin c$ as it follows from the original derivations in [18], although other choices remain [24]. In any case, the resulting Hamiltonian is no longer quadratic in the variables (c, p) and thus appears quite complicated. However, if we introduce the operators $\hat{p}$ and $\hat{J} := \hat{p}\,\widehat{\exp(ic)}$, the *linear* Hamiltonian operator $\hat{H} = -\frac{1}{2}i(\hat{J} - \hat{J}^\dagger)$ reduces to the classical Hamiltonian when curvature c is small, and it depends on c only through $\exp(ic)$ which is required by loop quantum cosmology. It can be viewed as a quantization of $p\sin c$ in a specific factor ordering.

A linear Hamiltonian usually simplifies the dynamics very much, but only if the basic variables in which it is linear form a closed algebra (see [6] for a discussion in the general context of effective equations). Since we transformed from canonical variables (c, p) to noncanonical ones (p, J), the system is not guaranteed to simplify even with a linear $\hat{H}$. Fortunately, as one can easily check the variables do satisfy the closed $\mathrm{sl}(2,\mathbb{R})$ algebra

$$[\hat{p}, \hat{J}] = \hbar\hat{J}, \qquad [\hat{p}, \hat{J}^\dagger] = -\hbar\hat{J}^\dagger, \qquad [\hat{J}, \hat{J}^\dagger] = -2\hbar\hat{p} - \hbar^2. \tag{2}$$

which includes the Hamiltonian as a linear combination of $\hat{J}$ and $\hat{J}^\dagger$ [25]. This is the reason for the exact solvability of the system which we will use below. Representations of $\mathrm{sl}(2,\mathbb{R})$ do not allow one to have a purely positive spectrum for $i(\hat{J} - \hat{J}^\dagger)$. Thus, any representation space contains positive and negative energy solutions and we will have to discuss appropriate restrictions on states to rule out superpositions of negative energy contributions to positive energy states if superpositions of expanding and collapsing universe branches are not to be allowed.

Before analyzing equations of motion we note that the system can be generalized by performing a canonical transformation such as $\pi := p^k c$, $v := p^{1-k}/(1-k)$ and using the new canonical variables (π, v) instead of (c, p) in the definition of $\hat{p}$ and $\hat{J}$. Properties of the system to be discussed below are not affected by this reinterpretation of the variables. Such transformations can be motivated by taking into account features of nearly isotropic but inhomogeneous lattice states of loop quantum gravity and possible dynamical refinements of the underlying lattice [26,27]. The special value $k = 0$ then corresponds to a fixed lattice as realized in [18,22], while $k = -1/2$, introduced independently in [27], corresponds to a refinement such that the number of lattice vertices increases linearly as a function of volume.

A. Equations of motion

As discussed before, the system can much more easily be understood if we do not first solve for wave functions, subject to $i\hbar\dot{\psi} = \hat{H}\psi$, and then compute expectation values and fluctuations from solutions. Instead, using a more algebraic point of view we derive equations of motion directly for expectation values, fluctuations and higher moments and solve them [6,16].

For expectation values, now simply denoted as $p := \langle\hat{p}\rangle$, $J := \langle\hat{J}\rangle$ and $\bar{J} := \langle\hat{J}^\dagger\rangle$ in an arbitrary normalized state, equations of motion follow immediately by taking expectation values of Heisenberg equations of motion, or by using the Schrödinger equation for the state appearing in the expectation value,

$$\dot{p} = \frac{1}{i\hbar}\langle[\hat{p}, \hat{H}]\rangle = -\frac{1}{2}(J + \bar{J}), \tag{3}$$

$$\dot{J} = \frac{1}{i\hbar}\langle[\hat{J}, \hat{H}]\rangle = -p - \frac{1}{2}\hbar = \dot{\bar{J}}. \tag{4}$$

That these equations form a closed system is a consequence of the linear nature of the variables and Hamiltonian. In general, the evolution of expectation values would also depend on all higher moments of the state: During evolution the state spreads and deforms, which then backreacts on the peak position of a wave packet. This backreaction is the dynamical essence of a quantum system, captured in effective equations.

In fact, a state is characterized not just by its expectation values of basic operators but also by the infinitely many quantum variables which specify fluctuations and higher moments of a state. Fluctuations (and correlations) can be defined, as usually, by

$$G^{pp} := \langle\hat{p}^2\rangle - p^2, \tag{5}$$

$$G^{JJ} := \langle\hat{J}^2\rangle - J^2, \tag{6}$$

$$G^{\bar{J}\bar{J}} := \langle\hat{J}^{\dagger 2}\rangle - \bar{J}^2, \tag{7}$$

$$G^{pJ} := \tfrac{1}{2}\langle\hat{p}\hat{J} + \hat{J}\hat{p}\rangle - pJ, \tag{8}$$

$$G^{p\bar{J}} := \tfrac{1}{2}\langle\hat{p}\hat{J}^\dagger + \hat{J}^\dagger\hat{p}\rangle - p\bar{J}, \tag{9}$$

$$G^{J\bar{J}} := \tfrac{1}{2}\langle\hat{J}\hat{J}^\dagger + \hat{J}^\dagger\hat{J}\rangle - |J|^2. \tag{10}$$

Since we use partially complex variables there are initially more than three independent fluctuations. However, reality conditions to be imposed later at the quantum level will lead to additional relations and reduce the number of independent degrees of freedom to the correct value. Higher moments are defined analogously, using totally symmetric orderings in expressions where both $\hat{p}$ and $\hat{J}$ are involved.

Fluctuations are not expectation values of a single operator and their equations of motion do not follow directly as before. But they can easily be derived using linearity and the Leibniz rule. We have, e.g.,

$$\dot{G}^{pp} = \frac{1}{i\hbar}\langle[\hat{p}^2, \hat{H}]\rangle - 2p\dot{p} \tag{11}$$

$$= -\tfrac{1}{2}\langle \hat{p}\,\hat{J} + \hat{J}\,\hat{p} + \hat{p}\hat{J}^{\dagger} + \hat{J}^{\dagger}\hat{p}\rangle + p(J + \bar{J}) \tag{12}$$

$$= -G^{pJ} - G^{p\bar{J}} \tag{13}$$

and similarly

$$\dot{G}^{JJ} = -2G^{pJ}, \qquad \dot{G}^{\bar{J}\bar{J}} = -2G^{p\bar{J}}, \tag{14}$$

$$\dot{G}^{pJ} = -\tfrac{1}{2}G^{JJ} - \tfrac{1}{2}G^{J\bar{J}} - G^{pp}, \tag{15}$$

$$\dot{G}^{p\bar{J}} = -\tfrac{1}{2}G^{\bar{J}\bar{J}} - \tfrac{1}{2}G^{J\bar{J}} - G^{pp}, \tag{16}$$

$$\dot{G}^{J\bar{J}} = -G^{pJ} - G^{p\bar{J}}. \tag{17}$$

Higher moments also are subject to equations of motion which follow analogously.

For our solvable system, all these equations of motion are linear in the quantum variables and only finitely many ones are coupled to each other. They can thus be solved straightforwardly, such as

$$p(\phi) = \tfrac{1}{2}(Ae^{-\phi} + Be^{\phi}) - \tfrac{1}{2}\hbar, \tag{18}$$

$$J(\phi) = \tfrac{1}{2}(Ae^{-\phi} - Be^{\phi}) + iH \tag{19}$$

(using $-\frac{1}{2}i(J - \bar{J}) = H := \langle\hat{H}\rangle$) for the expectation values. To decouple the six equations for fluctuations, we first note that $\dot{G}^{J\bar{J}} = \dot{G}^{pp}$ and $\frac{1}{2}(\dot{G}^{JJ} + \dot{G}^{\bar{J}\bar{J}}) = \dot{G}^{pp}$ implies

$$G^{J\bar{J}} - G^{pp} = c_1, \tag{20}$$

$$\tfrac{1}{2}G^{JJ} + \tfrac{1}{2}G^{\bar{J}\bar{J}} - G^{pp} = c_2, \tag{21}$$

with constants c_1 and c_2. Moreover, only two sets of two coupled equations, one for G^{pp} and $G^{pJ} + G^{p\bar{J}}$,

$$\dot{G}^{pp} = -G^{pJ} - G^{p\bar{J}},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(G^{pJ} + G^{p\bar{J}}) = -c_1 - c_2 - 4G^{pp}$$

and one for $G^{pJ} - G^{p\bar{J}}$ and $G^{JJ} - G^{\bar{J}\bar{J}}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(G^{pJ} - G^{p\bar{J}}) = -\frac{1}{2}(G^{JJ} - G^{\bar{J}\bar{J}}),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(G^{JJ} - G^{\bar{J}\bar{J}}) = -2(G^{pJ} - G^{p\bar{J}})$$

remain. They yield

$$G^{pp}(\phi) = \tfrac{1}{2}(c_3e^{-2\phi} + c_4e^{2\phi}) - \tfrac{1}{4}(c_1 + c_2), \tag{22}$$

$$G^{JJ}(\phi) = \tfrac{1}{2}(c_3e^{-2\phi} + c_4e^{2\phi}) + \tfrac{1}{4}(3c_2 - c_1), \tag{23}$$

$$- i(c_5e^{\phi} - c_6e^{-\phi}), \tag{24}$$

$$G^{\bar{J}\bar{J}} = \tfrac{1}{2}(c_3e^{-2\phi} + c_4e^{2\phi}) + \tfrac{1}{4}(3c_2 - c_1), \tag{25}$$

$$+ i(c_5e^{\phi} - c_6e^{-\phi}), \tag{26}$$

$$G^{pJ}(\phi) = \tfrac{1}{2}(c_3e^{-2\phi} - c_4e^{2\phi}) + \tfrac{i}{2}(c_5e^{\phi} + c_6e^{-\phi}), \tag{27}$$

$$G^{p\bar{J}}(\phi) = \tfrac{1}{2}(c_3e^{-2\phi} - c_4e^{2\phi}) - \tfrac{i}{2}(c_5e^{\phi} + c_6e^{-\phi}), \tag{28}$$

$$G^{J\bar{J}}(\phi) = \tfrac{1}{2}(c_3e^{-2\phi} + c_4e^{2\phi}) + \tfrac{1}{4}(3c_1 - c_2), \tag{29}$$

with further constants of integration.

B. Physical inner product properties

Although we are not dealing explicitly with states, properties of the Hilbert space must be reflected in the structure of our quantum variables. The equations of motion, for instance, are only valid if we understand the quantum variables to be defined with normalized states since $\langle|\rangle = 1$ has been used, for instance when taking an expectation value of $[\hat{J}, \hat{J}^{\dagger}] = -2\hbar\hat{p} - \hbar^2$ to compute $\dot{J}$ in (4). The system is, initially, defined through the Friedmann equation as a constraint on the space where both the metric and scalar field degrees of freedom are quantized. After quantizing this equation, it is reinterpreted as describing relational motion in ϕ. Loop quantum cosmology provides the kinematical inner product on the original Hilbert space where the constraint operator is defined, but not immediately one on the solution space. Such a physical inner product can be difficult to determine explicitly in a representation of states. It can be derived, for instance, by requiring that basic operators are self-adjoint if they correspond to real classical variables. This then directly implies that all quantum variables defined from self-adjoint operators must be real, which in our procedure is the analog of using the physical inner product in deriving expectation values through states. Since such reality conditions can be imposed directly for quantum variables, implementing physical inner product properties at this level can be much more straightforward than at the level of states. One reason is the representation independence of the formalism which allows one to avoid looking for a representation of states in which a computation of the physical inner product may be feasible. This is especially useful for quantum gravity where the general physical inner product problem is one of the major issues.

1. Reality conditions

In our case, we use one complex classical variable $J = pe^{ic}$, and thus cannot refer to a self-adjoint quantization of c directly since no such operator exists at all in a loop quantization. Reality conditions implementing the physical inner product must be formulated in a more complicated way: In addition to the simple adjointness relation $\hat{p}^{\dagger} = \hat{p}$ quantizing the real variable p, we have a nonlinear relation

$$\hat{J}\hat{J}^{\dagger} = \hat{p}^2 \tag{30}$$

which follows from the fact that e^{ic} must be quantized to a unitary operator for c to be real. (However, $\hat{J}^\dagger\hat{J} \neq \hat{p}^2$ in the ordering chosen for the definition of $\hat{J} = \hat{p}\,\widehat{e^{ic}}$.)

Taking expectation values of this equation, we obtain a relation between quantum variables and expectation values: using the commutation relations (2) in

$$G^{J\bar{J}} = \tfrac{1}{2}\langle\hat{J}\hat{J}^\dagger + \hat{J}^\dagger\hat{J}\rangle - |J|^2 = \langle\hat{J}\hat{J}^\dagger\rangle + \hbar p + \tfrac{1}{2}\hbar^2 - |J|^2, \tag{31}$$

we have

$$\langle\hat{J}\hat{J}^\dagger\rangle = G^{J\bar{J}} - \hbar p - \tfrac{1}{2}\hbar^2 + |J|^2 = \langle\hat{p}^2\rangle = G^{pp} + p^2 \tag{32}$$

and thus

$$|J|^2 - \left(p + \tfrac{1}{2}\hbar\right)^2 = G^{pp} - G^{J\bar{J}} + \tfrac{1}{4}\hbar^2 = \tfrac{1}{4}\hbar^2 - c_1. \tag{33}$$

This condition mixes expectation values and quantum variables, but depends on fluctuations only through the constant c_1. An initial state thus determines how the reality conditions between expectation values are realized. The relation (33) is then preserved in time since $G^{J\bar{J}} - G^{pp} = c_1$ is constant as derived in (20). If fluctuations are small, as in semiclassical states, we have $|J|^2 = p^2 + O(\hbar)$ which, as it should, is the classical relation satisfied up to quantum corrections.

2. *Sign of the energy*

Although not directly related to the physical inner product, we include in this section a discussion of the requirement of a definite sign for the Hamiltonian in superposed states. One may or may not wish to allow superpositions of expanding and contracting universe branches in quantum cosmology, but arguments using the physical inner product in analogy to the Klein–Gordon equation suggest that only energy eigenstates of a definite sign should be allowed in superpositions [15]. For a linear system, we did not explicitly take an absolute value of the Hamiltonian operator, which implies that in general we are not guaranteed that only positive energy solutions enter states corresponding to our solutions. In a language more suitable to quantum cosmology, "positive energy" means that at any fixed time ϕ we should not allow superpositions of expanding and contracting branches of a universe (while at different times the universe certainly does not need to be always expanding or always contracting). To rule out significant contributions from a superposed branch we have to pose further conditions for our quantum variables ensuring that they arise from expectation values in states which are superpositions of only positive energy eigenstates (or only negative energy eigenstates). In general, expressing the positivity of operators through expectation values can be complicated.

But for our purposes it is, fortunately, possible to proceed without technical complications. We will be interested in states which at some point (e.g. at late times) are semiclassical. This restricts the values that fluctuations can take compared to the magnitude of expectation values. It refers, in particular, to p_ϕ as one of the matter variables. We require that its fluctuation is small compared to its expectation value which, through the dynamical equation, implies the relation

$$G^{HH} := \langle\hat{H}^2\rangle - \langle\hat{H}\rangle^2 \ll \langle\hat{H}\rangle^2. \tag{34}$$

If this is realized and $H = \langle\hat{H}\rangle > 0$, a state in the H-representation, i.e. written as a superposition of $\hat{H}$-eigenstates, is sharply peaked at a large positive value of H. Thus, there are no significant contributions from negative energy states. (We will return to this issue in Sec. III D 3.) Since H and G^{HH} are constant during evolution, imposing (34) at one initial time ensures that it is satisfied at all times.

We can express this condition in terms of the integration constants derived before. From

$$G^{HH} = \langle\hat{H}^2\rangle - H^2 \tag{35}$$

$$= -\tfrac{1}{4}\langle\hat{J}^2 - \hat{J}\hat{J}^\dagger - \hat{J}^\dagger\hat{J} + \hat{J}^{\dagger 2}\rangle + \tfrac{1}{4}(J^2 - 2|J|^2 + \bar{J}^2)$$
$$= -\tfrac{1}{4}(G^{JJ} + G^{\bar{J}\bar{J}}) + \tfrac{1}{2}G^{J\bar{J}} = \tfrac{1}{2}(c_1 - c_2), \tag{36}$$

we see that $c_1 - c_2$ must be small compared to H^2. (But it cannot be zero since $\hat{H}$ has continuous spectrum.) This is the primary condition we have to impose not only for semiclassicality but also to ensure that a state is dominated by contributions of definite energy sign. Later, we will add further semiclassicality conditions to restrict also the fluctuations of other variables such as p.

C. Bouncing solutions

Our general solution (18) for p allows bouncing [28] solutions for $AB > 0$ as well as "singular" solutions for $AB < 0$ which reach $p = 0$ in finite time ϕ. (Although isotropic loop quantum cosmology is nonsingular for any solution [29,30], additional correction terms become manifest at small volume which are not included here in the solvable model. The model itself thus breaks down before $p = 0$ is reached. The singularity in our equations only indicates that a deep quantum geometry regime is reached, just as one commonly expects the general singularity problem to be resolved. Nevertheless, we keep solutions with $AB < 0$ for now since they will be ruled out even within our model shortly.) The internal time variable ϕ has just been chosen for convenience of the mathematical description, rather than referring to physical observers. For a solution reaching $p = 0$ to be considered singular one must also verify that *proper* time remains finite. We thus need to interpret our relational solution (p and J as functions of ϕ) as a space-time geometry subject to modified dynamics as it arises from the loop quantization.

We do not have any manifold picture, except for the homogeneous spatial manifold we started with to reduce the classical system. What is missing is a manifold for the time extension, which is indispensable if we want to compute a proper time interval. A time direction and coordinate can be introduced by reverting back to the Friedmann formulation of constrained dynamics. We interpret the effective Hamiltonian density $p^{-3/2}\langle\hat{H}\rangle^2 = p^{-3/2}(\mathrm{Im}J)^2 = \sqrt{p}\sin^2 c$ together with the matter contribution $\frac{1}{2}p^{-3/2}p_\phi^2$ as an effective constraint [31]

$$C = -\sqrt{p}\sin^2 c + \frac{4\pi G}{3}p^{-3/2}p_\phi^2$$

which generates coordinate evolution (in η) in a gauge specified by the lapse function N, $\mathrm{d}p/\mathrm{d}\eta = \{p, NC\}$. For proper time, $\eta = \tau$, we simply have $N = 1$ and thus

$$\mathrm{d}p/\mathrm{d}\tau = \{p, C\} = \sqrt{p}\sin(2c).$$

From the equation

$$\sin(2c(\phi)) = \frac{1}{\sqrt{p(\phi)}}\frac{\mathrm{d}p(\phi)}{\mathrm{d}\tau} = \frac{1}{\sqrt{p(\phi)}}\frac{\mathrm{d}p}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} \tag{37}$$

and our solutions (18) and (19) we can then compute $\phi(\tau)$ by integrating

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \sqrt{p(\phi)}\frac{2\sin(c(\phi))\cos(c(\phi))}{\mathrm{d}p/\mathrm{d}\phi} = \frac{-2\sqrt{2}H}{(Ae^{-\phi} + Be^{\phi} - \hbar)^{3/2}} \tag{38}$$

(using $J/p = \cos c + i\sin c$). We can always assume that either $A = B > 0$ (for a bouncing solution) or $A = -B > 0$ (for a nonbouncing one) since we only need to shift the origin of ϕ if $|A| \neq |B|$. This leaves us with two cases,

$$\tau(\phi) = -\frac{A^{3/2}}{H}\int^{\phi}\cosh^{3/2}(z)\mathrm{d}z \tag{39}$$

for $A = B$ and

$$\tau(\phi) = -\frac{A^{3/2}}{H}\int^{\phi}\sinh^{3/2}(z)\mathrm{d}z \tag{40}$$

for $A = -B$. The integrals can be determined in terms of elliptic functions, but we are only interested in the fact that $\tau(\phi)$ is finite at any finite value of ϕ which can be seen directly from the integrals. Thus, proper time remains finite when $p = 0$ is reached.

Singular solutions could thus be possible. But not all these solutions satisfy the reality condition (33) which still has to be imposed. From its general form we obtain

$$|J|^2 - (p + \tfrac{1}{2}\hbar)^2 = -AB + H^2 = \tfrac{1}{4}\hbar^2 - c_1$$

and thus

$$AB = H^2 + c_1 - \tfrac{1}{4}\hbar^2. \tag{41}$$

For macroscopic values of H and small (or positive) c_1 from fluctuations, we only have bouncing solutions with $AB > 0$. Singular solutions can only be obtained for large and negative c_1 which is never realized for states which are semiclassical at one time. Note, however, that c_1 can be large even if our condition $G^{HH} \ll H^2$ which is necessary for solutions to respect positivity is realized since the latter condition only constrains $c_1 - c_2$. Our discussion thus shows that it is crucial to know and use the reality conditions, or ultimately the physical inner product, to draw conclusions about bouncing solutions versus nonbouncing ones.

D. Uncertainty relations

Fluctuations cannot take arbitrary values but are restricted by uncertainty relations. For each pair of *self-adjoint* basic operators we have one uncertainty relation, which in our case implies three relations since $\hat{J} + \hat{J}^\dagger$ and $i(\hat{J} - \hat{J}^\dagger)$ are independent in addition to $\hat{p}$. For each pair $(\hat{A}, \hat{B})$ of self-adjoint operators we have the general form

$$G^{AA}G^{BB} - (G^{AB})^2 \geq \tfrac{1}{4}\langle -i[\hat{A}, \hat{B}]\rangle^2 \tag{42}$$

of uncertainty relations in terms of quantum variables whenever $\hat{A}$ and $\hat{B}$ are self-adjoint. The derivation is standard and recalled briefly in Appendix A1 for completeness.

Specifically, we have three pairs $(\hat{p}, \hat{J} + \hat{J}^\dagger)$, $(\hat{p}, i(\hat{J} - \hat{J}^\dagger))$ and $(\hat{J} + \hat{J}^\dagger, i(\hat{J} - \hat{J}^\dagger))$ of different self-adjoint basic operators. We obtain three inequalities involving G^{pp} and the fluctuations

$$G^{J+\bar{J},J+\bar{J}} := \langle(\hat{J} + \hat{J}^\dagger)^2\rangle - (J + \bar{J})^2 = G^{JJ} + 2G^{J\bar{J}} + G^{\bar{J}\bar{J}} = 4G^{pp} + 2(c_1 + c_2), \tag{43}$$

$$G^{i(J-\bar{J}),i(J-\bar{J})} := -\langle(\hat{J} - \hat{J}^\dagger)^2\rangle + (J - \bar{J})^2 = 4G^{HH} = 2(c_1 - c_2), \tag{44}$$

$$G^{p,J+\bar{J}} := \tfrac{1}{2}\langle\hat{p}(\hat{J} + \hat{J}^\dagger) + (\hat{J} + \hat{J}^\dagger)\hat{p}\rangle - p(J + \bar{J}) = G^{pJ} + G^{p\bar{J}}, \tag{45}$$

$$G^{p,i(J-\bar{J})} := \tfrac{i}{2}\langle\hat{p}(\hat{J} - \hat{J}^\dagger) + (\hat{J} - \hat{J}^\dagger)\hat{p}\rangle - ip(J - \bar{J}) = i(G^{pJ} - G^{p\bar{J}}), \tag{46}$$

$$G^{J+\bar{J},i(J-\bar{J})} := \tfrac{i}{2}\langle(\hat{J} + \hat{J}^\dagger)(\hat{J} - \hat{J}^\dagger) + (\hat{J} - \hat{J}^\dagger)(\hat{J} + \hat{J}^\dagger)\rangle - i(J + \bar{J})(J - \bar{J}) = i(G^{JJ} - G^{\bar{J}\bar{J}}). \tag{47}$$

Using the explicit solutions, they are

$$G^{pp}G^{J+\bar{J},J+\bar{J}} - (G^{p,J+\bar{J}})^2 = 4c_3c_4 - \tfrac{1}{4}(c_1+c_2)^2 \geq \hbar^2 H^2, \tag{48}$$

$$\begin{aligned} G^{pp}G^{i(J-\bar{J}),i(J-\bar{J})} - (G^{p,i(J-\bar{J})})^2 &= (c_1-c_2)(c_3e^{-2\phi}+c_4e^{2\phi}) + \tfrac{1}{2}(c_2^2-c_1^2) - c_5^2e^{2\phi} - 2c_5c_6 - c_6^2e^{-2\phi} \\ &\geq \tfrac{1}{4}\hbar^2(J+\bar{J})^2 = \tfrac{1}{4}\hbar^2(A^2e^{-2\phi} - 2AB + B^2e^{2\phi}), \end{aligned} \tag{49}$$

$$\begin{aligned} G^{J+\bar{J},J+\bar{J}}G^{i(J-\bar{J}),i(J-\bar{J})} - (G^{J+\bar{J},i(J-\bar{J})})^2 &= 4(c_1-c_2)(c_3e^{-2\phi}+c_4e^{2\phi}) - 2(c_2^2-c_1^2) - 4c_5^2e^{2\phi} + 8c_5c_6 - 4c_6^2e^{-2\phi} \\ &\geq \hbar^2(2p+\hbar)^2 = \hbar^2(A^2e^{-2\phi} + 2AB + B^2e^{2\phi}). \end{aligned} \tag{50}$$

These uncertainty relations are the equations which determine properties of semiclassical, near coherent states. We will later discuss these relations, in particular, their saturation, in more detail and give a complete analysis of coherent states of this system. Before doing so we can already note here that there are quite unfamiliar properties compared to what one knows from harmonic oscillator coherent states. The first relation (48) shows that there is a type of uncertainty relation between the constants of integration c_3 and c_4, one of which determines the p-fluctuation before and one after the bounce. Thus, if the uncertainties are very small at very late times, say, they must have been very large at early times. This relation also indicates that equally distributed fluctuations are typically of the size $\sqrt{c_3} \sim \sqrt{\hbar H}$ and thus $(\Delta p)/p = \sqrt{G^{pp}}/p \sim \sqrt{\hbar/H}$.

1. *Saturation*

Of particular interest is the case of coherent states which saturate the uncertainty relations. For the harmonic oscillator, such states are squeezed Gaussian states of the form $\psi(q) = \exp(-z_1q^2 + z_2q + z_3)$ with three complex numbers z_i such that $\mathrm{Re}z_1 > 0$ (see Appendix A 2 for a listing of fluctuations and correlations). While $\mathrm{Re}z_3$ is fixed by normalization and $\mathrm{Im}z_3$ is only a phase factor, $z_1 = \alpha_1 + i\beta_1$ and $z_2 = \alpha_2 + i\beta_2$ determine the peak and fluctuations of the state. As is well-known, these states even describe dynamical coherent states of the harmonic oscillator, i.e. their form is preserved during evolution. Saturation of the uncertainty relation leaves two free second moments such as one spread parameter G^{qq} and squeezing G^{qp}. Unsqueezed states imply $\beta_1 = 0 = G^{qp}$ and only one free parameter specifies the width of the Gaussian.

Our system is different and no operator for c (which would be an analog of $\hat{q}$) exists. We have to work with exponentials instead, and thus even kinematical coherent states change compared to the harmonic oscillator. For dynamical coherent states the form must anyway be different since we are not dealing with the harmonic oscillator Hamiltonian. Thanks to the solvability of our model we are still able to determine properties of dynamical coherent states explicitly. This provides an instructive example of how properties of coherent states, and physical implications, can change when a system is not closely related to a harmonic oscillator.

We thus look at saturation of our uncertainty relations (48)–(50), first removing the ϕ-dependence. Subtracting (50) divided by four from (49) yields

$$c_2^2 - c_1^2 - 4c_5c_6 = -\hbar^2 AB. \tag{51}$$

The coefficients of $e^{-2\phi}$ and $e^{2\phi}$ in (49) then yield two more independent relations

$$(c_1-c_2)c_3 - c_6^2 = \tfrac{1}{4}\hbar^2A^2, \tag{52}$$

$$(c_1-c_2)c_4 - c_5^2 = \tfrac{1}{4}\hbar^2B^2 \tag{53}$$

in addition to (48) which becomes

$$4c_3c_4 - \tfrac{1}{4}(c_1+c_2)^2 = \hbar^2H^2. \tag{54}$$

There are thus four equations for six variables, such that two remain free as in the case of Gaussians. Note, however, that the reality condition relates one of these, c_1, to the expectation values, or the constants A, B and H. Thus, only one combination of the uncertainty parameters is free, which we can take as $c_1 - c_2 = 2(\Delta H)^2$, and saturated states are more restricted than for the harmonic oscillator. (This is a consequence of the nonlinear reality condition which relates some fluctuations to expectation values.)

Without loss of generality we can assume $A = B$ for bounces or $A = -B$ for nonbouncing solutions since it simply amounts to choosing the origin of time ϕ such that the bounce (or the transition through $p = 0$) occurs at $\phi = 0$. Subtracting (52) and (53) then gives

$$(c_1-c_2)(c_3-c_4) = c_6^2 - c_5^2 \tag{55}$$

which shows that p-fluctuations are the same before and after the bounce, i.e. $c_3 = c_4$, if and only if also $|c_5| = |c_6|$ (recall that $c_1 - c_2 = 2(\Delta H)^2$ cannot be zero). This case will be discussed below.

Using $A = \pm B$, we can combine (51)–(53) to obtain a further A-independent relation

$$(c_1-c_2)(c_3+c_4) - (c_5 \pm c_6)^2 \pm \tfrac{1}{2}(c_2^2 - c_1^2) = 0. \tag{56}$$

Solving for $c_3 + c_4$ and combining it with (55) gives

$$c_3 = \frac{\pm c_5 + c_6}{c_1 - c_2}c_6 \pm \frac{1}{4}(c_1 + c_2), \tag{57}$$

$$c_4 = \frac{c_5 \pm c_6}{c_1 - c_2} c_5 \pm \frac{1}{4}(c_1 + c_2) \tag{58}$$

in terms of only c_1, c_2, c_5 and c_6. Using this in (54) and combining it with (51), we have

$$\begin{aligned} \hbar^2 H^2 + \frac{1}{4}(c_1 + c_2)^2 &= 4c_3 c_4 = \pm 4\left(\frac{c_5 \pm c_6}{c_1 - c_2}\right)^2 c_5 c_6 \pm \frac{c_1 + c_2}{c_1 - c_2}(c_5 \pm c_6)^2 + \frac{1}{4}(c_1 + c_2)^2 \\ &= \pm\left(\frac{c_5 \pm c_6}{c_1 - c_2}\right)^2 (4c_5 c_6 + c_1^2 - c_2^2) + \frac{1}{4}(c_1 + c_2)^2 = \hbar^2 A^2 \left(\frac{c_5 \pm c_6}{c_1 - c_2}\right)^2 + \frac{1}{4}(c_1 + c_2)^2 \end{aligned} \tag{59}$$

and thus (with $c_1 - c_2 = 2(\Delta H)^2 > 0$)

$$|c_5 \pm c_6| = \tfrac{H}{A}(c_1 - c_2). \tag{60}$$

This shows that it is impossible to have both c_5 and c_6 zero, i.e. $G^{p,i(J-\bar{J})} \neq 0$ and there are always correlations between p and H which evolve in time. Together with (51), (57), and (58), this last relation allows us immediately to express all parameters in terms of only c_1 and c_2, and all relations for saturation are solved.

We now ask whether it is possible to have a coherent state which behaves semiclassically at one (late) time and has identical fluctuations before and after the bounce. We thus focus on the case $c_5 = \pm c_6$. This assumption allows us to solve (60) directly for c_5,

$$|c_5| = |c_6| = \frac{H}{2A}(c_1 - c_2) \tag{61}$$

and to insert it in (57),

$$c_3 = c_4 = \frac{H^2}{2A^2}(c_1 - c_2) \pm \frac{1}{4}(c_1 + c_2). \tag{62}$$

Consistency with (54) implies

$$\begin{aligned} 4c_3^2 &= \hbar^2 H^2 + \frac{1}{4}(c_1 + c_2)^2 \\ &= \left(\frac{H^2}{A^2}(c_1 - c_2) \pm \frac{1}{2}(c_1 + c_2)\right)^2. \end{aligned} \tag{63}$$

Coherent states with identical fluctuations before and after the bounce are thus possible if and only if

$$\frac{H^4}{A^4}(c_1 - c_2)^2 \mp \frac{H^2}{A^2}(c_2^2 - c_1^2) = \hbar^2 H^2. \tag{64}$$

Using $c_1 - c_2 = 2(\Delta H)^2$ and $c_1 + c_2 = -2(\Delta H)^2 + 2c_1 = -2(\Delta H)^2 - 2H^2 \pm 2A^2 + \frac{1}{2}\hbar^2$ [imposing the reality condition (41)], we must then solve

$$\frac{H^2 \mp A^2}{H^2}(\Delta H)^4 \mp \frac{A^2(H^2 \mp A^2 - \frac{1}{4}\hbar^2)}{H^2}(\Delta H^2) - \frac{A^4}{H^2}\frac{\hbar^2}{4} = 0$$

giving

$$\begin{aligned} (\Delta H)^2 &= \pm \frac{A^2(H^2 \mp A^2 - \frac{1}{4}\hbar^2)}{2(H^2 \mp A^2)} \\ &\quad + \sigma \frac{\sqrt{A^4(H^2 \mp A^2 - \frac{1}{4}\hbar^2)^2 + A^4(H^2 \mp A^2)\hbar^2}}{2(H^2 \mp A^2)} \\ &= \frac{\mp A^2((H^2 \mp A^2 - \frac{1}{4}\hbar^2) + \sigma|H^2 \mp A^2 + \frac{1}{4}\hbar^2|)}{2(H^2 \mp A^2)}. \end{aligned} \tag{65}$$

Here we distinguished the two roots of the quadratic Eq. (64) by $\sigma = \pm 1$ since another $\pm$ has already been used for the two cases $A = \pm B$.

Depending on the signs involved there are four possibilities to have positive solutions for $(\Delta H)^2$:

(1) $A = B$, in which case there is a further distinction
- (a) $A^2 < H^2 + \frac{1}{4}\hbar^2$: Only $\sigma = 1$ is allowed, implying $(\Delta H)^2 = A^2$.
- (b) $A^2 > H^2 + \frac{1}{4}\hbar^2$: Both signs for σ are allowed,
  - (i) $\sigma = -1$ implies $(\Delta H)^2 = A^2$ as above;
  - (ii) $\sigma = 1$ implies

$$(\Delta H)^2 = \frac{A^2}{A^2 - H^2}\frac{\hbar^2}{4} \tag{66}$$

(2) $A = -B$, which allows only one choice of signs for a positive

$$(\Delta H)^2 = \frac{A^2}{H^2 + A^2}\frac{\hbar^2}{4}. \tag{67}$$

There are two cases where $(\Delta H)^2 = A^2$ which can satisfy the basic condition $(\Delta H)^2 \ll H^2$ only if $A^2 \ll H^2$. Thus c_1 must be of the order H^2. This can only happen if the bounce scale $p(0) = A - \frac{1}{2}\hbar$ is small compared to the total energy, i.e. the universe enters the deep Planck regime during the bounce. However, large c_1 (and thus large c_2 since $c_1 - c_2$ must remain small) of the order H implies, using (62), that c_3 is of the order H, too, and Δp is not small compared to p (in fact, not even smaller). This case does not give rise to semiclassical states at any time.

The case $A = -B$ allows small ΔH. However, as we already saw, the reality condition allows such solutions only if $c_1 = -H^2 - A^2 + \frac{1}{4}\hbar^2$ is large and negative. For states saturating the uncertainty relations, this implies that c_3 is of the same size as $-c_1$ and thus too large for the state to be semiclassical at any time.

For the last possibility (66), $(\Delta H)^2$ can be small compared to H^2. For instance, if c_1 is of the order $A\hbar$, which is allowed for semiclassical states, we have $(\Delta H)/H \sim \sqrt{A\hbar}/H \sim \sqrt{\hbar/H}$. Moreover, $(\Delta p)/p \sim \sqrt{\hbar/A}$ such that this final possibility does allow semiclassical states with equal spread before and after the bounce. This confirms the earlier indication that equally distributed fluctuations typically satisfy $c_3 \sim \hbar H$. Although there is a factor of $\hbar$, fluctuations are rather large due to the factor of H which, for a universe with large matter energy, is a large number. It is possible to have smaller fluctuations which are not magnified by the matter energy, but only at one side of the bounce and at the expense of having much larger fluctuations at the other side of the bounce.

There is an easier way to have symmetric fluctuations if one does not require that all uncertainty relations be saturated. One can argue that (49) is of primary interest since it determines the fluctuations of p and H, and that only this relation should be saturated. If this is done, symmetric fluctuations before and after the bounce are easily allowed. However, this can only be put in by assumption and not be inferred from conditions at one time after the bounce: only one of the relevant parameters c_3 or c_4 is controlled by the uncertainty relation at one late or early time ϕ while the other one would be suppressed exponentially by $e^{-2|\phi|}$. Symmetric fluctuations before and after the bounce can thus not be proven but only be assumed for coherent states of this system. Generically, even a universe which is semiclassical at late times can have been highly quantum before the big bang. Examples for one symmetric and the generic nonsymmetric bounces are shown in Fig. 2.

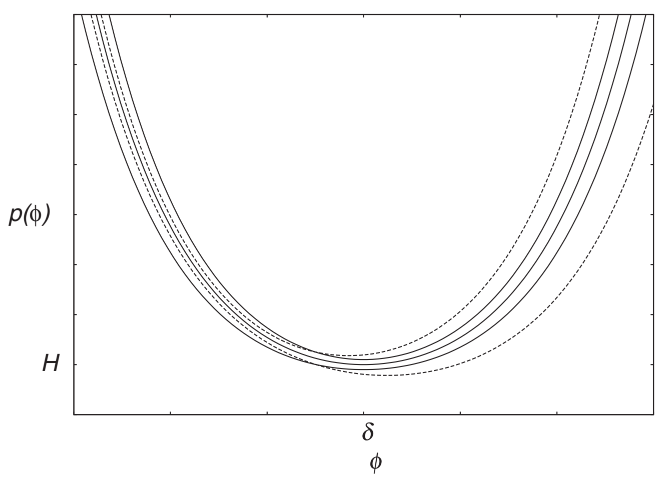

FIG. 2. Two bouncing solutions for the expectation value of $\hat{p}$ and the spread around it. Generic states have different spread before and after the bounce (dashed lines), while unsqueezed Gaussian initial states lead to solutions which are symmetric around the bounce not only in their expectation values but also in spreads (solid lines).

2. Gaussian states

This seems to be in conflict with recent numerical results [15,27,32] where the p-fluctuations turned out to be very close to each other before and after the bounce. Indeed, this gave rise to statements to the extent that the universe was as semiclassical before the big bang as it is now. To resolve the apparent contradiction we have to look at *unsqueezed Gaussian states* as they were used by construction in those numerical simulations (explicitly removing phase factors so as to desqueeze states), but are not covered by our preceding coherent state analysis. As emphasized before, Gaussian states do not saturate the uncertainty relations in this system. They are thus not coherent, and not distinguished as they would be for usual quantum mechanical systems. Nevertheless, their properties are quite interesting.

Let us thus assume that we have a state of the form $\psi(p) = N\exp(-z_1 p^2 + z_2 p)$, supported on integer p, with $\mathrm{Re}z_1 > 0$. We again refer to real and imaginary parts of z_1 and z_2 according to $z_1 = \alpha_1 + i\beta_1$, $z_2 = \alpha_2 + i\beta_2$. Although these states have the same form as in standard quantum mechanics, the representation of basic operators is different. For instance, we have

$$\langle\hat{p}\rangle = \sum_p p|\psi(p)|^2, \tag{68}$$

$$\langle\hat{J}\rangle = \sum_p p\bar{\psi}(p)\psi(p-\hbar), \tag{69}$$

where we sum over integers and the shift by $\hbar$ arises from the action of $\widehat{e^{ic}}$ which is a shift operator in p. For states which are nearly constant on the discrete scale of p, the expressions can be seen as Riemann sums and approximated by Gaussian integrals which one can compute explicitly.

We thus obtain

$$p \approx \frac{\alpha_2}{2\alpha_1}, \tag{70}$$

$$J \approx \frac{\alpha_2 + \alpha_1\hbar + i\beta_1\hbar}{2\alpha_1}\exp(-(\alpha_1^2 + \beta_1^2)\hbar^2/2\alpha_1 - i(\beta_2 - \alpha_2\beta_1/\alpha_1)\hbar), \tag{71}$$

$$G^{pp} \approx \frac{1}{4\alpha_1}, \tag{72}$$

$$G^{pJ} \approx \left(\frac{1}{4\alpha_1} - \frac{\beta_1^2}{4\alpha_1^2}\hbar^2 + \frac{i\beta_1\hbar}{4\alpha_1}(\hbar + \alpha_2/\alpha_1)\right) \times \exp(-(\alpha_1^2 + \beta_1^2)\hbar^2/2\alpha_1 - i\hbar(\beta_2 - \alpha_2\beta_1/\alpha_1)), \tag{73}$$

$$G^{JJ} \approx \left(\exp(-(\alpha_1^2+\beta_1^2)\hbar^2/\alpha_1)\left(\frac{1}{4\alpha_1}+\frac{\alpha_2^2}{4\alpha_1^2}-\frac{\beta_1^2}{\alpha_1^2}\hbar^2+\frac{\alpha_2\hbar}{2\alpha_1}+i\frac{\alpha_2\beta_1\hbar}{\alpha_1^2}\right)-\frac{\alpha_2^2}{4\alpha_1^2}-\frac{\hbar^2}{4}+\frac{\beta_1^2}{4\alpha_1^2}\hbar^2 -\frac{\alpha_2\hbar}{2\alpha_1}-i\frac{\beta_1\hbar}{2\alpha_1}(\hbar+\alpha_2/\alpha_1)\right)\exp(-(\alpha_1^2+\beta_1^2)\hbar^2/\alpha_1-2i(\beta_2-\alpha_2\beta_1/\alpha_1)\hbar)\approx\overline{G^{\bar{J}\bar{J}}}, \tag{74}$$

$$G^{J\bar{J}} \approx \frac{1}{4\alpha_1}+\frac{\hbar^2}{2}+\frac{\alpha_2\hbar}{2\alpha_1}+\frac{\alpha_2^2}{4\alpha_1^2}-\left(\frac{\alpha_2^2}{4\alpha_1^2}+\frac{\alpha_2\hbar}{2\alpha_1}+\frac{\hbar^2}{4}+\frac{\beta_1^2\hbar^2}{4\alpha_1^2}\right)\exp(-(\alpha_1^2+\beta_1^2)\hbar^2/\alpha_1). \tag{75}$$

These expressions involving J are much more messy than the corresponding ones for Gaussian states in standard quantum mechanics. This is a consequence of the fact that Gaussian states for the system considered here are not natural at all.

Nevertheless, the expressions simplify somewhat if one assumes that the state is unsqueezed, $\beta_1 = 0$. Moreover, for semiclassical states we can use $p \gg \Delta p \gg \hbar$ which implies $\alpha_2/\alpha_1 \gg \alpha_1^{-1/2} \gg \hbar$. Then, the leading order contribution to the energy fluctuations, derived using

$$\mathrm{Re}\, G^{JJ} = \left(\left(\frac{1}{4\alpha_1}+\frac{\alpha_2^2}{4\alpha_1^2}+\frac{\alpha_2}{2\alpha_1}\hbar\right)e^{-\alpha_1\hbar^2}-\frac{\alpha_2^2}{4\alpha_1^2}-\frac{\hbar^2}{4}-\frac{\alpha_2}{2\alpha_1}\hbar\right)e^{-\alpha_1\hbar^2}\cos(2\beta_2\hbar)\approx-\frac{\alpha_2^2}{4\alpha_1}\hbar^2\cos(2\beta_2\hbar)$$

and

$$G^{J\bar{J}} = \frac{1}{4\alpha_1}+\frac{\hbar^2}{2}+\frac{\alpha_2}{2\alpha_1}\hbar+\frac{\alpha_2^2}{4\alpha_1^2}-\left(\frac{\alpha_2^2}{4\alpha_1^2}+\frac{\alpha_2}{2\alpha_1}\hbar+\frac{\hbar^2}{4}\right)e^{-\alpha_1\hbar^2}\approx\frac{\alpha_2^2}{4\alpha_1}\hbar^2,$$

is

$$G^{HH} = -\frac{1}{4}(G^{JJ}+G^{\bar{J}\bar{J}})+\frac{1}{2}G^{J\bar{J}}\approx\frac{\alpha_2^2}{4\alpha_1}\hbar^2\cos^2(\beta_2\hbar) \tag{76}$$

and with $H = \mathrm{Im}J \approx -\alpha_2\sin(\beta_2\hbar)/2\alpha_1$ we have

$$\frac{G^{HH}}{H^2}\approx\alpha_1\hbar^2\cot^2(\beta_2\hbar) \tag{77}$$

whose left-hand side is constant throughout the whole evolution. At late and early times β_2 becomes small such that $\alpha_1 \propto \cot^{-2}(\beta_2\hbar) \approx \sin^2(\beta_2\hbar)$. This relates α_1 which determines the p-fluctuation to β_2 which determines the peak position of the wave packet. Thus, for unsqueezed Gaussians we prove that the fluctuations before and after the bounce are the same at times where $\sin^2(\beta_2\hbar)$ takes the same value. Since the expectation value solutions (18) are symmetric around the bounce point for any state, *fluctuations of unsqueezed Gaussian states are shown to be symmetric around the bounce.*

This reconciles our calculations with the numerical calculations of [15,27] and reinforces their validity. However, it also demonstrates that the result of identical fluctuations before and after the bounce is a consequence not of the generic dynamics of semiclassical states, but relies on the assumption that states are unsqueezed Gaussians. It is then not very surprising to find symmetric spreads since there is a single parameter determining the state, other than its expectation values. The fact that G^{HH} is constant then implies directly that there is a fixed relation between this parameter, α_1, and the peak position. Since expectation values are symmetric around the bounce for any solution, the spread must satisfy the same symmetry in this restricted case. As discussed before, there is no intrinsic basis in this model to restrict states to such a form. They do not saturate the uncertainty relations, and even if the Gaussian form is assumed but general squeezing is allowed, fluctuations before and after the bounce become independent of each other. There is then an additional parameter β_1 and only a certain function of spread α_1 and squeezing β_1 is fixed at the bounce-reflected point. This does not suffice to fix the spread to be symmetric. The precise relation follows by estimating G^{HH} as above, now keeping $\beta_1 \neq 0$. Then, $G^{HH}/H^2 = \mathrm{const}$ with

$$G^{HH}\approx\frac{\alpha_2^2}{4\alpha_1^3}(\alpha_1^2+\beta_1^2)\hbar^2\cos^2((\beta_2-\alpha_2\beta_1/\alpha_1)\hbar)$$

and

$$H \approx -\frac{\alpha_2}{2\alpha_1}\sin((\beta_2-\alpha_2\beta_1/\alpha_1)\hbar)+\frac{\beta_1}{2\alpha_1}\hbar\cos((\beta_2-\alpha_2\beta_1/\alpha_1)\hbar)$$

provides the relation between α_1 and β_1 in terms of the expectation values.

3. *The role of superposed branches*

One could suspect that nonsymmetric fluctuations are a consequence of small admixtures of superposed expanding

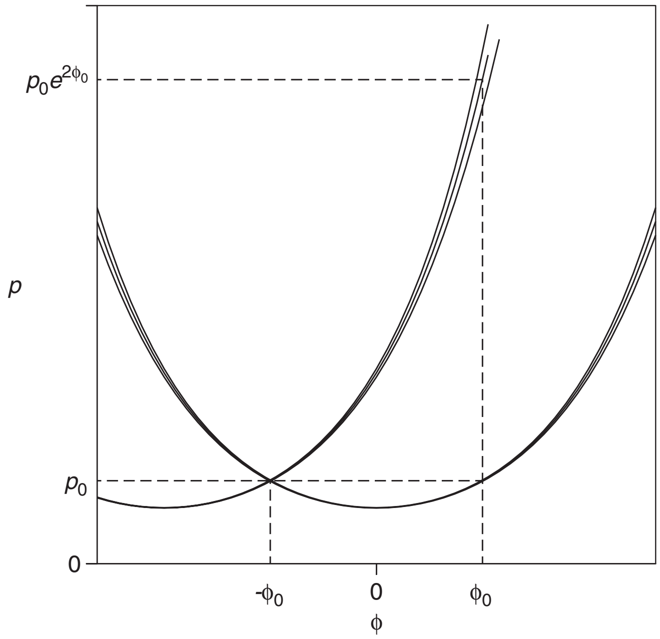

FIG. 3. Sketch of two wave functions with the same peak position p_0 at $-\phi_0$, where one function represents a collapsing branch, the other an expanding one. After the bounce of the first solution at $\phi = 0$, the wave packets deviate strongly at ϕ_0. Both wave packets are illustrated by their expectation values and spreads, assumed to be symmetric around the bounce for the sake of the argument.

branches on an initial contracting one, while superpositions of energy eigenstates of definite sign might always have symmetric fluctuations. As illustrated in Fig. 3, we choose two states $|\psi_1\rangle$ and $|\psi_2\rangle$ at $\phi = -\phi_0$ which are peaked at the same value p_0 of p with the same p-fluctuations. If the first state is expanding, $\langle\psi_1|\hat{H}|\psi_1\rangle =: H > 0$, while the second one is contracting, $\langle\psi_2|\hat{H}|\psi_2\rangle = -\langle\psi_1|\hat{H}|\psi_1\rangle < 0$, they are sharply peaked at different H, and thus $\langle\psi_1|\psi_2\rangle \ll 1$. Any superposition $|\psi\rangle = (|\psi_1\rangle + \alpha|\psi_2\rangle)/\sqrt{1+|\alpha|^2}$, where $|\psi_2\rangle$ presents an admixture of a negative-H state, is then sharply peaked at p_0. At $\phi = \phi_0$, expectation values of $\hat{p}$ in the two states of the superpositon have evolved away from each other and the combined spread Δp can be much larger than individual spreads $\Delta_1 p$ and $\Delta_2 p$ measured in $|\psi_1\rangle$ and $|\psi_2\rangle$, respectively.

This can be analyzed more quantitatively: For simplicity, we assume that the H-fluctuations of the two states are nearly equal, $\Delta_1 H \approx \Delta_2 H$. In the state $|\psi\rangle$, we have

$$\begin{aligned}(\Delta H)^2 &= \langle\psi|\hat{H}^2|\psi\rangle - \langle\psi|\hat{H}|\psi\rangle^2 \\ &\approx \frac{1}{1+|\alpha|^2}(\langle\hat{H}^2\rangle_1 + |\alpha|^2\langle\hat{H}^2\rangle_2) - H^2\left(\frac{1-|\alpha|^2}{1+|\alpha|^2}\right)^2 \\ &= \frac{1}{(1+|\alpha|^2)^2}((\Delta_1 H)^2(1+|\alpha|^2) \\ &\quad + (\Delta_2 H)^2|\alpha|^2(1+|\alpha|^2) + 4|\alpha|^2H^2),\end{aligned}$$

where subscripts of 1 and 2 at right brackets and Δ indicate which state is used for the expectation values. Now assuming $(\Delta_1 H)^2 \approx (\Delta_2 H)^2$, we obtain

$$\frac{(\Delta H)^2}{H^2} \approx \frac{(\Delta_1 H)^2}{H^2} + 4\frac{|\alpha^2|}{(1+|\alpha|^2)^2}. \tag{78}$$

Thus, the admixture changes the H-fluctuations only slightly and preserves $\Delta H \ll H$ if $|\alpha|$ is sufficiently small. It would thus be allowed in our approximation, although not in a superposition of only positive energy eigenstates. We will now see how the admixture can influence spreads of $\hat{p}$ before and after the bounce.

While $(\Delta H)/H$ is constant in time, $(\Delta p)/p$ changes and the initial value which by construction is close to $(\Delta_1 p)/p$ is not conserved. Using the behavior of exact solutions of expectation values, one can compute p-fluctuations at the bounce-reflected point of the initial state. At this time, the two states will, in the p-representation, have evolved away from each other since one state corresponds to an expanding branch and the other to a collapsing one. We have

$$\begin{aligned}(\Delta p)^2 &= \frac{1}{1+|\alpha|^2}(\langle\hat{p}^2\rangle_1 + |\alpha|^2\langle\hat{p}^2\rangle_2) - \left(\frac{\langle\hat{p}\rangle_1 + |\alpha|^2\langle\hat{p}\rangle_2}{1+|\alpha|^2}\right)^2 \\ &= \frac{1}{(1+|\alpha|^2)^2}(((\Delta_1 p)^2 + |\alpha|^2(\Delta_2 p)^2)(1+|\alpha|^2) + |\alpha|^2\langle\hat{p}\rangle_1^2 + |\alpha|^2\langle\hat{p}\rangle_2^2 - 2|\alpha|^2\langle\hat{p}\rangle_1\langle\hat{p}\rangle_2) \\ &= (\Delta_1 p)^2\frac{1+|\alpha|^2\langle\hat{p}\rangle_2^2/\langle\hat{p}\rangle_1^2}{1+|\alpha|^2} + \frac{|\alpha|^2}{(1+|\alpha|^2)^2}(\langle\hat{p}\rangle_1 - \langle\hat{p}\rangle_2)^2.\end{aligned}$$

In the last step we used the fact that by construction $|\psi_2\rangle$ is in the expanding branch at all times considered, thus $(\Delta p)/p \approx \text{const}$ away from the bounce and $(\Delta_2 p)^2 \approx \langle\hat{p}\rangle_2^2(\Delta_1 p)^2/\langle\hat{p}\rangle_1^2 > (\Delta_1 p)^2$. For instance, at the bounce-reflected ϕ_0 of $|\psi_1\rangle$ with large ϕ_0, we have

$$\langle\hat{p}\rangle_2(\phi_0) \approx p_0e^{2\phi_0} \approx \langle\hat{p}\rangle_1(\phi_0)(2p_0/H)^2$$

and thus

$$\frac{(\Delta p)(\phi_0)^2}{p(\phi_0)^2} = \frac{(\Delta_1 p)(\phi_0)^2}{p(\phi_0)^2} \frac{1 + 4|\alpha|^2 p_0^2/H^2}{1 + |\alpha|^2} + \frac{16|\alpha|^2}{(1 + |\alpha|^2)^2}\left(\frac{p_0}{H}\right)^4 \frac{\langle\hat{p}\rangle_1(\phi_0)^2}{p(\phi_0)^2}. \quad (79)$$

Because of the large factor $p_0^2/H^2 \gg 1$ for an initial state peaked at large volume, the p-fluctuation has grown much more than the H-fluctuation if we pass through the bounce.

To verify that this implies nonsymmetric fluctuations we have to compute Δp at the bounce-reflected point of $\langle\hat{p}\rangle$, which is not ϕ_0 due to the contribution from $|\psi_2\rangle$. From

$$\langle\hat{p}\rangle = \frac{\langle\hat{p}\rangle_1 + |\alpha|^2\langle\hat{p}\rangle_2}{1 + |\alpha|^2} \approx \langle\hat{p}\rangle_1 + |\alpha|^2\langle\hat{p}\rangle_2$$

for small $|\alpha|$, we have

$$\langle\hat{p}\rangle(\phi) \approx H\cosh\phi + \frac{1}{2}|\alpha|^2 H e^{\phi + 2\phi_0}$$

which has its minimum at $\phi_{\text{bounce}} = -\log\sqrt{1 + |\alpha|^2 e^{2\phi_0}}$. The bounce-reflected point for $\langle\hat{p}\rangle$ is thus at $\phi_{\text{reflected}} = \phi_{\text{bounce}} + (\phi_{\text{bounce}} - (-\phi_0)) = \phi_0 - \log(1 + |\alpha|^2 e^{2\phi_0})$. Evaluating (79) at this point, for simplicity assuming $|\alpha|^2 e^{2\phi_0} \gg 1$ which presents a characteristic example for an admixture at large initial volume, gives

$$\frac{(\Delta p)(\phi_{\text{reflected}})^2}{p(\phi_{\text{reflected}})^2} \approx \frac{(\Delta p)(-\phi_0)^2}{p(-\phi_0)^2}\frac{1}{|\alpha|^2} \times \frac{|\alpha|^4 + |\alpha|^{-4}e^{-2\phi_0}}{|\alpha|^4 + 2e^{-2\phi_0} + |\alpha|^{-4}e^{-2\phi_0}} + \frac{1}{|\alpha|^2(1 + |\alpha|^2)^2}$$

in terms of the initial values at $-\phi_0$. The additional inverse powers of the small $|\alpha|$ compared to the spread of H show the growth of p-fluctuations after the bounce, demonstrating that the superposition will not have symmetric spread.

One could thus suspect that such an admixture, which would not violate our condition $\Delta H \ll H$ if α is sufficiently small, could be the reason for unequal dispersions before and after the bounce, while solutions of exactly positive H would have symmetric dispersions as explicitly shown for unsqueezed Gaussians. This conjecture cannot be true, however, because the solutions we studied earlier only refer to expectation values and dispersions and, due to the decoupling in our solvable model, are completely independent of higher moments. The preceding construction of the admixture does provide states with suitable initial dispersions and expectation values, but the specific states $|\psi\rangle = (|\psi_1\rangle + \alpha|\psi_2\rangle)/\sqrt{1 + |\alpha|^2}$ also have fixed higher moments. There are many other states having the same expectation values and dispersions but different higher moments, not corresponding to what one obtains from an admixture of a negative energy state. Such states allow for positivity as well as nonsymmetric dispersions, which thus cannot be an artifact of a negative energy admixture. In short, the calculation confirms the intuitive expectation that an admixture does give nonsymmetric spreads even if each state in the superposition has symmetric spreads. But it does not show the converse, namely, that nonsymmetric fluctuations could only be caused by an admixture of a negative energy state.

IV. CONCLUSIONS

A solvable model such as the one discussed here allows a detailed analysis of dynamical coherent states which would otherwise be difficult to handle. This provides valuable information for quantum cosmology, just as the harmonic oscillator does traditionally for quantum optics (see, e.g., [33]). Compared to the harmonic oscillator, our system shows several new properties with implications for cosmology.

A. Spreading of states

Although the spreads are not constant for solutions to our system, ratios such as $(\Delta p)/p$ are nearly constant in each pre- and post-bounce branch. Nevertheless, this ratio can, and in general does, take different values in both branches. Dynamical coherent states which exactly saturate the uncertainty relations and have spreads symmetric around the bounce point do exist but are not generic. Nevertheless, they are distinguished in a certain sense and can thus be seen as analogs of the harmonic oscillator ground state (although no unique symmetric state exists). Indeed, for such states the scale of fluctuations, $(\Delta p)/p \approx \sqrt{\hbar/H}$, is determined more sharply than without the symmetry assumption.

Dynamical coherent states for the loop quantization are not Gaussians which turn out to have different and rather special properties. Nevertheless, Gaussian states can also be analyzed straightforwardly in this setting, with results being ultimately in agreement with recent numerical investigations. Interestingly, *unsqueezed* Gaussian states do have identical spread before and after the bounce. They cannot be coherent but may well serve as a special version of semiclassical states. This illustrates how differently coherent states in a new system can behave from those well-known for the harmonic oscillator. It also shows that a dynamical analysis of coherent and semiclassical states is always necessary even to select suitable initial states to be evolved. While dynamical coherent states are difficult to describe in most systems, solvable models make this possible which is now also available for cosmological systems.

As demonstrated, the techniques of [6] reviewed in Sec. II provide an efficient way to derive coherent state properties. Moreover, they allow one to see how properties can change under perturbations away from the solvable model, such as by including a scalar potential. Although this has not been dealt with in the present paper, our detailed analysis of dynamical coherent states of the solvable system provides crucial information for the zeroth order of such a perturbation theory.

B. Applications to cosmology

There is only scant information on properties of the present state of our universe other than that it is, to a high degree, semiclassical. But this does not tell us which semiclassical or even coherent state describes it best. From the harmonic oscillator or free quantum field theories one is used to unsqueezed Gaussians to represent the vacuum state. But the form of such states depends on the system being looked at, and quantum cosmology is not close to either the harmonic oscillator or free quantum field theory. Moreover, there is no obvious vacuum state for quantum cosmology, and even if a Hamiltonian should allow a ground state, it is unlikely to describe a universe able to expand to large volume. A dedicated analysis as done here shows which coherent states are available and what their generic properties are. These properties must be taken into account for robust statements about quantum cosmological systems.

For instance, we have seen that squeezing must be allowed for generic states, which crucially changes properties such as the symmetry of spreads before and after a bounce. In fact, squeezing of semiclassical states often plays a large role for decoherence or the transition to classical behavior (e.g. in the context of inflation [34–36]). Thus, a semiclassical state at large volume of a cosmological model should indeed be assumed to be highly squeezed. Lacking additional input, robust cosmological conclusions can be drawn only with reference to *generic* coherent states. Then, no strong restrictions on fluctuations of a state before the bounce are justified. For all we know, it could have been coherent but with large quantum fluctuations.

C. Effective equations and the possibility not to bounce

Although we have proven that the solvable quantum system is *exactly* described by the effective Hamiltonian

$$H = \langle \hat{H} \rangle = \frac{1}{2i}(J - \bar{J}) = p \sin c$$

(determining the same equations of motion for p and J as $\hat{H}$ does for $\langle \hat{p} \rangle$ and $\langle \hat{J} \rangle$) it is possible, depending on the initial state, that the system does not bounce but reaches $p = 0$ in finite proper time. This looks contradictory at first sight since such an effective Hamiltonian implies the effective constraint equation

$$H^2 = p^2 \sin^2 c \propto p_\phi^2.$$

With p_ϕ constant and $|\sin c| \leq 1$ there must be a nonzero lower bound for p, the bounce scale.

However, while the effective Hamiltonian of our linear system does not receive corrections from quantum backreaction of fluctuations and other quantum variables, the reality condition $\hat{J}\hat{J}^\dagger = \hat{p}^2$ is nonlinear. Classically, this condition implies that c is real and thus $|\sin c| \leq 1$. But the reality condition (33) now receives quantum corrections of second order in quantum variables which, for suitable states, can remove the bounce. A state can then enter the ‘‘classically forbidden region’’ where $|\sin c| > 1$ while still respecting the quantum reality condition.

As shown explicitly, zero volume is reached when the parameter c_1 is negative and large. While this can be achieved respecting the positivity condition $\Delta H \ll H$, such a state would never be semiclassical. Thus, *any state which is semiclassical at one time will give rise to a bounce.* These are all relevant states since a boundary condition for modeling our universe, however distantly, is always that there is at least one large volume regime in which the state is semiclassical. But the possibility of states which do not bounce demonstrates the nontriviality of the result. Simply replacing $\dot{a}^2$ in the Friedmann equation by a bounded function is not enough; any such replacement would have to be followed up by a coherent state analysis which is much more nontrivial than an analysis of the resulting effective Friedmann equation obtained by the naive replacement. Loop quantum cosmology with a free scalar passes this more stringent test and thus provides the first example in loop quantum gravity where complete effective equations have been computed.

We end by repeating that any physical statements derived from a single model have to be confirmed by a perturbation analysis around the model. This is feasible in our case, as it is for perturbations around any solvable model, but still requires detailed work which is now in progress. Only such an analysis could justify the transfer of results from single models to our own universe. It may well be that this removes the bounce through backreaction of quantum variables $G^{a,n}$ on the expectation values. In particular, it is then conceivable that a state starts out perfectly semiclassically at large volume where its expected volume collapses, evolves for a long time to small volume and all along picks up corrections from quantum backreaction. Since also quantum variables evolve, it cannot be ruled out without further analysis that the analog of c_1 does become negative and large close to the would-be bounce. If this happens, the bounce is avoided for the self-interacting state even if it starts out semiclassically by all possible conditions one could pose. This is only one possibility out of many which can only be ruled out by performing a comprehensive perturbation analysis for which the results of this paper present the zeroth order basis.

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APPENDIX A: UNCERTAINTY RELATIONS IN QUANTUM MECHANICS

1. General derivation

Starting from the Schwarz inequality

$$\langle\psi_1|\psi_1\rangle\langle\psi_2|\psi_2\rangle \geq |\langle\psi_1|\psi_2\rangle|^2$$

applied to $|\psi_1\rangle := \widehat{\Delta A}|\psi\rangle$ and $|\psi_2\rangle := \widehat{\Delta B}|\psi\rangle$ with $\widehat{\Delta A} := \hat{A} - \langle\hat{A}\rangle$ one obtains

$$\langle(\widehat{\Delta A})^2\rangle\langle(\widehat{\Delta B})^2\rangle \geq |\langle\widehat{\Delta A}\,\widehat{\Delta B}\rangle|^2. \quad \text{(A1)}$$

(Here, $\widehat{\Delta A}$ is not a linear operator due to the dependence on the state in $\langle\hat{A}\rangle$, but $|\psi_1\rangle$ is well-defined as a state obtained from $|\psi\rangle$.) Writing

$$\widehat{\Delta A}\,\widehat{\Delta B} = \frac{1}{2}(\widehat{\Delta A}\,\widehat{\Delta B} + \widehat{\Delta B}\,\widehat{\Delta A}) + i\frac{1}{2i}[\widehat{\Delta A}, \widehat{\Delta B}]$$

and

$$\tfrac{1}{2}\langle\widehat{\Delta A}\,\widehat{\Delta B} + \widehat{\Delta B}\,\widehat{\Delta A}\rangle = \tfrac{1}{2}\langle\hat{A}\,\hat{B} + \hat{B}\,\hat{A}\rangle - AB$$

$$[\widehat{\Delta A}, \widehat{\Delta B}] = [\hat{A}, \hat{B}]$$

we have

$$|\langle\widehat{\Delta A}\,\widehat{\Delta B}\rangle|^2 = \tfrac{1}{4}(\langle\hat{A}\,\hat{B} + \hat{B}\,\hat{A}\rangle^2 - 2\langle\hat{A}\rangle\langle\hat{B}\rangle)^2 + \tfrac{1}{4}\langle -i[\hat{A}, \hat{B}]\rangle^2,$$

where we used self-adjointness of the operators to compute the absolute square of the complex number $\langle\widehat{\Delta A}\,\widehat{\Delta B}\rangle$. In terms of quantum variables, we thus have the general form

$$G^{AA}G^{BB} - (G^{AB})^2 \geq \tfrac{1}{4}\langle -i[\hat{A}, \hat{B}]\rangle^2 \quad \text{(A2)}$$

of uncertainty relations whenever $\hat{A}$ and $\hat{B}$ are self-adjoint.

2. Saturation for Gaussian states

A general Gaussian state has the form $\psi(q) = \exp(-z_1q^2 + z_2q + z_3)$ with three complex numbers $z_1 = \alpha_1 + i\beta_1$, $z_2 = \alpha_2 + i\beta_2$ and $z_3 = \alpha_3 + i\beta_3$ such that $\mathrm{Re}z_1 > 0$. While β_3 can be dropped, determining only a phase factor, $\alpha_3 = \frac{1}{4}\log\frac{2\alpha_1}{\pi} - \frac{\alpha_2^2}{4\alpha_1}$ is fixed by normalization. The remaining parameters determine expectation values as well as fluctuations and correlations of q and p:

$$\langle\hat{q}\rangle = \frac{\alpha_2}{2\alpha_1}, \quad \text{(A3)}$$

$$\langle\hat{p}\rangle = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\alpha_1}\hbar, \quad \text{(A4)}$$

$$G^{qq} = \frac{1}{4\alpha_1}, \quad \text{(A5)}$$

$$G^{pp} = \alpha_1\hbar^2 + \frac{\beta_1^2}{\alpha_1}\hbar^2, \quad \text{(A6)}$$

$$G^{qp} = -\frac{\beta_1}{2\alpha_1}\hbar. \quad \text{(A7)}$$

With these values, any Gaussian state saturates the uncertainty relation,

$$G^{qq}G^{pp} - (G^{qp})^2 = \tfrac{1}{4}\hbar^2. \quad \text{(A8)}$$

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