PHYSICAL REVIEW D **75,** 123004 (2007)

Testing gravity at the second post-Newtonian level through gravitational deflection of massive particles

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(Received 19 October 2006; revised manuscript received 11 April 2007; published 14 June 2007)

Expression for second post-Newtonian level gravitational deflection angle of massive particles is obtained in a model independent framework. Comparison of theoretical values with the observationally constructed values of post-Newtonian parameters for massive particles offers the future possibility of testing at that level competing gravitational theories as well as the equivalence principle. Advantage of studying gravitational deflection of massive particles over that of massless particles in testing gravity is discussed.

DOI: 10.1103/PhysRevD.75.123004 PACS numbers: 95.30.Sf, 98.62.Sb

I. INTRODUCTION

A consequence of general relativity (GR) is that light rays are deflected by gravity. Historically, observations of this aspect of gravity provided one of the early proofs in favor of GR (the explanation of perihelion advance was the earliest.) The effect is nowadays routinely used as a tool to study various features of the Universe, such as viewing fainter or distant sources, estimating the masses of galaxies etc. [1].

Like photons, particles having masses are also deflected by gravity. The general relativistic corrections in the equation of motion of a massive test particle moving in bound orbits have been studied with great accuracy in the literature [2]. These studies have great relevance in comparing theory with observations of gravitational waves from compact binary systems. However, study of orbits of *unbound* massive particles in gravitational field has not received sufficient attention as of now. The main reason could be that the observational aspect of unbounded massive particles was not very practical: there was no known astrophysical source of free point particles that can be detected easily with good angular precision. The situation seems to have improved somewhat. Recent theoretical studies [3] favor the existence of local astrophysical sources of relativistic neutral particles like neutrons and neutrinos with observable fluxes. Besides, high energy neutrons are produced during solar flares [4]. Moreover, with the advent of new technology new experiments have been proposed [5], primarily to study gravitational deflection of light with high precision, in which laser interferometry will be employed between two spacecrafts/space stations whose line of sight pass close to the sun. Hence there might be a possibility that, in the future, neutron or some other neutral particle may be used in a similar experiment instead of photon, thus providing an opportunity for studying gravitational deflection of massive particles. Henceforth, we use the abbreviation for post-Newtonian as PN such that first-PN effect is of the order of $(1/\rho)$, second-PN effect is of the order of $(1/\rho^2)$ and so on.

The expected angular precision of the planned astrometric missions using optical interferometry is at the level of microarcseconds (μ arcsec) and hence these experiments would measure the effects of gravity on light at the second-PN order (c^{-4}). Though measurements with massive particles at the level of microarcsecond accuracy is way beyond the present technical capability, it can still be cautiously hoped that astrometric missions in the distant future using massive particle interferometry would have angular precisions close to that to be obtained using laser (optical) interferometry. Whatever be the technical scenario, a study of theoretical aspects of gravitational deflection of massive particles at the second-PN approximation is useful in its own right. (To our knowledge, the deflection angle for massive particles has been theoretically estimated in the literature with an accuracy of only first-PN order [6] so far.)

In the present article, we shall formulate the second-PN contribution to the gravitational deflection of massive particles in a *model independent* way but with a special emphasis on the sun as gravitating object. The corresponding PN parameters for light deflection then follow as a corollary. The key idea here is to exploit an advantage offered by the kinematics of massive particles over that of massless ones: The velocity of the probing massive particle can be altered. The investigation (i) helps us circumvent some difficulties related with photon deflection in the second-PN order, (ii) allows us to "construct" the coordinate solar radius from the particle deflection data itself and moreover, (iii) offers a possible further test of the equivalence principle. These issues are discussed at the end.

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II. GRAVITATIONAL DEFLECTION OF MASSIVE PARTICLES AT SECOND POST-NEWTONIAN ORDER

We consider the general static and spherically symmetric spacetime in isotropic coordinate which is given by (we use geometrized units i.e. $G = 1$, $c = 1$)

$$ds^2 = -B(\rho)dt^2 + A(\rho)(d\rho^2 + \rho^2 d\theta^2 + \rho^2 sin^2\theta d\phi^2). \quad (1)$$

Restricting to orbits in the equatorial plane ($\theta = \pi/2$), the expression for the deflection angle for particles moving with a velocity V as measured by an asymptotic rest observer can be written as [7]

$$\alpha(\rho_o) = I(\rho_o) - \pi \quad (2)$$

with (see Appendix)

$$I(\rho_o) = 2\int_{\rho_o}^{\infty} \frac{d\rho}{\rho^2} A^{-(1/2)}(\rho)\left[\frac{1}{J^2}\{B^{-1}(\rho) - E\} - \frac{1}{\rho^2 A}\right]^{-(1/2)}, \quad (3)$$

where ρ_o being the distance of the closest approach,

$$J = \rho_o[A(\rho_o)\{B^{-1}(\rho_o) - E\}]^{1/2} \quad (4)$$

and

$$E = 1 - V^2. \quad (5)$$

The PN formalism in some orders [7,8] is usually employed to describe the gravitational theories in the solar system and also to compare predictions of GR with the results predicted by an alternative metric theory of gravity. This method actually is an approximation for obtaining the dynamics of a particle (in a weak gravitational field under the influence of a slowly moving gravitational source) to one higher order in $\frac{M}{\rho}$ (M is the mass of the static gravitating object) than given by the Newtonian mechanics. Following the PN expansion method, we assume the metric tensor is equal to the Minkowski tensor $\eta_{\mu\nu}$ plus corrections in the form of expansions in powers of $\frac{M}{\rho}$ and considering up to the second-PN correction terms, we have

$$B(\rho) = 1 - 2\frac{M}{\rho} + 2\beta_i \frac{M^2}{\rho^2} - \frac{3}{2}\epsilon_i \frac{M^3}{\rho^3}, \quad (6)$$

$$A(\rho) = 1 + 2\gamma_i \frac{M}{\rho} + \frac{3}{2}\delta_i \frac{M^2}{\rho^2}. \quad (7)$$

β_i, γ_i are the PN parameters (also known as the Eddington parameters), δ_i and ϵ_i can be considered as the second-PN parameters, i stands for either γ or m denoting photons or massive particles, respectively. Several of these parameters are different for different theories. In GR, all of them are equal to 1 as can be readily checked by expanding the Schwarzschild metric.

We should note that the metric coefficients above are independent of any specific model; they result solely from the assumption of central symmetry. Starting with the expansion (6) and (7) *per se,* the expression for the angle of deflection for unbound particles up to the second-PN order follows from Eq. (3) and when $\frac{2M}{\rho} << V^2$, and it works out to

$$\alpha_m = a_m \frac{M}{\rho_o} + b_m\left(\frac{M}{\rho_o}\right)^2, \quad (8)$$

where

$$a_m = 2\left(\gamma_m + \frac{1}{V^2}\right), \quad (9)$$

$$b_m = \frac{3\delta_m \pi}{4} + (2 + 2\gamma_m - \beta_m)\frac{\pi}{V^2} - 2\left(\gamma_m + \frac{1}{V^2}\right)^2. \quad (10)$$

The above expressions are also valid for massless particles (α_γ, a_γ, b_γ) as may be seen under the substitution $V = 1$. Clearly the deflection angle would be larger for particles in comparison to that of photons. Our calculation shows that the term representing the second order effect (b_m) contains only the three parameters β_m, γ_m, δ_m and does not contain ϵ_m, a cubic order contribution. This implies that calculation of the deflection of *unbound* particle orbits (including photons) by gravity to any given order needs only the knowledge of every term to that order in the expansions. In other words, to second-PN order, one needs to consider both in g_{oo} and g_{ij} terms only up to $\frac{M^2}{\rho^2}$. Similarly, to third-PN order, which is not our interest here, we would need expansions of both the metric components up to order $\frac{M^3}{\rho^3}$ and so on. This is in contrast to the case of planetary dynamics (*bound* orbits) where the calculation typically requires knowledge of g_{oo} more accurately than g_{ij} (For instance, to calculate the planetary precession to the order of M, one expands g_{oo} up to $\frac{2\beta_m M^2}{\rho^2}$ while g_{ij} is expanded up to only $\frac{2\gamma_m M}{\rho}$; for next order accuracy, one would need to consider the complete expansion as given in Eqs. (6) and (7) above so that the parameters δ_m, ϵ_m become important in this case.) The deflection angle α_m for the Schwarzschild spacetime can be obtained by taking $\beta_m = \gamma_m = \delta_m = 1$.

The deflection angle also can be expressed in terms of coordinate independent variables, such as the impact parameter b which is the perpendicular distance from the center of the gravitating object to the tangent to the geodesic at the closest approach. In that case, ρ has to be replaced by b in Eq. (8), Eq. (9) would remain unaltered but Eq. (10) would change to

$$b_m = \frac{3\delta_m \pi}{4} + (2 + 2\gamma_m - \beta_m)\frac{\pi}{V^2} + 2\left(\gamma_m + \frac{1}{V^2}\right)\left(1 - \frac{1}{V^2}\right). \quad (11)$$

Since impact parameter is the ratio of the angular momentum and energy of the particle as measured by an observer at rest far from the gravitating object, it is a formally measurable quantity but is not very suitable for practical measurements [9].

III. OTHER SIGNIFICANT EFFECTS

In the present work, the mass distribution of the gravitating object is assumed to be mainly spherically symmetric; any deviation from such symmetry would produce their effects. The effect of quadrupole moment of the mass distribution on the deflection angle is proportional to $\frac{J_Q M R^2}{\rho_o^3}$, where R is the average radius of sun. Thus, even a small quadrupole moment parameter J_Q could produce significant contribution to deflection. However, the effect is limited largely to the first-PN order (~ 0.1 μ arcsec) while in the second-PN order the effect is too small ($\sim 10^{-7}$–10^{-8} μ arcsec). If the gravitating object also has angular momentum, its effect on the deflection angle contributes to the second-PN order but it can be separated out. All these are discussed below.

A. Effect of quadrupole moment of the mass distribution

Theoretical value of solar quadrupole moment J_Q, though it depends strongly on solar model used, is very small, of the order of 10^{-7} [5,8]. Since our study is aimed at sun as the gravitating object we have ignored higher order terms involving J_Q. Thus due to the quadrupole moment of the mass distribution the effective mass parameter becomes $M_{\mathrm{eff}} = M\left[1 + \frac{J_Q R^2}{2\rho^2}(3\cos^2\theta - 1)\right]$ which leads to the following corrections in the components of the metric tensors [5,8,10]:

$$\delta g_{oo}(\rho) = J_Q \frac{MR^2}{\rho^3}(3cos^2\theta - 1) \tag{12}$$

and

$$\delta g_{jk}(\rho) = -\delta_{jk}\gamma_i J_Q \frac{MR^2}{\rho^3}(3cos^2\theta - 1), \tag{13}$$

where R is the average radius of the mass distribution and θ is the angle between radius vector and the z-axis and hence in the equatorial plane $\theta = \pi/2$. In the equatorial plane, the deflection caused by the quadrupole moment calculates to

$$\alpha_{\mathrm{QM}} = \frac{2J_Q MR^2}{\rho_o^3}\left(\frac{\gamma_m}{3} + \frac{1}{V^2}\right). \tag{14}$$

Assuming $R \sim \rho_o$ at the closest approach to the sun and taking $V = 0.75$, $\beta_m = \gamma_m = \delta_m \sim 1$, for sun $\frac{M_\odot}{R_\odot} = 2.12 \times 10^{-6}$, this first-PN quadrupole term $\alpha_{\mathrm{QM}} \sim 0.1$ μarcsec. It is roughly 7 orders of magnitude less than the first-PN deflection $a_m \frac{M}{\rho_o} \sim 0.8$ sec and is more than 1 order of magnitude less than the second-PN contribution $b_m(\frac{M}{\rho_o})^2 \sim 3.4$ μarcsec. We have not displayed the next higher order quadrupole terms involving J_Q^2 and second order terms in $1/\rho^2$ containing J_Q here because they have magnitude in the range 10^{-7} to 10^{-8} μarcsec, too small to be of any practical significance. We can justifiably ignore these second-PN quadrupole contributions. The quadrupole contribution to the deflection of light is given in Ref. [11].

B. Effect of rotation

The angular momentum of the gravitating object is assumed small as in the case of sun. The resulting leading term of the relevant metric tensor is

$$g_{oi} = \frac{4Ma}{\rho}, \tag{15}$$

where a is the angular momentum per unit mass of the object. The contribution of the rotation to the deflection angle is then given by [6]

$$\alpha_{\mathrm{rot}} = \frac{4MaV}{\rho_o^2}. \tag{16}$$

The value of a can be positive or negative depending on the direction of rotation. When the angular momentum of the gravitating object is antiparallel with the direction of the incoming particle, a is positive and hence rotation causes larger deflection whereas for parallel angular momentum, a is negative and the deflection angle will be less. Thus the rotational effect can be easily separated out from other contributions by studying the deflection of particles at two opposite sides of the gravitating object.

The gravitational deflection angle of light with an accuracy up to second-PN order readily follows from Eqs. (8)–(10), (14), and (16) using $V = 1$.

IV. EXTRACTING POST-PN PARAMETERS FROM MEASUREMENTS

To extract post-PN parameters from gravitational deflection of massive particles, one first has to measure deflection angles α_m for different values of V of the probing massive particle grazing the sun. Then, a least square fitting of the recorded deflection angle data with Eq. (8) through Eqs. (9) and (10) will result in the PN values β_m, γ_m, δ_m *together* with the solar radius ρ_o in isotropic coordinate. If GR is a correct theory to second-PN order, then the best fit will give $\beta_m = \gamma_m = \delta_m = 1$, and if the weak equivalence principle holds then $\beta_m = \gamma_m = \delta_m = \beta_\gamma = \gamma_\gamma = \delta_\gamma = 1$. In the case of light, there is only one probe velocity available, namely, $V = c = 1$, and there is no option to fit β_γ, γ_γ, δ_γ separately. One just proceeds to check whether the measurement is consistent with GR prediction obtained by simply assumimg all the PN parameters as unity. But the difficulty with this procedure is that, for light grazing the sun, one needs the other parameter, the solar radius, whose value to the required level of accuracy is not available, nor can it be consistently obtained, together with other PN parameters, from the observed data itself. These points are illustrated later, in Sec. V.

It should be mentioned that the entire calculation of deflection angle could be performed in any other coordinates, standard or harmonic and so on. Expression for deflection angle in any other coordinates also can be obtained directly from Eqs. (8) to (10) just by applying appropriate transformations. The observable deflection angle α_m is of course coordinate choice independent. Thus change to another coordinate system will result merely in the corresponding functional changes in the expressions for the coefficients a_m, b_m, respectively, and in the value of the coordinate radius of the sun (say, to r_o if we change to standard system) [12]. Note that once used in a certain coordinate system, the PN values β_m, γ_m, δ_m are to remain fixed for a given theory of gravitation (GR or Brans-Dicke theory, etc.) in any other coordinate system.

V. DISCUSSION

The study of gravitational deflection of massive particles is important for several reasons which are discussed below.

First of all, observations of gravitational deflection of massive particles with μ arcsec precision could probe the gravitational theories at the second-PN level without any difficulty as explained at the end of Sec. II. The second order prediction for gravitational deflection of light as evolved from different studies is plagued by the following factors: When the light ray just grazes the limb of the sun, one needs a consistent value of coordinate solar radius to be put into the expression for deflection α_γ calculated in different coordinate systems (with fixed GR values of unity for PN parameters). Now there is a long known value for the solar radius $R_\odot$, ($R_\odot = 6.961 \times 10^8$ km [13]) measured under *Euclidean* approximation! But, even in the expression for α_γ in the Schwarzschild isotropic system, the radial coordinate ρ_o is erroneously identified with the same $R_\odot$. Then it gives a second-PN contribution of ~ 3.5 μ arcsec to deflection angle in GR [14]. If one uses the standard Schwarzschild system instead, the second-PN contribution to the deflection angle of light in GR following from Eqs. (8)–(10) would be $[\frac{15\pi}{16} - 1]\frac{4M^2}{r_o^2}$ which is numerically about 7 μarcsec, provided the standard coordinate distance r_o of closest approach is identified again with the same $R_\odot$. The deflection angle can also be expressed in terms of coordinate independent variables, such as the impact parameter b. In that case, the second-PN contribution to deflection angle in GR becomes $\frac{15\pi}{16}\frac{4M^2}{b^2}$, and when at closest approach b is identified with $R_\odot$, the magnitude of second-PN deflection angle is ~ 11 μarcsec [15]. Thus there exists difficulties about the interpretation of the prediction of GR (or in fact of any viable gravitational theory) at the second-PN order. The single fixed value for $R_\odot$ is used in all calculations because there cannot be any way to get consistent values for the solar radius from the higher order light deflection data due to its unique trajectory grazing the sun.

There is a more fundamental reason for these anomalies. It is that the measurements of solar radius usually employ Euclidean geometry as an approximation [6] whereas the angle of gravitational deflection or other GR effects are principally based on the consideration of curved spacetime. But, comparing points in two different geometries, i.e., in curved spacetime and flat spacetime, is totally meaningless [16]. The Euclidean approximation works tolerably well only up to the first order, that is, in weak field gravity caused by a source like the sun. The magnitude of the second order contribution is, however, of the same order as the error that arises due to such an approximation. Hence the numerical value of gravitational deflection angle of light cannot be unambiguously predicted at the level of second-PN order within the theoretical scheme currently in practice. Since the deflection angle for massive particles depends also on the *velocity* of the particle which gives us an extra freedom, the stated ambiguity can be easily avoided by measuring deflection angles for two or more velocities of the probing massive particle.

In GR, a coordinate length like ρ_o is not directly measurable, it can only can be indirectly constructed from the values of actual measurements. The PN parameters and also the otherwise unknown coordinate solar radius ρ_o (or equivalently, r_o in standard Schwarzschild coordinates) can be constructed through least square fitting with the measured deflection angles α_m and probing velocities V using the Eqs. (8)–(10). The idea is that the values of coordinates, ρ_o and r_o, which refer to the same radial point, should be treated more like other PN parameters (β_m, γ_m, δ_m) due to the fact that the "flat geometry" spacetime points cannot be algebraically identified in a curved spacetime [16]. Technically, however, the flat radial distances can be constructed by using metric gravity itself (Eddington expansion) in terms of a large set of unknown PN parameters (β_m, γ_m, δ_m) including ρ_o by fitting them with the observed data [7]. This method has been adopted, for example, by Shapiro and his group in the radar echo delay observations [7,17]. The resulting parameter values can then be compared with the theoretical predictions of deflection in GR as well as in other competing theories (like Brans-Dicke theory) in the second-PN order involving both massive and massless particles.

The study of gravitational deflection of massive particles is also important in the context of testing the weak equivalence principle which is one of the fundamental postulates of general relativity. The principle states that the trajectory of a freely falling object is independent of its internal structure and composition. In other words all particles are coupled with spacetime geometry universally. The principle has been tested with great accuracy through different experiments, notable among them are the Eötvös type experiments [10] where comparison of gravitational and inertial masses of objects are made by measuring their acceleration in a known gravitational field. For massless particles like photons, however, such measurements obviously cannot be performed. Instead, in such situation the principle is tested by examining whether the gravitational (second-PN) coupling parameter γ is universal for all particles, massive or massless. On the basis of supernova 1987 neutrino and optical data [18], a limit of $|\gamma_\gamma - \gamma_m| \leq 3.4 \times 10^{-3}$ has actually been found [19]. However, the mass of a neutrino m_{ν_e} is very small (if not zero), the present upper limit being $m_{\nu_e} \leq 3$ eV. Hence, a more conclusive experiment would be to examine whether the gravitational couplings for photon and massive particles (other than neutrinos) are the same or not. The observational value of $|\gamma_\gamma - \gamma_m|$ should provide a direct answer as to the degree of validity of the principle in question.

VI. REMARKS ON PARTICLE DEFLECTION EXPERIMENT

The main concern, which is still far from resolved, is whether realistic experiments for observing gravitational deflection of massive particles can be devised or not. Here, we only speculate on some possibilities. The most important requisite in this context is to generate a beam of suitable test particles. Charged particles like protons or electrons have to be excluded as test particles because they suffer electromagnetic interactions by the interplanetary magnetic field. Among neutral particles, neutrinos are unlikely to serve the purpose as their speeds are almost, if not exactly, the same as the speed of light. Thus, neutrons seem to be the only feasible candidate. They are known to be produced during solar flares but they can at best be used to study the gravitational deflection by an intermediate planet. If astrophysical sources of neutrons other than the sun are detected in future experiments, the problem of searching the test particle beam would be resolved automatically. Otherwise, one might hope to generate the beam only artificially. However, since neutrons are unstable with a mean lifetime of 886 sec, only neutrons with a minimum speed of 0.75 c can be used as test particles so that they do not decay during the travel from one micro-spacecraft to another. Though in (man-made) accelerator experiments (at earth) neutrons can be accelerated to such speeds, it seems improbable at the present stage of technology that neutrons can be accelerated to such high energies from a micro-spacecraft. This is a challenge for the future.

Alternatively, stable massive objects, such as a bullet, can also be used as test particles but they must have a minimum speed of $\sim 6 \times 10^7$ cm sec^{-1} so that its total energy remains positive throughout the path (from one micro-spacecraft/earth to another spacecraft) and would not be captured by solar gravity. The fastest man-made object (Helios 2 solar probe) has a speed of about 7×10^6 cm sec^{-1}.

Any meaningful information on second order effects can be extracted from measurements of gravitational deflection of massive particles at the level of second order accuracy only when the uncertainty of the first-PN contribution [Eq. (9)] is smaller than the second-PN contributions. Since radius is considered as free parameter in the proposed scheme, the uncertainty in the first-PN contribution of deflection angle is entirely due to uncertainty in the knowledge of the speed of the massive test particles. For solar gravity the ratio of second order to first order contributions of deflection angle is around 10^{-6}. Thus for a meaningful second order measurement of particle deflection, the relative uncertainty of first order deflection angle $\frac{\Delta\alpha_1}{\alpha_1}$ must be less than 10^{-6} which in turn requires $\frac{\Delta v}{v} < 10^{-6}$. This should not be a major problem as a comparable level of accuracy in measurement of particle velocity has already been achieved in different experiments [20].

Particle detectors with directional resolution at the level of μ arcsec accuracy is certainly beyond the present technical capability. The maximum directional accuracy of operating particle telescopes is limited to around 100 mili-arcsec [21]. Configuring neutron interferometer instrument $S18$ as a Bonse-Hart small angle scattering camera, an angular resolution of few($\sim$10)mili − arcsec has been achieved [22]. Maximum directional accuracy achieved so far using electromagnetic radiation based telescopes is also of the same order. However, currently planned astrometric missions employing optical interferometry have set their goal to achieve a directional accuracy at the level of μ arcsec. Thus, hopefully, achieving the required level of μ arcsec accuracy in particle detection might not be too far away.

VII. CONCLUSION

The subtlety of the observational meaning of coordinate distance is not unknown to the physics community [7]. One could live with the ambiguous predictions in higher order light deflection had the stakes been not high. It is imperative to test at a higher order level which theory of gravitation, GR or other theories, fits better. The far-reaching implications of the answer do not require any elaboration. What we analyzed above is a possible theoretical scheme for testing gravity at the second order.

For the scheme to work in practice, the level of experimental accuracy seems extremely demanding. But one should recall that when the second order deflection of light was first calculated theoretically in the early eighties [12], experimental verification of the result was completely beyond the then technical capability. Now, after 25 years, technology has been developed to the stage that measuring deflection angle due to solar gravity at the second-PN order appears feasible. However, as discussed in Sec. V, the numerical value of gravitational deflection angle of light cannot be uniquely predicted at the level of second-PN order within the existing theoretical scheme. As a result the proposed experiments with light are unlikely to provide any fruitful test of GR at that order. This is due to a question of principle related to the lack of a consistent parameter fitting procedure with light and not a question of attainable accuracy in experimental measurements. It has been shown that such a situation can be circumvented by using a kinematical freedom available with a massive test particle, viz., its velocity that can be altered at will unlike in the case with light. One can then measure the deflection angles for two or more velocities of the probing massive particle. However, it is understood that such measurements are completely beyond the present technical feasibility.

ACKNOWLEDGMENTS

The authors would like to thank an anonymous referee for insightful comments and suggestions.

APPENDIX

The standard equations for a geodesic, namely,

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \tag{A1}$$

for the general metric (1) become

$$\frac{d^2\rho}{ds^2} + \frac{A'}{2A}\left(\frac{d\rho}{ds}\right)^2 - \rho\left(1 + \frac{\rho A'}{2A}\right) = 0, \tag{A2}$$

$$\frac{d^2\theta}{ds^2} + \left(\frac{2}{\rho} + \frac{A'}{A}\right)\frac{d\rho}{ds}\frac{d\theta}{ds} - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0, \tag{A3}$$

$$\frac{d^2\phi}{ds^2} + \left(\frac{2}{\rho} + \frac{A'}{A}\right)\frac{d\rho}{ds}\frac{d\theta}{ds} + 2\cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} = 0, \tag{A4}$$

$$\frac{d^2t}{ds^2} + \frac{B'}{B}\frac{d\rho}{ds}\frac{dt}{ds} = 0 \tag{A5}$$

(primes denoting differentiation with respect to ρ). If we choose $\theta = \pi/2$ and $d\theta/ds = 0$ initially, Eq. (A3) warrants that they would remain the same always. Thus normalizing time coordinate suitably, one obtains for orbits in the equatorial plane from Eq. (A5)

$$\frac{dt}{ds} = B^{-1}. \tag{A6}$$

Integrating Eq. (A4)

$$\rho^2\frac{d\phi}{ds} = A^{-1}J^2, \tag{A7}$$

where J is a constant of integration. From Eqs. (A2), (A6), and (A7), one finally obtains

$$\frac{1}{A\rho^4}\left(\frac{d\rho}{d\phi}\right)^2 + \frac{1}{A\rho^2} - \frac{1}{J^2}\left(\frac{1}{B} - E\right) = 0, \tag{A8}$$

which leads to Eq. (3). J can be conveniently expressed in terms of distance at closest approach. At the point of closest approach, $d\rho/d\phi$ vanishes. Using this in Eq. (A8), one recovers Eq. (4).

[1] P. Schneider, J. Ehlers, and E. E. Falco, *Gravitational Lenses* (Springer-Verlag, Berlin, 1992).
[2] T. Ohta, H. Okamura, T. Kimura, and K. Hiida, Prog. Theor. Phys. **50**, 492 (1973); **51**, 1220 (1974); **51**, 1598 (1974); T. Damour, P. Jaranowski, and G. Schāfer, Phys. Rev. D **62**, 021501 (2000); L. Blanchet, G. Faye, and S. Nissanke, Phys. Rev. D **72**, 044024 (2005), and references therein.
[3] W. Bednarek, Mon. Not. R. Astron. Soc. **331**, 483 (2002); J. N. Capdevielle and Y. Muraki, Astropart. Phys. **11**, 335 (1999).
[4] Y. Muraki *et al.*, Astrophys. J. Lett. **400**, L75 (1992).
[5] S. G. Turyshev, M. Shao, and K. L. Nordtvedt, Jr., Nucl. Phys. B, Proc. Suppl. **134**, 171 (2004); , Astron. Nachr. **325**, 267 (2004); Space Interferometry Mission, Project Website at: http://planetquest.jpl.nasa.gov/SIM/sim_index.html; Galactic Census project GAIA, Project Website at: http://astro.estec.csa.nl/GAIA.
[6] P. M. Alsing, Am. J. Phys. **66**, 779 (1998).
[7] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[8] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[9] J. Bodemar and C. M. Will, Am. J. Phys. **71**, 770 (2003).
[10] C. M. Will, Living Rev. Relativity **9**, 3 (2006).
[11] G. W. Richter and R. A. Matzner, Phys. Rev. D **26**, 1219 (1982).
[12] Recall that the mass of a black hole can always be independently measured either by strong field gravitational lensing or by other methods. Once we know the mass M, we can *deduce* its coordinate horizon radius, e.g., $\rho_{\rm hor} = M/2$ or $r_{\rm hor} = 2M$. On the other hand, the invariant proper horizon radius $l_{\rm hor}$ of a Schwarzschild black hole of mass M can be obtained by integrating the proper radial distance in the interior metric of a homogeneous star, viz., $l = \int_0^R (1 - \frac{2Mr^2}{R^3})^{-(1/2)} dr$ where R is the coordinate radius of a star. If we formally put the extreme value for the radius $R = 2M$ and integrate, we get $l_{\rm hor} = \pi M$. But respecting the equilibrium condition for a star, one should take $R > 2.25M$. For example, with $R = 2.26M$, the proper radius l becomes $2.94M$. There is no direct way to measure this proper radius, neither does it appear as a

PN parameter. In practice, for an uncollapsed object like the sun, the radius is identified, *strictly technically*, with the fitted values of the observed parameters ρ_o or r_o depending on the employed coordinate system.

[13] W-M Yao *et al.*, J. Phys. G **33**, 1 (2006).

[14] E. Fischbach and B. S. Freeman, Phys. Rev. D **22**, 2950 (1980), See also Ref. [11].

[15] R. Epstein and I. I. Shapiro, Phys. Rev. D **22**, 2947 (1980).

[16] S. Ichinose and Y. Kaminaga, Phys. Rev. D **40**, 3997 (1989).

[17] I. I. Shapiro, G. H. Pettengill, M. E. Ash, M. L. Stone, W. B. Smith, R. P. Ingalls, and R. A. Brockelman, Phys. Rev. Lett. **20**, 1265 (1968); I. I. Shapiro, M. E. Ash, R. P. Ingalls, W. B. Smith, D. B. Campbell, R. B. Dyce, R. F. Jurgens, and G. H. Pettengill, Phys. Rev. Lett. **26**, 1132 (1971).

[18] K. Hirata *et al.*, Phys. Rev. Lett. **58**, 1490 (1987); R. M. Bionta *et al.*, Phys. Rev. Lett. **58**, 1494 (1987); I. Shelton, IAU Circular No. 4330 (1987).

[19] M. J. Longo, Phys. Rev. Lett. **60**, 173 (1988); L. M. Krauss and S. Tremaine, Phys. Rev. Lett. **60**, 176 (1988); I. Ciufolini and J. A. Wheeler, *Gravitation and Inertia* (Princeton University Press, New York, 1995).

[20] J. Bailey *et al.*, Nature (London) **268**, 301 (1977); D. Buskulic *et al.*, Nucl. Instrum. Methods Phys. Res., Sect. A **360**, 481 (1995).

[21] O. Adriani *et al.*, Nucl. Instrum. Methods Phys. Res., Sect. A **488**, 209 (2002).

[22] G. Kroupa *et al.*, Nucl. Instrum. Methods Phys. Res., Sect. A **440**, 604 (2000).