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Baryon currents in QCD with compact dimensions

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On a compact space with nontrivial cycles, for sufficiently small values of the radii of the compact dimensions, SU(N) gauge theories coupled with fermions in the fundamental representation spontaneously break charge conjugation, time reversal, and parity. We show at one loop in perturbation theory that a physical signature for this phenomenon is a nonzero baryonic current wrapping around the compact directions. The persistence of this current beyond the perturbative regime is checked by lattice simulations.

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Quantum Chromodynamics (QCD) is the theory of strong interactions. Experimental evidence suggests that the theory is invariant under charge conjugation (C), parity (P), and time reversal (T) (see [1] for a recent account of experimental data). The invariance of QCD under P has been rigorously proved in [2]. One of the assumptions of the proof is Lorentz invariance, which holds in an infinite volume, but it is manifestly broken at finite temperature or in compact space, where parity can be spontaneously broken [3]. Although convincing arguments exist [4], a proof of the invariance of QCD under T and C is still lacking.

Recently, it has been pointed out by the authors of [5] that C, P, and T are spontaneously broken in a geometry with one compact dimension with toroidal topology for sufficiently small values of the radius of the torus when periodic boundary conditions are imposed on fermion fields. This provides a controllable mechanism for testing the consequences of the breaking of those symmetries in QCD. The order parameter is the vacuum expectation value (VEV) of the Wilson loop winding in the compact direction

$$W = \mathrm{Tr} P e^{i \int_0^L A_\alpha dx^\alpha}, \tag{1}$$

with α the compact direction of size L, g the coupling, and A_μ the vector potential. In pure gauge SU(N) $\langle W \rangle \propto e^{i(2\pi/N)n}$ with $0 \leq n < N$ for $L < L_c$ and $\langle W \rangle = 0$ for $L > L_c$, where L_c is the critical value of the length of the compact direction [6]. Modulo relabeling of the axes, the Euclidean rotated system corresponds to the theory at finite temperature, and the transition that takes place at L_c is the well-known confinement-deconfinement phase transition [7,8].

When fermions in the fundamental representation are considered, the structure of the ground state changes radically. At small radius, if the fermions have antiperiodic boundary conditions in the compact direction, $\langle W \rangle \propto 1$ (again this is the case for a system at finite temperature in the deconfined phase), while for periodic boundary conditions the Wilson loop can take two values with a nonzero imaginary part. These VEV are related by complex conjugation. Each one of the two values identifies a possible vacuum of the theory. The effect of C, P, and T is to interchange the vacua. Hence, in this system those symmetries are broken. For orientifold gauge theories in the large N limit, which are related to QCD [9], on a $S^3 \times S$ space as the radius of the S is increased above a critical value keeping the radius of the S^3 small, the system regains invariance under C, P, and T [10].

The arguments from which the phase structure of QCD on a finite volume is determined are based on perturbative calculations. Their validity beyond the perturbative regime has been proved by lattice simulations [11].

While the Wilson loop wrapping around the compact direction proves to be useful to characterize the phases, it is not a quantity that can be accessed directly in experiments. Physically, we expect a symmetry breaking to determine a detectable change in the properties of the system. Hence, at least one measurable quantity that is not invariant under the broken symmetries should acquire a VEV. The spatial components in the compact directions of the baryonic current $j_i = \sum_{n=1}^{N_f} \bar{\psi}_n \mathbb{I} \otimes \gamma_i \psi_n$, where ψ_n is the fermion field for flavor n, the sum runs over the flavor index, and $\mathbb{I}$ is the identity in color space, satisfy this requirement [12]. Moreover, like the system, they are invariant under CP, CT, and PT. This makes the j_i suitable candidates as detectors of the symmetry breaking. If $\langle j_\alpha \rangle \neq 0$ for the compact direction α, an observer will see a nonzero flux of baryons in that direction.

Using a similar ansatz to the one of [5], we shall now show at one loop in perturbation theory that indeed the VEV of the spatial current in a compact direction is different from zero. The Lagrangian for a SU(N) gauge theory coupled with N_f degenerate flavors of fermions of mass m in the fundamental representation is

$$\mathcal{L} = -\frac{1}{2g^2}\,\mathrm{Tr}(G_{\mu\nu}(x)G^{\mu\nu}(x)) + \sum_{n=1}^{N_f} \bar{\psi}_n(x)(i\not{\partial} - \not{A} - m)\psi_n(x), \qquad (2)$$

where $G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. The corresponding partition function is

$$Z = \int(\mathcal{D}A_\mu)\det(i\not{\partial} - \not{A} - m)^{N_f} e^{-(i/2g^2)\int d^4x\,\mathrm{Tr}(G_{\mu\nu}G^{\mu\nu})}. \qquad (3)$$

We consider the system on a $T \times L^3$ manifold, in which the T direction corresponds to time and the three spatial compact directions L are equal. We impose periodic boundary conditions in space, while T is assumed to be large enough for the choice of boundary conditions in that direction to be irrelevant. The path integral (3) can be evaluated at one loop, by fixing a diagonal background gauge. This gives an effective one loop potential for the diagonal components of the gauge field

$$\vec{A} = \begin{pmatrix} \frac{\vec{v}_1}{L} & & \\ & \ddots & \\ & & \frac{\vec{v}_N}{L} \end{pmatrix}, \qquad \sum_i \vec{v}_i = 0 \bmod(2\pi), \qquad (4)$$

which reads [13]

$$V(\vec{v}_1, \ldots, \vec{v}_N) = \left[\sum_{i,j=1}^{N} f(0, \vec{v}_i - \vec{v}_j) - 2N_f \sum_{i=1}^{N} f(m, \vec{v}_i)\right]. \qquad (5)$$

The first sum comes from the integration of the fluctuations of the gauge and ghost fields, while the second sum comes from the fermion determinant. The function f is defined as

$$f(m, \vec{v}) = \frac{1}{L}\left(\frac{mL}{\pi}\right)^2 \sum_{\vec{k}\neq 0} \frac{K_2(mLk)}{k^2} \sin^2\left(\frac{1}{2}\vec{k}\cdot\vec{v}\right), \qquad (6)$$

with the sum running over vectors in $\mathbb{Z}^3 - \vec{0}$ and K_2 the order two modified Bessel function of the second kind. From the asymptotic behavior of $K_2(x)$ at small x, $K_2(x) = 2/x^2$, we get

$$f(0, \vec{v}) = \frac{2}{\pi^2 L} \sum_{\vec{k}\neq 0} \frac{\sin^2(\frac{1}{2}\vec{k}\cdot\vec{v})}{k^4}. \qquad (7)$$

For $m \gg L^{-1}$, $K_2(mLk) \approx e^{-mLk}\sqrt{\pi/(2mLk)}$ and the sum in f is dominated by terms with $k = 1$. This is true also in general, the higher frequencies in the sum being quickly oscillating with amplitude suppressed at least as $1/k^4$. With the constraints in Eq. (4), the minima of the effective potential are located at

$$v_1^j = v_2^j = \ldots = v_N^j = \begin{cases} \pm\frac{N-1}{N}\pi & \text{for } N \text{ odd} \\ \pi & \text{for } N \text{ even.} \end{cases} \qquad (8)$$

There are eight degenerate minima for odd N and one minimum for even N. In the former case, $\langle W\rangle$ develops a VEV with an imaginary part, and the spontaneous symmetry breaking occurs. The baryonic current can be computed adding a source to the Lagrangian (2). Defining

$$\mathcal{L}(\vec{\mu}) = \mathcal{L} + \vec{\mu}\cdot\vec{j}, \qquad (9)$$

we obtain

$$\langle j_i\rangle = -i\frac{1}{L^3 T}\left(\frac{\partial}{\partial\mu_i}\log Z[\vec{\mu}]\right)_{\vec{\mu}=0}, \qquad (10)$$

with $Z[\vec{\mu}]$ the partition function in the presence of a source $\vec{\mu}$. The source has the effect of shifting $\vec{v}_i \to \vec{v}_i + L\vec{\mu}$ in the expression for the effective potential (5). This does not change the gauge contribution. Since

$$Z[\vec{\mu}] = e^{iTV(\vec{v}_1 + L\vec{\mu}, \ldots, \vec{v}_N + L\vec{\mu})}, \qquad (11)$$

at the minima (8) we get

$$\langle\vec{j}\rangle = -\frac{N_f N}{L^3}\left(\frac{mL}{\pi}\right)^2 \sum_{\vec{k}\neq 0} \frac{K_2(mLk)}{k^2} \sin(\vec{k}\cdot\vec{v})\vec{k}. \qquad (12)$$

$\langle j_i\rangle$ is zero when $v_i = 0$ or $v_i = \pi$ (i.e. when the symmetry breaking does not occur in direction i), is odd under $v_i \to -v_i$, and goes to zero when $m \to \infty$. Hence it fulfills all the natural requirements in the current scenario. In particular, we expect a nonzero current for an odd number of colors. In order to get a better handle on the properties of the baryonic current in the broken phase beyond perturbation theory, we have performed a lattice simulation using four flavors of staggered quarks coupled to an SU(3) gauge field. The number of flavors has been fixed as the minimal one for which the staggered action has an undoubtedly well-defined continuum limit. For the pure gauge action we have used the standard Wilson form $S_G = \beta\sum_P(1 - \frac{1}{3}\mathrm{Tr}U_P)$, where $\beta = 2N/g_0^2$ is the coupling of the theory, U_P is the path-ordered product of link variables around the elementary plaquette P, and the sum runs over all plaquettes P. For the fermionic part we have used the simple staggered action

$$S_F = \sum_{x,\mu}\eta_\mu(x)\frac{1}{2}(\bar{\chi}(x)U_\mu(x)\chi(x+\hat{\mu}) - \text{c.c.}) + am\sum_x \bar{\chi}(x)\chi(x), \qquad (13)$$

with $\eta_\mu(x) = (-1)^{\sum_{\nu=0}^{\mu-1} x_\nu}$ $(\eta_0(x) = 1)$, χ a complex three-vector, am the mass in lattice units (a is the lattice spacing), and c.c. stands for the complex conjugate term to the first one in parentheses. More complicated formulations of the action or choice of another discretized form for the fermionic fields would have added extra complication with very little payback for the problem at hand.

Using as a base the publicly available MILC code [14], we have performed a simulation for $\beta = 5.5$ and $am = 0.1$. The physical scale has been determined by measuring the Sommer parameter r_0 [15] on a 24×16^3 lattice, where the three equal spatial directions N_s have been closed with periodic boundary conditions and the temporal direction N_t with antiperiodic boundary conditions for the fermions, while the gauge fields are periodic in all directions. We find $ar_0 = 4.0(1)$; since the Sommer scale is $\simeq 0.5$ fm, the lattice spacing is $a \simeq 0.125$ fm, which means that $L_s = aN_s \simeq 2$ fm and $L_t = aN_t \simeq 3$ fm. Hence, in physical units the lattice is large enough for the calculation to be reliable and the spatial volume is such that C, P, and T are not broken. We then studied the system with the same β and m on a 24×4^3 lattice, with the same boundary conditions as above. The spatial geometry is a three-torus, while the size of the temporal direction is large enough for the system to be confined. In this setup, $L_s \simeq 0.5$ fm. A quick check of the VEV of the spatial Wilson loops shows that the system is in the broken symmetry phase. An example of the obtained distribution for $\langle W \rangle$ is displayed in Fig. 1, which shows $\langle W \rangle$ clustering around $e^{i(2/3)\pi}$.

The baryonic current can be obtained for staggered fermions via the Noether theorem, like in the continuous case. Defining the massless Dirac operator as

$$D^{x,y} = \sum_\mu \eta_\mu(x)(U_\mu(x)\delta_{y,x+\hat{\mu}} - U^\dagger_\mu(x-\hat{\mu})\delta_{y,x-\hat{\mu}}) \quad (14)$$

and the four matrices

$$K^{x,y}_\mu = \eta_\mu(x)(U_\mu(x)\delta_{y,x+\hat{\mu}} + U^\dagger_\mu(x-\hat{\mu})\delta_{y,x-\hat{\mu}}) \quad (15)$$

the current reads

$$\langle \vec{j} \rangle = \frac{1}{TL^3}\,\mathrm{Tr}((D+m)^{-1}\vec{K}). \quad (16)$$

In order to evaluate the current on a given configuration, we have taken 100 stochastic estimates. Since the current is an anti-Hermitian operator in the Euclidean space, we expect its imaginary part to develop a VEV, while the real part should average to zero. In Fig. 2 we show the behavior of the imaginary part of the baryonic current in a compact direction as a function of the Monte Carlo sweeps, and we contrast such behavior with that of the imaginary part of the Wilson line in the same direction. Not only does

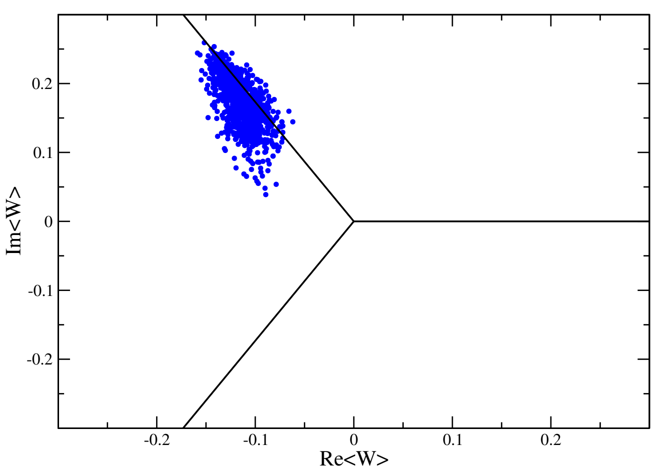

FIG. 1 (color online). Scatter plot for 1000 measurements of the Wilson line in one compact direction on a 24×4^3 lattice at $\beta = 5.5$ and $am = 0.1$. The directions corresponding to the three cubic roots of the unity are indicated by the solid lines.

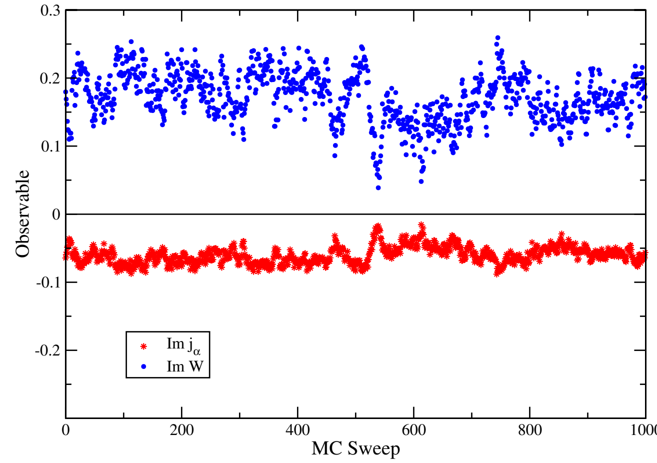

FIG. 2 (color online). The imaginary part of the current and the imaginary part of the Polyakov loop in one compact direction as a function of the Monte Carlo sweeps.

the plot show that the baryonic current is different from zero, but it also strongly suggests that there is a correlation between the value of the current and the value of the Wilson line. In particular, the modulus of the imaginary part of the current grows when the modulus of the imaginary part of the Wilson line grows, the sign being opposite between the two. This is better shown by Fig. 3, which displays the behavior of the current in another compact direction. In this case, the system makes a transition between the vacuum identified by the phase of the Wilson line being $\frac{2}{3}\pi$ to the other vacuum and then back. Noticeably, the current changes sign exactly at the points in which the imaginary part of the Wilson line changes sign, with its magnitude always tracking closely the magnitude of the phase of $\langle W \rangle$. The sum of the terms with $|\vec{k}| = 1$ in the current (12) is proportional to $\langle W \rangle$. The strong correlation between the two quantities suggests that the nonleading terms in (12) do not affect significantly the

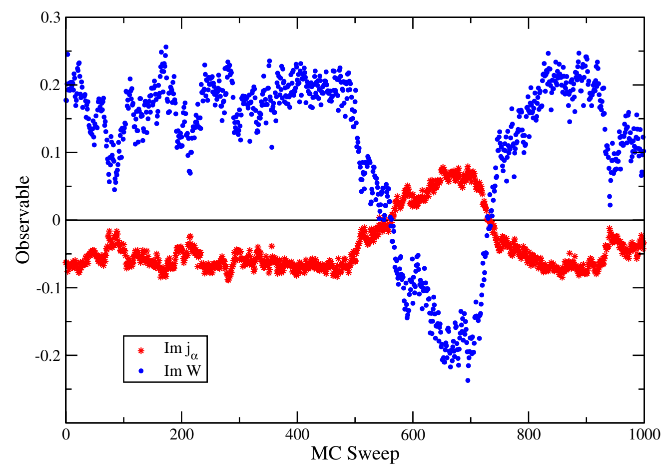

FIG. 3 (color online). As in Fig. 2, but in another compact direction. The system shows a transition between two vacua. The transition probability is finite, due to the finite lattice extension.

behavior of j_α. Transitions between the different vacua like those shown in Fig. 3 are possible because of the finite spatial size of the system. We have verified that increasing β at fixed lattice extensions N_s and N_t, which corresponds to decreasing the physical volume, the frequency of the transitions increases. Likewise, decreasing β decreases the likelihood of a transition taking place. Since the baryonic current is zero for symmetry reasons in the symmetric phase, its behavior in the broken symmetry phase makes it legitimate to use that current as an order parameter for the symmetry breaking. For consistency, we have also checked that the real part of the current in the compact directions and the zero component are zero also in the broken symmetry phase. In order to evaluate the magnitude of the current, we averaged over directions for which no tunneling between the two vacua took place. We find

$$|\mathrm{Im}\langle j_\alpha\rangle| = 0.060 \pm 0.002. \tag{17}$$

It is instructive to compare this number with the one loop expression, Eq. (12), which gives $\langle j_\alpha\rangle \simeq 0.037473(4)$, where the error is a conservative estimate for the truncation of the sum. Hence, quite remarkably the one loop calculation pins down the correct order of magnitude even for a compact dimension with size of the order of $1/\Lambda_{\mathrm{QCD}}$. Nonperturbative effects could explain the discrepancy between the perturbative formula and the measured value. Besides, our calculation being at one single lattice spacing, we do not have any handle on the size of discretization errors. For this reason, a careful comparison between the perturbative expression and the lattice result should be the subject of a more detailed study, which is beyond the scope of this paper. Our preliminary Monte Carlo results for the current closer to the continuum limit show substantial agreement between the measured value and the perturbative formula.

In conclusion, we have shown that QCD on small compact dimensions with nontrivial cycles is characterized by a flow of current whose sign depends on the vacuum selected by the system. The persistent baryonic current reminds us of the supercurrent observed in superconductors. However, there is a fundamental difference: unlike the case of superconductors, in QCD in compact not simply connected space the current is still conserved, since the U(1) baryon symmetry (which in the case of QCD is a global symmetry) remains unbroken. The persistent flow is induced by the spontaneous breaking of a discrete symmetry, charge conjugation. The baryonic current can be used as an order parameter for the spontaneous breaking of charge conjugation in SU(N) gauge theories. Over the Wilson line, it has the advantage of being an observable quantity. Moreover, unlike the Wilson line, which is ultraviolet divergent on the lattice, the baryonic current is a well-defined observable. This makes it better suited for numerical studies of the physics of C parity spontaneous breaking close to the continuum limit. A similar investigation is currently in progress.

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