# Vacuum birefringence from the effective Lagrangian of the electromagnetic field

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The propagation of a linearly polarized laser beam in the external transverse magnetic field is studied. We explore the effective Lagrangian of the electromagnetic field. With the help of the effective Lagrangian, Stokes parameters, induced ellipticity, and the angular rotation of the polarization plane of the beam are evaluated. Ellipticity measured in the PVLAS experiment allows us to obtain the relation between two parameters in the effective Lagrangian.

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The results of the PVLAS experiment [1], measuring the vacuum birefringence and dichroism, cannot be explained by quantum electrodynamics [2,3]. This indicates physics beyond the standard model. Two possible scenarios to treat the PVLAS experiment were suggested (for a last review, see [4]): the existence of a new axionlike (spin-0) particle [5] and millicharged particles[6]. Parameters of axionlike (spin-0) particles exclude the identity of a new neutral light spin-0 boson and QCD axion [4].

In this paper, we study vacuum birefringence on the base of the effective Lagrangian of the electromagnetic field. The Lorentz-invariant effective Lagrangian can be taken in the form

$$\mathcal{L}_{\text{eff}} = \tfrac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) + a(\mathbf{E}^2 - \mathbf{B}^2)^2 + b(\mathbf{E}\mathbf{B})^2, \qquad (1)$$

where $a$ and $b$ are free parameters. We do not include in Eq. (1) the term $c(\mathbf{E}^2 - \mathbf{B}^2)(\mathbf{E}\mathbf{B})$ because it violates the $P$ and $CP$ invariance. If it exists (for example, from electric dipole moments of particles), the parameter $c$ is much smaller compared to $a$ and $b$ in Eq. (1). The effective Lagrangian (1) generalizes the Heisenberg-Euler Lagrangian [7–9] on the case of arbitrary parameters $a$, $b$. In the case of quantum electrodynamics (QED), parameters $a$, $b$ are given by

$$a_{\text{QED}} = \frac{2\alpha^2}{45m_e^4}, \qquad b_{\text{QED}} = \frac{14\alpha^2}{45m_e^4}, \qquad (2)$$

where $\alpha = e^2/(4\pi) \simeq 1/137$ is the fine structure constant, and the $m_e$ being the electron mass. The rationalized (Heaviside-Lorentz) units and $\hbar = c = 1$ are used here. We mention that the Lagrangian of the form (1) appears also in [10,11] as a result of vacuum polarization of arbitrary spin particles.

From Eq. (1), one obtains the electric and magnetic permeability tensors of the vacuum

$$\begin{aligned}\varepsilon_{ik} &= \delta_{ik}[1 + 4a(\mathbf{E}^2 - \mathbf{B}^2)] + 2bB_iB_k,\\ \mu_{ik} &= \delta_{ik}[1 + 4a(\mathbf{B}^2 - \mathbf{E}^2)] + 2bE_iE_k,\end{aligned} \qquad (3)$$

so that

$$D_i = \varepsilon_{ik}E_k, \qquad B_i = \mu_{ik}H_k. \qquad (4)$$

Let us consider the case of the plane electromagnetic wave $(\mathbf{e}, \mathbf{b})$ traveling in $z$ direction and perpendicular to the external constant and uniform magnetic field $\bar{\mathbf{B}} = (\bar{B}, 0, 0)$. Then $\mathbf{E} = \mathbf{e}$, $\mathbf{B} = \mathbf{b} + \bar{\mathbf{B}}$. In the PVLAS experiment the external magnetic field rotates with angular velocity $\Omega$, but the rotation of the magnetic field does not influence vacuum birefringence within QED calculations [2,3]. The same situation occurs in our case, and therefore, we ignore the rotation of the external magnetic field. The difference between effective Lagrangian (1) and the Heisenberg-Euler Lagrangian is in the parameters $a$ and $b$ in Eq. (2). Therefore, all calculations here are similar to QED calculations. Linearizing Eq. (3) around the background magnetic induction field $\bar{\mathbf{B}}$, we find the polarization tensors (in matrix notations)

$$\begin{aligned}\varepsilon &= (1 - 4a\bar{B}^2)I + 2b\bar{B}^2\hat{\mathbf{B}} \cdot \hat{\mathbf{B}},\\ \mu^{-1} &= (1 - 4a\bar{B}^2)I - 8a\bar{B}^2\hat{\mathbf{B}} \cdot \hat{\mathbf{B}},\end{aligned} \qquad (5)$$

where $\hat{\mathbf{B}}$ is a unit vector along the magnetic field, $I$ being unit $3 \times 3$ matrix, $\hat{\mathbf{B}} \cdot \hat{\mathbf{B}}$ is matrix-dyad with matrix elements $(\hat{\mathbf{B}} \cdot \hat{\mathbf{B}})_{ik} = \hat{B}_i\hat{B}_k$. Maxwell equations are given by

$$k_id_i = k_ib_i = 0, \qquad \mathbf{k} \times \mathbf{e} = \omega\mathbf{b}, \qquad \mathbf{k} \times \mathbf{h} = -\omega\mathbf{d}, \qquad (6)$$

with $\mathbf{k}$ being the wave vector. Replacing vectors $\mathbf{d} = \varepsilon\mathbf{e}$ and $\mathbf{h} = \mu^{-1}\mathbf{b}$ into Eq. (6), one obtains

$$\varepsilon_{ijk}k_j(\mu^{-1})_{kl}b_l = -\omega\varepsilon_{ik}e_k, \qquad (7)$$

where $\varepsilon_{ijk}$ is the antisymmetric tensor, $\varepsilon_{123} = 1$. Substituting the vector $\mathbf{b}$ from Eq. (6) into Eq. (7), we find the wave equation for the electric field $\mathbf{e}$:

$$[\varepsilon_{ijp}\varepsilon_{lab}k_j(\mu^{-1})_{pl}k_a + \omega^2\varepsilon_{ib}]e_b = 0. \qquad (8)$$

Equation (8) can be transformed into the matrix equation as follows:

$$\Lambda\mathbf{e} = 0, \qquad \Lambda = A + C\hat{\mathbf{B}} \cdot \hat{\mathbf{B}}, \qquad (9)$$

 

$$A = 12a\bar{B}^2 - 1 + \frac{1}{n^2}(1 - 4a\bar{B}^2),$$
$$C = \frac{1}{n^2}2b\bar{B}^2 - 8a\bar{B}^2, \tag{10}$$

with $n = k/\omega$ index of refraction. It is easy to verify that the matrix $\Lambda$ obeys the equation

$$(\Lambda - A)(\Lambda - A - C) = 0. \tag{11}$$

From Eq. (11), one obtains the eigenvalues of the matrix $\Lambda$: $\lambda_1 = A$, $\lambda_2 = A + C$. The homogeneous matrix Eq. (9) has nontrivial solutions when the determinant of the matrix $\Lambda$ vanishes, or equivalently $\lambda_1 = 0$, $\lambda_2 = 0$. As a result, we have two modes defining dispersion relations:

$$A = 0, \qquad n_\perp = 1 + 4a\bar{B}^2, \tag{12}$$

$$A + C = 0, \qquad n_\| = 1 + b\bar{B}^2. \tag{13}$$

In Eqs. (12) and (13), we use the expansion in small parameters $a\bar{B}^2$, $b\bar{B}^2$ ($a\bar{B}^2 \ll 1$, $b\bar{B}^2 \ll 1$). The solution (12) corresponds to the electric field of the plane wave $\mathbf{e}$ perpendicular to the background magnetic induction field, $\mathbf{e}\perp\bar{\mathbf{B}}$, and the solution (13) corresponds to the case $\mathbf{e} \parallel \bar{\mathbf{B}}$. So, two different polarizations of the electromagnetic wave travel with different velocities. Let the polarization vector at $z = 0$ be $\mathbf{e}|_{z=0} = E_0(\cos\theta, \sin\theta)\exp(-i\omega t)$ so that the angle between the polarization vector $\mathbf{e}$ and the external magnetic induction field $\bar{\mathbf{B}}$ is $\theta$. Then the components of the polarization vector at arbitrary $z$ are given by

$$e_\perp = E_0 \sin\theta \exp i(k_\perp z - \omega t),$$
$$e_\| = E_0 \cos\theta \exp i(k_\| z - \omega t), \tag{14}$$

where $k_\perp = n_\perp\omega$, $k_\| = n_\|\omega$. From Eq. (14), using the notations of [12], we find

$$\alpha = \theta, \qquad \delta = (k_\perp - k_\|)z = (4a - b)\omega\bar{B}^2 z,$$
$$\sin 2\chi = (\sin 2\alpha)\sin\delta. \tag{15}$$

Since $\delta$ is small, one obtains from Eq. (14) ellipticity (the ratio of the axes of the ellipse)

$$\Psi \equiv \tan\chi \simeq \chi \simeq \tfrac{1}{2}\delta \sin 2\theta = \tfrac{1}{2}(4a - b)\omega\bar{B}^2 z \sin 2\theta, \tag{16}$$

where $\omega = 2\pi/\lambda$, $\lambda$ is a wave length. Initially a linearly polarized wave after traveling the distance $L$ becomes an elliptically polarized wave. As the angle of rotation of the ellipse $\psi$ is given by $\tan 2\psi = (\tan 2\alpha)\cos\delta$, we obtain to first order in the small parameter $\delta$: $\psi \simeq \theta$. So, there is no rotation of the polarization axis of the beam. With the usual QED values for $a$ and $b$ (2), results (16) reduce to the known ones for QED [2].

We write down also the Stokes parameters [12]

$$s_1 = s_0 \cos 2\chi \cos 2\psi \simeq s_0 \cos 2\theta,$$
$$s_2 = s_0 \cos 2\chi \sin 2\psi \simeq s_0 \sin 2\theta, \tag{17}$$
$$s_3 = s_0 \sin 2\chi \simeq s_0(4a - b)\omega\bar{B}^2 z \sin 2\theta,$$

where $s_0$ is proportional to the intensity of the wave. The Stokes parameters, Eq. (17), can be measured from experiments [12]. The angular rotation of the polarization plane of the electromagnetic wave, traveling a path length $L$ perpendicular to the magnetic field, is given by the relation

$$\Delta\varphi = (k_\perp - k_\|)L = (4a - b)\omega\bar{B}^2 L. \tag{18}$$

With the help of the data [1]

$$\Psi = (-3.4 \pm 0.3) \times 10^{-12}\frac{\text{rad}}{\text{pass}}, \qquad L = 1\ \text{m},$$
$$\lambda = 1064\ \text{nm}, \qquad \theta = \frac{\pi}{4}, \qquad \bar{B} = 5.5\ \text{T}, \tag{19}$$

and using the value $e\bar{B} = 3.25 \times 10^{-10}\ (\text{MeV})^2$ (for $\bar{B} = 5.5$ T), we find from Eq. (16) the relation for parameters $a$, $b$ contributed to ellipticity observed

$$b - 4a \simeq 1\ (\text{MeV})^{-4}. \tag{20}$$

The experimental induced ellipticity allows us to fix only the combination of parameters $a$ and $b$. For the QED case, the parameters obtained from Eq. (2) lead to the relationship

$$b_{\text{QED}} - 4a_{\text{QED}} \simeq 10^{-4}\ (\text{MeV})^{-4}. \tag{21}$$

So, the magnitude (20) found from the experiment is $10^4$ times greater than expected from QED (21).

It follows from Eqs. (16) and (18) that for the data (19), the angle of the polarization plane rotation is $|\Delta\varphi| = 2|\Psi| \simeq 6.8 \times 10^{-12}$ rad/pass. An improvement in the PVLAS experiment would correct the relation between parameters $a$ and $b$. So, based on the effective Lagrangian, we have obtained the relation between the parameters $a$, $b$ contributed to ellipticity observed in the PVLAS experiment. We notice that light higher spin ($s \geq 1$) charged particles can contribute to parameters $a$, $b$ resulting in Eq. (20) from "box" diagrams [10]. However, if such particles exist, they should have been produced in electron-positron collisions. We do not discuss here models resulting in the appearance of the relation (20) for parameters of the effective Lagrangian (1). If results of the PVLAS experiment will be confirmed it would require new physics.

*Added note.*—After this paper was prepared, we studied the manuscript [13], where results obtained agree with ours.

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