Decay constants of the heavy-light mesons from the field correlator method

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Meson Green's functions and decay constants f_Γ in different channels Γ are calculated using the field correlator method. Both, spectrum and f_Γ, appear to be expressed only through universal constants: the string tension σ, α_s, and the pole quark masses. For the S-wave states the calculated masses agree with the experimental numbers within ± 5 MeV. For the D and D_s mesons the values of $f_P(1S)$ are equal to 210(10) and 260(10) MeV, respectively, and their ratio $f_{D_s}/f_D = 1.24(3)$, agrees with a recent CLEO experiment. The values $f_P(1S) = 182, 216, 438$ MeV are obtained for the B, B_s, and B_c mesons with the ratio $f_{B_s}/f_B = 1.19(2)$ and $f_D/f_B = 1.14(2)$. The decay constants $f_P(2S)$ for the first radial excitations as well as the decay constants $f_V(1S)$ in the vector channel are also calculated. The difference of about 20% between f_{D_s} and f_D, f_{B_s} and f_B directly follows from our analytical formulas.

I. INTRODUCTION

The decay constants f_P in the pseudoscalar (P) channel, being important characteristics of mesons, in many cases can be measured directly in experiment, and therefore they can provide a precise manner to compare different theoretical approaches and check their accuracy. During the last decade the constants f_P have been studied by many authors in potential models [1–8], in the QCD sum rule method [9,10], in lattice simulations [11–14], as well as in experiment [15–19]. (The papers [2–7] contain references and a summary of numerous calculations of decay constants).

The present article is devoted to the systematic derivation of the meson Green's functions in QCD and study of the decay constants for channels with arbitrary quantum numbers Γ, of which we specifically consider P and vector (V) channels. For the decay constant f_Γ transparent analytical expressions will be obtained and, in particular, using those, the difference between f_{D_s} and f_D, f_{B_s} and f_B can be explained easily.

This paper is an improvement and extension of the earlier paper [20] devoted to the heavy-light (HL) pseudoscalars. Reference [20] appeared before the systematic formulation of the field correlator method (FCM) [21], in particular, before the derivation of the string Hamiltonian [22], therefore some steps in [20] were not accurately proved. In this paper we give a consistent and general treatment of the meson Green's function and its spectral properties. The main problem, which one encounters when addressing the spectral properties in QCD, is the necessity to include quantitative nonperturbative (NP) methods, which are responsible for the main dynamical phenomena: confinement and chiral symmetry breaking (CSB).

In the FCM, introduced in [23], one derives the effective Hamiltonian, which comprises both confinement and relativistic effects and contains only universal quantities: the string tension σ, the strong coupling α_s, and the current (pole) quark masses m_i. We use here the pole quark masses

which correspond to the conventional current (Lagrangian) masses $\bar{m}_q(\bar{m}_q)$ [16]. The simple local form of this Hamiltonian, which will be called the string Hamiltonian (SH), has been derived recently and used for light mesons [24], heavy quarkonia [25], and heavy-light mesons [26], giving good agreement with experimental masses, leptonic widths, and fine-structure effects.

As compared to QCD sum rules and lattice QCD this method has an essential advantage, because the radial and orbital excitations can be considered in this approach on the same grounds as the ground states.

The calculation of spectral coefficients, like decay constants f_Γ, needs an additional step, namely, besides using the SH the computation of all coefficients in the Green's function, including the Dirac spinor structure etc. Moreover, for π and K mesons CSB is vitally important. Recently the FCM was extended to include the effects of CSB [27], where it was shown that the phenomenon of CSB occurs due to confinement and two characteristic parameters of CSB—the chiral condensate and f_π—were computed in terms of σ. The calculation of f_π and f_K can be done using a simple extension of the same general expression (23), derived in Ref. [27], while here we concentrate on the calculations of the masses and f_Γ of HL mesons; our method also enables one to calculate f_Γ for excited states and here decay constants will be calculated for the 2S states.

One important technical problem, which is solved in this paper and allows one to calculate f_Γ in all channels, is the accurate einbein reformulation of the Fock-Feynman-Schwinger representation (FFSR), or the worldline representation, where the dynamical quark mass ω_q appears as an integration variable instead of the proper time. The previous step in this direction [22] has enabled one to write only the string Hamiltonian, but the whole Green's function was not attainable. Below in the FFSR we derive the explicit einbein form of the meson Green's function.

The paper is organized as follows. In Sec. II the general einbein form of the meson Green's function is presented, while in Appendices A, B, and C the details of the derivation are given. In Sec. III the masses of the heavy-light pseudoscalar mesons are considered. The decay constants are calculated in Sec. IV, while auxiliary variables are given in Appendices D and E. Section V is devoted to the approximations used in our calculations and Sec. VI contains our concluding remarks.

II. THE MESON GREEN'S FUNCTION IN THE FFSR

Here we derive the einbein form of the meson Green's function written as the path integral in the FFSR [28–30]. We start with the FFSR for the quark Green's function in the gluonic field A_μ, which contains both perturbative and NP contributions and Euclidean space-time is assumed everywhere:

$$S(x, y) = (m + \hat{D})^{-1} = (m - \hat{D})(m^2 - \hat{D}^2)^{-1}$$
$$= (m - \hat{D}) \int_0^\infty ds (Dz)_{xy} e^{-K} \Phi_\sigma(x, y), \quad (1)$$

with

$$\hat{D} = D_\mu \gamma_\mu, \qquad D_\mu = \frac{\partial}{\partial x_\mu} - ig A_\mu,$$

$$K = m^2 s + \frac{1}{4} \int_0^s \left(\frac{dz_\mu}{d\tau}\right)^2 d\tau,$$

$$\Phi_\sigma(x, y) = P \exp\left(ig \int_y^x A_\mu dz_\mu\right) \exp\left(g \int_0^s \sigma_{\mu\nu} F_{\mu\nu} d\tau\right), \quad (2)$$

where

$$(Dz)_{xy} = \lim_{N\to\infty} \prod_{k=1}^N \frac{d^4 \Delta z(k)}{(4\pi\varepsilon)^2} \Bigg|_{\sum_k \Delta z(k) = x - y}, \quad (3)$$

$$N\varepsilon = s, \qquad |_{a=b} \equiv \int \frac{d^4 p}{(2\pi)^4} e^{ip(a-b)}.$$

In Eq. (1) the role of the evolution parameter is played by the proper time s, whereas in $(Dz)_{xy}$ there is an integration over the fourth component $z_4(\tau)$, $(0 \le \tau \le s)$, which is the Euclidean time of the particle. The crucial point now is to go over in Eq. (1) to the Euclidean time $z_4 \equiv t$ as an evolution parameter. To get rid of the proper time s, one can use the so-called einbein method [31], which was applied to the FFSR in Ref. [22,30], and here it is developed and used for the correlator of the currents.

To this end the so-called dynamical mass (variable) $\omega(t)$ can be introduced via the relation between the proper time τ and Euclidean time t:

$$d\tau = \frac{dt}{2\omega(t)}, \qquad \bar{\omega} = \frac{1}{s} \int_0^s \omega(\tau) d\tau = \frac{1}{N} \sum_{k=1}^N \omega(k). \quad (4)$$

The integrals in Eq. (1) can now be identically rewritten as

$$ds(Dz)_{xy} \equiv (D^3 z)_{xy}(D\omega)$$
$$= \lim_{N\to\infty} \prod_{k=1}^N \frac{d^3 \Delta z(k)}{2\bar{\omega} l^3(k)} \Bigg|_{\sum_k \Delta z = x - y} \frac{d\omega(k)}{l_\omega(k)}, \quad (5)$$

where

$$l(k) = \sqrt{\frac{2\pi\Delta t}{\omega(k)}}, \qquad l_\omega(k) = \sqrt{\frac{2\pi\omega(k)}{\Delta t}}, \quad (6)$$

$$N\Delta t = x_4 - y_4 \equiv T.$$

In Appendix A the representations (1) and (5) are illustrated by calculating the free quark propagator, where the meaning of $\omega(k)$ and $\bar{\omega}$ in the momentum representation appears to be very simple: $\bar{\omega} = \omega(k) = \sqrt{p^2 + m^2}$.

We now turn to the meson (quark-antiquark) case and consider the correlator $G_\Gamma(x)$ of the currents $j_\Gamma(x)$:

$$j_\Gamma(x) = \bar{\psi}_1(x)\Gamma\psi_2(x),$$
$$\Gamma = t^a \otimes (1, \gamma_5, \gamma_\mu, i\gamma_\mu\gamma_5)$$
$$\text{for S, P, V, and A channels,} \quad (7)$$
$$t^a = \frac{\lambda^a}{2}, \qquad \text{tr}(t^a t^b) = \frac{1}{2}\delta_{ab},$$

and

$$G_\Gamma(x) \equiv \langle j_\Gamma(x) j_\Gamma(0) \rangle_v$$
$$= 4N_c \int Y_\Gamma(D^3 z)_{x0}(D^3 \bar{z})_{x0}(D\omega_1)(D\omega_2)$$
$$\times \exp(-K_1 - K_2) W_\sigma. \quad (8)$$

In (8) we have defined the new quantity Y_Γ:

$$4Y_\Gamma = \text{tr}_L(m_1 - \hat{D}_1)\Gamma(m_2 - \hat{D}_2)\Gamma$$
$$\to \text{tr}_L(m_1 + \omega_1\hat{\dot{z}})\Gamma(m_2 - \omega_2\hat{\dot{\bar{z}}})\Gamma, \quad (9)$$

which also can be written in the operator form:

$$4Y_\Gamma = \text{tr}_L((m_1 - i\hat{p}_1)\Gamma(m_2 + i\hat{p}_2)\Gamma), \quad (10)$$

with $p_\mu^{(i)}$ ($\hat{p}_i = p_\mu\gamma_\mu$) the momentum of particle i as it is derived in Appendix B for the convenience of the reader (tr_L means the trace over Dirac indices). The resulting expressions for Y_Γ in the V, A, S, and P channels are given below in Eqs. (35).

In (8) the symbol $W_\sigma = \Phi_{\sigma_1}(x, y)\Phi_{\sigma_2}(y, x)$ stands for the average value of the Wilson loop with the insertions of the operator

$$\Lambda_\sigma^{(1)} \equiv \exp\left(g\sigma_{\mu\nu}^{(1)} \int_0^s F_{\mu\nu} d\tau \right)$$
$$= \exp\left(g\sigma_{\mu\nu}^{(1)} \int_0^T F_{\mu\nu}(z(t_1)) \frac{dt_1}{2\omega_1(t_1)} \right) \quad (11)$$

for the quark line and of the operator $\Lambda_\sigma^{(2)}$ for the antiquark line:

$$\Lambda_\sigma^{(2)} \equiv \exp\left(-g\sigma_{\mu\nu}^{(2)} \int_0^s F_{\mu\nu} d\tau \right)$$
$$= \exp\left(-g\sigma_{\mu\nu}^{(2)} \int_0^T F_{\mu\nu}(\bar{z}(t_2)) \frac{dt_2}{2\omega_2(t_2)} \right). \quad (12)$$

Since in HL mesons we will consider spin-effects as a perturbation, in the first approximation both factors $\Lambda_\sigma^{(1)}$ and $\Lambda_\sigma^{(2)}$ are replaced by 1 and W_σ is simplified.

In Appendix C one can find all explicit steps for the derivation of the correlator (8) in the simplest case when the gluon interaction is absent, i.e., for $W_\sigma \equiv 1$. One can see there that the quark-loop contribution is reconstructed with the correct coefficients. Now we turn to the case when the NP interaction is included in W_σ.

In general, W_σ contains all effects of the interaction which include: (i) the perturbative static gluon exchange; (ii) the radiative corrections to G_Γ in the form of the operator anomalous dimension and corrections to f_Γ; (iii) the NP contributions to G_Γ. To calculate all of them one can use the background perturbation theory [32,33] and the FCM. For all hadrons of interest to us (with a size larger than $T_g \approx 0.2$ fm) the use of the FCM reduces to the appearance of the area-law factor in W_σ, which is accompanied by Coulomb, radiative, and spin-dependent factors. (For more discussion see the reviews [30,34]). As was shown in Refs. [33,34] this produces (apart from radiative corrections) the local interaction $\hat{V}(r)$, which for not so large angular momentum ($L \leq 4$) can be presented as $W_\sigma = \exp(-\int_0^T dt \hat{V})$, where the interaction,

$$\hat{V}(r) = V_0(r) + V_{\text{SD}} + \Delta V_{\text{string}} + V_{\text{SE}}, \quad (13)$$

contains the static potential:

$$V_0(r) = \sigma r - \frac{4}{3} \frac{\alpha_{\text{st}}(r)}{r}, \quad (14)$$

and the spin-dependent part $V_{\text{SD}}(r)$ given by

$$V_{\text{SD}} = V_{\text{SS}} + V_{\text{LS}} + V_{\text{T}}, \quad (15)$$

with spin-spin, spin-orbit, and tensor terms; the self-energy contribution V_{SE} [35], and also a "string correction" (occurring only for the states with $L \neq 0$ [22,24]). For the S-wave mesons, considered here,

$$\hat{V}(r) = V_0(r) + V_{\text{SS}} + V_{\text{SE}}, \quad (16)$$

while for the spin-averaged masses only two terms are left in the potential, $V(r) = V_0(r) + V_{\text{SE}}$. In Sec. V we shall also take into account radiative corrections and the operator anomalous dimension.

One can now rewrite G_Γ in Eq. (8), separating c.m. and relative distance coordinates,

$$\boldsymbol{\eta} = \Delta z_1 - \Delta z_2, \qquad \boldsymbol{\rho} = \frac{\omega_1 \Delta z_1 + \omega_2 \Delta z_2}{\omega_1 + \omega_2},$$
$$\omega_r = \frac{\omega_1 \omega_2}{\omega_1 + \omega_2}, \quad (17)$$

where all coordinates are labeled with the index k as in Eq. (6).

Integrating out the c.m. coordinate $d\boldsymbol{\rho}$ and $d\omega_+ = \omega_1 + \omega_2$, as shown in Appendix C, one arrives at the path integral in the relative coordinate $d\boldsymbol{\eta}$, which can be expressed through the SH. Indeed, from the path-integral formalism [29,36] it is known that a general equivalence relation holds:

$$\int \frac{(D^3\eta)_{xy}}{(2\pi\Delta t/\omega_r(k))^{3/2}} e^{-\sum_k (((\omega_r(k)\eta^2(k))/2\Delta t) + \hat{V}(k)\Delta t)}$$
$$= \langle x|e^{-\hat{H}T}|y\rangle, \quad (18)$$

where

$$\hat{H} = \frac{p_\eta^2}{2\omega_r} + \hat{V}(\eta), \qquad p_\eta = \frac{1}{i} \frac{\partial}{\partial \boldsymbol{\eta}}. \quad (19)$$

Taking into account the integral (see Appendix C):

$$\int \frac{2d\omega_r(k)}{\sqrt{\omega_r(k)}\sqrt{\frac{2\pi}{\Delta t}}} e^{-2\omega_r(k)\Delta t - (((p^2+m^2)\Delta t)/(2\omega_r(k)))}$$
$$= e^{-2\sqrt{p^2+m^2}\Delta t}, \quad (20)$$

one obtains the important relation:

$$\int G_\Gamma(x) d^3\boldsymbol{x} = N_c \left\langle 0 \left| \frac{Y_\Gamma}{\bar{\omega}_1 \bar{\omega}_2} e^{-\hat{H}T} \right| 0 \right\rangle, \quad (21)$$

where the Hamiltonian is

$$\hat{H} = \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} + \hat{V}(r). \quad (22)$$

Then with the use of the spectral expansion the expression (21) can be presented as

$$\left\langle 0 \left| \frac{Y_\Gamma}{\bar{\omega}_1 \bar{\omega}_2} e^{-\hat{H}T} \right| 0 \right\rangle = \sum_n \frac{\langle Y_\Gamma \rangle_n |\varphi_n(0)|^2}{\langle \bar{\omega}_1 \rangle_n \langle \bar{\omega}_2 \rangle_n} e^{-M_n T}, \quad (23)$$

where φ_n and M_n are the eigenfunction and the eigenvalue (e.v.) of the Hamiltonian (22). It follows from the extremum conditions (D7) that the variables $\bar{\omega}_1$ and $\bar{\omega}_2$ in Eq. (21) are defined as the operators,

$$\bar{\omega}_i = \sqrt{p^2 + m_i^2}, \qu(24)$$

while in Eq. (23) $\langle \bar{\omega}_i \rangle_n$ is the matrix element over this operator for the nL state.

On the other hand for the left-hand side of Eq. (21) one can also use the conventional spectral decomposition:

$$\int G_\Gamma(x) d^3\boldsymbol{x} = \sum_n \int d^3\boldsymbol{x} \langle 0|j_\Gamma|n\rangle$$

$$\times \langle n|j_\Gamma|0\rangle e^{i\boldsymbol{P}\cdot\boldsymbol{x} - M_n T} \frac{d^3\boldsymbol{P}}{2M_n(2\pi)^3}$$

$$= \sum_n \varepsilon_\Gamma \otimes \varepsilon_\Gamma \frac{(M_n f_\Gamma^n)^2}{2M_n} e^{-M_n T}. \qquad (25)$$

Here we have used the standard definition for $f_\Gamma^n \equiv f_\Gamma(nS)$:

$$\langle 0|j_\Gamma|n, \boldsymbol{P} = 0\rangle = \varepsilon_\Gamma M_n f_\Gamma^n, \qquad (26)$$

where $\varepsilon_\Gamma = \varepsilon_\mu^{(k)}$ for V and A channels, $\varepsilon_\Gamma = 1$ for S and P channels, while the polarization vector $\varepsilon_\mu^{(k)}$ satisfies the normalization condition:

$$\sum_{k=1,2,3} \varepsilon_\mu^{(k)}(q) \varepsilon_\nu^{(k)}(q) = \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}. \qquad (27)$$

Inserting the expression (23) into (21) and using the relation (25), one obtains the decay constant f_Γ^n in the channel Γ for the nS state:

$$(f_\Gamma^n)^2 = \frac{2N_c\langle Y_\Gamma\rangle|\varphi_n(0)|^2}{\langle\bar\omega_1\rangle_n\langle\bar\omega_2\rangle_n M_n} = \frac{6|\varphi_n(0)|^2}{M_n} \frac{\langle Y_\Gamma\rangle}{\langle\bar\omega_1\rangle_n\langle\bar\omega_2\rangle_n}. \qquad (28)$$

To derive the relations (23) and (28) we have used the essential property of the SH Eq. (22) that the average value $\langle\omega_i(k)\rangle$ of the operator $\omega_i(k)$ is equal to the average of the operator $\sqrt{\boldsymbol{p}^2 + m_i^2}$, and hence the average $\bar\omega_i = \frac{1}{N} \times \sum_{k=1}^N \omega_i(k)$, denoted as $\langle\bar\omega_i\rangle_n$ is

$$\langle\bar\omega_i\rangle_n = \langle\omega_i(k)\rangle_n = \langle\sqrt{\boldsymbol{p}^2 + m_i^2}\rangle_n, \qquad (29)$$

where the average is assumed to be taken over the eigenstate $\varphi_n(r)$:

$$\langle\bar\omega_i\rangle_n = \langle\varphi_n|\sqrt{\boldsymbol{p}^2 + m_i^2}|\varphi_n\rangle. \qquad (30)$$

In the most general case this average may differ from the path-integral average $\langle\bar\omega_i\rangle$ in Eqs. (4), (5), and (8). However, the analysis of this problem, done in Ref. [37], shows that the difference between the two definitions is small ($\lesssim 3\%$ for the lowest states) and we assume here that the same accuracy holds for our basic relation (28).

In Eq. (28) the mass of the $^1S_0$-wave meson $M_n(^1S_0) = M_{0n} - \frac{3}{4}\Delta_{\rm HF} + \Delta_{\rm SE}$, where M_{0n} is the e.v. of the Hamiltonian (22) with the static interaction (14):

$$H_0 = \sqrt{m_1^2 + \boldsymbol{p}^2} + \sqrt{m_2^2 + \boldsymbol{p}^2} + V_0(r),$$
$$H_0\varphi_n = M_{\rm on}\varphi_n. \qquad (31)$$

The general expression of the self-energy correction to the

meson mass $M(nS)$ is calculated in [35] (see Appendix E) (it comes from the NP contributions to the quark and antiquark mass):

$$\Delta_{\rm SE} = \sum_{i=1,2}\left\{\left(-\frac{1.5\sigma\eta_{f_i}}{\pi\langle\bar\omega_i\rangle}\right)\right.$$

$$\left.+ \frac{\sigma^2}{4\langle\bar\omega_i\rangle[m_i + \langle\bar\omega_i\rangle/2 + \varepsilon(\omega_r)]^2}\right\}, \qquad (32)$$

where the factor $\eta_f(i)$ depends on the flavor of the ith quark (antiquark) and its analytical expression (E2) is deduced in Ref. [35]. For the $u(d)$, s, c, and b quarks the values

$$\eta_{u(d)} = 1.0, \qquad \eta_s = 0.65, \qquad \eta_c = 0.35, \qquad \eta_b = 0.03 \qquad (33)$$

are obtained in Appendix E.[1]

It is worthwhile to notice that for the b quark η_b is small and its contribution to $\Delta_{\rm SE}$ is small (~ 1 MeV) and can be neglected. Therefore, for the B, B_s, and B_c mesons we have to use in Eq. (32) only a contribution which comes from the lighter quark (antiquark) $q_1(\bar q_1)$ denoted later by the index 1. For the D and D_s mesons both terms (with $i = 1$ and $i = 2$) are taken into account, although the contribution from the c-quark term is small, around -20 MeV.

Thus, the scheme of our calculations of f_Γ is as follows:
(1) First, one calculates the e.v. M_{0n} of the SH (22) with the static interaction (14) and then takes into account the self-energy (32) and the HF corrections:

$$M_n(^3S_1) = M_{0n} + \Delta_{\rm SE} + \frac{1}{4}\Delta_{\rm HF},$$
$$M_n(^1S_0) = M_{0n} + \Delta_{\rm SE} - \frac{3}{4}\Delta_{\rm HF}. \qquad (34)$$

(2) The values $\langle\bar\omega_1\rangle$ and $\langle\bar\omega_2\rangle$ in Eqs. (28) and (32) are the matrix elements (m.e.) of the kinetic energy term defined in Eq. (24).
(3) The factor $\langle Y_\Gamma\rangle$ (in the channel Γ) can be computed in terms of the momenta of a quark and an antiquark, or in the c.m. system in terms of the relative momentum $\boldsymbol{p}$, with the following results for the m.e. $\langle Y_\Gamma\rangle$ (see Appendix B):

$$\langle Y_V\rangle = m_1 m_2 + \langle\bar\omega_1\rangle\langle\bar\omega_2\rangle + \frac{1}{3}\langle\boldsymbol{p}^2\rangle,$$
$$\langle Y_P\rangle = m_1 m_2 + \langle\bar\omega_1\rangle\langle\bar\omega_2\rangle - \langle\boldsymbol{p}^2\rangle = \langle Y_{A_4}\rangle. \qquad (35)$$

Here we use for the operators Y_Γ the notations:

$$Y_V = \frac{1}{3}\sum_i {\rm tr}[(m_1 - \hat D_1)\gamma_i(m_2 - \hat D_2)\gamma_i],$$

$$Y_P = Y_{A_4} = -[{\rm tr}(m_1 - \hat D_1)\gamma_4\gamma_5(m_2 - \hat D_2)\gamma_4\gamma_5]. \qquad (36)$$

In the case of the P channel with $m_1, m_2 \to 0$ due to

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[1]Note, that these values of $\eta_{f(i)}$ in (33) are not fitting parameters but calculated through the same input m_i and σ.

CSB there appears an additional mass term in Eq. (35), which can be computed through field correlators. (The π and K mesons will be considered later [27]).

For the calculations of $\langle Y_\Gamma \rangle$ one needs also to know the m.e. $\langle \boldsymbol{p}^2 \rangle_{nS}$ and the wave function (w.f.) at the origin, $\varphi_n(0) = R_n(0)/\sqrt{4\pi}$; they are given in Appendix D.

(4) Our calculations are done with the relativistic SH (22) which was derived in einbein approximation (EA); therefore, the w.f. at the origin must also be calculated with the use of the EA (see Appendix D), which provides an accuracy of $\lesssim 5\%$ [26,37]. In the nonrelativistic limit, $m_i \gg \sqrt{\sigma}$, one can easily find that $\langle \bar{\omega}_i \rangle \approx m_i$, while $\langle \boldsymbol{p}^2 \rangle \sim O(\sigma)$ can be neglected in Eqs. (35), and therefore

$$
\begin{aligned}
\langle Y_V \rangle_{NR} &\approx 2m_1 m_2 + O(\sigma), \\
\langle Y_P \rangle_{NR} &= 2m_1 m_2 + O(\sigma).
\end{aligned}
\tag{37}
$$

Thus in the nonrelativistic limit for $f_\Gamma^{(n)}$ in the V and P channels one obtains the well-known result [1]:

$$
(f_\Gamma^n)_{NR}^2 = \frac{4N_c}{M_n} |\varphi_n(0)|^2 \qquad (\Gamma = V, P),
\tag{38}
$$

while in the S channel $f_S \to 0$.

(5) As a final step one needs to compute the radiative corrections to f_Γ^n, which come from the short-distance (large momentum) perturbative one-gluon exchange (OGE) contributions. Neglecting interference terms they can be written as in [20,21],

$$
\langle W_\sigma \rangle = \langle W_{OGE} \rangle \langle W_{nonpert} \rangle
\tag{39}
$$

with

$$
\langle W_{OGE} \rangle = Z_m \exp\left(-\frac{4}{3\pi} \iint \frac{dz_4 dz_4' \alpha_s(z-z')}{(z-z')^2} \right),
\tag{40}
$$

where Z_m is a regularization factor. After separating the Coulomb interaction in $\hat{H}$ in this way, one gets the correction to $\langle W_\sigma \rangle$, and f_Γ^2 can be written in the form:

$$
f_\Gamma^2 \to \xi_\Gamma f_\Gamma^2, \qquad \xi_\Gamma = 1 + c_\Gamma \alpha_s + O(\alpha_s^2).
\tag{41}
$$

Another important contribution from perturbative gluon exchanges (GE) is the account of asymptotic freedom (AF) in the coupling constant α_s in (40), which is especially important for the value of $\varphi_n(0)$ in the S-wave channels. In our calculations we use the GE interaction where in the strong coupling $\alpha_B(r)$ the AF behavior is taken into account. Nevertheless, it is of interest to compare the w.f. at the origin with and without AF behavior in the GE term, introducing the factor $\rho_{AF} = |\varphi_n^{(AF)}(0)/\varphi_n(0)|^2$, which appears to be around 0.80

for the 1S states [20]. Then f_Γ^2 (with AF taken into account) can be expressed through $\tilde{f}_\Gamma^2$ (where AF is neglected) as $f_\Gamma^2 = \tilde{f}_\Gamma^2 \xi_\Gamma \rho_{AF}$.

(6) We conclude this section with the discussion of the input parameters m_i, α_s, and σ. We take $\sigma = 0.18$ GeV^2 for all HL mesons (as in light mesons and in heavy quarkonia [38]); m_i are the conventional pole masses which are defined through the Lagrangian (current) masses in the $\overline{MS}$-scheme (see [16,38] and references therein):

$$
\begin{aligned}
m_i = \bar{m}_{\overline{MS}}(\bar{m}_{\overline{MS}}) &\left\{ 1 + \frac{4}{3} \frac{\alpha_s(\bar{m}_{\overline{MS}})}{\pi} \right. \\
&\left. + \eta_2 \left(\frac{\alpha_s}{\pi} \right)^2 + O(\alpha_s^3) \right\}.
\end{aligned}
\tag{42}
$$

We use here the pole masses presented in Table I. They correspond to the conventional current masses $\bar{m}_c = 1.18$ GeV and $\bar{m}_b = 4.20$ GeV while for the strange quark the pole mass, $m_s = 180$ MeV is taken at the scale $\mu \approx 0.5$ GeV ($r_0 \approx 0.5$ fm) and therefore it should be larger than the standard $m_s(2 \text{ GeV}) \approx 100$ MeV [16]. For the u and d quarks $m_u = 5$ MeV, $m_d = 8$ MeV are taken. One can see the essential difference between the input masses in our SH and the constituent masses used in the relativistic instantaneous Bethe-Salpeter method [5], and also in the spinless Salpeter equation [2]. The b, c, and s quark masses from [4], where the relativistic Dirac Hamiltonian is used for a light quark, are not large and can be considered as the pole masses, while $m_u = m_d = 71$ MeV seem to be too large.

(7) The coupling $\alpha_{st}(r)$ in the GE term is taken here as the vector coupling $\alpha_B(r)$ in background perturbation theory (in two-loop approximation) from [39]. At large distances this theory predicts saturation: $\alpha_B(r) \to \alpha_{crit} = \text{const}$ (this coupling $\alpha_B(r)$ is in good agreement with lattice potential at small distances [40]):

$$
\alpha_B(r) = \frac{2}{\pi} \int_0^\infty dq \frac{\sin qr}{q} \alpha_B(q),
$$

$$
\alpha_B(q) = \frac{4\pi}{\beta_0 t_B} \left\{ 1 - \frac{\beta_1}{\beta_0^2} \frac{\ln t_B}{t_B} \right\},
\tag{43}
$$

$$
t_B = \ln \frac{q^2 + M_B^2}{\Lambda_B^2}.
$$

TABLE I. The pole quark masses m_q (pole) (in GeV), used in this paper, and the constituent masses m_q^C from [2,4,5].

Quark	b	c	s	u	d
m_q (pole) this paper	4.78	1.40	0.180	0.005	0.008
m_q from [4]	4.655	1.511	0.216	0.071	0.071
m_q^C from [2]	4.977	1.628	0.419	0.220	0.220
m_q^C from [5]	5.158	1.755	0.535	0.371	0.377

Here the background mass $M_B = 1$ GeV can be expressed through the $\sqrt{\sigma}$ [38] while the QCD constant Λ_B is given by

$$\Lambda_B = \Lambda_{\overline{MS}} \exp\left(\frac{\frac{31}{3} - \frac{10}{9} n_f}{2\beta_0}\right). \qquad (44)$$

It is of interest to notice that in our calculations of HL meson properties the value $n_f = 3$ is strongly preferable.

In this way the problem is uniquely defined and no fitting parameters are introduced. The resulting masses and decay constants of HL mesons appear to be unbiased theoretical predictions which will be compared with recent experiments, lattice data, and other theoretical predictions.

III. MASSES OF HEAVY-LIGHT MESONS

In the relativistic SH a correct choice of the static interaction $V_0(r)$, which defines the e.v. of the unperturbed Hamiltonian H_0 (31), is of great importance. Here, for all HL mesons we take the static potential from [39]:

$$V_B(r) = \sigma r - \frac{4}{3} \frac{\alpha_B(r)}{r} \qquad (45)$$

with $\sigma = 0.18$ GeV^2, the number of flavors $n_f = 3$, the vector coupling $\alpha_B(r)$ (43) from [38,39], which also contains

$$\Lambda_B(n_f = 3) = 360 \text{ MeV}, \qquad M_B = 1.0 \text{ GeV}. \qquad (46)$$

For this choice the saturated (critical) value of the vector coupling, reached at large r, is $\alpha_{\text{crit}}(n_f = 3) = 0.495$, while at $r \approx r_0 = 0.5$ fm, $\alpha_B(r_0) \cong 0.43$.

It is very convenient to start the calculations with the spin-averaged masses $M_{\text{cog}}(nL)$,

$$M_{\text{cog}}(nS) = M_0(nS) + \Delta_{\text{SE}}(nS),$$
$$M_{\text{cog}}(nL) = M_0(nL) + \Delta_{\text{SE}}(nL) + \Delta_{\text{str}}(nL) \qquad (L \neq 0),$$
$$\qquad (47)$$

since these masses do not depend on any additional parameter. In contrast to M_{cog} the masses of the singlet and triplet states depend also on the parameters defining the HF interaction, or for states with $L \neq 0$ the masses $M(J^{PC})$ depend also on the coupling $\alpha_{\text{FS}}(\mu_{\text{FS}})$, which defines the fine-structure splittings, and these couplings are not well determined. In Eq. (47) $M_0(nL)$ is the e.v. of Eq. (31) (see Tables II and V). The self-energy term $\Delta_{\text{SE}}(nL)$ (32) and the string correction $\Delta_{\text{str}}(nL) = \langle H_{\text{str}} \rangle$ (D12) are defined by analytical expressions and discussed in Appendices D and E.

The calculated $M_{\text{cog}}(1S)$, as seen from Table II, agree with the experimental numbers with an accuracy better than 5 MeV. It is of interest to notice that the difference $M_{\text{cog}}(D_s) - M_{\text{cog}}(D)$, which in experiment is 101 MeV, is only partly due to dynamical reasons: $M_0(D_s) - M_0(D) =$

TABLE II. The spin-averaged masses $M_{\text{cog}}(1S)$ (in MeV)[a].

Meson multiplet	$M_0(1S)$	Δ_{SE}	$M_{\text{cog}}(1S)$	$M_{\text{cog}}(\text{exp})$[b]
$D - D^*$	2139	-164	1975	1974.8(4)
$D_s - D_s^*$	2177	-105	2072	2076.0(6)
$B - B^*$	5433	-120	5313	5313.5(6)
$B_s - B_s^*$	5468	-72	5396	5400.7(32)
$B_c - B_c^*$	6332	-17	6315	-

[a]The Δ_{SE} are calculated in Appendix E.
[b]The experimental numbers are taken from PDG [16,41].

38 MeV, but mostly occurs due to the difference in the SE contributions, equal to 60 MeV.

With the use of our number $M_{\text{cog}}(B_c) = 6315$ MeV and $M(B_c)_{\text{exp}} = 6275(7)$ MeV [41] one can predict the mass of the vector state $B_c^*(1^3S_1)$:

$$M(B_c^*) = 6328(7) \text{ MeV}, \quad \text{if } M(B_c) = 6275(7) \text{ MeV}. \qquad (48)$$

The masses of the triplet and singlet HL mesons are calculated taking into account the HF interaction which in general contains both perturbative and NP contributions:

$$\Delta_{\text{HF}}(nS) = \Delta_{\text{HF}}^{\text{P}} + \Delta_{\text{HF}}^{\text{NP}}, \qquad (49)$$

where the P term with the one-loop correction is taken from [42]:

$$\Delta_{\text{HF}}^{\text{P}}(nS) = \frac{8}{9} \frac{\alpha_{\text{HF}}(\mu)}{\omega_1 \omega_2} |R_n(0)|^2 \left(1 + \frac{\alpha_{\text{HF}}}{\pi} \rho\right). \qquad (50)$$

Here $\rho = \frac{5}{12}\beta_0 - \frac{8}{3} - \frac{3}{4} \ln 2$ and for $n_f = 3$, $\rho(n_f = 3) = 0.5635$, so that the one-loop correction is about 6%. From here on for simplicity we use the notation ω_1, ω_2 instead of $\langle \bar{\omega}_1 \rangle_n$, $\langle \bar{\omega}_2 \rangle_n$.

The NP contribution to the HF splitting was derived in [43] and with good accuracy it is given by the m.e.:

$$\Delta_{\text{HF}}^{\text{NP}}(nS) \approx 1.20 \frac{\pi^2}{18} \frac{G_2}{\omega_1 \omega_2} \left\langle r K_1\left(\frac{r}{T_g}\right) \right\rangle_{nS}. \qquad (51)$$

Here G_2 is the gluonic condensate, for which we take the value $G_2 = 0.043$ GeV^4 [43], which provides the correct value of the string tension $\sigma = 0.18$ GeV^2. For the D and D_s mesons the NP term (51) appears to be not small (as for the $J/\psi - \eta_c$ splitting): around 10 MeV if the gluonic (vacuum) correlation length is $T_g = 0.2$ fm [44]. However, at present the value of T_g is not known with high accuracy and $T_g = 0.28$ fm was obtained from unquenched lattice data [45]. Because of this uncertainty in T_g we have two possibilities to describe the HF effects in HL mesons:

(A) In the case $T_g \approx 0.2$ fm, $\Delta_{\text{HF}}^{\text{NP}} \approx 10$ MeV is not large. Then in the perturbative term, Eq. (50), we need to take a rather large coupling: $\alpha_{\text{HF}} = 0.40$ for the D and D_s mesons and $\alpha_{\text{HF}} = 0.32$ for the B and

B_s mesons, to reach agreement with experiment.

(B) In the case of large gluonic length, $T_g = 0.3$ fm, the NP contribution Δ_{HF}^{NP} appears to be larger, about 22 MeV, and therefore the coupling α_{HF} can be taken smaller. In this case for the D and D_s mesons the value $\alpha_{HF} = 0.365$ (rather close to that for the $J/\Psi - \eta_c$ splitting [46]) gives rise to agreement with the experimental HF splitting. In Table III the calculated HF splittings both for $T_g = 0.2$ fm and $T_g = 0.3$ fm are given.

Taking the HF splittings from Table III the masses of the singlet and triplet states can be calculated; they are presented in Table IV ($M_{cog}(1S)$ are taken from Table II).

The calculated triplet and singlet masses for the ground states appear to be in good agreement with the experimental numbers (with an accuracy $\lesssim 5$ MeV). In our analysis we have observed that for the D and D_s mesons the value $M_{cog}(1S)$ is very sensitive to the pole mass of the c quark and high accuracy can be reached only for $m_c(\text{pole}) = 1.40$ GeV; at the same time we cannot exclude that for the B_c meson the choice of $m_c = 1.38$ GeV would be preferable.

For the B_c meson, with the use of $\alpha_{HF} = 0.26$ and $M_{cog} = 6313$ MeV from Table II, we obtain $\Delta_{HF}(B_c) = 44$ MeV, which gives $M(B_c) = 6280$ MeV close to

TABLE III. The hyperfine splittings (in MeV) in HL mesons. Case A: $T_g = 0.20$ fm, $\alpha_{HF} = 0.40$ for the D and D_s mesons and $\alpha_{HF} = 0.32$ for the B and B_s mesons. Case B: $T_g = 0.30$ fm, $\alpha_{HF} = 0.365$ for the D and D_s mesons and $\alpha_{HF} = 0.305$ for the B and B_s mesons.

Multiplet	Δ_{HF}^{P}	Δ_{HF}^{NP}	Δ_{HF} (tot)	Δ_{HF} (exp)
		$T_g = 0.20$ fm		
$D^* - D$	131.6	8.9	140.5	140.64 ± 0.10
$D_s^* - D_s$	130.7	8.9	139.6	143.8 ± 0.4
$B^* - B$	43.5	1.5	45.0	45.78 ± 0.35
$B_s^* - B_s$	44.2	1.4	45.8	44.2 ± 1.8
		$T_g = 0.30$ fm		
$D^* - D$	119.4	21.8	141.2	140.6 ± 0.1
$D_s^* - D_s$	118.6	21.6	140.2	143.8 ± 0.4
$B^* - B$	41.3	3.7	45.0	45.8 ± 0.4
$B_s - B$	42.0	3.4	45.4	44.2 ± 1.8

TABLE IV. The masses (in MeV) of the ground states (for $T_g = 0.3$ fm).

	$D^{\pm}$	$D^{*\pm}$	D_s	D_s^*	B	B^*	B_s	B_s^*
this paper	1869.1	2010.3	1966.8	2107.1	5279.3	5324.2	5362.0	5407.4
Exper.[a]	1869.3	2010.0	1968.2	2112.0	5279.0	5325.0	5367.7	5411.7
	± 0.4	± 0.4	± 0.4	± 0.6	± 0.5	± 0.6	± 1.8	± 3.2

[a]The experimental numbers are taken from PDG [16] and $M(B_s^*)$ from Ref. [19].

TABLE V. The masses $M_{cog}(2S)$, $M(2^1S_0)$, and $M(2^3S_1)$ (in MeV) for heavy-light mesons ($\alpha_{HF} = 0.30$).

		$D(2S)$	$D_s(2S)$	$B(2S)$	$B_s(2S)$	$B_c(2S)$[a]
this paper	$M_0(2S)$	2758	2797	5998	6034	6868
	$M_{cog}(2S)$	2615	2702	5888	5966	6852
	$M(2^1S_0)$	2560	2646	5864	5941	6821
	$M(2^3S_1)$	2633	2721	5896	5974	6862
from Ref. [4]	$M(2^1S_0)$	2589	2700	5886	5985	
	$M(2^3S_1)$	2692	2806	5920	6019	
from Ref. [2]	$M(2^1S_0)$	2580	2670	5900	5980	6855
	$M(2^3S_1)$	2640	2730	5930	6010	6887

[a]For the $B_c(2S)$ meson we use the value $\alpha_{HF} = 0.26$.

$M(B_c)(\text{exp}) = 6275(7)$ MeV   [41]   and $M(B_c^*) = 6324$ MeV.

Finally, in Table V we give the calculated masses $M(2S)$ for the first radial excitations, which are not yet found in experiment. Our prediction for the singlet and triplet masses of the radially excited HL mesons strongly depends on the value of α_{HF} taken. If for the $D(2S)$ and $D_s(2S)$ mesons one takes the value $\alpha_{HF}(\mu_2) = 0.30$, as in the case of $\eta_c(2S)$ [46], then the values $M(2^1S_0)$ and $M(2^3S_1)$, given in Table V, are obtained.

The triplet and singlet masses, calculated here, are rather close to those from [2,4], nevertheless for the 2^1S_0 states the numbers obtained are systematically lower by ~ 30–50 MeV.

In this paper we do not consider orbital excitations of HL mesons; it will be done in our next paper. Still, we would like to notice that our values of $M(B_{s1})$ and $M(B_{s2})$ lie in the region 5.82–5.83, i.e., approximately 100 MeV lower than in Ref. [4] and close to the recent experimental data [19].

IV. DECAY CONSTANTS OF HEAVY-LIGHT MESONS

The general formula for the decay constants f_Γ^2 (23) can be rewritten (later on we shall use the notation ω_1, ω_2 for $\langle \omega_1 \rangle$, $\langle \omega_2 \rangle$) as follows:

$$(f_\Gamma^n)^2(nS) = \frac{3\langle Y_\Gamma \rangle}{2\pi\omega_1\omega_2 M_n}|R_n(0)|^2, \qquad (52)$$

which contains the w.f. at the origin $R_n(0) = \varphi_n(0)/\sqrt{4\pi}$, the average kinetic energies ω_1, ω_2, the meson mass $M_n = M(nS)$, and also the m.e. $\langle p^2 \rangle$ in the factor $\langle Y_\Gamma \rangle$. All these auxiliary values are given in Appendix D (Tables IX and X). Then the values of f_P can be calculated easily. They are given in Table VI, together with the experimental, unquenched lattice data, and some other theoretical analyses.

From Table VI one can see that our central value f_{B_d} is 15% smaller than the one in unquenched lattice QCD [13], but rather close to f_B in relativistic models [6,7].

TABLE VI.   The decay constants f_P (in MeV).

	D	D_s	B	B_s	B_c
Ref. [20]	206	252	174	-	-
Ref. [7]	230(25)	248(27)	196(29)	216(32)	322(42)
Ref. [6]	234	268	189	218	433
lattice [11,14] quenched	201(20)	249(20)	216(28)	240(30)	420(20)
lattice [11,13] $n_f = 2 + 1$	201(20)	249(19)	216(38)	259(32)	440(2)
this paper	210(10)	260(10)	182(8)	216(8)	438(10)
experiment	222.6(20)[a]	280(23)[a]	160^{+50}_{-80}[b] $229(70)$[c]		

[a]The experimental values are taken from Ref. [17].
[b]*BABAR* data [18].
[c]Belle data [18].

For the analysis of experimental data on direct measurements of the leptonic decay, $P \rightarrow l\nu$, it is important to know the ratios of the decay constants, which in our calculations are

$$\frac{f_{D_s}}{f_D} = 1.22(4), \qquad \frac{f_{B_s}}{f_B} = 1.19(3),$$
$$\frac{f_{D_s}}{f_{B_s}} = 1.20; \qquad \frac{f_D}{f_B} = 1.14(2). \qquad (53)$$

These ratios are in good agreement with recent lattice data (unquenched) [11,13] and close to the experimental number obtained by the CLEO collaboration $f_{D_s}/f_D = 1.27(14)$ [17].

It is of interest to compare these ratios with other theoretical calculations which are typically smaller than our numbers and also with recent lattica data (see Table VII).

Because of the large theoretical errors in the ratios $\zeta_D = f_{D_s}/f_D$ and $\zeta_B = f_{B_s}/f_B$ (see Table VII) one cannot judge what is the true value of ζ_D and ζ_B; however, from our general formula (52) for f_P it follows that f_{D_s} and f_D as well as f_{B_s} and f_B have to differ by $\sim 20\%$. This happens due to the presence in Eq. (52) through $\langle Y_\Gamma \rangle$ of a term proportional to $m_1 m_2$: $m_1 = 0.008$ GeV for D^+ and $m_1 = 0.18$ GeV for D_s^+, $m_2 = 1.40$ in both cases. If one neglects this term then the value $f_{D_d} = f_{D_u} = 208$ MeV, practically does not change, while for the D_s meson one obtains

TABLE VII.   The ratios f_{D_s}/f_D, f_{B_s}/f_B, and f_{D_s}/f_{B_s}.

	f_{D_s}/f_D	f_{B_s}/f_B	f_{D_s}/f_{B_s}[a]
RPM [6]	1.15	1.15	1.23
BS [7]	1.08(1)	1.10(1)	1.15(1)
lattice [13] unquenched	1.24(8)	1.20(4)	1.01(8)
this work	1.24(4)	1.19(3)	1.20
experiment	1.27(14)		

[a]The ratio of the central values is taken for f_{D_s} and f_{B_s}.

the essentially smaller number 215 MeV, instead of $f_{D_s} = 260$ MeV in the case with $m_1 = m_s = 0.18$ GeV. Thus, the ratios ζ_D and ζ_B appear to be very sensitive to what one takes for the pole mass of the s quark and can be used as a convenient criterium to choose m_s at a low renormalization scale. Note that the factor $|R_1(0)|^2/(\omega_1 \omega_2) = 0.35(1)$ turns out to be the same both for the D and D_s mesons. Our number for $\zeta_D = 1.24(3)$ (with $m_s = 0.18$ GeV) is in agreement with the experimental number $\zeta_D(\exp) = 1.27(14)$ [17]. The important role of this relatively large ratio for the theory and its sensitivity to the mass of the s-quark at low scale is discussed in detail in Ref. [47].

The first radial excitations of HL mesons are considered here neglecting open channels which can decrease the w.f. at the origin. In this approximation

$$f_P(D(2S)) = 167 \text{ MeV}, \qquad f_P(D_s(2S)) = 201 \text{ MeV},$$
$$f_P(B(2S)) = 168 \text{ MeV}, \qquad f_P(B_s(2S)) = 194 \text{ MeV},$$
$$f_P(B_c(2S)) = 347 \text{ MeV}. \qquad (54)$$

Our calculations of the $2S$ states are done in single-channel approximation, while the wave functions at the origin in some cases decrease if open channels are taken into account; therefore the values of $f_P(2S)$ in Eq. (54) can be considered as upper limits for the $2S$ decay constants.

From these numbers it follows that for the $2S$ states $f_{D_s}/f_D = 1.20$ and $f_{B_s}/f_B = 1.15$ are changed only by about $\approx 4\%$, although the values of $f_P(2S)$ appear to be $\approx 20\%$ smaller than for the ground states.

One can compare our results for f_P with other theoretical calculations (see Table VI). The agreement with potential-model results [6,7] is evident with the only exception for $f_P(B_c)$: our number $f_P(B_c) = 439$ MeV is about 30% higher and is close to the one calculated in unquenched lattice QCD, where $f_P(B_c)(\text{lattice}) = 420(20)$ MeV in [14]. A detailed analysis of f_{B_c} (in the framework of the potential model and QCD sum rule approach with radiative corrections taken into account)

TABLE VIII.   The vector decay constants f_V (in MeV) and the ratio f_V/f_P for the $1S$ states in heavy-light mesons.

	D^*	D_s^*	B^*	B_s^*	B_c^*
$\langle \boldsymbol{p}^2 \rangle$ in GeV^2	0.273	0.291	0.359	0.383	0.784
f_V [5]	340(23)	375(24)	238(18)	272(20)	418(24)
f_V, this work	273(13)	307(18)	200(10)	230(12)	453(20)
f_V/f_P [5]	1.48(26)	1.51(26)	1.21(27)	1.26(28)	1.30(24)
f_V/f_P this paper	1.27(5)	1.17(4)	1.08(4)	1.07(4)	1.03(3)

[48] gives the value $f_{B_c} \approx 400$ MeV, which is approximately 10% lower than our number.

Finally, some remarks about decay constants in V channels. As seen from the expressions for $\langle Y_V \rangle$ and $\langle Y_P \rangle$, f_V^2 has to be larger than f_P^2 (since M_V and M_P are close to each other), while $\langle \boldsymbol{p}^2 \rangle$ enters with different signs

$$\frac{f_V^2(nS)}{f_P^2(nS)} = \frac{(m_1 m_2 + \omega_1 \omega_2 + \frac{1}{3}\langle \boldsymbol{p}^2 \rangle_{nS})}{(m_1 m_2 + \omega_1 \omega_2 - \langle \boldsymbol{p}^2 \rangle_{nS})} \frac{M_P(nS)}{M_V(nS)} > 1. \tag{55}$$

These ratios (for the ground states) are given in Table VIII. For the heavy B_c meson this ratio is approaching unity.

From Table VIII one can see that our numbers for the vector decay constants are systematically lower than f_V (central values) from Ref. [5]: by $\approx 20\%$ for the D and D_s mesons and by $\approx 15\%$ for the B and B_s mesons, although their values lie within the large theoretical errors.

V. APPROXIMATIONS

We discuss here the approximations we made and the accuracy of our results. The starting expression for the current correlator (8) is exact, because the FFSR is an exact representation of the meson Green's function. The main approximation refers to the transition from the FFSR path integral to the local Hamiltonian formalism (19) neglecting quark pair and hybrid production and taking spin-dependent interactions as a perturbation. The accuracy of this approximation is determined by several factors:

(1) The smallness of the gluon correlation length $T_g \approx 0.2$ fm as compared to the average size of the hadrons under consideration. For the ground states of HL mesons $R_h \sim (0.5-0.7)$ fm, which can yield a correction to the self-energy terms of the order $O(T_g^2/R_h^2) \sim 4-10\%$, or $\lesssim 10$ MeV in mass values.

(2) Neglect of hybrid excitations in the Wilson loop, which actually leads to a multichannel Hamiltonian. The hybrid admixture was shown to be small, of the order of a few percent for ground state mesons [49,50].

(3) The use of the EA to define the w.f. at the origin and some m.e. The corresponding accuracy was checked in [37] and shown to be around 5%.

One should stress that the SH contains only well-defined fundamental parameters: the pole quark masses, Λ_{QCD}, and

the universal string tension σ, and does not contain any fitting parameters, in particular, there is no overall constant often used in the static potential, or in the mass. The accuracy of the SH was checked for light mesons [24], heavy quarkonia [25], heavy-light mesons [26], hybrids [50], and glueballs [51]. In all cases meson masses are in agreement with experimental and lattice data with an accuracy of a few percent.

Concerning the decay constants f_P, a special sensitivity occurs in the w.f. at the origin $|\varphi_n(0)|^2$, i.e., to the behavior of the GE potential (or the vector coupling $\alpha_{st}(r)$) at small distances, not only in the AF region ($r \lesssim 0.1$ fm) but also in the region 0.1 fm $\lesssim r \lesssim 0.3$ fm, in [39] called the intermediate region. This behavior is known quite well for the perturbative part, where Λ_{QCD} is well known: for $n_f = 5$, $\Lambda_{\overline{MS}}^{(5)} = 217^{+25}_{-23}$ MeV [16]. The major uncertainty comes from two sources:

(i) The behavior of the spin-dependent part, in particular, the HF interaction, where smearing of the δ function can drastically change the wave function for systems of small size, $R \lesssim 0.4$ fm. For HL mesons with $R \gtrsim 0.6$ fm this effect is becoming less important (for more discussion of the influence of the HF interaction, see [52]).

(ii) We estimate the accuracy of the resulting $|\varphi_n(0)|^2$ better than $\pm 5\%$. Thus the accuracy of our computed values of f_P is expected to be $\lesssim 8\%$, while the ratio of decay constant has better accuracy, $\lesssim 4\%$.

These accuracy estimates are given for the chosen SH and do not include the accuracy of the method itself. Such an accuracy may be different for different states and physical characteristics and needs further investigations. From our numerous calculations of low-lying states in HL mesons, for which additional pairs creation is not important, it can be estimated to be 10% or better.

At this point one should discuss the effects which were unaccounted for till now. First of all, this concerns the radiative corrections due to transverse gluon exchanges and higher loops. These corrections contain the (pseudo) evolution factor, governed by the operator anomalous dimension, and considered in [53] two decades ago,

$$X_M = \left(\frac{\alpha_s(m_b)}{\alpha_s(m_c)} \right)^{-6/(33-2n_f)} \equiv x^{6/(33-2n_f)}. \tag{56}$$

One can estimate that when going from B to D mesons this factor changes only by 4%.

Nowadays the radiative corrections are done within heavy-quark effective theory (HQET) and known to three-loop accuracy [54], e.g. for f_D one has

$$f_D = \frac{1}{\sqrt{m_c}} \left[1 + c_1 \frac{\alpha_s^{(4)}(m_c)}{4\pi} + c_2^{(3)} \left(\frac{\alpha_s^{(4)}(m_c)}{4\pi} \right)^2 \right.$$
$$\left. + \dots \right] F^{(3)}(m_c) + O\left(\frac{\Lambda_{QCD}}{m_c} \right), \tag{57}$$

where $F^{(3)}(m_c)$ is subject to the operator anomalous dimension correlations, and $c_1 = -2C_f$ etc. As a result, e.g. for the ratio f_B/f_D, one has [54]

$$\frac{f_B}{f_D} = \sqrt{\frac{m_c}{m_b}} X_M \left[1 + r_1(x-1)\frac{\alpha_s^{(4)}(m_b)}{4\pi} + O\left(\left(\frac{\alpha_s^{(4)}}{4\pi}\right)^2\right)\right]$$
$$+ O\left(\alpha_s^3, \frac{\Lambda_{QCD}}{m_{c,b}}\right) \qquad (58)$$

with $r_1 = \frac{56}{75}\zeta_2 + \frac{4403}{1875} \cong 3.58$, and $x \approx 1.56$. One can see that loop corrections contribute less than 4% and can be neglected within the accuracy of HQET and our approximations.

From Eq. (58) it follows that in HQET the ratio $f_B/f_D < 1$ as it happens in relativistic models [2,5–7] and also in our calculations where $f_B/f_D \approx 0.87(2)$, while in unquenched lattice data this ratio is larger than unity, but has large computational error ($\sim 20\%$).

We now turn again to Table VI and discuss our results in comparison to other calculations. The first important point, which should be stressed, is that our input is minimal and fundamental, e.g. the pole masses in Table I correspond to the current masses quoted by PDG [16]. This is in contrast to many relativistic potential models with spinless Salpeter [2] or Bethe-Salpeter Hamiltonian [5,7], where the constituent quark masses are used as input. In addition an overall constant is usually introduced in the interaction. The advantage of our approach is that excited states can be considered as well as the ground states. For excited states a quark (antiquark) of a given nL state has its characteristic "constituent" mass $\langle \bar{\omega} \rangle_{nL}$, which grows for higher nL. Because of this effect, and also owing to the negative string correction, in our approach the masses of the radial and orbital excitations appear to be smaller (~ 20–40 MeV) than in other relativistic models (see Table V).

Looking at Table VI one notices a good agreement (within 6%–10%) of our results with unquenched lattice data and experiment (for f_D and f_{D_s}). Thus one can conclude that the field correlator method and its essential part, the effective string Hamiltonian, appears to be successful in the prediction of f_Γ as well as in other tests done so far [21–26].

VI. CONCLUSIONS

In this paper we have studied the current correlator $G_\Gamma(x) = \langle j_\Gamma(x)j_\Gamma(0)\rangle$ and the integral $J_\Gamma = \int G_\Gamma(x)dx$ in an arbitrary channel Γ with the use of the FFS path-integral representation. This method allows one to express decay constants $f_\Gamma(nS)$ for HL mesons through well-defined characteristics of the relativistic SH which was successfully used before in light mesons and heavy quarkonia.

It is essential that the SH does not contain any fitting parameters, being fully defined by universal fundamental values: the conventional pole masses, the string tension σ, and the strong vector coupling $\alpha_{st}(r)$.

The analytical expressions, obtained here for f_P, show that the decay constants f_{D_s} and f_{B_s} for the ground states have to be always larger by 20%–25% than f_D and f_B because of the large difference between the pole (current) masses of the strange and light $u(d)$ quarks. This theoretical statement is supported by recent experimental data [17].

In our analysis we have observed that

(i) The calculated masses $M(1^1S_0)$ and $M(1^3S_1)$ of all HL mesons agree with the experimental numbers within ± 5 MeV. Our prediction for B_c^* is $M(B_c^*) = 6325(10)$ MeV.

(ii) The calculated masses of the first radial excitations $M(2^1S_0)$ for the D and D_s mesons appear to be ≈ 40 MeV lower and for the B, B_s, and B_c mesons ≈ 20–30 MeV lower than the numbers from Refs. [2,4].

(iii) For the decay constants the values $f_D = 210(10)$ MeV and $f_{D_s} = 260(10)$ MeV are obtained. Their ratio $f_{D_s}/f_D = 1.24(3)$ is close to the experimental number 1.27(14) [17].

(iv) Our decay constants $f_B = 182(8)$ MeV and $f_{B_s} = 216(8)$ MeV give the ratio $f_{B_s}/f_B = 1.19(3)$ which agrees with a recent unquenched lattice number [13].

(v) In the V channel the ratio f_V/f_P is monotonically decreasing while going from the D meson to the heavier mesons: it is equal to 1.27(6), 1.17(4), 1.08(4), 1.07(4), 1.03(3) for the D, D_s, B, B_s, and B_c mesons, respectively. For the D_s^* and B_s^* mesons calculated here, f_V turned out to be $\approx 20\%$ smaller than in Ref. [5].

APPENDIX A: FREE QUARK PROPAGATOR IN THE EINBEIN PATH-INTEGRAL REPRESENTATION

One starts with the FFSR for the free quark propagator, which can be written as

$$S(x, y) = (m - \hat{\partial}) \int_0^\infty ds(Dz)_{xy} \exp(-K), \qquad (A1)$$

and introduces the einbein variable, or the dynamical mass, $\omega(t)$ as in (4), such that the function K can be rewritten as

$$K = m^2 s + \frac{1}{4}\int_0^s \left(\frac{dz_\mu(\tau)}{d\tau}\right)^2 d\tau$$
$$= \int_0^T dt\left\{\frac{m^2}{2\omega(t)} + \frac{\omega(t)}{2} + \frac{\omega(t)}{2}\left(\frac{dz_i(t)}{dt}\right)^2\right\}. \qquad (A2)$$

In $(Dz)_{xy}$, Eq. (3), there is an integration over the time components of the path, namely,

$$(Dz_4) \equiv \prod_k \frac{d\Delta z_4(k)}{(4\pi\varepsilon)^{1/2}} \delta\left(\sum \Delta z_4 - T\right), \qquad (A3)$$

where $T \equiv x_4 - y_4$. With the use of (4) one can rewrite the integration element in (A3) as follows ($t \equiv z_4$):

$$\frac{d\Delta z_4(k)}{\sqrt{4\pi\varepsilon}} = 2d\omega(k)\sqrt{\frac{\varepsilon}{4\pi}} = \frac{d\omega(k)\sqrt{\Delta t}}{\sqrt{2\pi\omega(k)}},$$
$$\sqrt{\varepsilon} = \sqrt{\frac{\Delta t}{2\omega(k)}}. \qquad (A4)$$

Moreover, the δ function in (A3) acquires the form

$$\delta\left(\sum \Delta z_4 - T\right) = \delta(2\bar{\omega}s - T), \qquad (A5)$$

where we have defined

$$\bar{\omega} = \frac{1}{s}\int_0^s \omega(\tau)d\tau. \qquad (A6)$$

As a result, in (A1) one can integrate over ds using the δ function (A5), and rewrite $ds(Dz)_{xy}$ as it is shown in Eq. (5) of the main text.

Then one can write the Green's function as follows:

$$S(x, y) = (m - \hat{\partial})\int \prod \frac{d^3\Delta z_i(k)}{l^3} e^{-K}\frac{d\omega(k)}{l_\omega(k)}$$
$$\times \frac{d^3p}{(2\pi)^3} e^{ip\cdot(x-y-\sum \Delta z(k))}, \qquad (A7)$$

where K is given in (A2), and l, l_μ in Eq. (6).

The integration over $d^3\Delta z_i(k)$ yields

$$S(x, y) = (m - \hat{\partial})\int \frac{d^3p}{(2\pi)^3}$$
$$\times e^{ip(x-y)-(1/2)\int_0^T dt\omega(t)(1+(p^2+m^2)/\omega^2(t))}\frac{1}{2\bar{\omega}}(D\omega). \qquad (A8)$$

Taking into account the relation,

$$\int_0^\infty \frac{d\omega(k)}{\sqrt{\omega(k)}} e^{-(\Delta t/2)(\omega(k)+(p^2+m^2)/\omega(k))} = \sqrt{\frac{2\pi}{\Delta t}} e^{-\Delta t\sqrt{p^2+m^2}}, \qquad (A9)$$

one has for the scalar part $G(x, y)$, defined by $S = (m - \hat{\partial})G$, the following expression:

$$G(x, y) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip\cdot(x-y)-\int_0^T dt\sqrt{p^2+m^2}}}{2\sqrt{p^2+m^2}}, \qquad (A10)$$

where we have used the relation following from the stationary point in the integral (A9):

$$\bar{\omega} = \frac{1}{s}\int_0^s \omega(\tau)d\tau = \frac{1}{N}\sum_{k=1}^N \omega(k) = \sqrt{p^2 + m^2}. \quad (A11)$$

The expression (A10) can be compared with the conventional integral,

$$G(r, T) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot r+ip_4 T}}{p_4^2 + p^2 + m^2}, \qquad r = x - y,$$
$$(A12)$$

which reduces to (A10) after integrating over dp_4 for $T > 0$. The expression (A12) is the standard form of the free propagator.

APPENDIX B: CALCULATION OF THE FACTOR $\langle Y_\Gamma \rangle$

Here we use and extend Appendix 1 of Ref. [20], considering the quark propagator (1) in the gluonic field A_μ: $S(x, y) \equiv [m - \gamma_\mu(\partial/\partial x_\mu - igA_\mu)]G(x, y)$. The derivative $\partial/\partial x_\mu$, acting on $G(x, y)$, differentiates only the δ function, $\delta(x_\mu - y_\mu - \sum_{k=1}^N \Delta z_\mu)$, and can be rewritten as a derivative in $\Delta z_\mu(n)$:

$$\frac{\partial}{\partial x_\mu}\delta\left(x - y - \sum_{k=1}^N \Delta z(k)\right) = -\frac{\partial}{\partial\Delta z_\mu(N)}\delta\left(x - y\right.$$
$$\left. - \sum_{k=1}^N \Delta z(k)\right). \qquad (B1)$$

Integrating by parts in the expression for $G(x, y)$, one obtains

$$D_\mu G(x, y) = \int_0^\infty ds e^{-ms}(Dz)_{xy}e^{-K}\Phi_\sigma(x, y)$$
$$\times \left(-\frac{\Delta z_\mu(N)}{2\varepsilon} + O(\sqrt{\varepsilon})\right). \qquad (B2)$$

In the limit $\varepsilon \to 0$, $N \to \infty$ one has $\frac{\Delta z_\mu(k)}{2\varepsilon} \to \frac{dz_\mu(\tau)}{2d\tau}|_{\tau=s}$ and using the relation (4), one obtains

$$D_\mu G(x, y) = -\omega(\tau = s)\frac{dz_\mu(t)}{dt}G(x, y), \qquad (B3)$$

where the right-hand side of (B3) is a symbolic writing implying that $\omega\frac{dz_\mu}{dt}$ should be under the integral in $G(x, y)$. Finally, when $G(x, y)$ is expressed via the Hamiltonian $\hat{H}_\omega$ [as in the definitions (19) and (21)], then one realizes that $\omega_i\dot{z}_\mu^{(i)}(t) = p_\mu^{(i)}$ and finds the relation:

$$(m - \hat{D})G = (m - i\hat{p})G, \qquad (B4)$$

which will be used throughout the paper.

As a check one can see that (B4) yields the correct form of the free quark propagator in Euclidean space-time.

Consider now a meson in the c.m. system and take into account that for a quark one has

$$D_\mu^{(1)} \Rightarrow -\omega_1 \dot{z}_\mu^{(1)} \Rightarrow i p_\mu^{(1)}, \qquad D_\mu^{(2)} = \omega_2 \dot{z}_\mu^{(2)} = -i p_\mu^{(2)},$$

$$\text{(B5)}$$

and $p_4^{(1)} = i\omega_1$, $p_4^{(2)} = i\omega_2$, so that $D_4^{(1)} \Rightarrow -\omega_1$, $D_4^{(2)} \Rightarrow \omega_2$, while with the 3-momentum $\boldsymbol{p}$, $p_i^{(1)} = -p_i^{(2)} = p_i$

$$D_i^{(1)} \Rightarrow +i p_i, \qquad D_i^{(2)} \Rightarrow i p_i. \qquad \text{(B6)}$$

For the factor Y_Γ (9) one obtains

$$Y_\Gamma = \tfrac{1}{4} \text{tr}(\Gamma(m_1 + \omega_1 \gamma_4 - i p_k \gamma_k)\Gamma(m_2 - \omega_2 \gamma_4 - i p_i \gamma_i)).$$

$$\text{(B7)}$$

Inserting $\Gamma = 1$, γ_5, $i\gamma_\mu \gamma_5$, γ_μ in (B7) one arrives at the expressions (35) for different channels.

APPENDIX C: CALCULATION OF THE PATH INTEGRAL FOR G_Γ, [EQ. (8)], IN THE FREE QUARK CASE

To find the solution in the general case with $W_\sigma \neq 1$ and an interaction depending only on the relative quark-antiquark coordinates, we separate here the relative and c.m. coordinates for any path-integral index k as follows:

$$\Delta z_1 - \Delta z_2 = \boldsymbol{\eta}, \qquad \frac{\omega_1 \Delta z_1 + \omega_2 \Delta z_2}{\omega_1 + \omega_2} = \boldsymbol{\rho},$$

$$\omega_+ = \omega_1 + \omega_2, \qquad \omega_r = \frac{\omega_1 \omega_2}{\omega_+}, \qquad \text{(C1)}$$

with the Jacobians:

$$d^3 \Delta z_1 d^3 z_2 = d^3 \eta d^3 \rho,$$

$$\frac{d\omega_1 d\omega_2}{\sqrt{\omega_1 \omega_2}} = \frac{d\omega_r}{\sqrt{\omega_r}} \frac{d\omega_+}{2\sqrt{\omega_+ - 4\omega_r}}. \qquad \text{(C2)}$$

Integrating (8) for $W_\sigma \equiv 1$ over $\prod_k d\boldsymbol{\rho}(k)$ one obtains

$$G_\Gamma(x) = 4 N_c Y_\Gamma \int \frac{(D^3 \eta)_{00}}{l_\eta^3 4 \bar{\omega}_1 \bar{\omega}_2} \prod_k \frac{d\omega_+(k)}{\sqrt{\frac{2\pi}{\Delta t}} 2 \sqrt{\omega_+ - 4\omega_r(k)}}$$

$$\times \frac{d\omega_r(k)}{\sqrt{\omega_r(k) \frac{2\pi}{\Delta t}}} e^{-\mathcal{F}_1}. \qquad \text{(C3)}$$

Here the notation $(D^3 \eta)_{ab}$ means the initial a and final b value of relative coordinate; in our case, evidently, for considered current correlator $a = b = 0$. Also in Eq. (C3) we use the notation l_η which was given in Eq. (6). The quantity $\mathcal{F}_1$ is defined as

$$\mathcal{F}_1 = \sum_k \left\{ \frac{\omega_+(k)\Delta t}{2} + \frac{m^2 \Delta t}{2\omega_r(k)} + \frac{\omega_r(k)\eta^2(k)}{2\Delta t} \right\}. \quad \text{(C4)}$$

Integration over $d\omega_+(k)$ can be done easily,

$$\prod_k \int \frac{d\omega_+(k)}{2\sqrt{\omega_+ - 4\omega_r(k)}} \frac{\exp\{-\sum \omega_+(k)\frac{\Delta t}{2}\}}{\sqrt{2\pi \Delta t}}$$

$$= \prod_k 2 e^{-\sum\limits_k 2\omega_r(k)\Delta t}. \qquad \text{(C5)}$$

Thus at zero c.m. momentum one finds

$$\int G_\Gamma(x) d^3 \boldsymbol{x} = \int \frac{N_c Y_\Gamma}{\bar{\omega}_1 \bar{\omega}_2} \frac{(D^3 \eta)_{00}}{l_\eta^3} \prod_{k=1}^N \frac{2 d\omega_r(k)}{\sqrt{\frac{\omega_r(k) 2\pi}{\Delta t}}} e^{-\mathcal{F}_2},$$

$$\text{(C6)}$$

where

$$\mathcal{F}_2 = \sum_k \left\{ 2\omega_r(k)\Delta t + \frac{m^2}{2\omega_r(k)} \Delta t + \frac{\omega_r(k)}{2\Delta t} \eta^2(k) \right\}. \quad \text{(C7)}$$

At this point one can use the general relation [30,36]

$$\int \frac{(D^3 \eta)_{xy}}{l_\eta^3} e^{-\sum\limits_k ((\omega_r(k)\eta^2(k)/2\Delta t) + \hat{V}(k)\Delta t)} = \langle x | e^{-\hat{H}T} | y \rangle \quad \text{(C8)}$$

with

$$\hat{H} = \frac{\boldsymbol{p}_\eta^2}{2\omega_r} + \hat{V}(\eta), \qquad \boldsymbol{p}_\eta = \frac{1}{i} \frac{\partial}{\partial \boldsymbol{\eta}}. \qquad \text{(C9)}$$

Then for the free case, $\hat{V}(\eta) \equiv 0$, the right-hand side of (C8) yields

$$\left\langle 0 \left| \exp\left(-\frac{\boldsymbol{p}^2}{2\omega_r} T\right) \right| 0 \right\rangle = \frac{d^3 p}{(2\pi)^3} \exp\left(-\frac{\boldsymbol{p}^2}{2\omega_r} T\right). \quad \text{(C10)}$$

Integrating over $d\omega_r(k)$ and performing the steepest descent method (stationary point analysis) one obtains

$$\int \frac{2 d\omega_r(k)}{\sqrt{\omega_r(k)} \sqrt{\frac{2\pi}{\Delta t}}} e^{-2\omega_r(k)\Delta t - ((\boldsymbol{p}^2 + m^2)\Delta t / 2\omega_r(k))} = e^{-2\sqrt{\boldsymbol{p}^2 + m^2}\Delta t},$$

$$\text{(C11)}$$

and

$$\int G_\Gamma(x) d^3 x = \int \frac{d^3 \boldsymbol{p}}{(2\pi)^3} \frac{N_c Y_\Gamma}{\bar{\omega}_1 \bar{\omega}_2} e^{-2T\sqrt{\boldsymbol{p}^2 + m^2}}. \quad \text{(C12)}$$

For equal current masses $m_1 = m_2 = m$ one has evidently $\bar{\omega}_1 = \bar{\omega}_2 = 2\omega_r$, and from (C11) it follows that for the stationary point (for any k) $2\omega_r = \sqrt{\boldsymbol{p}^2 + m^2}$.

To compare (C12) with the standard Feynman amplitude for the free quark loop, one can go to the momentum space,

$$G_\Gamma(\mathbf{q}=0,q_4) = \int_{-\infty}^{\infty} dT e^{iq_4 T} \int G_\Gamma(x) d^3 x$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{N_c Y_\Gamma}{\mathbf{p}^2 + m^2} \left(\frac{1}{2\sqrt{\mathbf{p}^2 + m^2} - iq_4} \right.$$

$$\left. + \frac{1}{2\sqrt{\mathbf{p}^2 + m^2} + iq_4} \right)$$

$$= \int 4N_c Y_\Gamma \frac{d^3 p}{(2\pi)^3 \sqrt{\mathbf{p}^2 + m^2} [4(\mathbf{p}^2 + m^2) + q_4^2]}$$

$$= \int \frac{4N_c Y_\Gamma d^4 p}{(2\pi)^4 p^2 (p-q)^2} \qquad (C13)$$

This result proves our sequence of equations for the free case. For the case of interacting quarks, one should use in (C8) the spectral decomposition in the infinite set of bound states, as it is done in the main text, Eqs. (23), (25), and (28).

APPENDIX D: THE RELATIVISTIC STRING HAMILTONIAN

We start with the relativistic SH $\hat{H}_\omega$ for a meson $q_1 \bar{q}_2$, taken in the most general form [22]:

$$\hat{H}_\omega = \sum_{i=1,2} \left(\frac{\omega_i}{2} + \frac{m_i^2}{2\omega_i} \right) + \frac{p_r^2}{2\omega_r} + V_{\mathrm{GE}}(r)$$

$$+ \int_0^1 d\beta \left(\frac{\sigma^2 r^2}{2\nu} + \frac{\nu}{2} \right) + \frac{\mathbf{L}^2}{2r^2} \frac{1}{g(\omega_1, \omega_2, \sigma r)}. \qquad (D1)$$

Here $\omega_r = \omega_1 \omega_2 / (\omega_1 + \omega_2)$ and m_i $(i = 1, 2)$ are the pole masses of a quark (antiquark) and

$$g(\omega_1, \omega_2, \sigma r) = \omega_1 (1 - \zeta)^2 + \omega \zeta^2$$

$$+ \int_0^1 d\beta (\beta - \zeta)^2 \nu d\beta, \qquad (D2)$$

with

$$\zeta = \frac{\omega_1 + \frac{1}{2}\sigma r}{\omega_1 + \omega_2 + \sigma r}. \qquad (D3)$$

In the SH (D1) taken from [22] we have added the GE potential, $V_{\mathrm{GE}} = -\frac{4}{3} \frac{\alpha_{\mathrm{st}}(r)}{r}$. This can be done due to the property of additivity of the static potential in QCD [40,41]. $\hat{H}_\omega$ depends on the variables ω_1, ω_2, and ν, which have been shown to be the canonical variables of the SH [26], and therefore they can be defined from the extremum conditions. But first, instead of the operator p_r^2 in (D1) we introduce $\mathbf{p}^2 = p_r^2 + \mathbf{L}^2/r^2$ and present $\hat{H}_\omega$ as the sum of two terms:

$$\hat{H}_\omega = H_\omega^{(0)} + H_{\mathrm{str}}, \qquad (D4)$$

where

$$H_\omega^{(0)} = \sum_{i=1,2} \left(\frac{\omega_i}{2} + \frac{m_i^2}{2\omega_i} \right) + \frac{\mathbf{p}^2}{2\omega_r} + V_{\mathrm{GE}}(r)$$

$$+ \int_0^1 d\beta \left(\frac{\sigma^2 r^2}{2\nu} + \frac{\nu}{2} \right), \qquad (D5)$$

and the "string" part of the SH is

$$H_{\mathrm{str}} = -\frac{\mathbf{L}^2}{2r^2 \omega_r} \left[1 - \frac{\omega_r}{g(\omega_1, \omega_2, r)} \right]. \qquad (D6)$$

This term occurs only for the states with $L \neq 0$ and can be considered as a perturbation, because it gives corrections $\lesssim 5\%$ to the e.v. of the unperturbed Hamiltonian $H_\omega^{(0)}$ (D5). (For light mesons this correction is becoming rather large only for the states with $L \gtrsim 5$). Then to determine the variables ω_1, ω_2, and ν we use the following extremum conditions applied to $H_\omega^{(0)}$:

$$\frac{\partial H_\omega^{(0)}}{\partial \omega_i} = 0, \quad (i = 1, 2); \qquad \frac{\partial H_\omega^{(0)}}{\partial \nu} = 0. \qquad (D7)$$

From (D7) it follows that

$$\omega_i^2 = \mathbf{p}^2 + m_i^2, \qquad \nu = \sigma r, \qquad (D8)$$

and therefore $H_\omega^{(0)}$ can be rewritten as

$$H_\omega^{(0)} = \sum_{i=1,2} \sqrt{m_i^2 + \mathbf{p}^2} + \sigma r + V_{\mathrm{GE}}(r) \equiv T_R + V_0(r), \qquad (D9)$$

where $V_0(r)$ is just the same potential as in (14). From (D9) one can see that the kinetic term T_R coincides with the one in the spinless Salpeter equation (SSE). The equation

$$(H_\omega^{(0)} + V_0(r))\varphi_{nL}(r) = M_0 \varphi_{nL}(r) \qquad (D10)$$

defines the e.v. $M_0(nL)$ and eigenfunction $\varphi_{nL}(r)$. There are two essential differences between our SH $H_\omega^{(0)}$ with the kinetic part

$$T_R = \sqrt{m_1^2 + \mathbf{p}^2} + \sqrt{m_2^2 + \mathbf{p}^2} \qquad (D11)$$

and many other papers where the SSE is used.

The first one is that in (D11) the quark (antiquark) mass is the pole (current) mass and it is not considered to be a fitting parameter. In Table I we compare the input masses m_1, m_2 used in (D11) and the constituent masses from [2,4,5].

The second difference refers to the string correction H_{str} which can be rewritten as

$$H_{\mathrm{str}} = -\frac{\mathbf{L}^2 \sigma}{2\omega_r} \left\{ \frac{g_1(\omega_1, \omega_2, r)}{r} \right.$$

$$\left. + \frac{1}{4} \sigma \frac{(\omega_1 - \omega_2)^2}{(\omega_1 + \omega_2)^2} g_2(\omega_1, \omega_2, r) \right\}. \qquad (D12)$$

Here

$$g_1 = (\tfrac{1}{3} - \zeta + \zeta^2)(\omega_1 + \omega_2 + \sigma r)^2 \mathcal{F}^{-1}(r, \omega_1, \omega_2),$$
$$g_2 = \mathcal{F}^{-1}(r, \omega_1, \omega_2), \qquad (D13)$$

with

$$\mathcal{F}(r_1, \omega_1, \omega_2) = \omega_1(\omega_2 + \tfrac{1}{2}\sigma r)^2 + \omega_2(\omega_1 + \tfrac{1}{2}\sigma r)^2$$
$$+ \sigma r(\omega_1 + \omega_2 + \sigma r)^2(\tfrac{1}{3} - \zeta + \zeta^2). \qquad (D14)$$

The string correction to a meson mass, $\Delta_{\text{str}}(nL) = \langle H_{\text{str}}\rangle$, can be calculated with the use of the expressions (D12)–(D14), in which ω_1, ω_2 can be replaced by their averaged values with good accuracy.

For light mesons with $m_1 = m_2 = 0$, $\omega_1 = \omega_2 = \omega$, $\sigma\langle r\rangle = 2\omega$, and $\zeta = \tfrac{1}{2}$, the second term in (D12) is absent and

$$\langle \mathcal{F}(r, \omega)\rangle = \frac{32}{3}\omega^3, \qquad \langle g_1\rangle = \frac{1}{8\omega}; \qquad \langle g_2\rangle = \frac{3}{32\omega^3}, \qquad (D15)$$

so that in this case ($m_1 = m_2 = 0$) the string correction,

$$\Delta_{\text{str}} = \langle H_{\text{str}}\rangle = -\frac{L(L+1)}{\omega}\frac{\sigma\langle r^{-1}\rangle}{8\omega}, \qquad (D16)$$

just coincides with the string correction obtained in [24] for light mesons. It is important that due to the negative sign of the string correction the masses of P- and D-wave heavy-light mesons appear to be 30–50 MeV smaller in our calculations than in other relativistic models which use the SSE equation [2].

In Table IX we give the values of the average kinetic energies $\bar{\omega}_1$ and $\bar{\omega}_2$, and the excitation energy $\varepsilon(\omega_{\text{r}})$, which are needed to determine the self-energy contribution (32) to $M_{\text{cog}}(nS)$.

To calculate the decay constants f_P and f_V we need also to know the w.f. at the origin $\varphi_n(0) = R_n(0)/\sqrt{4\pi}$ and the m.e. $\langle p^2\rangle_{nS}$. Their values are given in Table X.

As seen from Table IX, the constituent masses $\omega_1(2S)$ and $\omega_1(1S)$ of the lighter quark q_1(or $\bar{q}_1$) differ by $\sim$130 MeV for the D and D_s mesons and $\sim$150 MeV for

TABLE IX. The average energies $\bar{\omega}_i(nS) = \langle\sqrt{m_i^2 + p^2}\rangle_{nS}$ ($i = 1, 2$), the reduced mass ω_{r}, and the excitation energy $\varepsilon_n(\omega_{\text{r}})$ (in MeV) for the 1S and 2S heavy-light mesons.

Meson	D	D_s	B	B_s	B_c
$\bar{\omega}_1(1S)$	507	559	587	639	1662
$\bar{\omega}_2(1S)$	1509	1515	4827	4830	4869
$\omega_{\text{r}}(1S)$	379	408	523	564	1238
$\varepsilon_1(\omega_{\text{r}})$	541	534	432	406	149
$\bar{\omega}_1(2S)$	643	692	741	789	1732
$\bar{\omega}_2(2S)$	1585	1590	4862	4865	4898
$\omega_{\text{r}}(2S)$	457	482	643	679	1279
$\varepsilon_2(\omega_{\text{r}})$	1164	1124	985	959	687

TABLE X. The w.f. at the origin $|R_n(0)|^2$ (in GeV^3) and $\langle p^2\rangle_{nS}$ (in GeV^2) in einbein approximation for the 1S and 2S states of heavy-light mesons.

Meson	1S		2S	
	$|R_1(0)|^2$	$\langle p^2\rangle_{1S}$	$|R_2(0)|^2$	$\langle p^2\rangle_{2S}$
D	0.268	0.273	0.266	0.464
D_s	0.291	0.290	0.284	0.482
B	0.410	0.359	0.410	0.599
B_s	0.455	0.383	0.439	0.624
B_c	1.470	0.784	1.032	1.023

the B and B_s mesons. This difference in $\omega_1(nS)$ illustrates the statement that in a relativistic approach there does not exist a universal constituent mass ω_i for different nL states, but only the pole (current) quark masses may be considered as universal input.

At this point we remind that the expressions (28) and (35) for the decay constants have been derived in the EA approximation and therefore to have a consistent description one needs to take also $|R_n(0)|^2$ and $\langle p^2\rangle_{nS}$ as calculated in the EA. In this approximation, instead of the extremum conditions (D7), different conditions may be used, [26], namely,

$$\frac{\partial M^{\text{EA}}}{\partial \tilde{\omega}_1} = 0, \qquad \frac{\partial M^{\text{EA}}}{\partial \tilde{\omega}_2} = 0, \qquad (D17)$$

where in the EA we use the notations $\tilde{\omega}_1$ and $\tilde{\omega}_2$ instead of $\bar{\omega}_1$ and $\bar{\omega}_2$. The mass $M^{\text{EA}}(nL)$ satisfies the equation,

$$H_0^{\text{EA}}\tilde{\varphi}_{nL}(r) = M_0^{\text{EA}}(nL)\tilde{\varphi}_{nL}(r), \qquad (D18)$$

where the EA Hamiltonian [26] is given by,

$$H_0^{\text{EA}} = \sum_{i=1,2}\left(\frac{\tilde{\omega}_i}{2} + \frac{m_i^2}{2\tilde{\omega}_i}\right) + \frac{p^2}{2\omega_{\text{r}}} + V_0(r), \qquad (D19)$$

and has the same interaction $V_0(r)$ and the reduced mass is given by $\tilde{\omega}_{\text{r}} = (\tilde{\omega}_1\tilde{\omega}_2)/(\tilde{\omega}_1 + \tilde{\omega}_2)$. Then writing

$$M^{\text{EA}}(nL) = \sum_{i=1,2}\left(\frac{\tilde{\omega}_i}{2} + \frac{m_i^2}{2\tilde{\omega}_i}\right) + \varepsilon_{nL}(\omega_r), \qquad (D20)$$

the excitation energy $\varepsilon(\omega_r)$ satisfies the equation

$$\left\{\frac{p^2}{2\tilde{\omega}_{\text{r}}} + V_0(r)\right\}\tilde{\varphi}_{nL}(r) = \varepsilon_{nL}(\omega_{\text{r}})\tilde{\varphi}_{nL}(r). \qquad (D21)$$

With the use of (D20) the extremum conditions (D17) reduce to the equations:

$$\tilde{\omega}_i^2 = m_i^2 - 2\tilde{\omega}_r^2\frac{\partial\varepsilon}{\partial\tilde{\omega}_r} \qquad (i = 1, 2). \qquad (D22)$$

The derivative $\partial\varepsilon(\tilde{\omega}_{\text{r}})/\partial\tilde{\omega}_{\text{r}}$ in (D22) is a very smooth function of the variable $\tilde{\omega}_{\text{r}}(nL)$ and can be calculated easily. Equation (D21) formally coincides with the Schrödinger equation but differs from the physical point

of view: the masses $\tilde{\omega}_i(nL)$ ($i = 1, 2$) and $\tilde{\omega}_r(nL)$), being the average kinetic energy of a quark (antiquark), are different for every nL state. In this way relativistic corrections are taken into account in Eq. (D21) through the increase of $\tilde{\omega}_1$ and $\tilde{\omega}_2$.

In the EA the values of $\tilde{\omega}_i$ appear to be very close to $\bar{\omega}_i$, determined from the SSE (26) or (D10). Therefore, one can define $\omega_i(nL)$, solving the SSE once and calculating the m.e.

$$\bar{\omega}_i(nL) = \langle\sqrt{\boldsymbol{p}^2 + m^2}\rangle_{nL}, \qquad (D23)$$

instead of calculations of a variety of m.e. in the solutions of (D21) for different nL states. However, even for $\tilde{\omega}_i = \omega_i$ the e.v. $M_0^{EA}(nL)$ (D18) slightly differs from $M_0(nL)$ for the SSE with the same quark pole mass m_q. The differences between them may be called the relativistic correction δ_R:

$$M_0(nL) \equiv M_0^{EA}(nL) - \delta_R, \qquad (D24)$$

which can be approximately calculated,

$$\delta_R \approx \frac{\langle\boldsymbol{p}^2\rangle_R + m_1^2 - \omega_1^2}{2\omega_1}. \qquad (D25)$$

In (D25) the index "1" refers to the lighter quark (antiquark), its pole mass m_1 and kinetic energy ω_1 (D23). Such a difference between M_0 and M_0^{EA} occurs due to the fact that

$$\sqrt{\langle\boldsymbol{p}^2 + m_i^2\rangle_{nL}} \neq \langle\sqrt{\boldsymbol{p}^2 + m_i^2}\rangle_{nL}. \qquad (D26)$$

Surprisingly, for the D and D_s and B and B_s mesons δ_R remains almost constant, being approximately equal to 70 MeV for the ground states and about 100 MeV for the $2S$ states. For the heavy B_c meson $\delta_R \approx 20$ MeV is essentially smaller.

APPENDIX E: THE SELF-ENERGY CORRECTION TO THE MESON MASS

The self-energy correction to the meson mass originates from the NP contribution to the squared quark mass m_q^2 as a result of the spin interaction of a quark with NP background gluonic field [35]. It was shown that only due to the presence of this correction in the meson mass it is possible to obtain a linear Regge trajectory for light mesons [24]. With the use of the old result from [35] and the recent result from [55] it can be presented as

$$\Delta_{SE}(nL) = \sum_{i=1,2} \left(-\frac{1.5\sigma\eta_f^i}{\pi\bar{\omega}_i} + \frac{\sigma^2}{4\bar{\omega}_i[m + \bar{\omega}_i/2 + \varepsilon(\omega_r)]^2}\right). \qquad (E1)$$

Here m_i ($i = 1, 2$) is the pole mass of ith quark (antiquark), $\bar{\omega}_i$ is determined by (D7), ω_r is the reduced mass, and $\varepsilon(\omega_r)$ is defined by the solution of Eq. (D21). The factor η_f^i depends on the flavor of a given quark (antiquark) $q_i(\bar{q}_i)$. The first, negative term in (E1) was calculated in Simonov's paper [35] while the second, positive term is

rather small and has been derived recently [55]. The analytical expression for η_f^i differs for a heavy quark with mass $m_q > T_g^{-1}$ and $m_q < T_g^{-1}$, where T_g is the gluonic (vacuum) correlation length which defines the behavior of the bilocal vacuum correlators $D(x)$ and $D_1(x)$ [21]. The value T_g has been measured in lattice QCD and in the quenched approximation $T_g \lesssim 0.2$ fm [44] while in the unquenched case T_g is larger ($T_g \approx 0.30$ fm) [45].

Introducing the variable $y = m_q T_g$ (for c, b quarks $y > 1$ and for $u(d)$, s quarks $y < 1$) η_f is given by the following expressions [35]:

$$\eta_f = \frac{1 + 2y^2}{(y^2 - 1)^2} - \frac{3y^2}{(y^2 - 1)^{5/2}} \arctan\sqrt{y^2 - 1} \quad \text{for } y \geq 1,$$

$$\eta_f = \frac{1 + 2y^2}{(1 - y^2)^2} - \frac{3y^2}{(1 - y^2)^{5/2}} \ln\frac{1 + \sqrt{1 - y^2}}{y} \quad \text{for } y < 1. \qquad (E2)$$

For $m_u, m_d \to 0$ the quantity η_f tends to 1, while for $m_q \to \infty$ the value $\eta_f \to 0$. Here for $m_u = m_d \approx 0$ and $m_b = 4.78$ GeV we have

$$\begin{aligned} \eta_f &= 1.0, \quad \text{for } u(d) \text{ quarks} \\ \eta_b &= 0.03, \quad \text{for } b \text{ quarks.} \end{aligned} \qquad (E3)$$

However, since the value of T_g and the mass of the s quark are not known with a good accuracy, the value of η_s may essentially differ. Then from (E2) we obtain

$$\begin{aligned} \eta_s &= 0.87 \\ &\text{for } y = 0.20 \ (T_g = 0.22 \text{ fm}, m_s = 0.18 \text{ GeV}), \\ \eta_s &= 0.657 \\ &\text{for } y = 0.28 \ (T_g = 0.30 \text{ fm}, m_s = 0.18 \text{ GeV}). \end{aligned} \qquad (E4)$$

and also for c quarks

$$\begin{aligned} \eta_c &= 0.30 \ (T_g = 0.2 \text{ fm}, m_c = 1.4 \text{ GeV}), \\ \eta_c &= 0.18 \ (T_g = 0.3 \text{ fm}, m_c = 1.4 \text{ GeV}). \end{aligned} \qquad (E5)$$

In our analysis here we use the following numbers:

$$\eta_{u(d)} = 1.0, \qquad \eta_s = 0.65, \qquad \eta_c = 0.30, \qquad \eta_b = 0.025. \qquad (E6)$$

The contribution of the b quark to Δ_{SE} [(E1)] is about -1 MeV and can be neglected.

Finally, we notice that in the first term in (E1) we use the number 1.5 instead of the number 2.0 derived in [35] (where $\eta_{u(d)} = 0.9$ was taken, as well as in [24]). This change is made because here we do not neglect the contribution of the correlator $D_1(x)$ as in [35(a)]. Instead at $x = 0$ we use the relation $D(0) + D_1(0) = \frac{\pi^2}{18}G_2$, where G_2 is the gluonic condensate (the details are given in [55(b)]).

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