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Van Stockum-Bonnor spacetimes of rigidly rotating dust

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Stationary, axisymmetric, and asymptotically flat spacetimes of dust of which trajectories are integral curves of the time translation Killing vector are investigated. The flow has no Newtonian limit. Asymptotic flatness implies the existence of singularities of the curvature scalar that are distributions and that are not isolated from regularity regions of the flow. The singularities are closely related to the presence of additional stresses that contribute negative active mass to the total (Komar) mass, which is zero for asymptotically flat spacetimes. Several families of solutions were constructed.

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I. INTRODUCTION

We study stationary, axisymmetric and asymptotically flat spacetimes of dust in free fall along integral curves of the time translation Killing vector. For reasons which will become clear later, we shall call this motion ''van Stockum flow.'' Despite the flow is a rigid rotation with zero angular velocity with respect to asymptotically static observers, the squared vorticity scalar (which for the particular flow equals, up to a constant factor, the proper energy density of dust) in contrast to Newtonian physics, does not vanish. Angular velocity of the flow with respect to locally nonrotating observers numerically equals the angular velocity of dragging of inertial frames, while angular velocity of matter in linearized gravity is many orders of magnitude greater. This shows that van Stockum flow is ultrarelativistic even in the limit of negligible density.

We demonstrate that asymptotic flatness implies the existence of curvature singularities that have distributional character, spatial measure zero, and are not isolated from regularity regions. The singularities are closely related to additional weird stresses with negative active mass. Total mass of such spacetimes is necessarily zero, which has already been conjectured by Bonnor [1], and total angular momentum is determined by the amplitude of the dipole in a multipole expansion.

We construct two classes of asymptotically flat solutions and a class of asymptotically nonflat solutions. Bonnor's solutions [1,2] belong to the first two, and the van Stockum solution [3] to the third.

II. VAN STOCKUM FLOW

We shall focus on axisymmetric, stationary and asymptotically flat spacetimes of dust flowing along opened integral curves of the time translation Killing vector $\boldsymbol{\xi}$. By asymptotic flatness the axial symmetry Killing vector $\boldsymbol{\eta}$, of which integral lines are closed, vanishes on the symmetry axis at least for radii sufficiently large. Consequently, (i) $\eta_{[\alpha}\xi_{\beta}\xi_{\mu;\nu]}=0$ and $\xi_{[\alpha}\eta_{\beta}\eta_{\mu;\nu]}=0$ at least at one point. The energy-momentum tensor is proportional to $\xi_{\mu}\xi_{\nu}$, hence (ii) $\xi^{\mu}R_{\mu}{}^{[\nu}\xi^{\alpha}\eta^{\beta]}=0$ and $\eta^{\mu}R_{\mu}{}^{[\nu}\xi^{\alpha}\eta^{\beta]}=0$ on the basis of Einstein's equations. In addition, we assume (iii) $[\boldsymbol{\xi},\boldsymbol{\eta}]=0$. Under these three assumptions a theorem proved in Ref. [4] guarantees that, maybe apart from isolated points (ρ,z), there exist a coordinate system in which the line element takes the general form

$$\mathrm{d}s^2=-V(\mathrm{d}t-K\mathrm{d}\phi)^2+V^{-1}\rho^2\mathrm{d}\phi^2+e^{2\Psi}(\mathrm{d}\rho^2+\Lambda\mathrm{d}z^2), \tag{2.1}$$

where V, K, Ψ, and Λ are structure functions of two variables ρ and z. In these coordinates $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ attain the particularly simple form $\xi^{\mu}=\delta^{\mu}_{t}$ and $\eta^{\mu}=\delta^{\mu}_{\phi}$.

The four-velocity of van Stockum flow reads $\boldsymbol{u}=Z\boldsymbol{\xi}$, thus $Z^{-2}=-\xi^{\mu}\xi_{\mu}$ and $\xi^{\mu}Z_{,\mu}=0$. Killing equations imply $u_{(\mu;\nu)}=\xi_{(\mu}Z_{,\nu)}$, hence the expansion scalar $u^{\mu}{}_{;\mu}=0$. On projecting $u_{(\mu;\nu)}$ onto the $\boldsymbol{u}$-orthogonal subspace and taking the traceless part, one infers the shear tensor vanishes identically, as well. van Stockum flow is therefore rigid. Despite the fact and that angular velocity of the flow also vanishes ($\boldsymbol{u}\propto\boldsymbol{\xi}$), the vorticity scalar ω does not, where (we use the notation $\boldsymbol{xy}=x^{\mu}y_{\mu}$)

$$\omega^2=\frac{1}{4}\frac{(\boldsymbol{\xi\xi})^2(\boldsymbol{\nabla}S)^2}{(\boldsymbol{\xi\eta})^2-\boldsymbol{\xi}^2\boldsymbol{\eta}^2},\qquad S=\frac{\boldsymbol{\xi\eta}}{\boldsymbol{\xi\xi}}.$$

A. Equations of van Stockum flow

By definition of the flow, the energy-momentum tensor reads $T_{\mu\nu}=\mathcal{D}Z^2\xi_{\mu}\xi_{\nu}$, where $\mathcal{D}$ is the proper energy density such that $\xi^{\mu}\mathcal{D}_{,\mu}=0=\eta^{\mu}\mathcal{D}_{,\mu}$. Einstein's equations and the contracted Bianchi identity imply the flow is continuous and geodesic. As $u^{\mu}{}_{;\mu}=0$, the continuity equation $(\mathcal{D}Z\xi^{\mu})_{;\mu}=0$ is satisfied identically. In addition, $u^{\nu}u_{\mu;\nu}=-u^{\nu}Z\xi_{\nu;\mu}$. The geodesic equation $u^{\nu}u_{\mu;\nu}=0$ will be satisfied if $0=-u^{\nu}u_{\nu;\mu}+u^{\nu}u_{\nu}(\ln Z)_{,\mu}$. As $u^{\mu}u_{\mu}\equiv-1$, Z must be constant. This,

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in turn, implies V is also constant and, without loss of generality, we may set $V \equiv 1$. On defining $K^{\mu\nu} = \boldsymbol{\xi}^2\eta^\mu\eta^\nu + 2\boldsymbol{\xi\eta}\xi^\mu\eta^\nu + \boldsymbol{\eta}^2\xi^\mu\xi^\nu$ we obtain for dust $K^{\mu\nu}(T_{\mu\nu} - Tg_{\mu\nu}/2) = 0$, then Einstein's equations imply $K^{\mu\nu}R_{\mu\nu} = 0$ or $\rho e^{-2\Psi}\partial_\rho \ln\sqrt{|\Lambda|} = 0$ in coordinates, hence $\Lambda = \Lambda(z)$. If so, the form of (2.1) allows us to set $\Lambda(z) \equiv 1$. We have thus shown that the line element of van Stockum flow reads

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + 2K(\rho, z)\mathrm{d}t\mathrm{d}\phi + (\rho^2 - K^2(\rho, z))\mathrm{d}\phi^2 + e^{2\Psi(\rho,z)}(\mathrm{d}\rho^2 + \mathrm{d}z^2). \tag{2.2}$$

Let $E^\mu{}_\nu = R^\mu{}_\nu - \frac{1}{2}R\delta^\mu{}_\nu - 8\pi T^\mu{}_\nu$, then $E^\rho{}_\rho = 0$ and $E^\rho{}_z = 0$ yield the following relations

$$\Psi_{,\rho} = \frac{K_{,z}^2 - K_{,\rho}^2}{4\rho}, \qquad \Psi_{,z} = -\frac{K_{,\rho}K_{,z}}{2\rho}. \tag{2.3}$$

The integrability condition $\Psi_{,\rho z} = \Psi_{,z\rho}$ imposes on K the elliptic constraint

$$\mathcal{L}K = 0, \qquad \mathcal{L} = \partial_\rho^2 - \frac{1}{\rho}\partial_\rho + \partial_z^2. \tag{2.4}$$

Provided (2.3) and (2.4) are satisfied, the other components of $E^\mu{}_\nu$, but $E^t{}_t$ and $E^t{}_\phi$, vanish identically. The latter two will also vanish for $\mathcal{C}^2$ solutions (we stress, the reservation 'for $\mathcal{C}^2$ solutions' is necessary) if only

$$\mathcal{D} = e^{-2\Psi}\frac{K_{,\rho}^2 + K_{,z}^2}{8\pi\rho^2}. \tag{2.5}$$

III. ASYMPTOTIC FLATNESS AND CURVATURE SINGULARITIES

Let $\mathcal{L}K = 0$ almost everywhere in an open subset $\mathcal{V}$ of the plane (ρ, z) and let $K^\epsilon \in \mathcal{C}^\infty(\mathcal{V})$ tend pointwise to K as $\epsilon \to 0$. We shall refer to K^ϵ as a regularized profile of K. Let $\mathcal{I} \in \mathbb{R}^3$ be the set of points where $\mathcal{L}K$ does not exist in the sense that $\lim_{\epsilon\to 0}\int_{\mathcal{I}_\delta}(\mathcal{L}K^\epsilon)f\mathrm{d}m \neq 0$ for any $\delta > 0$, where $\mathcal{I} \subset \mathcal{I}_\delta$, $0 < \mathrm{dist}(\partial\mathcal{I}_\delta, \mathcal{I}) < \delta$, $e^{2\Psi}f = \rho^{-1}K^\epsilon_{,\rho}$ or $\rho^{-2}K^\epsilon$, and $\mathrm{d}m = e^{2\Psi}\rho\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}z$. We recall from the theory of elliptic equations that $\mathcal{I}/\mathcal{S}^1$ has measure zero in the plane (ρ, z) and $K \in \mathcal{C}^2$ elsewhere.

In what follows we shall prove that $\mathcal{I} \neq \varnothing$ for asymptotically flat (starlike) van Stockum spacetimes. For if we suppose for contradiction that $\mathcal{I} = \varnothing$, then inside a ball $\mathcal{B}_\mathsf{R} \subset \mathbb{R}^3$ bounded by a two-sphere $\mathcal{S}_\mathsf{R}$ of radius R and centered at the origin

$$\int_{\mathcal{B}_\mathsf{R}} \mathcal{D}e^{2\Psi}\rho\mathrm{d}\rho \wedge \mathrm{d}\phi \wedge \mathrm{d}z \overset{\mathcal{L}K=0}{=} \frac{1}{8\pi}\int_{\mathcal{S}_\mathsf{R}} \frac{K}{\rho}(K_{,z}\mathrm{d}\rho - K_{,\rho}\mathrm{d}z) \wedge \mathrm{d}\phi \equiv \frac{1}{8\pi}\int_{\mathcal{S}_\mathsf{R}} \frac{K\partial_r K}{\sin\theta}\mathrm{d}\phi \wedge \mathrm{d}\theta \tag{3.1}$$

in virtue of the Stokes theorem, provided $(K^2)_{,r} = o(\sin\theta)$, $(r\sin\theta = \rho,\ r\cos\theta = z)$. By asymptotic flatness $K \sim 2Jr^{-1}\sin^2\theta$ as $r \to \infty$, hence, for R sufficiently large, the surface integral on the right-hand side is negative and tends to 0 as $\mathsf{R} \to \infty$, while the volume integral on the left-hand side is positive, a contradiction, thus indeed $\mathcal{I} \neq \varnothing$.

The surface integral in Eq. (3.1) coincides in the limit $\mathsf{R} \to \infty$ with the total (Komar) mass M which vanishes for asymptotically flat spacetimes with metric (2.2) and reads

$$M = -\frac{1}{8\pi}\lim_{\mathsf{R}\to\infty}\int_{\mathcal{S}_\mathsf{R}} \frac{1}{2}\sqrt{-g}\epsilon_{\alpha\beta\mu\nu}\nabla^\mu\xi^\nu\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \equiv \lim_{\mathsf{R}\to\infty}\frac{1}{8\pi}\int_{\mathcal{S}_\mathsf{R}} \frac{K\partial_r K}{\sin\theta}\mathrm{d}\phi \wedge \mathrm{d}\theta = 0. \tag{3.2}$$

We conclude, therefore, that the integral $\lim_{\mathsf{R}\to\infty}\int_{\mathcal{B}_\mathsf{R}}\mathrm{d}m(8\pi\rho^2)^{-1}e^{-2\Psi}K\mathcal{L}K$, which we were justified to omit in Eq. (3.1) for $\mathcal{C}^2$ solutions, does not vanish and as $\epsilon \to 0$ it tends for regularized profiles K^ϵ to minus the total mass $\int_{\mathbb{R}^3\setminus\mathcal{I}}\mathcal{D}\mathrm{d}m$ of the regularity region $\mathbb{R}^3 \setminus \mathcal{I}$. Putting this in other words, asymptotically flat van Stockum spacetimes contain additional sources of negative active mass located in $\mathcal{I}$ that balances positive masses smoothly distributed in regular regions.

To be more explicit, the proper energy density $\tilde{\mathcal{D}} = T_{\mu\nu}u^\mu u^\nu$, the trace of spatial stresses $\tilde{\mathcal{S}} = T_{\mu\nu}(u^\mu u^\nu + g^{\mu\nu})$, Tolman's active mass density on a hypersurface of constant time $\tilde{\mathcal{D}}_T = (8\pi)^{-1}R^t{}_\mu\xi^\mu$, and the curvature scalar $\tilde{R} = R^\mu{}_\mu$ of a spacetime with metric (2.2) read

$$\begin{aligned}
\tilde{\mathcal{D}} &= \frac{3}{4}e^{-2\Psi}\frac{K_{,\rho}^2 + K_{,z}^2}{8\pi\rho^2} - \frac{1}{8\pi}e^{-2\Psi}(\Psi_{,\rho\rho} + \Psi_{,zz}),\\
\tilde{\mathcal{D}}_T &= e^{-2\Psi}\frac{K_{,\rho}^2 + K_{,z}^2}{8\pi\rho^2} + \frac{1}{8\pi}e^{-2\Psi}\frac{K}{\rho^2}\mathcal{L}K,\\
\tilde{\mathcal{S}} &= \frac{1}{4}e^{-2\Psi}\frac{K_{,\rho}^2 + K_{,z}^2}{8\pi\rho^2} + \frac{1}{8\pi}e^{-2\Psi}(\Psi_{,\rho\rho} + \Psi_{,zz}),\\
\tilde{\mathcal{R}} &= 8\pi(\tilde{\mathcal{D}} - \tilde{\mathcal{S}}).
\end{aligned} \tag{3.3}$$

For any smooth Ψ and K satisfying (2.3), the above definitions reduce to

$$\begin{aligned}
\tilde{\mathcal{D}} &= \mathcal{D} + e^{-2\Psi}\frac{K_{,\rho}}{16\pi\rho}\mathcal{L}K = \mathcal{D} - \tilde{\mathcal{S}},\\
\tilde{\mathcal{D}}_T &= \mathcal{D} + e^{-2\Psi}\frac{K}{8\pi\rho^2}\mathcal{L}K,\\
\tilde{\mathcal{S}} &= -e^{-2\Psi}\frac{K_{,\rho}}{16\pi\rho}\mathcal{L}K,\\
\tilde{R} &= 8\pi\mathcal{D} + e^{-2\Psi}\frac{K_{,\rho}}{\rho}\mathcal{L}K = 8\pi(\mathcal{D} - 2\tilde{\mathcal{S}}).
\end{aligned}$$

In particular, in the regularity region $\mathbb{R}^3 \setminus \mathcal{I}$ we obtain $\tilde{\mathcal{D}} = \tilde{\mathcal{D}}_T = D = (8\pi)^{-1}\tilde{R}$ and $\tilde{\mathcal{S}} = 0$, like for dust, and

Einstein's equations are indeed equivalent there to (2.3), (2.4), and (2.5). However, for regularized profiles K^ϵ we get $\int_{\mathcal{I}_\delta} \tilde{S}\mathrm{d}m \neq 0$ outside $\mathcal{I}_\delta$ in the limit $\epsilon \to 0$ for arbitrarily small δ. Note, that $\mathcal{I}$ is the set of scalar curvature singularity as $\tilde{R}$ is a distribution on $\mathcal{I}$. Indeed, if K_ϵ is a regularized profile of an asymptotically flat solution with integrable $\mathcal{D}$ then $\int_{\mathbb{R}^3} \tilde{R} \neq \int_{\mathbb{R}^3\setminus\mathcal{I}} \tilde{R} = 8\pi \int_{\mathbb{R}^3} \mathcal{D}$ in the limit $\epsilon \to 0$. Thus $\tilde{R} = 8\pi\mathcal{D} + \gamma_{\mathcal{I}}$ where D is smooth and integrable, and $\gamma_{\mathcal{I}}$ is a distribution localized on $\mathcal{I}$. The singularity is not isolated from regularity regions.[1]

Here is an example illustrating these statements. The Bonnor solution [1] with an embedded surface layer of negative mass and integrable $\mathcal{D}$, which we shall denote by K_B^0, can be regularized by defining $K_B^\epsilon(\rho, z) = \sqrt{8a^3\mu} \cdot \rho^2 \cdot ((a + \sqrt{z^2 + \epsilon^2})^2 + \rho^2)^{-3/2}$, $a > 0$, $\mu > 0$. Although K_B^ϵ is globally $\mathcal{C}^\infty$, its limit K_B^0 is not even differentiable in $\mathcal{I}$, which for K_B^0 is the plane $z = 0$. On integrating over $\mathbb{R}^3$ and taking the limit $\epsilon \to 0$ we obtain $\int_{\mathbb{R}^3} D\mathrm{d}m = \int_{\mathbb{R}^3\setminus\mathcal{I}} D\mathrm{d}m = \mu$, $\int_{\mathbb{R}^3} \tilde{D}\mathrm{d}m = 3\mu/4$, $\int_{\mathcal{I}} \tilde{S}\mathrm{d}m = \mu/4$, $\int_{\mathbb{R}^3} \tilde{R}\mathrm{d}m = 4\pi\mu$ and $\int_{\mathbb{R}^3} \tilde{D}_T\mathrm{d}m \equiv 0$. The latter holds identically as $8\pi\sqrt{-g}\tilde{D}_T\mathrm{d}\rho \wedge \mathrm{d}\phi \wedge \mathrm{d}z = \mathrm{d}(\rho^{-1}KK_{,z}\mathrm{d}\rho \wedge \mathrm{d}\phi + \rho^{-1}KK_{,\rho}\mathrm{d}\phi \wedge \mathrm{d}z)$ for $\mathcal{C}^2$ functions, hence $\int_{\mathbb{R}^3} \tilde{D}_T\mathrm{d}m \equiv M = 0$ in general for regularized asymptotically flat profiles. Since $8\pi D = \tilde{R}$ for K_B^0 only outside $\mathcal{I}$, and since $\int_{\mathbb{R}^3\setminus\mathcal{I}} D\mathrm{d}m = \int_{\mathbb{R}^3} D\mathrm{d}m = \mu \neq \mu/2 = (8\pi)^{-1}\int_{\mathbb{R}^3} \tilde{R}\mathrm{d}m$, the curvature scalar is a distribution, and $\tilde{R}$ is smooth and bounded only outside $\mathcal{I}$.

IV. THREE CLASSES OF SOLUTIONS

A. Solutions with a layer of negative mass

Solutions to Eq. (2.4) can be sought via integral transforms, for example,

$$K(\rho, z) = \rho \int_0^\infty \lambda \hat{K}_{\mathsf{J}}(\lambda) e^{-\lambda|z|} \mathsf{J}_1(\lambda\rho)\mathrm{d}\lambda, \quad \text{or}$$

$$\rho \int_0^\infty \lambda \hat{K}_{\mathsf{K}}(\lambda) \cos(\lambda z) \mathsf{K}_1(\lambda\rho)\mathrm{d}\lambda$$

generate z-symmetric solutions; J_1 and K_1 are Bessel functions. To give an example, the solution $\rho^2 r^{-3}$ discussed in Ref. [2] has $\hat{K}_{\mathsf{J}}(\lambda) = 1$ and $\hat{K}_{\mathsf{K}}(\lambda) = 2/\pi$, while the Bonnor solution K_B^0 has $\hat{K}_{\mathsf{J}}(\lambda) \propto e^{-a\lambda}$. The latter belongs to a class of solutions defined by specifying $\hat{K}_{\mathsf{J}}(\lambda) = \frac{l^{2n+2}\lambda^{2n}}{(2n+1)!} e^{-a\lambda}$, $n \in \mathbb{N}$, which yields

$$K_n(\rho, z) = \frac{(n+1)l^{2n+2}\rho^2}{(a+|z|)^{2n+3}} \cdot {}_2F_1\left(\frac{3}{2} + n, 2 + n; 2, -\frac{\rho^2}{(a+|z|)^2}\right).$$

Apart from the plane $z = 0$ the solutions are smooth everywhere. One can show that $|K_n| < \rho(l/(a + |z|))^{2n+2}$, thus, at least for $l < a$, hypersurfaces of constant t are globally spacelike as then $|K_n| < \rho$. Since $K_n\partial_r K_n(r, \theta) \sim r^{-(4n+3)}\sin^4\theta$ times a bounded geometrical factor, the spacetimes are asymptotically flat with $M = 0$. Function $\mathcal{D}e^{2\Psi}$ is finite for $z \neq 0$ and for r sufficiently large behaves as $r^{-(4n+6)}$. The plane $z = 0$ is thus a curvature singularity with finite and negative active mass. Only the Bonnor solution K_B^0 ($n = 0$) has nonzero angular momentum.

B. External and internal multipolar solutions and a multipole expansion

Let $K(\rho, z) = W(r)Y(\cos\theta)$, where $\rho = r\sin\theta$, $z = r\cos\theta$. There exist three families of solutions to Eq. (2.4) satisfying $r^2W''(r) = \lambda W(r)$ and $(1 - x^2)Y''(x) + \lambda Y(x) = 0$, $(x = \cos\theta)$; with (i) $\lambda = \alpha(\alpha + 1)$, $\alpha \geq 0$; (ii) $-\cos^2(\alpha)/4$, $0 \leq \alpha < \pi/2$ and (iii) $-\cosh^2(\alpha)/4$, $\alpha > 0$. The (i) class contains x-analytic external ($W = r^{-n}$) and internal ($W = r^{n+1}$) solutions. In this way we obtain external $K_E^{(n)}$ and internal $K_I^{(n)}$ multipolar solutions

n	$0, 2, 4, 6, \ldots$	$1, 3, 5, 7, \ldots$
$K_E^{(n)}(\rho, z)$	$\frac{z}{(\rho^2+z^2)^{(n+1)/2}} A_n(\frac{z^2}{\rho^2+z^2})$	$\frac{1}{(\rho^2+z^2)^{n/2}} B_n(\frac{z^2}{\rho^2+z^2})$
$K_I^{(n)}(\rho, z)$	$z(\rho^2 + z^2)^{n/2} A_n(\frac{z^2}{\rho^2+z^2})$	$(\rho^2 + z^2)^{(n+1)/2} B_n(\frac{z^2}{\rho^2+z^2})$

where $A_n(y) = {}_2F_1(\frac{1}{2} + \frac{n}{2}, -\frac{n}{2}; \frac{3}{2}, y)$ and $B_n(y) = {}_2F_1(-\frac{1}{2} - \frac{n}{2}, \frac{n}{2}; \frac{1}{2}, \frac{z^2}{\rho^2+z^2})$. Internal solutions $K_I^{(n)}$ (of which element is the van Stockum solution [3]) give rise to spacetimes that are not asymptotically flat. With the exception of the monopole $K_E^{(0)}$, $K_E^{(n)}$ yield asymptotically flat spacetimes that contain in the center pathological singularities with nonintegrable $\mathcal{D}$ (e.g. $e^{2\Psi}\mathcal{D} > a^4/(2\pi r^6)$ for the dipole $K_E^{(1)} = a^2\sin^2(\theta)r^{-1}$). For the solutions $M = 0$, thus contributions to M from the singularities are formally $-\infty$. Another nonphysical property of $K_E^{(n)}$ is that $|K(\rho, z)| > \rho$ in the vicinity of the center. In the region the axial symmetry Killing vector $\boldsymbol{\eta}$ is timelike, as so, the region contains closed timelike curves (e.g. $K_E^{(1)}$ is such inside the region bounded by $\rho = a\sqrt{\sin\alpha}\sin\alpha$, $z = a\cos\alpha\sqrt{\sin\alpha}$, $\alpha \in (0, \pi)$). However, external multipoles $K_E^{(n>0)}$ appear in multipolar expansions of asymptotically flat solutions. We shall illustrate this by giving an example below.

It should be clear that $K_a = -\int_{-a}^{a} a^{-1}s\mathrm{d}sf(\rho, z - s)$, $a > 0$, where $f(\rho, z) = K_E^{(0)} = zr^{-1}$, is a z-symmetric solution such that $0 \leq K_a \leq a$. The conformal mapping $z + i\rho = a\cosh(u + iv)$ $(u \geq 0,\ 0 \leq v \leq \pi)$ is invertible apart from two points $(\rho, z) = (0, \pm a)$ and transforms $S_a = \{(\rho, z):\rho = 0, z \in [-a, a]\}$ to the segment $u = 0$, $v \in [0, \pi]$. In this map the solution reads

$$K_a(u, v) = a\sin^2(v)\left(\cosh(u) + \frac{1}{2}\sinh^2(u)\ln\left[\tanh^2\left(\frac{u}{2}\right)\right]\right).$$

[1] In this sense we cannot agree with the statement of paper [1] that a solution found therein has no curvature singularity.

The resulting spacetime is asymptotically flat as $K_a \sim (4/3)ae^{-u}\sin^2 v$ for large u. In the vicinity of $u = 0$, $K_a \sim u^2 \ln u$, therefore $\mathcal{I} = \mathcal{S}_a$. Multipolar expansion of K_a in the base of functions $K_E^{(n)}$, $n = 1, 3, 5, \ldots$, reads

$$K_a(\rho, z) = \frac{2}{3}\frac{\rho^2}{r^3}a^2 - \frac{1}{5}\frac{\rho^2(\rho^2 - 4z^2)}{r^7}a^4 + \frac{3}{28}\frac{\rho^2(\rho^4 - 12\rho^2 z^2 + 8z^4)}{r^{11}}a^6 + \ldots,$$

and $\Psi_a(r, \theta) \sim -(a^4/72)r^{-4}(7 + 9\cos 2\theta)\sin^2\theta$ as $r \to \infty$. Asymptotically, (2.2) reduces to

$$ds^2 \sim -dt^2 - \frac{4}{3}\frac{a^2}{r}\sin^2\theta dt d\phi + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Comparison with the asymptotic expansion of the Kerr metric gives total mass $M = 0$ and total angular momentum $J = a^2/3$ in agreement with Eq. (3.2) and with the analogous expression for the total angular momentum of asymptotically flat van Stockum flow

$$J = \lim_{r\to\infty}\frac{1}{16\pi}\iint\left[\frac{2K}{r} - \left(1 + \frac{K^2}{r^2\sin^2\theta}\right)\partial_r K\right]r^2 \sin\theta d\theta d\phi.$$

V. SUMMARY

We investigated stationary, axisymmetric and asymptotically flat spacetimes of self-gravitating dust moving on integral curves of the time translation Killing vector. The flow is rigid with the proper energy density proportional to the square of vorticity scalar, and is ultrarelativistic even in the limit of negligible density. Geometry of the spacetime is described by van Stockum metric [3], hence the name "van Stockum spacetimes."

We demonstrated that positive definiteness and integrability of proper energy density excludes regular asymptotically flat van Stockum spacetimes. Asymptotical flatness implies the existence of spatial measure zero sets of scalar curvature singularities. The singularities have distributional character and are not isolated from regularity regions. Closely related to the singularities are stresses (distinct from dust) that contribute negative active masses to the total Komar mass (ADM and Komar mass agree for the spacetimes). In effect, owing to the presence of the singularities, total mass of asymptotically flat van Stockum spacetimes is zero. We illustrated our statements by discussing Bonnor solution [1] with a negative layer of mass and with integrable energy density. We constructed also a class of similar solutions and a class of external (asymptotically flat) and internal (not asymptotically flat) multipolar solutions. The external multipoles appear in multipolar expansions of asymptotically flat solutions. The use of multipolar expansions is the other way around to see that total mass of asymptotically flat solutions is zero and that the total angular momentum is proportional to the amplitude of the dipole momentum.

Because of the mathematically singular properties examined in Sec. III and peculiarities of the corresponding flows discussed in Secs. I and II, global and asymptotically flat van Stockum spacetimes, or spacetimes of which line elements asymptotically tend to or are matched on to asymptotically flat van Stockum metrics (then total mass would be zero), are not viable physically and thus seem to be of no importance to astrophysics. However, there is still a possibility that van Stockum spacetime can be part of a regular spacetime.

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