Progress toward the determination of complete vertex operators for the IIB matrix model

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We report on progress in determining the complete form of vertex operators for the IIB matrix model. The exact expressions are obtained for those emitting massless IIB supergravity fields up to sixth order in the light-cone superfield, in which the conjugate gravitino and conjugate two-form vertex operators are newly determined. We also provide a consistency check by computing the kinematical factor of a four-point graviton amplitude in a D-instanton background. We conjecture that the low-energy effective action of the IIB matrix model at large N is given by tree-level supergravity coupled to the vertex operators.

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I. INTRODUCTION

Realizing string theory as a matrix model is an efficient way for investigating its nature beyond perturbation theory. Among others, the IIB matrix model is in a sense the simplest proposal, and there are several good reasons to believe that it is a natural framework for type IIB string theory. The first direct link between them is that the former may be regarded as a matrix regularization of the latter [1]. Another suggestive fact is that, defined as a zero-dimensional reduced model of the maximal supersymmetric Yang-Mills theory, the IIB matrix model can be thought of as an effective theory of D instantons. Since its has twice as many supercharges as the original Yang-Mills theory, it must also contain a graviton. Indeed the analysis based on the one-loop effective action has revealed already an interaction due to a graviton exchange [1]. Furthermore, the continuum limit of the loop equation was shown to reproduce the type IIB light-cone superstring field theory [2].

One of the issues of the IIB matrix model is how it describes the coupling to the supergravity multiplet. The first study of this question was undertaken by one of the present authors [3], who constructed a set of vertex operators for the massless supergravity multiplet to leading order in momentum k by repeatedly applying the $\mathcal{N} = 2$ SUSY transformations to the straight Wilson line operator. There was also found a set of "wave functions" which are dual to the vertex operators and linearly transform under the $\mathcal{N} = 2$ supersymmetry (SUSY) transformations.

To fully determine the complete form of them is a hard task. In [4] it was noticed that the supersymmetric Wilson loop operators were useful tools in determining the complete form of vertex operators. In general, Wilson loops are basic gauge invariant objects in any gauge theory. In particular, in the IIB matrix model, they describe creations and annihilations of fundamental strings [1,2], of which the lowest excitations are the massless fields of IIB supergravity multiplet. Therefore, it is natural that Wilson loop operators come into play in the construction of vertex operators in the IIB matrix model. Expanding a supersymmetric Wilson loop operator as a polynomial of the "mean field" fermion variable λ, which may be identified as the light-cone superfield variable, exact vertex operators have been systematically derived [4] up to fourth order in λ.

If one goes beyond that order, one is faced with the enormous task of algebras and Fierz arrangements. In this paper, to simplify the computations, we develop some alternative ways of deriving them, as we will explain shortly. Combining information obtained through these as well as other available means, we determine the complete form of vertex operators up to *sixth* order in λ. The expressions for vertex operators emitting a conjugate gravitino (Ψ^c_μ) and a conjugate two-form field ($B^c_{\mu\nu}$) are the new results. To compute the remaining two most complicated ones of $\mathcal{O}(\lambda^7)$ and $\mathcal{O}(\lambda^8)$ (corresponding to the conjugate dilatino and the conjugate dilaton) still require another enormous amount of labor, and are not determined so far.

One of the goals that we hope to achieve through these computations is to rederive various known scattering amplitudes obtained in type IIB string theory. Despite the evidence of the relation between the two, there is no proof yet that the former is a constructive definition of the latter. Therefore it will be important to evaluate amplitudes defined in terms of vertex operators and verify an agreement. Another goal we work toward is to gain insight into the low-energy effective action of the IIB matrix model. As we briefly comment in the last section, the form of the effective action is severely constrained by symmetries. The precise knowledge of it may be useful to understand the dynamics of the IIB matrix model.

The contents of this paper are as follows. In Sec. II, we briefly review the general concept of vertex operators in the IIB matrix model and the idea of how to compute them. In Sec. III, we describe the method adopted in [4] as well as our improved ones which simplify the calculation. In Sec. IV, we present the new expressions for the conjugate gravitino and conjugate two-form vertex operators. We also display other vertex operators known so far for completeness. In Sec. V a consistency check is given by computing the kinematical factor of the four-point graviton amplitude. Section VI is devoted to the conclusions. In two appendices we collect useful formulas and summarize our results.

II. VERTEX OPERATORS FOR THE IIB MATRIX MODEL

The action of the IIB matrix model is [1]

$$S = -\tfrac{1}{4}\mathrm{tr}[A_\mu, A_\nu]^2 - \tfrac{1}{2}\mathrm{tr}(\bar{\psi}\Gamma^\mu[A_\mu, \psi]), \tag{2.1}$$

where A_μ and ψ are $N \times N$ Hermitian matrices which transform as a $D = 10$ Lorentz vector and a Majorana-Weyl spinor. Vertex operators in the IIB matrix model are given by functions of them and are characterized by the properties (i) they describe linear couplings to the background fields. and (ii) they are related with each other by the supersymmetry transformations. Let $V_i(A, \psi)$ be the vertex operator associated with the background field f_i which is any of the members of type IIB supergravity multiplet. Then the property (i) implies that the interaction terms are given by

$$S_{\mathrm{int}} = \sum_i V_i(A, \psi) f_i, \tag{2.2}$$

and (ii) asserts that the following equation holds [5]:

$$\sum_i V_i(\delta A, \delta\psi) f_i = \sum_i V_i(A, \psi)\delta f_i, \tag{2.3}$$

where δ is a supersymmetry variation. The relation (2.3) ensures the super invariance of correlation functions

$$W(f_i) \equiv \left\langle \sum_i V_i(A, \psi) f_i \right\rangle, \tag{2.4}$$

and, in principle, determines the form of vertex operators completely [3]. Indeed, the vertex operators for all members of the IIB supergravity multiplet were determined in this way to leading order in momentum k [3]. The derivation of the complete form becomes, however, more complicated as the vertex operator comes to include more fermions.

The authors of [4] have developed, by utilizing a supersymmetric Wilson loop operator, a more systematic way of determining the exact form of vertex operators in the IIB matrix model. Let us focus on the operator

$$w(A, \psi; \lambda) = e^{\bar{\lambda}Q_1}\mathrm{tr}e^{ik\cdot A}e^{-\bar{\lambda}Q_1}, \tag{2.5}$$

where λ is a Majorana-Weyl spinor and Q_1 is one of the $\mathcal{N} = 2$ SUSY generators of the IIB matrix model

$$\bar{\epsilon}_1 Q_1 = i(\bar{\epsilon}_1\Gamma_\mu\psi)\frac{\delta}{\delta A_\mu} - \frac{i}{2}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon_1\frac{\delta}{\delta\psi}, \tag{2.6}$$

$$\bar{\epsilon}_2 Q_2 = \epsilon_2\frac{\delta}{\delta\psi}, \tag{2.7}$$

where $F_{\mu\nu} = [A_\mu, A_\nu]$. $w(A, \psi; \lambda)$ is the simplest supersymmetric Wilson loop operator first introduced in [6]. One of the nice properties of $w(A, \psi; \lambda)$ is that it transforms under the $\mathcal{N} = 2$ SUSY as [4]

$$[\bar{\epsilon}Q_1, w(A, \psi; \lambda)] = \epsilon\frac{\delta}{\delta\lambda}w(A, \psi; \lambda), \tag{2.8}$$

$$[\bar{\epsilon}Q_2, w(A, \psi; \lambda)] = (\bar{\epsilon}\not{k}\lambda)w(A, \psi; \lambda). \tag{2.9}$$

In other words, the operations (2.6) and (2.7) acting on the space of functions of matrices A_μ and ψ amounts to the operations

$$\delta^{(1)} = \epsilon\frac{\delta}{\delta\lambda}, \tag{2.10}$$

$$\delta^{(2)} = \bar{\epsilon}\not{k}\lambda \tag{2.11}$$

on the space of polynomials of λ. Since (2.10) and (2.11) realize the $\mathcal{N} = 2$ SUSY algebra, one may construct a representation in this space. In the following we take $k^2 = 0$, then the irreducible subspace is spanned by monomials of λ of at most degree eight. Such a polynomial can be regarded as a light-cone superfield for the massless type IIB supergravity multiplet [7].

The basic strategy of deriving vertex operators adopted by [4] is as follows. We first find a set of homogeneous polynomial of λ such that each of them corresponds to some supergravity field and transforms as a linearized SUSY multiplet by the operations (2.10) and (2.11). Such a polynomial is called a *wave function*. We expand the supersymmetric Wilson loop $w(A, \psi; \lambda)$ as a polynomial of λ in terms of wave functions

$$w(A, \psi; \lambda) = \sum_i V_i(A, \psi) f_i(\lambda). \tag{2.12}$$

One immediately finds

$$\sum_i [\bar{\epsilon}Q_j, V_i(A, \psi)] f_i(\lambda) = \sum_i V_i(A, \psi)\delta^{(j)} f_i(\lambda) \tag{2.13}$$

for $j = 1, 2$. Thus the supersymmetric Wilson loop realizes the relation (2.3), and the coefficient of each wave function can be identified as the corresponding vertex operator. The fermionic variable λ may be regarded as an isolated eigenvalue of the matrix ψ representing the effect of the background as a "mean field" [4,8]. Indeed the SUSY transformations for such a single eigenvalue are generated by (2.10) and (2.11) if the off-diagonal interactions are neglected. Thus we see that the wave functions for the external fields are realized as condensations of particular spinor eigenvalues of the matrix model.

III. THE METHODS

In this section we will show how we determine the form of vertex operators in some detail. We begin by rewriting the supersymmetric Wilson loop as

$$w(A, \psi; \lambda) = \mathrm{tr} e^{ik\cdot A + \Sigma_{n=1}^{8} G_n}, \tag{3.1}$$

where G_n is defined by

$$G_n = \frac{1}{n!}[\bar{\lambda}Q_1, \cdots [\bar{\lambda}Q_1, ik\cdot A]\cdots] \quad (n \text{ commutators}), \tag{3.2}$$

and hence contains $n\lambda$'s. The sum in the exponent of (3.1) terminates at G_8 because $\not{k}\lambda$ has only eight independent components. Each term can be evaluated as

$$G_0 = ik\cdot A, \tag{3.3}$$

$$G_1 = \bar{\psi}\not{k}\lambda, \tag{3.4}$$

$$G_2 = \frac{i}{4}b^{\mu\nu}[A_\mu, A_\nu], \tag{3.5}$$

$$G_3 = -\frac{1}{3!}b^{\mu\nu}[\bar{\lambda}\Gamma_\mu, A_\nu], \tag{3.6}$$

$$G_4 = \frac{1}{4!}\left(\frac{i}{2}b^{\mu\nu}(\bar{\lambda}\Gamma_{\mu\rho\sigma}\lambda)[[A^\rho, A^\sigma], A_\nu] - ib^{\mu\nu}[\bar{\lambda}\Gamma_\mu\psi, \bar{\lambda}\Gamma_\nu\psi]\right), \tag{3.7}$$

$$G_5 = -\left(\frac{1}{5!}b^{\mu\nu}(\bar{\lambda}\Gamma_{\mu\rho\sigma}\lambda)[[\bar{\lambda}\Gamma^\rho\psi, A^\sigma], A_\nu] + \frac{3}{2}b^{\mu\nu}(\bar{\lambda}\Gamma_{\mu\rho\sigma}\lambda)[[A^\rho, A^\sigma], \bar{\lambda}\Gamma_\nu\psi]\right), \tag{3.8}$$

$$\vdots$$

Expanding (3.1) and collecting terms with the same powers of λ, we get

$$w(A, \psi; \lambda) = \mathrm{Str} e^{ik\cdot A}\Big[1 + G_1 + \left(\frac{1}{2!}(G_1)^2 + G_2\right) + \left(\frac{(G_1)^3}{3!} + G_1\cdot G_2 + G_3\right) + \left(\frac{(G_1)^4}{4!} + \frac{1}{2}(G_1)^2\cdot G_2 + \frac{1}{2}(G_2)^2 + G_1\cdot G_3 + G_4\right) + \left(\frac{(G_1)^5}{5!} + \frac{1}{3!}(G_1)^3\cdot G_2 + \frac{1}{2}G_1\cdot(G_2)^2 + \frac{1}{2}(G_1)^2\cdot G_3 + G_2\cdot G_3 + G_1\cdot G_4 + G_5\right) + \cdots\Big], \tag{3.9}$$

where "Str" means the symmetrized trace whose definitions and some properties are given in the appendix. If we let i in (2.12) denote the power of λ (but see below for $i = 4$), we obtain

$$V_0(A, \psi)f_0(\lambda) = \mathrm{Str} e^{ik\cdot A}, \tag{3.10}$$

$$V_1(A, \psi)f_1(\lambda) = \mathrm{Str} G_1 e^{ik\cdot A}, \tag{3.11}$$

$$V_2(A, \psi)f_2(\lambda) = \mathrm{Str}\left(\frac{1}{2!}(G_1)^2 + G_2\right)e^{ik\cdot A}, \tag{3.12}$$

$$V_3(A, \psi)f_3(\lambda) = \mathrm{Str}\left(\frac{(G_1)^3}{3!} + G_1\cdot G_2 + G_3\right)e^{ik\cdot A}, \tag{3.13}$$

$$\vdots$$

$$V_n(A, \psi)f_n(\lambda) = \mathrm{Str}\left(\frac{(G_1)^n}{n!} + \cdots + G_n\right)e^{ik\cdot A}, \tag{3.14}$$

$$\vdots$$

If we rearrange the right-hand side in terms of wave functions, we obtain corresponding vertex operators [4]. Note that there are two independent wave functions (graviton and self-dual four-form field) at $i = 4$. Therefore, the terms of $\mathcal{O}(\lambda^4)$ split into a linear combination of them; each coefficient is a vertex operator in this case.

Although this method offers a systematic derivation, it requires too many calculations for orders higher than four. In order to reduce the amount of labor in the derivation, we have developed the following two improved methods. The first one uses a different expansion of the supersymmetric Wilson loop as

$$\begin{aligned} w(A, \psi; \lambda) &= e^{\bar{\lambda}Q_1}\mathrm{tr} e^{ik\cdot A}e^{-\bar{\lambda}Q_1} \\ &= \mathrm{Str}\Big(e^{ik\cdot A} + [\bar{\lambda}Q_1, e^{ik\cdot A}] \\ &\quad + \frac{1}{2!}[\bar{\lambda}Q_1, [\bar{\lambda}Q_1, e^{ik\cdot A}]] + \cdots \\ &\quad + \frac{1}{8!}[\bar{\lambda}Q_1, \cdots [\bar{\lambda}Q_1, e^{ik\cdot A}]\cdots]\Big). \end{aligned} \tag{3.15}$$

Again, the sum terminates at $\mathcal{O}(\lambda^8)$. From (3.15) we obtain

$$V_n(A,\psi)f_n(\lambda) = \mathrm{Str}\frac{1}{n!}[\bar{\lambda}Q_1, \cdots[\bar{\lambda}Q_1, e^{ik\cdot A}]\cdots] = \mathrm{Str}\frac{1}{n!}\left(i(\bar{\lambda}\Gamma_\mu\psi)\frac{\delta}{\delta A_\mu} - \frac{i}{2}F_{\mu\nu}\Gamma^{\mu\nu}\lambda\frac{\delta}{\delta\psi}\right)^n e^{ik\cdot A}. \quad (3.16)$$

This equation relates the $\mathcal{O}(\lambda^n)$ term with the $\mathcal{O}(\lambda^{n-1})$ term as

$$V_n(A,\psi)f_n(\lambda) = \frac{1}{n}\left(i(\bar{\lambda}\Gamma_\mu\psi)\frac{\delta}{\delta A_\mu} - \frac{i}{2}F_{\mu\nu}\Gamma^{\mu\nu}\lambda\frac{\delta}{\delta\psi}\right)V_{n-1}(A,\psi)f_{n-1}(\lambda). \quad (3.17)$$

Therefore, once we know the exact expression for $V_{n-1}(A,\psi)$, we can determine $V_n(A,\psi)$ by using this relation. Note that the operator in the right-hand side is the homogeneous supersymmetry transformation (2.6) with parameter λ. Thus, $V_n(A,\psi)$ is determined by supersymmetry. This is close to the original method adopted in [3], but what is new here is that we can now use the precise expressions of wave functions obtained in [4] as we present in the next subsection.

The second alternative method uses another recursion relation [9]

$$\bar{\epsilon}\frac{\delta}{\delta\psi}V_n(A,\psi)f_n(\lambda) = (\bar{\epsilon}\not{k}\lambda)V_{n-1}(A,\psi)f_{n-1}(\lambda), \quad (3.18)$$

which can be shown easily by comparing the $\mathcal{O}(\lambda^n)$ terms on the left- and right-hand sides of (2.9). Using this formula, we can also determine the form of $V_n(A,\psi)$ if $V_{n-1}(A,\psi)$ is known. In this case $V_n(A,\psi)$ is transformed by inhomogeneous supersymmetry transformation (2.7).

The advantage of the first method is that we do not need to calculate the polynomial of G_i's, which becomes longer as the power of λ gets higher. The disadvantage is that it is not easy to reexpress the polynomials of λ in terms of wave functions. On the other hand, the advantage of the second method is that the right-hand side can be written readily in terms of the next wave function by using the SUSY transformation formula. The disadvantage is that it is not a trivial problem to integrate the left-hand side by ψ. Anyway, by combining information obtained in various ways above, we have successfully obtained new expressions for the vertex operators of $\mathcal{O}(\lambda^5)$ and $\mathcal{O}(\lambda^6)$, as we now show.

IV. THE EXPLICIT FORMS OF VERTEX OPERATORS

We will now display our results. The expressions for the conjugate gravitino and conjugate two-form vertex operators are new, while other vertex operators known so far are also shown for completeness.

A. Wave functions

The wave functions corresponding to the massless IIB supergravity multiplet have already obtained [3,4][1]

$$\Phi(\lambda) = 1, \quad (4.1)$$

$$\tilde{\Psi}(\lambda) = \not{k}\lambda, \quad (4.2)$$

$$B_{\mu\nu}(\lambda) = -\tfrac{1}{2}b_{\mu\nu}, \quad (4.3)$$

$$\Psi_\mu(\lambda) = \frac{i}{24}\Gamma^\alpha\not{k}\lambda b_{\alpha\mu}, \quad (4.4)$$

$$h_{\mu\nu}(\lambda) = \tfrac{1}{96}b_\mu{}^\rho b_{\rho\nu}, \quad (4.5)$$

$$A_{\mu\nu\rho\sigma}(\lambda) = -\frac{i}{32\cdot(4!)^2}b_{[\mu\nu}b_{\rho\sigma]} = -\frac{i}{4\cdot(4!)^2}(b_{\mu\nu}b_{\rho\sigma} + b_{\mu\rho}b_{\sigma\nu} + b_{\mu\sigma}b_{\nu\rho}), \quad (4.6)$$

$$\Psi^c_\mu(\lambda) = -\frac{i}{4\cdot 5!}\Gamma_\rho\not{k}\lambda b^{\rho\sigma}b_{\sigma\mu}, \quad (4.7)$$

$$B^c_{\mu\nu}(\lambda) = \frac{1}{6!}b_{\mu\rho}b^{\rho\sigma}b_{\sigma\nu}, \quad (4.8)$$

$$\tilde{\Psi}^c(\lambda) = \frac{1}{8!}\Gamma^{\mu\nu}\not{k}\lambda b_{\mu\rho}b^{\rho\sigma}b_{\sigma\nu}, \quad (4.9)$$

$$\Psi^c(\lambda) = -\frac{1}{8\cdot 8!}b_\mu{}^\nu b_\nu{}^\lambda b_\lambda{}^\sigma b_\sigma{}^\mu. \quad (4.10)$$

One may check the following SUSY transformations:

$$\delta\Phi = \bar{\epsilon}_2\tilde{\Phi}, \quad (4.11)$$

$$\delta\tilde{\Phi} = \not{k}\epsilon_1\Phi - \frac{i}{24}\Gamma^{\mu\nu\rho}\epsilon_2 H_{\mu\nu\rho}, \quad (4.12)$$

$$\delta B_{\mu\nu} = -\bar{\epsilon}_1\Gamma_{\mu\nu}\tilde{\Phi} + 2i(\bar{\epsilon}_2\Gamma_{[\mu}\Psi_{\nu]} + k_\mu\Lambda_{\nu]}), \quad (4.13)$$

$$\delta\Psi_\mu = \frac{1}{24\cdot 4}[9\Gamma^{\nu\rho}\epsilon_1 H_{\mu\nu\rho} - \Gamma_{\mu\nu\rho\sigma}\epsilon_1 H^{\nu\rho\sigma}] + \frac{i}{2}\Gamma^{\nu\rho}k_\rho h_{\mu\nu}\epsilon_2 + \frac{i}{4\cdot 5!}\Gamma^{\rho_1\cdots\rho_5}\Gamma_\mu\epsilon_2 F_{\rho_1\cdots\rho_5} + k_\mu\xi, \quad (4.14)$$

[1] Signs are corrected in (4.7), (4.8), (4.9), and (4.10).

$$\delta h_{\mu\nu} = -\frac{i}{2}\bar{\epsilon}_1\Gamma_{(\mu}\Psi_{\nu)} + \frac{i}{2}\bar{\epsilon}_2\Gamma_{(\mu}\Psi^c_{\nu)} + k_{(\mu}\xi_{\nu)}, \quad (4.15)$$

$$\delta A_{\mu\nu\rho\sigma} = -\frac{1}{(4!)^2}\bar{\epsilon}_1\Gamma_{[\mu\nu\rho}\Psi_{\sigma]} - \frac{1}{(4!)^2}\bar{\epsilon}_2\Gamma_{[\mu\nu\rho}\Psi_{\sigma]} + k_{[\mu}\xi_{\nu\rho\sigma]}, \quad (4.16)$$

$$\delta\Psi^c_\mu = -\frac{i}{2}\Gamma^{\nu\rho}k_\rho h_{\mu\nu}\epsilon_1 + \frac{i}{4\cdot 5!}\Gamma^{\rho_1\cdots\rho_5}\Gamma_\mu\epsilon_1 F_{\rho_1\cdots\rho_5} + \frac{1}{24\cdot 4}[9\Gamma^{\nu\rho}\epsilon_2 H^c_{\mu\nu\rho} - \Gamma_\mu{}^{\nu\rho\sigma}\epsilon_2 H^c_{\nu\rho\sigma}] + k_\mu\xi^c, \quad (4.17)$$

$$\delta B^c_{\mu\nu} = 2i(\bar{\epsilon}_1\Gamma_{[\mu}\Psi^c_{\nu]} + k_{[\mu}\Lambda^c_{\nu]}) - \bar{\epsilon}_2\Gamma_{\mu\nu}\tilde{\Phi}^c, \quad (4.18)$$

$$\delta\tilde{\Phi}^c = -\frac{i}{24}\Gamma^{\mu\nu\rho}\epsilon_1 H^c_{\mu\nu\rho} + \not{k}\epsilon_2\Phi^c, \quad (4.19)$$

$$\delta\Phi^c = \bar{\epsilon}_1\tilde{\Phi}^c, \quad (4.20)$$

where ξ, ξ_μ, $\xi_{\mu\nu\rho}$, and Λ_μ are gauge parameters.

B. Vertex operator for the dilaton Φ [3,4]

The dilaton vertex operator V^Φ is simply given by (3.10)

$$V^\Phi(A, \psi) = \mathrm{Str} e^{ik\cdot A}, \quad (4.21)$$

where $k^2 = 0$. Since there is no operator other than $e^{ik\cdot A}$ in the symmetrized trace, Str is equivalent to the ordinary trace.

C. Vertex operator for dilatino $\tilde{\Phi}$ [3,4]

The dilatino vertex operator $V^{\tilde{\Phi}}$ is read off from the term with a single λ (3.11) as

$$V^{\tilde{\Phi}}(A, \psi)\tilde{\Phi}(\lambda) = \mathrm{Str} e^{ik\cdot A}G_1 = \mathrm{Str} e^{ik\cdot A}\bar{\psi}\not{k}\lambda = \mathrm{Str} e^{ik\cdot A}\bar{\psi}\,\tilde{\Phi}(\lambda). \quad (4.22)$$

We obtain

$$V^{\tilde{\Phi}}(A, \psi) = \mathrm{Str} e^{ik\cdot A}\bar{\psi}. \quad (4.23)$$

The symmetrized trace is identical to ordinary trace.

D. Vertex operator of antisymmetric tensor field $B_{\mu\nu}$ [3,4]

The vertex operator for the antisymmetric tensor $B_{\mu\nu}$ can be obtained from the terms containing two λ's in the expansions of the supersymmetric Wilson loop. Using (3.17), we have

$$\begin{aligned} V^B_{\mu\nu}(A, \psi)B^{\mu\nu}(\lambda) &= \mathrm{Str}\frac{1}{2}\left((\bar{\lambda}\Gamma_\mu\psi)\frac{\delta}{\delta A_\mu} - \frac{i}{2}F_{\mu\nu}\Gamma^{\mu\nu}\lambda\frac{\delta}{\delta\psi}\right)V^{\tilde{\Phi}}(A, \psi)\tilde{\Phi}(\lambda) \\ &= \mathrm{Str}\frac{1}{2}\left((\bar{\lambda}\Gamma_\mu\psi)\frac{\delta}{\delta A_\mu} - \frac{i}{2}F_{\mu\nu}\Gamma^{\mu\nu}\lambda\frac{\delta}{\delta\psi}\right)(\bar{\psi}\not{k}\lambda)e^{ik\cdot A} = \mathrm{Str}\left(\frac{1}{2}(\bar{\psi}\not{k}\lambda)\cdot(\bar{\psi}\not{k}\lambda) + \frac{i}{4}F_{\mu\nu}b^{\mu\nu}\right)e^{ik\cdot A} \\ &= \mathrm{Str} e^{ik\cdot A}\left(\frac{1}{16}\bar{\psi}\cdot\not{k}\Gamma_{\mu\nu}\psi - \frac{i}{2}F_{\mu\nu}\right)B_{\mu\nu}(\lambda). \end{aligned} \quad (4.24)$$

Hence the vertex operator for the antisymmetric tensor field is given by [3,4]

$$V^B_{\mu\nu}(A, \psi) = \mathrm{Str} e^{ik\cdot A}\left(\frac{1}{16}\bar{\psi}\cdot\not{k}\Gamma_{\mu\nu}\psi - \frac{i}{2}F_{\mu\nu}\right). \quad (4.25)$$

The vertex operator satisfies

$$k^\mu V^B_{\mu\nu}(A, \psi) = 0. \quad (4.26)$$

This implies that the coupling of the vertex operator with the background field $V^B_{\mu\nu}(A, \psi)B^{\mu\nu}(\lambda)$ is gauge invariant.

E. Vertex operator for gravitino Ψ_μ [4]

The third order terms give the gravitino Ψ_μ vertex operator

$$V^\Psi_\mu(A, \psi) = \mathrm{Str} e^{ik\cdot A}\left(-\frac{i}{12}(\bar{\psi}\cdot\not{k}\Gamma_{\mu\nu}\psi) - 2F_{\mu\nu}\right)\cdot\bar{\psi}\Gamma^\nu. \quad (4.27)$$

It can be shown that (4.27) satisfies (3.18):

$$\epsilon\frac{\delta}{\delta\psi}V^\Psi_\mu(A, \psi)\Psi^\mu(\lambda) = V^B_{\mu\nu}(A, \psi)(\bar{\epsilon}\not{k}\lambda)B^{\mu\nu}(\lambda). \quad (4.28)$$

Indeed, the left-hand side is evaluated as follows:

$$\begin{aligned} \epsilon\frac{\delta}{\delta\psi}V^\Psi_\mu(A, \psi)\Psi^\mu(\lambda) &= \mathrm{Str} e^{ik\cdot A}\left(-\frac{i}{6}k^\rho(\bar{\epsilon}\Gamma_{\mu\nu\rho}\psi)\cdot\bar{\psi}\Gamma^\nu - \frac{i}{12}k^\rho(\bar{\psi}\cdot\Gamma_{\mu\nu\rho}\psi)\bar{\epsilon}\Gamma^\nu - 2F_{\mu\nu}\bar{\epsilon}\Gamma^\nu\right)\Psi^\mu(\lambda) \\ &= \mathrm{Str} e^{ik\cdot A}\left(-\frac{i}{4}(\bar{\psi}\cdot\not{k}\Gamma_{\mu\nu}\psi)\bar{\epsilon}\Gamma^\nu\Psi^\mu(\lambda) - 2F_{\mu\nu}\bar{\epsilon}\Gamma^\nu\Psi^\mu(\lambda)\right), \end{aligned} \quad (4.29)$$

whereas the right-hand side becomes

$$V^B_{\mu\nu}(A,\psi)(\bar{\epsilon}\not{k}\lambda)B^{\mu\nu}(\lambda) = \mathrm{Str}e^{ik\cdot A}\Big(\frac{1}{16}\bar{\psi}\cdot\not{k}\Gamma_{\mu\nu}\psi - \frac{i}{2}F_{\mu\nu}\Big)2i\bar{\epsilon}\Gamma^{[\mu}\Psi^{\nu]}(\lambda). \quad (4.30)$$

Thus we have established the Eq. (4.28).

This gravitino vertex operator is shown to satisfy

$$k^\mu V^\Psi_\mu(A,\psi) = 0. \quad (4.31)$$

The first term of $V^\Psi_\mu(A,\psi)$ trivially satisfies this relation, while the second term is calculated as

$$\begin{aligned} k^\mu V^\Psi_\mu(A,\psi) &= \mathrm{Str}e^{ik\cdot A}2i[ik\cdot A, A_\nu]\bar{\psi}\Gamma^\nu \\ &= \mathrm{Str}[e^{ik\cdot A}, A_\nu]2i\bar{\psi}\Gamma^\nu \\ &= -\mathrm{Str}e^{ik\cdot A}2i[\bar{\psi}, A_\nu]\Gamma^\nu = 0. \end{aligned} \quad (4.32)$$

In the last line we have used the equation of motion for ψ:

$$\Gamma^\mu[A_\mu,\psi] = 0. \quad (4.33)$$

F. Vertex operators for graviton $h_{\mu\nu}$ and fourth-rank antisymmetric tensor field $A_{\mu\nu\rho\sigma}$ [4]

The next terms with four λ 's give the vertex operators for the graviton $h_{\mu\nu}$ and the fourth-rank antisymmetric tensor $A_{\mu\nu\rho\sigma}$. In order to derive them, we use (3.17) with $n = 4$:

$$\begin{aligned} &V^h_{\mu\nu}(A,\psi)h^{\mu\nu}(\lambda) + V^A_{\mu\nu\rho\sigma}(A,\psi)A^{\mu\nu\rho\sigma}(\lambda) \\ &= \frac{1}{4}\Big(i(\bar{\lambda}\Gamma_\mu\psi)\frac{\delta}{\delta A_\mu} - \frac{i}{2}F_{\mu\nu}\Gamma^{\mu\nu}\lambda\frac{\delta}{\delta\psi}\Big)V^\Psi_\mu(A,\psi)\Psi^\mu(\lambda). \end{aligned} \quad (4.34)$$

In evaluating the right-hand side we use many Fierz identities and properties of symmetrized trace. Here we only write down the final results. The vertex operator for the graviton is given by

$$\begin{aligned} V^h_{\mu\nu} = \mathrm{Str}e^{ik\cdot A}\Big(&-\frac{1}{96}k^\rho k^\sigma(\bar{\psi}\cdot\Gamma_{\mu\rho}{}^\beta\psi)\cdot(\bar{\psi}\cdot\Gamma_{\nu\sigma\beta}\psi) \\ &-\frac{i}{4}k^\rho\bar{\psi}\cdot\Gamma_{\rho\beta(\mu}\psi\cdot F_{\nu)}{}^\beta + \frac{1}{2}\bar{\psi}\cdot\Gamma_{(\mu}[A_{\nu)},\psi] \\ &+ 2F_\mu{}^\rho\cdot F_{\nu\rho}\Big). \end{aligned} \quad (4.35)$$

The vertex operator for the fourth-rank antisymmetric tensor field is given by

$$\begin{aligned} V^A_{\mu\nu\rho\sigma} = \mathrm{Str}e^{ik\cdot A}\Big(&\frac{i}{8\cdot 4!}k_\alpha k_\gamma(\bar{\psi}\cdot\Gamma_{[\mu\nu}{}^\alpha\psi)\cdot(\bar{\psi}\cdot\Gamma_{\rho\sigma]}{}^\gamma\psi) \\ &+\frac{i}{3}\bar{\psi}\cdot\Gamma_{[\nu\rho\sigma}[\psi,A_{\mu]}] + \frac{1}{4}F_{[\mu\nu}\cdot(\bar{\psi}\cdot\Gamma_{\rho\sigma]}{}^\gamma\psi)k_\gamma \\ &- iF_{[\mu\nu}\cdot F_{\rho\sigma]}\Big). \end{aligned} \quad (4.36)$$

These vertex operators satisfy the conservation laws

$$k^\mu V^h_{\mu\nu}(A,\psi) = 0, \quad (4.37)$$

$$k^\mu V^A_{\mu\nu\rho\sigma}(A,\psi) = 0. \quad (4.38)$$

The first term of the graviton vertex operator trivially satisfies this equation, while the second term is evaluated as

$$\begin{aligned} &\mathrm{Str}\, e^{ik\cdot A}\Big(-\frac{1}{4}k^\rho(\bar{\psi}\cdot\Gamma_{\rho\beta\nu}\psi)[ik\cdot A, A^\beta]\Big) \\ &= \mathrm{Str}e^{ik\cdot A}\Big(\frac{1}{2}\bar{\psi}\cdot\Gamma_{\rho\beta\nu}[\psi, A^\beta]\Big) \\ &= \mathrm{Str}e^{ik\cdot A}\Big(+\frac{i}{2}\bar{\psi}\cdot\Gamma_\nu[\psi, ik\cdot A] + \frac{1}{2}\bar{\psi}\cdot\not{k}[\psi, A_\nu]\Big), \end{aligned} \quad (4.39)$$

where we used the equation of motion (4.33). Thus we obtain

$$\begin{aligned} k^\mu V^h_{\mu\nu}(A,\psi) &= \mathrm{Str}e^{ik\cdot A}(-i\bar{\psi}\cdot\Gamma_\nu[ik\cdot A,\psi] \\ &\quad + 2i[F_{\nu\rho}, A^\rho]) \\ &= \mathrm{Str}e^{ik\cdot A}(+i\{\psi_\alpha,\psi_\beta\}(\Gamma_0\Gamma_\nu)_{\alpha\beta} \\ &\quad + 2i[F_{\nu\rho}, A^\rho]) \\ &= 0. \end{aligned} \quad (4.40)$$

where we have used another equation of motion

$$[A^\nu,[A_\mu,A_\nu]] - \tfrac{1}{2}(\Gamma_0\Gamma_\mu)_{\alpha\beta}\{\psi_\alpha,\psi_\beta\} = 0. \quad (4.41)$$

We can similarly show that the vertex operator for the antisymmetric tensor field satisfies the conservation law; multiplying k^μ, the first term trivially vanishes while the second and third terms cancel with each other. The fourth term vanishes due to the Jacobi identity. Thus the couplings to a graviton and a fourth-rank antisymmetric tensor $V^h_{\mu\nu}(A,\psi)h^{\mu\nu}(\lambda)$, $V^A_{\mu\nu\rho\sigma}(A,\psi)A^{\mu\nu\rho\sigma}(\lambda)$ are, respectively, gauge invariant.

G. Vertex operator for gravitino Ψ^c_μ

The vertex operator for the gravitino Ψ^c_μ can be obtained from the terms with five λ's by using the following relation:

$$\begin{aligned} V^{\Psi^c}_\mu(A,\psi)\Psi^{c\mu}(\lambda) = \frac{1}{5}\Big(&i(\bar{\lambda}\Gamma_\mu\psi)\frac{\delta}{\delta A_\mu} - \frac{i}{2}F_{\mu\nu}\Gamma^{\mu\nu}\lambda\frac{\delta}{\delta\psi}\Big) \\ &\times(V^h_{\mu\nu}(A,\psi)h^{\mu\nu}(\lambda) \\ &+ V^A_{\mu\nu\rho\sigma}(A,\psi)A^{\mu\nu\rho\sigma}(\lambda)). \end{aligned} \quad (4.42)$$

Many identities are needed in order to derive the vertex operator. Carrying out the complicated calculation, we finally obtain

$$V_\mu^{\Psi^c} = \mathrm{Str} e^{ik\cdot A}\Big(-\frac{i}{2\cdot 5!}k^\lambda k^\tau (\bar\psi\cdot\Gamma_{\mu\lambda}{}^\sigma\psi)\cdot(\bar\psi\cdot\Gamma_{\nu\tau\sigma}\psi)\cdot\bar\psi\Gamma^\nu + \frac{1}{24}k^\lambda(\bar\psi\cdot\Gamma_{\lambda\mu\nu}\psi)\cdot\bar\psi\Gamma^\nu\Gamma_{\rho\sigma}\cdot F^{\rho\sigma} - \frac{1}{6}k^\lambda(\bar\psi\cdot\Gamma_{\lambda\alpha\beta}\psi)\cdot\bar\psi\Gamma^\beta\cdot F^\alpha{}_\mu + \frac{i}{3}(\bar\psi\cdot\Gamma_\mu[A_\nu,\psi])\cdot\bar\psi\Gamma^\nu + \frac{i}{3}(\bar\psi\cdot\Gamma_\nu[A_\mu,\psi])\cdot\bar\psi\Gamma^\nu + \frac{i}{6}(\bar\psi\cdot\Gamma_{\alpha\beta\mu}\psi)\cdot[A^\alpha,\bar\psi]\Gamma^\beta - iF_{\mu\nu}\cdot F_{\rho\sigma}\cdot\bar\psi\Gamma^\nu\Gamma^{\rho\sigma}\Big). \tag{4.43}$$

It can be shown that this gravitino vertex operator satisfies

$$k^\mu V^{\Psi^c}(A,\psi) = 0. \tag{4.44}$$

This assures the gauge invariance of the coupling with gravitino field $V_\mu^{\Psi^c}(A,\psi)\Psi^{c\mu}(\lambda)$. Using the following identity

$$(\bar\psi\Gamma_\mu[A_\nu,\psi])\bar\psi\Gamma^\nu\Psi^{c\mu}(\lambda) = \tfrac{1}{2}(\bar\psi\Gamma_\alpha[A_\mu,\psi])\bar\psi\Gamma^\alpha\Psi^{c\mu}(\lambda) + \tfrac{1}{4}(\bar\psi\Gamma^{\nu\alpha\mu}\psi)[A_\nu,\bar\psi]\Gamma_\alpha\Psi^c_\mu(\lambda), \tag{4.45}$$

we can rewritten the vertex operator

$$V_\mu^{\Psi^c} = \mathrm{Str} e^{ik\cdot A}\Big(-\frac{i}{2\cdot 5!}k^\lambda k^\tau (\bar\psi\cdot\Gamma_{\mu\lambda}{}^\sigma\psi)\cdot(\bar\psi\cdot\Gamma_{\nu\tau\sigma}\psi)\cdot\bar\psi\Gamma^\nu + \frac{1}{24}k^\lambda(\bar\psi\cdot\Gamma_{\lambda\mu\nu}\psi)\cdot\bar\psi\Gamma^\nu\Gamma_{\rho\sigma}\cdot F^{\rho\sigma} - \frac{1}{6}k^\lambda(\bar\psi\cdot\Gamma_{\lambda\alpha\beta}\psi)\cdot\bar\psi\Gamma^\beta\cdot F^\alpha{}_\mu + i\bar\psi\Gamma_\mu[A_\nu,\psi]\bar\psi\Gamma^\nu - iF_{\mu\nu}\cdot F_{\rho\sigma}\cdot\bar\psi\Gamma^\nu\Gamma^{\rho\sigma}\Big). \tag{4.46}$$

Since the vertex operator for gravitino is the supercurrent, we obtain the supercurrent for IIB matrix model

$$J_\mu^{(2)}(A,\psi) = (4.43)\ (\text{or } (4.46)). \tag{4.47}$$

This is conserved due to (4.44).

H. Vertex operator for antisymmetric tensor field $B^c_{\mu\nu}$

The sixth order terms give the antisymmetric tensor field $B^c_{\mu\nu}$ vertex operator. After lengthy calculations we finally obtained the following new expression:

$$\begin{aligned} V^{B^c}_{\mu\nu}(A,\psi) = \mathrm{Str} e^{ik\cdot A}\Big(&-\frac{1}{8\cdot 6!}k^\lambda k^\tau k^\alpha(\bar\psi\cdot\Gamma_{\mu\lambda}{}^\sigma\psi)\cdot(\bar\psi\cdot\Gamma_{\gamma\tau\sigma}\psi)\cdot(\bar\psi\cdot\Gamma^\gamma{}_{\alpha\nu}\psi) + \frac{i}{64}(\bar\psi\cdot\not{k}\Gamma_{\mu\alpha}\psi)\cdot F^{\alpha\beta}(\bar\psi\cdot\not{k}\Gamma_{\beta\nu}\psi) \\ &+\frac{i}{16\cdot 4!}(\bar\psi\cdot\not{k}\Gamma_{[\mu\alpha}\psi)\cdot(\bar\psi\cdot\not{k}\Gamma^{\alpha\sigma}\psi)\cdot F_{\sigma\nu]} - \frac{1}{32}\bar\psi\cdot\Gamma_{[\mu}[A^\sigma,\psi]\cdot(\bar\psi\cdot\not{k}\Gamma_{\sigma\nu]}\psi) \\ &-\frac{1}{64}(\bar\psi\cdot\not{k}\Gamma_{[\mu\alpha}\psi)\cdot\bar\psi\Gamma^\alpha[A_{\nu]},\psi] + \frac{i}{4!\cdot 32}\Xi_{\mu\nu\alpha\beta\gamma}\cdot(\bar\psi\cdot\Gamma^{\alpha\beta\gamma}\psi) - \frac{i}{64}[A_\alpha,F^{\alpha\tau}]\cdot(\bar\psi\cdot\Gamma_{\tau\mu\nu}\psi) \\ &+\frac{1}{64}(\bar\psi\cdot\Gamma_{\mu\nu\rho\sigma\lambda\tau}\not{k}\psi)\cdot F^{\rho\sigma}\cdot F^{\lambda\tau} + \frac{1}{16}(\bar\psi\cdot\Gamma_{\rho\sigma}\not{k}\psi)\cdot F^{\rho\sigma}\cdot F^{\mu\nu} - \frac{1}{8}(\bar\psi\cdot\Gamma_{\rho\sigma}\not{k}\psi)\cdot F^{\mu\rho}\cdot F^{\nu\sigma} \\ &+\frac{1}{8}(\bar\psi\cdot\Gamma_{[\mu\sigma}\not{k}\psi)\cdot F^{\sigma\alpha}\cdot F_{\alpha\nu]} - \frac{1}{32}(\bar\psi\cdot\Gamma^{\mu\nu}\not{k}\psi)\cdot F^{\rho\sigma}\cdot F_{\sigma\rho} + \frac{i}{4}\bar\psi\cdot\Gamma_{\mu\nu\alpha}[A_\beta,\psi]\cdot F^{\alpha\beta} \\ &+\frac{i}{8}\bar\psi\cdot\Gamma_{\rho\sigma[\mu}[A_{\nu]},\psi]\cdot F^{\rho\sigma} + \frac{i}{4}\bar\psi\cdot\Gamma_{(\mu}[A_{\rho)},\psi]\cdot F^\rho{}_\nu \\ &-\frac{i}{4}\bar\psi\cdot\Gamma_{(\nu}[A_{\rho)},\psi]\cdot F^{\rho\mu} - iF_{\mu\rho}\cdot F^{\rho\sigma}\cdot F_{\sigma\nu} + \frac{i}{4}F_{\mu\nu}\cdot F^{\rho\sigma}\cdot F_{\sigma\rho}\Big), \end{aligned} \tag{4.48}$$

where $\Xi_{\mu\nu\rho\sigma\tau}$ is defined by

$$\Xi_{\mu\nu\rho\sigma\tau} = \{\psi_\alpha,\psi_\beta\}(\Gamma_0\Gamma_{\mu\nu\rho\sigma\tau})_{\alpha\beta}. \tag{4.49}$$

This vertex operator satisfies

$$k^\mu V^{B^c}_{\mu\nu}(A,\psi) = 0. \tag{4.50}$$

Hence the coupling $V^{B^c}_{\mu\nu}(A,\psi)B^{c\mu\nu}(\lambda)$ is gauge invariant.

I. Other vertex operators

The next terms with seven λ's give the vertex operator for the dilatino $\tilde\Phi$. The calculation becomes more complicated. We have not yet derived the complete expression so far. Here we give a part of our result:

$$V^{\tilde{\Phi}^c}(A,\psi) = \mathrm{Str}e^{ik\cdot A}\bigg(\frac{1}{8!}(\bar{\psi}\cdot\Gamma^{\alpha\gamma}\not{k}\psi)\cdot(\bar{\psi}\cdot\Gamma_{\gamma\delta}\not{k}\psi)\cdot(\bar{\psi}\cdot\Gamma^{\delta\beta}\not{k}\psi)\cdot\bar{\psi}\Gamma_{\alpha\beta} - \frac{i}{2\cdot 5!}F^{\mu\alpha}\cdot(\bar{\psi}\cdot\not{k}\Gamma_{\alpha\beta}\psi)\cdot(\bar{\psi}\cdot\not{k}\Gamma^{\beta\nu}\psi)\cdot\bar{\psi}\Gamma_{\mu\nu} + \cdots\cdots - \frac{1}{8\cdot 4!}F^{\mu\nu}\cdot F^{\rho\sigma}(\bar{\psi}\cdot\Gamma_{\mu\nu\rho\sigma\lambda\alpha\beta}\psi)k^\lambda\cdot\bar{\psi}\Gamma^{\alpha\beta} - \frac{1}{12}F^{\mu\alpha}\cdot F_{\alpha\beta}\cdot(\bar{\psi}\cdot\not{k}\Gamma^{\beta\nu}\psi)\cdot\bar{\psi}\Gamma_{\mu\nu} - \frac{1}{24}F^{\mu\alpha}\cdot(\bar{\psi}\cdot\not{k}\Gamma_{\alpha\beta}\psi)\cdot F^{\beta\nu}\cdot\bar{\psi}\Gamma_{\mu\nu} - \frac{1}{48}F^{\rho\sigma}\cdot(\bar{\psi}\cdot\not{k}\Gamma_{\rho\sigma}\psi)\cdot F^{\mu\nu}\cdot\bar{\psi}\Gamma_{\mu\nu} + \cdots + \cdots + \frac{i}{24}\bar{\psi}\cdot\Gamma_{\mu\nu\rho\sigma\lambda\tau}F^{\mu\nu}\cdot F^{\rho\sigma}\cdot F^{\lambda\tau} + i\bar{\psi}\cdot\Gamma^{\mu\nu}\bigg(F_{\mu\rho}\cdot F^{\rho\sigma}\cdot F_{\sigma\nu} - \frac{1}{4}F^{\rho\sigma}\cdot F_{\sigma\rho}\cdot F_{\mu\nu}\bigg)\bigg). \tag{4.51}$$

The $\mathcal{O}(\lambda^8)$ terms give the vertex operator for the dilaton Φ^c. It also has partly been obtained as follows:

$$V^{\Phi^c}(A,\psi) = \mathrm{Str}e^{ik\cdot A}\bigg(\frac{1}{8\cdot 8!}(\bar{\psi}\cdot\Gamma^{\alpha\gamma}\not{k}\psi)\cdot(\bar{\psi}\cdot\Gamma_{\gamma\delta}\not{k}\psi)\cdot(\bar{\psi}\cdot\Gamma_{\delta\beta}\not{k}\psi)\cdot(\bar{\psi}\cdot\Gamma_{\alpha\beta}\not{k}\psi) + \cdots\cdots + \frac{i}{48}(\bar{\psi}\cdot\Gamma_{\mu\nu\rho\sigma\lambda\tau}\not{k}\psi)\cdot F^{\mu\nu}\cdot F^{\rho\sigma}\cdot F^{\lambda\tau} + [A_\mu,\bar{\psi}]\cdot\Gamma_{\rho\sigma}\Gamma_\nu\psi\cdot F^{\mu\nu}\cdot F^{\rho\sigma} + \frac{i}{2}(\bar{\psi}\cdot\Gamma^{\mu\nu}\not{k}\psi)\cdot\bigg(F_{\mu\rho}\cdot F^{\rho\sigma}\cdot F_{\sigma\nu} - \frac{1}{4}F^{\rho\sigma}\cdot F_{\sigma\rho}\cdot F_{\mu\nu}\bigg) - \bigg(F_{\mu\nu}\cdot F^{\nu\rho}\cdot F_{\rho\sigma}\cdot F^{\sigma\mu} - \frac{1}{4}F_{\mu\nu}\cdot F^{\nu\mu}\cdot F_{\rho\sigma}\cdot F^{\sigma\rho}\bigg)\bigg). \tag{4.52}$$

We summarize our results in Appendix B.

V. KINEMATIC FACTOR OF FOUR-GRAVITON AMPLITUDE: A CONSISTENCY CHECK

As we already mentioned, the IIB matrix model was originally proposed as a matrix regularization of the Schild-gauge GS type IIB superstring, and may also be regarded as composed of the D-instanton degrees of freedom of type IIB string theory. We have constructed a set of massless vertex operators of the IIB matrix model by requiring the covariance under supersymmetry, and therefore they form a massless type IIB supergravity multiplet by construction. They also appear as factors of one-loop block-block interactions, which serves as a consistency check. As a further check on the validity, we will show that the kinematic factor of the R^4 term may be derived by using our vertex operators, precisely in the same manner as was done by using the light-cone supermembrane vertex operators [5,10].

Suppose that we consider the following four-graviton amplitude

$$\bigg\langle\prod_{r=1}^{4}V^h_{\mu_r\nu_r}(k_r)h^{\mu_r\nu_r}(k_r)\bigg\rangle. \tag{5.1}$$

Using the zeromode saturation argument, we find that only the ψ^4 term can contribute to the amplitude

$$V^h_{\mu\nu}(k) \sim -\frac{1}{4\cdot 4!}k^\lambda k^\tau\mathrm{Str}e^{ikA}(\bar{\psi}\cdot\Gamma^{\sigma}_{\mu\lambda}\psi)\cdot(\bar{\psi}\cdot\Gamma_{\nu\tau\sigma}\psi). \tag{5.2}$$

Thus (5.1) contains the factor

$$\int d\psi_0^{16}\prod_{r=1}^{4}k_r^{\lambda_r}k_r^{\tau_r}(\bar{\psi}_0\Gamma_{\mu_r\lambda_r}{}^{\sigma_r}\psi_0)\cdot(\bar{\psi}_0\Gamma_{\nu_R\tau_r\sigma_r}\psi_0), \tag{5.3}$$

where ψ_0 is the zeromode. This is precisely the factor which arises in the four-graviton amplitude in the D-instanton background computed in [11]. Indeed, our construction of the IIB matrix-model vertex operators shares some similarity with their construction of the closed string vertices in the D-instanton background. Equation (5.3) is therefore proportional to $t_8t_8R^4$, which is a consistent result.

VI. CONCLUSIONS

In this paper we have reported progress in determining the complete form of massless supergravity vertex operators in the IIB matrix model. In principle, they are determined by supersymmetry, but it becomes harder to carry out the actual computation for higher vertex operators. We have developed two new methods to lighten our work. After dozens of pages of Fierz arrangements we have finally reached new formulas for the vertex operators emitting two higher component fields in the supergravity multiplet.

While it is important to fix the exact form in its own right, vertex operators may be useful in computing correlation functions in the IIB matrix model. We have taken the same step as in [5] to compute a four-point graviton amplitude to find a consistent result.

We have extensively used the fact that the simplest kind of supersymmetric Wilson loop operator satisfies the same equation as the generating function of the Wilson loop correlation function does. The one-loop analysis already has revealed that the effective action has a form expressed in terms of bilinear of vertex operators. In view of these facts, we conjecture that the low-energy effective action of the IIB matrix model in the large-N limit is precisely given by tree-level supergravity coupled to the vertex operators presented here. It is because they possess the identical $\mathcal{N}=2$ supersymmetry.

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APPENDIX A: NOTATION

In Appendix A, we present the notation used in this paper. Basically, we follow the same notation as in Ref. [4].

1. 10D gamma matrices

We use the Minkowskian spacetime metric

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \cdots, +1). \tag{A1}$$

Gamma matrices are defined by

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}. \tag{A2}$$

We use the *Majorana-Weyl representation*. In this representation Γ_0 has the following properties:

$$(\Gamma_0)^T = \Gamma_0, \tag{A3}$$

$$(\Gamma_0)^2 = -1, \tag{A4}$$

$$\Gamma_0 \Gamma_\mu \Gamma_0 = -(\Gamma_\mu)^T. \tag{A5}$$

The chirality matrix Γ_{11} is defined by

$$\Gamma_{11} \equiv \Gamma_0 \Gamma_1 \cdots \Gamma_9, \tag{A6}$$

and satisfies

$$(\Gamma_{11})^2 = 1, \tag{A7}$$

$$(\Gamma_{11})^\dagger = \Gamma_{11}. \tag{A8}$$

We denote the antisymmetrized gamma matrix as follows:

$$\Gamma_{\mu_1\mu_2\cdots\mu_n} = \underbrace{\Gamma_{\mu_1}\Gamma_{\mu_2}\cdots\Gamma_{\mu_n}} = \frac{1}{n!}\Gamma_{[\mu_1}\Gamma_{\mu_2}\cdots\Gamma_{\mu_n]}. \tag{A9}$$

a. Symmetry and antisymmetry

In our representation $\Gamma_0\Gamma_{\mu_1\cdots\mu_n}$ is either symmetric or antisymmetric. For example,

$\Gamma_0 \cdots$ symmetric $\qquad \Gamma_0\Gamma_\mu \cdots$ symmetric

$\Gamma_0\Gamma_{\mu_1\mu_2} \cdots$ anti-symmetric

$\Gamma_0\Gamma_{\mu_1\mu_2\mu_3} \cdots$ anti-symmetric

$\Gamma_0\Gamma_{\mu_1\mu_2\mu_3\mu_4} \cdots$ symmetric

$\Gamma_0\Gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} \cdots$ symmetric.

In general,

$$(\Gamma_0\Gamma_{\mu_1\mu_2\cdots\mu_n})_{\alpha\beta} = (-1)^{(n(n-1))/2}(\Gamma_0\Gamma_{\mu_1\mu_2\cdots\mu_n})_{\beta\alpha}. \tag{A10}$$

b. Duality

The following relations hold between $\Gamma^{\mu_1\cdots\mu_n}$ and $\Gamma^{\mu_{n+1}\cdots\mu_{10}}$:

$$\Gamma^{\mu_1\cdots\mu_n} = \frac{(-1)^{(k(k-1))/2}}{(10-k)!}\epsilon^{\mu_1\mu_2\cdots\mu_{10}}\Gamma_{\mu_{k+1}\cdots\mu_{10}}\Gamma_{11}. \tag{A11}$$

c. Multiplication law

The product of two antisymmetrized gamma matrices can be decomposed in terms of antisymmetrized matrices. For instance, the product of Γ_μ and Γ_ν is decomposed as

$$\Gamma_\mu\Gamma_\nu = \Gamma_{\mu\nu} + \eta_{\mu\nu}, \tag{A12}$$

and the product of $\Gamma_{\mu\nu}$ and Γ_λ decomposed as

$$\Gamma_{\mu\nu}\Gamma_\lambda = \Gamma_{\mu\nu\lambda} - 2\eta_{\mu\underbrace{\lambda}}\Gamma_{\underbrace{\nu}}. \tag{A13}$$

Generally, the following multiplication law of the gamma matrices holds:

$$\Gamma^{\mu_1\mu_2\cdots\mu_p}\Gamma^{\nu_1\nu_2\cdots\nu_q} = \sum_{k=0}^{\min(p,q)}(-1)^{\frac{1}{2}k(2p-k-1)} \times \frac{p!q!}{(p-k)!(q-k)!k!} \times \eta^{\mu_1\nu_1}\cdots\eta^{\mu_k\nu_k}\Gamma^{\mu_{k+1}\cdots\mu_p\nu_{k+1}\cdots\nu_q}. \tag{A14}$$

The above two cases correspond to $(p, q) = (1, 1)$, $(p, q) = (2, 1)$ respectively.

d. Commutators and anticommutators

Using the multiplication law, we can derive the following relations:

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \tag{A15}$$

$$[\Gamma^{\alpha\beta}, \Gamma^\mu] = -4\eta^{\mu\alpha}\Gamma^{\beta} \tag{A16}$$

$$\{\Gamma^{\alpha\beta\gamma}, \Gamma^\mu\} = 6\eta^{\mu\alpha}\Gamma^{\beta\gamma} \tag{A17}$$

$$[\Gamma^{\alpha\beta\gamma\delta}, \Gamma^\mu] = -8\eta^{\mu\alpha}\Gamma^{\beta\gamma\delta} \tag{A18}$$

$$\{\Gamma^{\alpha\beta\gamma\delta\lambda}, \Gamma^\mu\} = 10\eta^{\mu\alpha}\Gamma^{\beta\gamma\delta\lambda} \tag{A19}$$

$$[\Gamma_{\mu\nu\rho}, \Gamma_{\alpha\beta}] = 12\eta_{\mu\alpha}\Gamma_{\nu\rho\beta} \tag{A21}$$

$$[\Gamma_{\mu\nu\rho}, \Gamma_{\alpha\beta}] = 12\eta_{\mu\alpha}\Gamma_{\nu\rho\beta} \tag{A21}$$

$$[\Gamma_{\mu\nu\rho\sigma}, \Gamma_{\alpha\beta}] = -16\eta_{\mu\alpha}\Gamma_{\nu\rho\sigma\beta} \tag{A22}$$

$$[\Gamma_{\mu\nu\rho\sigma\gamma}, \Gamma_{\alpha\beta}] = 20\eta_{\mu\alpha}\Gamma_{\nu\rho\sigma\beta} \tag{A23}$$

$$\{\Gamma_{\mu\nu\rho}, \Gamma_{\alpha\beta\gamma}\} = 18\eta_{\mu\alpha}\Gamma_{\nu\rho\beta\gamma} - 12\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\rho\gamma}. \tag{A24}$$

e. Contractions

The following relations among the gamma matrices hold:

$$\Gamma^\alpha\Gamma_\alpha = 10, \tag{A25}$$

$$\Gamma^\alpha\Gamma_\mu\Gamma_\alpha = -8\Gamma_\mu, \tag{A26}$$

$$\Gamma^\alpha\Gamma_{\mu\nu}\Gamma_\alpha = 6\Gamma_{\mu\nu}, \tag{A27}$$

$$\Gamma^\alpha\Gamma_{\mu\nu\rho}\Gamma_\alpha = -4\Gamma_{\mu\nu\rho}, \tag{A28}$$

$$\Gamma^\alpha\Gamma_{\mu\nu\rho\sigma}\Gamma_\alpha = 2\Gamma_{\mu\nu\rho\sigma}, \tag{A29}$$

$$\Gamma^\alpha\Gamma_{\mu\nu\rho\sigma\lambda}\Gamma_\alpha = 0, \tag{A30}$$

$$\Gamma^\alpha\Gamma_{\mu\nu\rho\sigma\lambda\tau}\Gamma_\alpha = -2\Gamma_{\mu\nu\rho\sigma\lambda\tau}, \tag{A31}$$

$$\Gamma^\alpha\Gamma_{\mu\nu\rho\sigma\lambda\tau\eta}\Gamma_\alpha = 4\Gamma_{\mu\nu\rho\sigma\lambda\tau\eta}, \tag{A32}$$

$$\Gamma^{\alpha\beta}\Gamma_{\alpha\beta} = -90, \tag{A33}$$

$$\Gamma^{\alpha\beta}\Gamma_\mu\Gamma_{\alpha\beta} = -54\Gamma_\mu, \tag{A34}$$

$$\Gamma^{\alpha\beta}\Gamma_{\mu\nu}\Gamma_{\alpha\beta} = -26\Gamma_\nu, \tag{A35}$$

$$\Gamma^{\alpha\beta}\Gamma_{\mu\nu\rho}\Gamma_{\alpha\beta} = -6\Gamma_{\mu\nu\rho}, \tag{A36}$$

$$\Gamma^{\alpha\beta}\Gamma_{\mu\nu\rho\sigma}\Gamma_{\alpha\beta} = 6\Gamma_{\mu\nu\rho\sigma}, \tag{A37}$$

$$\Gamma^{\alpha\beta}\Gamma_{\mu\nu\rho\sigma\lambda}\Gamma_{\alpha\beta} = 10\Gamma_{\mu\nu\rho\sigma\lambda}, \tag{A38}$$

$$\Gamma^{\alpha\beta\gamma}\Gamma_{\alpha\beta\gamma} = -720, \tag{A39}$$

$$\Gamma^{\alpha\beta\gamma}\Gamma_\mu\Gamma_{\alpha\beta\gamma} = 288\Gamma_\mu, \tag{A40}$$

$$\Gamma^{\alpha\beta\gamma}\Gamma_{\mu\nu}\Gamma_{\alpha\beta\gamma} = -48\Gamma_{\mu\nu}, \tag{A41}$$

$$\Gamma^{\alpha\beta\gamma}\Gamma_{\mu\nu\rho}\Gamma_{\alpha\beta\gamma} = -48\Gamma_{\mu\nu\rho}, \tag{A42}$$

$$\Gamma^{\alpha\beta\gamma}\Gamma_{\mu\nu\rho\sigma}\Gamma_{\alpha\beta\gamma} = 48\Gamma_{\mu\nu\rho\sigma}, \tag{A43}$$

$$\Gamma^{\alpha\beta\gamma}\Gamma_{\mu\nu\rho\sigma\lambda}\Gamma_{\alpha\beta\gamma} = 0. \tag{A44}$$

2. Majorana-Weyl spinors and fermion bilinears

In ten dimensions, a spinor ψ can be both Weyl and Majorana. *Weyl* means that ψ is an eigenstate of the chirality operator Γ_{11},

$$\Gamma_{11}\psi = \pm\psi, \tag{A45}$$

while *Majorana* means that the charge conjugate of ψ is ψ itself:

$$\psi^c = \psi. \tag{A46}$$

In our representation of gamma matrices we have

$$\psi^c = \psi^*. \tag{A47}$$

Hence the Majorana condition becomes

$$\psi^* = \psi. \tag{A48}$$

Let us consider a fermion bilinear $\bar{\psi}_1\Gamma_{\mu_1\cdots\mu_n}\psi_2$. If ψ_1 and ψ_2 are Weyl spinors with positive chirality, they satisfy the following relations:

$$\begin{aligned}\bar{\psi}_1\Gamma_{\mu_1\cdots\mu_n}\psi_2 &= \psi_1^\dagger\Gamma_0\Gamma_{\mu_1\cdots\mu_n}\Gamma_{11}\psi_2 \\ &= (-1)^{n+1}\psi_1^\dagger\Gamma_{11}\Gamma_0\Gamma_{\mu_1\cdots\mu_n}\psi_2 \\ &= (-1)^{n+1}\bar{\psi}_1\Gamma_{\mu_1\cdots\mu_n}\psi_2.\end{aligned} \tag{A49}$$

Hence, bilinear forms vanish unless n is odd.

Next, let ψ_1 and ψ_2 be Majorana spinors, then

$$\begin{aligned}\bar{\psi}_1\Gamma_{\mu_1\cdots\mu_n}\psi_2 &= (\psi_1)_\alpha(\Gamma_0\Gamma_{\mu_1\cdots\mu_n})_{\alpha\beta}(\psi_2)_\beta \\ &= -(-1)^{(n(n-1))/2}(\psi_2)_\beta(\Gamma_0\Gamma_{\mu_1\cdots\mu_n})_{\beta\alpha}(\psi_1)_\alpha \\ &= -(-1)^{(n(n-1))/2}\bar{\psi}_2\Gamma_{\mu_1\cdots\mu_n}\psi_1.\end{aligned} \tag{A50}$$

If $\psi_1 = \psi_2$, the bilinear form vanishes unless $n = 2, 3, 6, 7, 10$.

In summary, if ψ is a Majorana-Weyl spinor, the bilinear form $\bar{\psi}\Gamma_{\mu_1\cdots\mu_n}\psi$ vanishes unless $n = 3, 7$.

3. The Fierz identity

The Fierz identity is given by [12]

$$\begin{aligned}(\bar{\psi}M\chi)(\bar{\lambda}N\psi) = &-\frac{1}{32}\sum_{n=0}^{5} C_n(\bar{\psi}\Gamma_{\mu_1\cdots\mu_n}\phi) \\ &\times(\bar{\lambda}N\Gamma^{\mu_1\cdots\mu_n}M\chi),\end{aligned} \tag{A51}$$

$$C_0 = 2, \qquad C_1 = 2, \qquad C_2 = -1,$$
$$C_3 = -\tfrac{1}{3}, \qquad C_4 = \tfrac{1}{12}, \qquad C_5 = \tfrac{1}{120}. \tag{A52}$$

We can derive many identities using the above Fierz rearrangement formula; we present some of them. In the following formulas λ is a Majorana-Weyl spinor and $k^2 = 0$. $f^{\mu_1\cdots\mu_n}$ is an antisymmetric tensor.

a. $\boldsymbol{\lambda^2}$

The following relations hold:

$$(\bar\epsilon \not{k}\lambda)\bar\lambda = \tfrac{1}{96}(\bar\lambda\Gamma_{\alpha\beta\gamma}\lambda)\bar\epsilon\not{k}\Gamma^{\alpha\beta\gamma}$$
$$= \tfrac{1}{16}b_{\alpha\beta}\bar\epsilon\Gamma^{\alpha\beta} - \tfrac{1}{96}(\bar\lambda\Gamma_{\alpha\beta\gamma}\lambda)\bar\epsilon\Gamma^{\alpha\beta\gamma}\not{k}, \tag{A53}$$

$$(\bar\epsilon_1\Gamma^\rho\lambda)(\bar\lambda\Gamma_\rho\epsilon_2) = -\tfrac{1}{24}(\bar\lambda\Gamma_{\alpha\beta\gamma}\lambda)(\bar\epsilon_1\Gamma^{\alpha\beta\gamma}\epsilon_2), \tag{A54}$$

$$(\bar\epsilon\Gamma^\nu\lambda)\bar\lambda\not{k}\Gamma_\nu = \tfrac{1}{8}b_{\alpha\beta}\bar\epsilon\Gamma^{\alpha\beta} + \tfrac{1}{48}(\bar\lambda\Gamma_{\alpha\beta\gamma}\lambda)\bar\epsilon\Gamma^{\alpha\beta\gamma}\not{k}, \tag{A55}$$

$$(\bar\epsilon_1\not{k}\lambda)(\bar\lambda\not{k}\epsilon_2) = \tfrac{1}{16}b^{\alpha\beta}(\bar\epsilon_1\not{k}\Gamma_{\alpha\beta}\epsilon_2), \tag{A56}$$

$$(\bar\psi\not{k}\lambda)^2 = -\tfrac{1}{16}\tilde b^{\mu\nu}b_{\mu\nu}, \tag{A57}$$

where $b_{\mu\nu} = (\bar\lambda\not{k}\Gamma_{\mu\nu}\lambda)$, and $\tilde b_{\mu\nu} = (\bar\psi\not{k}\Gamma_{\mu\nu}\psi)$.

b. $\boldsymbol{\lambda^3}$

The following identities hold:

$$\Gamma^{\alpha\beta}\lambda(\bar\lambda\Gamma_{\alpha\beta\gamma}\lambda) = 0, \tag{A58}$$

$$\Gamma^{\alpha\beta\gamma}\lambda(\bar\lambda\Gamma_{\alpha\beta\gamma}\lambda) = 0, \tag{A59}$$

$$(\bar\lambda\Gamma^{\alpha\beta\gamma}\lambda)\bar\lambda = \tfrac{1}{2}(\bar\lambda\Gamma_\tau{}^{\overline{\alpha\beta}}\lambda)\bar\lambda\Gamma^\tau\Gamma^{\overline{\gamma}} \tag{A60}$$

$$\Gamma_{\mu\nu}\not{k}\lambda b^{\mu\nu} = 0, \tag{A61}$$

$$(\bar\psi\not{k}\lambda)^3 = -\tfrac{1}{48}\tilde b^{\mu\nu}(\bar\psi\Gamma_\mu\Gamma_\alpha\not{k}\lambda)b^\alpha{}_\nu. \tag{A62}$$

c. $\boldsymbol{\lambda^4}$

The following identities hold:

$$b_{\mu\nu}b^{\mu\nu} = 0, \tag{A63}$$

$$(\bar\lambda\Gamma^{\tau\overline{\alpha\beta}}\lambda)b_\tau{}^{\overline{\gamma}} = 0 \tag{A64}$$

$$b_{\mu\nu}b_{\rho\sigma} = \tfrac{1}{3}(b_{\mu\nu}b_{\rho\sigma} + b_{\mu\rho}b_{\sigma\nu} + b_{\mu\sigma}b_{\nu\rho})$$
$$- \tfrac{2}{3}\eta_{\underline{\mu}\rho}\overline{b_{\underline{\nu}}{}^\alpha b_{\alpha\sigma}} - \tfrac{1}{3}k_{\underline{\mu}}b_{\underline{\nu}}{}^\alpha(\bar\lambda\Gamma_{\rho\sigma\alpha}\lambda)$$
$$- \tfrac{1}{3}k_{\underline{\rho}}b_{\underline{\sigma}}{}^\alpha(\bar\lambda\Gamma_{\mu\nu\alpha}\lambda) \tag{A65}$$

$$b_{\mu\nu}b_{\rho\sigma} = b_{\mu\overline{\rho}}b_{\overline{\sigma}\nu} - \eta_{\underline{\mu}\overline{\rho}}b_{\underline{\nu}}{}^\alpha b_{\alpha\overline{\sigma}}$$
$$- \tfrac{1}{2}k_{\underline{\mu}}b_{\underline{\nu}}{}^\alpha(\bar\lambda\Gamma_{\rho\sigma\alpha}\lambda)$$
$$- \tfrac{1}{2}k_{\underline{\rho}}b_{\underline{\sigma}}{}^\alpha(\bar\lambda\Gamma_{\mu\nu\alpha}\lambda) \tag{A66}$$

$$b^{\nu\tau}(\bar\lambda\Gamma_\nu\Gamma_{\alpha\beta\gamma}\Gamma_\tau\lambda)f^{\alpha\beta\gamma} = 0, \tag{A67}$$

$$b^{\tau\nu}(\bar\lambda\Gamma_\nu\Gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5}\Gamma_\tau\lambda)f^{\mu_1\mu_2\mu_3\mu_4\mu_5}$$
$$= 40b_{\mu_1\mu_2}(\bar\lambda\Gamma_{\mu_3\mu_4\mu_5}\lambda)f^{\mu_1\mu_2\mu_3\mu_4\mu_5}, \tag{A68}$$

$$(\bar\epsilon\not{k}\lambda)\Gamma^\alpha\not{k}\lambda b_{\alpha\mu} = \tfrac{1}{8}\Gamma^\alpha\not{k}\epsilon b_{\mu\beta}b^\beta{}_\alpha + \tfrac{1}{16}\Gamma^{\alpha\beta\gamma}\not{k}\epsilon b_{\mu\alpha}b_{\beta\gamma}, \tag{A69}$$

$$(\bar\psi\not{k}\lambda)^4 = \frac{1}{4\cdot 96}\tilde b^{\mu\alpha}\tilde b_\alpha{}^\nu b_{\mu\beta}b^\beta{}_\nu$$
$$+ \frac{1}{8\cdot 96}\tilde b^{\mu\nu}\tilde b^{\rho\sigma}(b_{\mu\nu}b_{\rho\sigma} + b_{\mu\rho}b_{\sigma\nu} + b_{\mu\sigma}b_{\nu\rho}), \tag{A70}$$

$$\frac{1}{5!}\epsilon^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\beta_1\beta_2\beta_3\beta_4\beta_5}k_{\beta_1}b_{\beta_2\beta_3}b_{\beta_4\beta_5} = k^{\alpha_1}b^{\alpha_2\alpha_3}b^{\alpha_4\alpha_5}. \tag{A71}$$

d. $\boldsymbol{\lambda^5}$

The following identities hold:

$$\Gamma^\alpha\Gamma^\beta\not{k}\lambda b_{\mu\alpha}b_{\beta\nu} = \tfrac{3}{5}\not{k}\Gamma_\mu\Gamma^\alpha\lambda b_{\alpha\beta}b^\beta{}_\nu - \tfrac{2}{5}\not{k}\Gamma_\nu\Gamma^\alpha\lambda b_{\alpha\beta}b^\beta{}_\mu, \tag{A72}$$

$$\lambda b_{\mu\alpha}b^\alpha{}_\nu = \tfrac{1}{10}\Gamma_{(\mu}\Gamma^\alpha\lambda b_\alpha{}^\beta b_{\beta\nu)} + \tfrac{1}{10}\not{k}\Gamma^\alpha\lambda(\bar\lambda\Gamma_{\alpha(\mu\beta}\lambda)b^\beta{}_{\nu)}$$
$$= \tfrac{1}{6}\Gamma_{(\mu}\Gamma^\alpha\lambda b_\alpha{}^\beta b_{\beta\nu)} - \tfrac{1}{6}\Gamma^\alpha\not{k}\lambda(\bar\lambda\Gamma_{\alpha(\mu\beta}\lambda)b^\beta{}_{\nu)}, \tag{A73}$$

$$(\bar\epsilon\not{k}\lambda)b_{\mu\alpha}b^\alpha{}_\nu = \tfrac{1}{10}(\bar\epsilon\not{k}\Gamma_{(\mu}\Gamma^\beta\lambda)b_{\beta\alpha}b^\alpha{}_{\nu)} \tag{A74}$$

$$= \tfrac{1}{10}(\bar\epsilon\Gamma_{(\mu}\Gamma_\alpha\not{k}\lambda)b^\alpha{}_\beta b^\beta{}_{\nu)} + \tfrac{1}{5}k_{(\mu}(\bar\epsilon\Gamma^\alpha\lambda)b_{\alpha\beta}b^\beta{}_{\nu)}, \tag{A75}$$

$$\Gamma^{\alpha\beta}\lambda b_{\alpha\nu}(\bar\lambda\Gamma_{\rho\sigma\beta}\lambda)f^{\nu\rho\sigma} = -2\Gamma_\nu\Gamma^\alpha\lambda(\bar\lambda\Gamma_{\alpha\rho\beta}\lambda)b^\beta{}_\sigma f^{\nu\rho\sigma}, \tag{A76}$$

$$\Gamma^{\alpha\beta}\lambda b_{\rho\alpha}b_{\beta\sigma}f^{\rho\sigma} = \Gamma_\rho\Gamma^\alpha\lambda b_{\alpha\beta}b^\beta{}_\sigma f^{\rho\sigma}$$
$$+ \not{k}\Gamma^\alpha\lambda(\bar\lambda\Gamma_{\alpha\rho\beta}\lambda)b^\beta{}_\sigma f^{\rho\sigma}$$
$$= \Gamma_\rho\Gamma^\alpha\lambda b_{\alpha\beta}b^\beta{}_\sigma f^{\rho\sigma}$$
$$- \Gamma^\alpha\not{k}\lambda(\bar\lambda\Gamma_{\alpha\rho\beta}\lambda)b^\beta{}_\sigma f^{\rho\sigma}, \tag{A77}$$

$$\Gamma_{\alpha\beta}\not{k}\lambda b^{\mu\alpha}b^{\beta\nu} = \not{k}\Gamma^{\overline{\mu}}\Gamma^\alpha\lambda b_{\alpha\beta}b^{\beta\overline{\nu}} \tag{A78}$$

$$\Gamma^{\alpha}\lambda b_{\alpha\nu}b_{\rho\sigma}f^{\nu\rho\sigma} = -\tfrac{1}{5}\Gamma_{\nu}\Gamma^{\alpha\beta}\lambda b_{\rho\alpha}b_{\beta\sigma}f^{\nu\rho\sigma} - \tfrac{1}{10}\not{k}\Gamma^{\alpha}\Gamma^{\beta}\lambda b_{\alpha\nu}(\bar{\lambda}\Gamma_{\rho\sigma\beta}\lambda)f^{\nu\rho\sigma} = -\tfrac{1}{5}\Gamma_{\nu\rho}\Gamma^{\alpha}\lambda b_{\alpha\beta}b^{\beta}{}_{\sigma}f^{\nu\rho\sigma} - \tfrac{2}{5}k_{\nu}\Gamma^{\alpha}\lambda(\bar{\lambda}\Gamma_{\alpha\rho\beta}\lambda)b^{\beta}{}_{\sigma}f^{\nu\rho\sigma} + \tfrac{2}{5}\not{k}\Gamma_{\nu}\Gamma^{\alpha}\lambda(\bar{\lambda}\Gamma_{\alpha\rho\beta}\lambda)b^{\beta}{}_{\sigma}f^{\nu\rho\sigma}, \quad \text{(A79)}$$

$$(\bar{\epsilon}\not{k}\lambda)b_{\mu\nu}b_{\rho\sigma}f^{\mu\nu\rho\sigma} = -\tfrac{1}{15}(\bar{\epsilon}\not{k}\Gamma_{\mu\nu\rho}\Gamma^{\alpha}\lambda)b_{\alpha\beta}b^{\beta}{}_{\sigma}f^{\mu\nu\rho\sigma} - \tfrac{2}{5}k_{\mu}(\bar{\epsilon}\Gamma_{\nu}\not{k}\Gamma_{\alpha}\lambda)(\bar{\lambda}\Gamma^{\alpha}{}_{\rho\gamma}\lambda) \times b^{\gamma}{}_{\sigma}f^{\mu\nu\rho\sigma}, \quad \text{(A80)}$$

$$\Gamma^{\mu\nu\rho}\Gamma_{\alpha\beta}\lambda b_{\mu\nu}b_{\rho\sigma}f^{\alpha\beta} = -\frac{2}{5}\Gamma_{\alpha\beta}\Gamma^{\delta}\lambda b_{\delta\eta}b^{\eta\sigma}f^{\alpha\beta} + \frac{24}{5}\eta_{\alpha\sigma}\Gamma^{\delta}\lambda b_{\delta\eta}b^{\eta}{}_{\beta}f^{\alpha\beta} - \frac{4}{5}\Gamma_{\sigma}\Gamma_{\alpha}\Gamma^{\delta}\lambda b_{\delta\eta}b^{\eta}{}_{\beta}f^{\alpha\beta} - \frac{8}{5}\Gamma_{\alpha}\not{k}\Gamma^{\delta}\lambda(\bar{\lambda}\Gamma_{\delta\beta\eta}\lambda)b^{\eta}{}_{\sigma}f^{\alpha\beta} - \frac{16}{5}k_{\sigma}\Gamma^{\delta}\lambda(\bar{\lambda}\Gamma_{\delta\alpha\eta}\lambda)b^{\eta}{}_{\beta}f^{\alpha\beta} - \frac{32}{5}k_{\alpha}\Gamma^{\delta}\lambda(\bar{\lambda}\Gamma_{\delta\beta\eta}\lambda)b^{\eta}{}_{\sigma}f^{\alpha\beta} + \frac{16}{5}\Gamma_{\sigma}\not{k}\Gamma^{\delta}\lambda(\bar{\lambda}\Gamma_{\delta\alpha\eta}\lambda)b^{\eta}{}_{\beta}f^{\alpha\beta} \quad \text{(A81)}$$

$$\not{k}\lambda b_{\tau}{}^{\nu}(\bar{\lambda}\Gamma^{\alpha\beta\tau}\lambda) = -\frac{1}{10}\not{k}\Gamma^{\nu}\Gamma_{\delta}\lambda b^{\delta\tau}(\bar{\lambda}\Gamma_{\tau}{}^{\alpha\beta}\lambda) + \frac{1}{5}\not{k}\Gamma^{\alpha}\Gamma_{\delta}\lambda(\bar{\lambda}\Gamma^{\delta\beta\tau}\lambda)b_{\tau}{}^{\nu} \quad \text{(A82)}$$

$$\Gamma_{\alpha}\Gamma_{\nu}\not{k}\lambda b_{\tau}{}^{\nu}(\bar{\lambda}\Gamma^{\alpha\beta\tau}\lambda) = -2\Gamma^{\nu}\lambda b_{\nu}{}^{\tau}b_{\tau}{}^{\beta}, \quad \text{(A83)}$$

$$(\bar{\psi}\not{k}\lambda)^5 = \frac{1}{4\cdot 5\cdot 48}\tilde{b}^{\mu\alpha}\tilde{b}_{\alpha}{}^{\nu}(\bar{\psi}\Gamma_{\mu}\Gamma_{\beta}\not{k}\lambda)b^{\beta}{}_{\gamma}b^{\gamma}{}_{\nu}. \quad \text{(A84)}$$

e. λ^6

The following identities hold:

$$b_{\alpha\beta}b^{\beta\gamma}b_{\gamma}{}^{\alpha} = 0, \quad \text{(A85)}$$

$$b_{\rho\sigma}b_{\mu\alpha}b^{\alpha}{}_{\nu}S^{\mu\nu}f^{\rho\sigma} = -\tfrac{8}{15}\eta_{\rho\mu}b_{\sigma\beta}b^{\beta}{}_{\alpha}b^{\alpha}{}_{\nu}S^{\mu\nu}f^{\rho\sigma} - \tfrac{1}{15}\eta_{\mu\nu}b_{\rho}{}^{\beta}b_{\beta\alpha}b^{\alpha}{}_{\sigma}S^{\mu\nu}f^{\rho\sigma} + \tfrac{7}{15}k_{\mu}b_{\nu}{}^{\alpha}(\bar{\lambda}\Gamma_{\alpha\rho\beta}\lambda)b^{\beta}{}_{\sigma}S^{\mu\nu}f^{\rho\sigma} - \tfrac{2}{15}k_{\mu}b_{\rho}{}^{\alpha}(\bar{\lambda}\Gamma_{\alpha\nu\beta}\lambda)b^{\beta}{}_{\sigma}S^{\mu\nu}f^{\rho\sigma} + \tfrac{1}{15}k_{\mu}b_{\rho}{}^{\alpha}(\bar{\lambda}\Gamma_{\alpha\sigma\beta}\lambda)b^{\beta}{}_{\nu}S^{\mu\nu}f^{\rho\sigma} + \tfrac{8}{15}k_{\rho}b_{\mu}{}^{\alpha}(\bar{\lambda}\Gamma_{\alpha\nu\beta}\lambda)b^{\beta}{}_{\sigma}S^{\mu\nu}f^{\rho\sigma}, \quad \text{(A86)}$$

$$(\bar{\lambda}\not{k}\epsilon)\Gamma_{\alpha}\not{k}\lambda b^{\alpha\beta}b_{\beta\mu} = \tfrac{1}{8}k^{\alpha}\Gamma_{\alpha\beta}\epsilon b^{\beta}{}_{\gamma}b^{\gamma\delta}b_{\delta\mu} - \tfrac{1}{48}\Gamma_{\mu\nu\rho\sigma}\epsilon k^{\nu}b^{\rho\alpha}b_{\alpha\beta}b^{\beta\sigma} - \tfrac{1}{48}k_{\mu}\Gamma^{\nu\rho\sigma\lambda}\epsilon k_{\nu}b_{\rho}{}^{\alpha}(\bar{\lambda}\Gamma_{\alpha\sigma\beta}\lambda)b^{\beta}{}_{\lambda}, \quad \text{(A87)}$$

$$(\bar{\lambda}\not{k}\Gamma^{\mu}\Gamma^{\theta}\epsilon)\Gamma_{\alpha}k/\lambda b^{\alpha\beta}b_{\beta\mu} = \tfrac{1}{4}\Gamma^{\nu\beta\mu}\Gamma^{\theta}\epsilon k_{\nu}b_{\beta}{}^{\gamma}b_{\gamma\delta}b^{\delta}{}_{\mu}, \quad \text{(A88)}$$

$$(\bar{\lambda}\Gamma^{\alpha}\epsilon_1)(\bar{\epsilon_2}\Gamma^{\rho}\Gamma_{\beta}\not{k}\lambda)b^{\beta}{}_{\delta}b^{\delta\mu} = \frac{1}{96}(\bar{\epsilon_2}\Gamma^{\rho}k^{\lambda}\Gamma_{\beta\lambda\mu_1\mu_2\mu_3}\Gamma^{\alpha}\epsilon_1)(\bar{\lambda}\Gamma^{\mu_1\mu_2\mu_3}\lambda)b^{\beta}{}_{\delta}b^{\delta\mu} - \frac{3}{32}(\bar{\epsilon_2}\Gamma^{\rho}k^{\lambda}\Gamma_{\lambda\mu_2\mu_3}\Gamma^{\alpha}\epsilon_1)b^{\mu_2}{}_{\delta}(\bar{\lambda}\Gamma^{\delta\mu_3\eta}\lambda)b_{\eta}{}^{\mu} + \frac{1}{96}k^{\mu}(\bar{\epsilon_2}\Gamma^{\rho}\Gamma_{\mu_1\mu_2\mu_3}\Gamma^{\alpha}\epsilon_1)b^{\mu_1}{}_{\delta}(\bar{\lambda}\Gamma^{\delta\mu_2\eta}\lambda)b_{\eta}{}^{\mu_3} - \frac{1}{24}(\bar{\epsilon_2}\Gamma^{\rho}k^{\lambda}\Gamma_{\lambda\mu_2\mu_3}\Gamma^{\alpha}\epsilon_1)b^{\mu_2}{}_{\delta}(\bar{\lambda}\Gamma^{\delta\mu\eta}\lambda)b_{\eta}{}^{\mu_3} - \frac{1}{96}(\bar{\epsilon_2}\Gamma^{\rho}\Gamma^{\mu}{}_{\mu_2\mu_3}\Gamma^{\alpha}\epsilon_1)b^{\mu_2\delta}b_{\delta\eta}b^{\eta\mu_3} - \frac{1}{16}(\bar{\epsilon_2}\Gamma^{\rho}\Gamma_{\mu_3}\Gamma^{\alpha}\epsilon_1)b^{\mu_3}{}_{\beta}b^{\beta}{}_{\delta}b^{\delta\mu} \quad \text{(A89)}$$

$$(\bar{\lambda}\Gamma_{\mu}\epsilon_1)(\bar{\epsilon}_2\Gamma^{\rho}\Gamma_{\beta}\not{k}\lambda)b^{\beta}{}_{\delta}b^{\delta\mu} = -\tfrac{1}{8}(\bar{\epsilon}_2\Gamma^{\rho}k^{\lambda}\Gamma_{\lambda\mu_1\mu_2\mu_3}\epsilon_1)b^{\mu_1}{}_{\delta}(\bar{\lambda}\Gamma^{\delta\mu_2\eta}\lambda)b_{\eta}{}^{\mu_3} - \tfrac{1}{8}(\bar{\epsilon}_2\Gamma^{\rho}\Gamma_{\mu_2\mu_3}\epsilon_1)b^{\mu_2\delta}b_{\delta\eta}b^{\eta\mu_3}, \quad \text{(A90)}$$

$$b_{\rho\sigma}b_{\lambda}{}^{\beta}b_{\beta\mu}f^{\rho\sigma\lambda} = -\tfrac{1}{3}\eta_{\mu\rho}b_{\sigma}{}^{\alpha}b_{\alpha\beta}b^{\beta}{}_{\lambda}f^{\rho\sigma\lambda} + \tfrac{1}{3}k_{\mu}b_{\rho}{}^{\alpha}(\bar{\lambda}\Gamma_{\alpha\sigma\beta}\lambda)b^{\beta}{}_{\lambda}f^{\rho\sigma\lambda} - \tfrac{1}{3}k_{\rho}b_{\sigma}{}^{\alpha}(\bar{\lambda}\Gamma_{\alpha\mu\beta}\lambda)b^{\beta}{}_{\lambda}f^{\rho\sigma\lambda}, \quad \text{(A91)}$$

$$(\bar{\psi}\not{k}\lambda)^6 = -\frac{1}{8\cdot 6!}\tilde{b}^{\mu\alpha}\tilde{b}_{\alpha\beta}\tilde{b}^{\beta\nu}b_{\mu\gamma}b^{\gamma\delta}b_{\delta\nu}, \quad \text{(A92)}$$

$$\frac{1}{5!}\epsilon^{\alpha_1\alpha_2\alpha_5\beta_1\beta_2\beta_3\beta_beta_5\alpha_3\alpha_4}k_{\beta_1}b_{\beta_2\beta_3}b_{\beta_4\beta_5}b_{\alpha_3\alpha_4} = k^{\alpha_1}b^{\alpha_2\alpha_3}b_{\alpha_3\alpha_4}b^{\alpha_4\alpha_5}, \quad (A93)$$

where $S^{\mu_1\cdots\mu_n}$ is a symmetric tensor.

f. λ^7

The following identities hold:

$$(\bar\epsilon \not k\lambda)b_{\mu\rho}b^{\rho\sigma}b_{\sigma\nu} = \tfrac{1}{3}(\bar\epsilon\Gamma^\rho\Gamma_\alpha\not k\lambda)b_{\underline{\mu\rho}}b^{\alpha\sigma}\underline{b_{\sigma\nu}} \quad (A94)$$

$$(\bar\epsilon\Gamma_\rho\Gamma_\alpha\not k\lambda)b^\alpha{}_\mu b^{\rho\sigma}b_{\sigma\nu} = (\bar\epsilon\not k\lambda)b_{\mu\rho}b^{\rho\sigma}b_{\sigma\nu}, \quad (A95)$$

$$(\bar\epsilon\not k\lambda)b_{\mu\rho}b^{\rho\sigma}b_{\sigma\nu}f^{\mu\nu} = \tfrac{1}{7}(\bar\epsilon\not k\Gamma_\mu\Gamma_\alpha\lambda)b^\alpha{}_\rho b^{\rho\sigma}b_{\sigma\nu}f^{\mu\nu}, \quad (A96)$$

$$\begin{aligned}\Gamma^\mu\lambda b_{\mu\alpha}b^{\alpha\beta}b_{\beta\nu} &= -\tfrac{1}{8}\Gamma_\nu\Gamma^{\alpha\beta}\lambda b_{\alpha\gamma}b^{\gamma\delta}b_{\delta\beta} - \tfrac{1}{8}\not k\Gamma^{\alpha\beta}\lambda b_{\alpha\gamma}b^{\gamma\delta}(\bar\lambda\Gamma_{\delta\nu\beta}\lambda)\\ &= -\tfrac{1}{8}\Gamma_\nu\Gamma^{\alpha\beta}\lambda b_{\alpha\gamma}b^{\gamma\delta}b_{\delta\beta} - \tfrac{1}{8}\not k\Gamma^{\alpha\beta}\lambda b_\alpha{}^\gamma(\bar\lambda\Gamma_{\gamma\nu\delta}\lambda)b^\delta{}_\beta, \quad (A97)\end{aligned}$$

$$\begin{aligned}\Gamma_\mu\not k\lambda b^{\mu\rho}b_{\rho\sigma}b^{\sigma\nu} &= \tfrac{1}{8}\not k\Gamma_\nu\Gamma^{\alpha\beta}\lambda b_{\alpha\gamma}b^{\gamma\delta}b_{\delta\beta}\\ &= -\tfrac{1}{8}\Gamma^\nu\Gamma_{\alpha\beta}\not k\lambda b^\alpha{}_\rho b^{\rho\sigma}b_\sigma{}^\beta + \tfrac{1}{4}k_\nu\Gamma_{\alpha\beta}\lambda b^\alpha{}_\rho b^{\rho\sigma}b_\sigma{}^\beta \quad (A98)\end{aligned}$$

$$\begin{aligned}(\bar\epsilon\not k\lambda)b_{\mu\rho}b^{\rho\sigma}b_{\sigma\nu}f^{\mu\nu} &= -\tfrac{1}{56}(\bar\epsilon\Gamma_{\mu\nu}\Gamma_{\alpha\beta}\not k\lambda)b^\alpha{}_\rho b^{\rho\sigma}b_\sigma{}^\beta f^{\mu\nu} - \tfrac{1}{28}k_\mu(\bar\epsilon\Gamma_\nu\Gamma_{\alpha\beta}\lambda)b^\alpha{}_\rho b^{\rho\sigma}b_\sigma{}^\beta f^{\mu\nu} + \tfrac{2}{7}k_\mu(\bar\epsilon\Gamma_\alpha\lambda)b^\alpha{}_\rho b^{\rho\sigma}b_{\sigma\nu}f^{\mu\nu}\\ &= -\tfrac{1}{56}(\bar\epsilon\Gamma_{\mu\nu}\Gamma_{\alpha\beta}\not k\lambda)b^\alpha{}_\rho b^{\rho\sigma}b_\sigma{}^\beta f^{\mu\nu} - \tfrac{1}{14}k_\mu(\bar\epsilon\Gamma_\nu\Gamma_{\alpha\beta}\lambda)b^\alpha{}_\rho b^{\rho\sigma}b_\sigma{}^\beta f^{\mu\nu} - \tfrac{1}{28}k_\mu(\bar\epsilon\not k\Gamma^{\alpha\beta}\lambda)b_\alpha{}^\gamma(\bar\lambda\Gamma_{\gamma\nu\delta}\lambda)\times b^\delta{}_\beta f^{\mu\nu}, \quad (A99)\end{aligned}$$

$$\Gamma_{\alpha\delta}\lambda b^\delta{}_\mu b^{\alpha\beta}b_{\beta\nu}f^{\mu\nu} = \tfrac{1}{3}\Gamma_{\mu\delta}\lambda b^{\delta\theta}b_{\theta\beta}b^\beta{}_\nu f^{\mu\nu} - \tfrac{1}{3}k^\delta\Gamma_{\alpha\delta}\lambda b_\mu{}^\theta(\bar\lambda\Gamma_{\theta\alpha\beta}\lambda)b^\beta{}_\nu f^{\mu\nu} + \tfrac{1}{3}k^\alpha\Gamma_{\alpha\delta}\lambda(\bar\lambda\Gamma^\delta{}_{\mu\theta}\lambda)b^\theta{}_\beta b^\beta{}_\nu f^{\mu\nu}, \quad (A100)$$

$$\Gamma^\delta\not k\lambda(\bar\lambda\Gamma_{\delta\mu\alpha}\lambda)b^{\alpha\beta}b_{\beta\nu}f^{\mu\nu} = \tfrac{1}{20}\not k\Gamma_\nu\Gamma_{\alpha\beta}\lambda b^{\alpha\delta}(\bar\lambda\Gamma_{\delta\mu\tau}\lambda)b^{\tau\beta}f^{\mu\nu} - \tfrac{11}{5}\lambda b_{\mu\delta}b^\delta{}_\tau b^\tau{}_\nu f^{\mu\nu} + \tfrac{3}{5}\Gamma_\mu\Gamma^\eta\lambda b_{\eta\tau}b^{\tau\delta}b_{\delta\nu}f^{\mu\nu}, \quad (A101)$$

$$\Gamma_\alpha\not k\lambda b^{\alpha\delta}(\bar\lambda\Gamma_{\delta\mu\tau}\lambda)b^\tau{}_\nu f^{\mu\nu} = \tfrac{3}{20}\not k\Gamma_\nu\Gamma_{\alpha\beta}\lambda b^{\alpha\delta}(\bar\lambda\Gamma_{\delta\mu\tau}\lambda)\times b^{\tau\beta}f^{\mu\nu} + \tfrac{7}{5}\lambda b_{\mu\delta}b^\delta{}_\tau b^\tau{}_\nu f^{\mu\nu} - \tfrac{1}{5}\Gamma_\mu\Gamma^\eta\lambda b_{\eta\tau}b^{\tau\delta}b_{\delta\nu}f^{\mu\nu}, \quad (A102)$$

$$\Gamma_{\tau\eta}\lambda b^\eta{}_\mu b^{\tau\delta}b_{\delta\nu}f^{\mu\nu} = -\tfrac{1}{20}\not k\Gamma_\nu\Gamma_{\alpha\beta}\lambda b^{\alpha\delta}(\bar\lambda\Gamma_{\delta\mu\tau}\lambda)b^{\tau\beta}f^{\mu\nu} - \tfrac{4}{5}\lambda b_{\mu\delta}b^\delta{}_\tau b^\tau{}_\nu f^{\mu\nu} + \tfrac{2}{5}\Gamma_\mu\Gamma^\eta\lambda b_{\eta\tau}b^{\tau\delta}b_{\delta\nu}f^{\mu\nu}, \quad (A103)$$

$$(\bar\psi\not k\lambda)^7 = \frac{1}{8\cdot 8!}\tilde b^{\alpha\gamma}\tilde b_{\gamma\delta}\tilde b^{\delta\beta}(\bar\psi\Gamma_{\alpha\beta}\Gamma_{\mu\nu}\not k\lambda)b^{\mu\rho}b_{\rho\sigma}b^{\sigma\nu}. \quad (A104)$$

g. λ^8

The following identities hold:

$$b_{\alpha\beta}b^{\beta\gamma}b_{\gamma\delta}b^{\delta\alpha} = -b_{\mu\nu}b^{\mu\rho}b_{\rho\sigma}b^{\sigma\nu}, \quad (A105)$$

$$\Gamma^{\alpha\mu\nu\beta\gamma}\epsilon k_\alpha b_{\mu\nu}b_{\beta\rho}b^{\rho\sigma}b_{\sigma\gamma} = 0, \quad (A106)$$

$$(\bar\epsilon\not k\lambda)\Gamma^{\mu\nu}\not k\lambda b_{\mu\rho}b^{\rho\sigma}b_{\sigma\nu} = -\tfrac{1}{8}\not k\epsilon b_{\mu\rho}b^{\rho\sigma}b_{\sigma\nu}b^{\nu\mu}, \quad (A107)$$

$$(\bar\psi\not k\lambda)^8 = -\frac{1}{8^2\cdot 8!}\tilde b^{\alpha\gamma}\tilde b_{\gamma\delta}\tilde b^{\delta\beta}\tilde b_{\alpha\beta}b_{\mu\rho}b^{\rho\sigma}b_{\sigma\nu}b^{\nu\mu}. \quad (A108)$$

4. The symmetrized trace

The symmetrized trace Str is defined by

$$\begin{aligned}\mathrm{Str}e^{ik\cdot A}B_1\cdot B_2\cdots B_n &= \int_0^1 dt_1\int_{t_1}^1 dt_2\cdots\int_{t_{n-2}}^1 dt_{n-1}\,\mathrm{tr}\times e^{ik\cdot At_1}B_1e^{ik\cdot A(t_2-t_1)}B_2\cdots\\ &\quad\times e^{ik\cdot A(t_{n-1}-t_{n-2})}B_{n-1}e^{ik\cdot A(1-t_{n-1})}B_n + (\text{permutations of } B_i' s\times(i=2,3\cdots n)). \quad (A109)\end{aligned}$$

The dot on the left-hand side indicates that the operators B_i are symmetrized. If we set $k=0$, the symmetrized trace becomes

$$\mathrm{Str}\,(B_1\cdot B_2\cdots B_n) = \frac{1}{n!}\sum_{\text{perm.}}\mathrm{tr}(B_{i_1}B_{i_2}\cdots B_{i_n}). \quad (A110)$$

The explicit forms with two and three inserted operators are written as

$$\mathrm{Str}\,(e^{ik\cdot A}B\cdot C) = \mathrm{tr}\int_0^1 dt e^{ik\cdot At}Be^{ik\cdot A(1-t)}C, \quad (A111)$$

$$\mathrm{Str}(e^{ik\cdot A}B\cdot C\cdot D)=\mathrm{tr}\int_0^1 dt_1\int_{t_1}^1 dt_2 e^{ik\cdot At_1}Be^{ik\cdot A(t_2-t_1)}C\times e^{ik\cdot A(1-t_2)}D+(C\leftrightarrow D),\qquad (A112)$$

where all the matrices are bosonic. The definitions for fermionic matrices can be similarly obtained by replacing the bosonic matrices with the fermionic ones.

The relations below follow from the definition:

$$\mathrm{Str}\,(e^{ik\cdot A}B\cdot C)=\mathrm{Str}(e^{ik\cdot A}C\cdot B),\qquad (A113)$$

$$\mathrm{Str}\,(e^{ik\cdot A}B\cdot C\cdot D)=\mathrm{Str}(e^{ik\cdot A}C\cdot B\cdot D)=\mathrm{Str}(e^{ik\cdot A}C\cdot D\cdot B)=\cdots.\qquad (A114)$$

That is, we can permute matrices in the symmetrized trace. In particular, there are no ordering ambiguities in symmetrized trace. For fermionic matrices, an appropriate change of sign must be included.

Another useful equation related to the symmetrized trace is

$$\begin{aligned}&\mathrm{Str}(e^{ik\cdot A}[ik\cdot A,B]C_1\cdot C_2\cdots C_n)\\&=-\mathrm{Str}(e^{ik\cdot A}[C_1,B]\cdot C_2\cdots C_n)\\&\quad-\mathrm{Str}(e^{ik\cdot A}C_1\cdot[C_2,B]\cdots C_n)-\cdots\\&\quad-\mathrm{Str}(e^{ik\cdot A}C_1\cdot C_2\cdots[C_n,B]).\end{aligned}\qquad (A115)$$

These relations above are frequently used in this paper to derive the vertex operators.

The Ξ term

Let us consider $\mathrm{Str}(e^{ik\cdot A}\psi_\alpha\cdot\psi_\beta)(\Gamma_0\Gamma_{\mu_1\cdots\mu_n})_{\alpha\beta}$ and $\mathrm{Str}(e^{ik\cdot A}\{\psi_\alpha,\psi_\beta\})(\Gamma_0\Gamma_{\mu_1\cdots\mu_n})_{\alpha\beta}$, where ψ is an $n\times n$ Majorana-Weyl fermionic matrix. Since they vanish if n is even (because ψ is Weyl), we only need to consider the case for odd n. Note that $\mathrm{Str}(e^{ik\cdot A}\psi_\alpha\cdot\psi_\beta)$ is antisymmetric in α and β, whereas $\mathrm{Str}(e^{ik\cdot A}\{\psi_\alpha,\psi_\beta\})$ is symmetric. Because of (A10), $\mathrm{Str}(e^{ik\cdot A}\psi_\alpha\cdot\psi_\beta)(\Gamma_0\Gamma_{\mu_1\cdots\mu_n})_{\alpha\beta}$ is zero unless $n=3,7$, whereas $\mathrm{Str}(e^{ik\cdot A}\{\psi_\alpha,\psi_\beta\})\,(\Gamma_0\Gamma_{\mu_1\cdots\mu_n})_{\alpha\beta}$ vanishes unless $n=1,5,9$. $\Gamma_{\mu_1\cdots\mu_9}$ can be described by Γ_{ν_1} using the duality relation of gamma matrices (A11), and hence reduces to $n=1$. In order to deal with $\{\psi_\alpha,\psi_\beta\}(\Gamma_0\Gamma_\mu)_{\alpha\beta}$, we can make use of the equation of motion. For $n=5$, we introduce a new notation Ξ as

$$\Xi_{\mu_1\mu_2\mu_3\mu_4\mu_5}\equiv\{\psi_\alpha,\psi_\beta\}(\Gamma_0\Gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5})_{\alpha\beta}.\qquad (A116)$$

Although unfamiliar in the literature, it necessarily appears in the expressions of the vertex operators.

Finally, we give a useful Fierz identity related to $\{\psi_\alpha,\psi_\beta\}$:

$$\begin{aligned}\mathrm{Str}(e^{ik\cdot A}\bar\psi\cdot\Gamma_\mu\{\psi,\bar\psi\}\Gamma_\nu\lambda)&=\mathrm{Str}(e^{ik\cdot A}\psi_\alpha\cdot(\Gamma_0\Gamma_\mu)_{\alpha\beta}\{\psi_\beta,\psi_\gamma\}(\Gamma_0)_{\gamma\delta}(\Gamma_\nu)_{\delta\epsilon}\lambda_\epsilon)\\&=\mathrm{Str}\Big(e^{ik\cdot A}\frac{1}{16}\{\psi_\gamma,\psi_\beta\}(\Gamma_0\Gamma^\tau)_{\gamma\beta}\cdot(\bar\psi\Gamma_\mu\Gamma_\tau\Gamma_\nu\lambda)\Big)\\&\quad+\mathrm{Str}\Big(e^{ik\cdot A}\frac{1}{32}\cdot\frac{1}{5!}\Xi^{\tau_1\tau_2\tau_3\tau_4\tau_5}\cdot(\bar\psi\Gamma_\mu\Gamma_{\tau_1\tau_2\tau_3\tau_4\tau_5}\Gamma_\nu\lambda)\Big).\end{aligned}\qquad (A117)$$

If $k=0$ and $\mu=\nu$, the above formula becomes

$$\mathrm{tr}\,\bar\psi\Gamma_\mu\{\psi,\bar\psi\}\Gamma^\mu\lambda=-\mathrm{tr}\tfrac{1}{2}(\bar\psi\Gamma^\tau\lambda)\{\bar\psi\Gamma_\tau,\psi\}.\qquad (A118)$$

APPENDIX B: 10D VERTEX OPERATORS

$$V^\Phi(A,\psi)=\mathrm{tr}e^{ik\cdot A},$$

$$V^{\tilde\Phi}(A,\psi)=\mathrm{Str}e^{ik\cdot A}\bar\psi,$$

$$V^B_{\mu\nu}(A,\psi)=\mathrm{Str}e^{ik\cdot A}\Big(\frac{1}{16}k^\rho(\bar\psi\cdot\Gamma_{\mu\nu\rho}\psi)-\frac{i}{2}[A_\mu,A_\nu]\Big),$$

$$V^\Psi_\mu(A,\psi)=\mathrm{Str}e^{ik\cdot A}\Big(-\frac{i}{12}k^\rho(\bar\psi\cdot\Gamma_{\mu\nu\rho}\psi)-2[A_\mu,A_\nu]\Big)\cdot\bar\psi\Gamma^\nu,$$

$$\begin{aligned}V^h_{\mu\nu}(A,\psi)=\mathrm{Str}e^{ik\cdot A}\Big(&-\frac{1}{96}k^\rho k^\sigma(\bar\psi\cdot\Gamma_{\mu\rho}{}^\beta\psi)\cdot(\bar\psi\cdot\Gamma_{\nu\sigma\beta}\psi)-\frac{i}{4}k^\rho\bar\psi\cdot\Gamma_{\rho\beta(\mu}\psi\cdot F_{\nu)}{}^\beta\\&+\frac{1}{2}\bar\psi\cdot\Gamma_{(\mu}[A_{\nu)},\psi]+2F_\mu{}^\rho\cdot F_{\nu\rho}\Big),\end{aligned}$$

$$\begin{aligned}V^A_{\mu\nu\rho\sigma}(A,\psi)=\mathrm{Str}e^{ik\cdot A}\Big(&\frac{i}{8\cdot 4!}k_\alpha k_\gamma(\bar\psi\cdot\Gamma_{[\mu\nu}{}^\alpha\psi)\cdot(\bar\psi\cdot\Gamma_{\rho\sigma]}{}^\gamma\psi)+\frac{i}{3}\bar\psi\cdot\Gamma_{[\nu\rho\sigma}[\psi,A_{\mu]}]\\&+\frac{1}{4}F_{[\mu\nu}\cdot(\bar\psi\cdot\Gamma_{\rho\sigma]}{}^\gamma\psi)k_\gamma-iF_{[\mu\nu}\cdot F_{\rho\sigma]}\Big),\end{aligned}$$

$$V_{\mu}^{\Psi^c}(A,\psi) = \mathrm{Str}e^{ik\cdot A}\bigg(-\frac{i}{2\cdot 5!}k^\lambda k^\tau(\bar\psi\cdot\Gamma_{\mu\lambda}{}^{\sigma}\psi)\cdot(\bar\psi\cdot\Gamma_{\nu\tau\sigma}\psi)\cdot\bar\psi\Gamma^\nu + \frac{1}{24}k^\lambda(\bar\psi\cdot\Gamma_{\lambda\mu\nu}\psi)\cdot\bar\psi\Gamma^\nu\Gamma_{\rho\sigma}\cdot F^{\rho\sigma}$$
$$-\frac{1}{6}k^\lambda(\bar\psi\cdot\Gamma_{\lambda\alpha\beta}\psi)\cdot\bar\psi\Gamma^\beta\cdot F^{\alpha}{}_{\mu} + \frac{i}{3}(\bar\psi\cdot\Gamma_\mu[A_\nu,\psi])\cdot\bar\psi\Gamma^\nu + \frac{i}{3}(\bar\psi\cdot\Gamma_\nu[A_\mu,\psi])\cdot\bar\psi\Gamma^\nu$$
$$+\frac{i}{6}(\bar\psi\cdot\Gamma_{\alpha\beta\mu}\psi)\cdot[A^\alpha,\bar\psi]\Gamma^\beta - iF_{\mu\nu}\cdot F_{\rho\sigma}\cdot\bar\psi\Gamma^\nu\Gamma^{\rho\sigma}\bigg),$$

$$V_{\mu\nu}^{B^c}(A,\psi) = \mathrm{Str}e^{ik\cdot A}\bigg(-\frac{1}{8\cdot 6!}k^\lambda k^\tau k^\alpha(\bar\psi\cdot\Gamma_{\mu\lambda}{}^{\sigma}\psi)\cdot(\bar\psi\cdot\Gamma_{\gamma\tau\sigma}\psi)\cdot(\bar\psi\cdot\Gamma^{\gamma}{}_{\alpha\nu}\psi) + \frac{i}{64}(\bar\psi\cdot\not{k}\Gamma_{\mu\alpha}\psi)\cdot F^{\alpha\beta}(\bar\psi\cdot\not{k}\Gamma_{\beta\nu}\psi)$$
$$+\frac{i}{16\cdot 4!}(\bar\psi\cdot\not{k}\Gamma_{[\mu\alpha}\psi)\cdot(\bar\psi\cdot\not{k}\Gamma^{\alpha\sigma}\psi)\cdot F_{\sigma\nu]} - \frac{1}{32}\bar\psi\cdot\Gamma_{[\mu}[A^\sigma,\psi]\cdot(\bar\psi\cdot\not{k}\Gamma_{\sigma\nu]}\psi) - \frac{1}{64}(\bar\psi\cdot\not{k}\Gamma_{[\mu\alpha}\psi)$$
$$\cdot\bar\psi\Gamma^\alpha[A_{\nu]},\psi] + \frac{i}{4!\cdot 32}\Xi_{\mu\nu\alpha\beta\gamma}\cdot(\bar\psi\cdot\Gamma^{\alpha\beta\gamma}\psi) - \frac{i}{64}[A_\alpha,F^{\alpha\tau}]\cdot(\bar\psi\cdot\Gamma_{\tau\mu\nu}\psi) + \frac{1}{64}(\bar\psi\cdot\Gamma_{\mu\nu\rho\sigma\lambda\tau}\not{k}\psi)\cdot F^{\rho\sigma}$$
$$\cdot F^{\lambda\tau} + \frac{1}{16}(\bar\psi\cdot\Gamma_{\rho\sigma}\not{k}\psi)\cdot F^{\rho\sigma}\cdot F^{\mu\nu} - \frac{1}{8}(\bar\psi\cdot\Gamma_{\rho\sigma}\not{k}\psi)\cdot F^{\mu\rho}\cdot F^{\nu\sigma} + \frac{1}{8}(\bar\psi\cdot\Gamma_{[\mu\sigma}\not{k}\psi)\cdot F^{\sigma\alpha}\cdot F_{\alpha\nu]}$$
$$-\frac{1}{32}(\bar\psi\cdot\Gamma^{\mu\nu}\not{k}\psi)\cdot F^{\rho\sigma}\cdot F_{\sigma\rho} + \frac{i}{4}\bar\psi\cdot\Gamma_{\mu\nu\alpha}[A_\beta,\psi]\cdot F^{\alpha\beta} + \frac{i}{8}\bar\psi\cdot\Gamma_{\rho\sigma[\mu}[A_{\nu]},\psi]\cdot F^{\rho\sigma}$$
$$+\frac{i}{4}\bar\psi\cdot\Gamma_{(\mu}[A_{\rho)},\psi]\cdot F^{\rho}{}_{\nu} - \frac{i}{4}\bar\psi\cdot\Gamma_{(\nu}[A_{\rho)},\psi]\cdot F^{\rho\mu} - iF_{\mu\rho}\cdot F^{\rho\sigma}\cdot F_{\sigma\nu} + \frac{i}{4}F_{\mu\nu}\cdot F^{\rho\sigma}\cdot F_{\sigma\rho}\bigg),$$

$$V^{\tilde\Phi^c}(A,\psi) = \mathrm{Str}e^{ik\cdot A}\bigg(\frac{1}{8!}(\bar\psi\cdot\Gamma^{\alpha\gamma}\not{k}\psi)\cdot(\bar\psi\cdot\Gamma_{\gamma\delta}\not{k}\psi)\cdot(\bar\psi\cdot\Gamma^{\delta\beta}\not{k}\psi)\cdot\bar\psi\Gamma_{\alpha\beta} - \frac{i}{2\cdot 5!}F^{\mu\alpha}\cdot(\bar\psi\cdot\not{k}\Gamma_{\alpha\beta}\psi)\cdot(\bar\psi\cdot\not{k}\Gamma^{\beta\nu}\psi)$$
$$\cdot\bar\psi\Gamma_{\mu\nu} + \cdots\cdots - \frac{1}{8\cdot 4!}F^{\mu\nu}\cdot F^{\rho\sigma}(\bar\psi\cdot\Gamma_{\mu\nu\rho\sigma\lambda\alpha\beta}\psi)k^\lambda\cdot\bar\psi\Gamma^{\alpha\beta} - \frac{1}{12}F^{\mu\alpha}\cdot F_{\alpha\beta}\cdot(\bar\psi\cdot\not{k}\Gamma^{\beta\nu}\psi)\cdot\bar\psi\Gamma_{\mu\nu}$$
$$-\frac{1}{24}F^{\mu\alpha}\cdot(\bar\psi\cdot\not{k}\Gamma_{\alpha\beta}\psi)\cdot F^{\beta\nu}\cdot\bar\psi\Gamma_{\mu\nu} - \frac{1}{48}F^{\rho\sigma}\cdot(\bar\psi\cdot\not{k}\Gamma_{\rho\sigma}\psi)\cdot F^{\mu\nu}\cdot\bar\psi\Gamma_{\mu\nu} + \cdots + \cdots + \frac{i}{24}\bar\psi$$
$$\cdot\Gamma_{\mu\nu\rho\sigma\lambda\tau}F^{\mu\nu}\cdot F^{\rho\sigma}\cdot F^{\lambda\tau} + i\bar\psi\cdot\Gamma^{\mu\nu}\bigg(F_{\mu\rho}\cdot F^{\rho\sigma}\cdot F_{\sigma\nu} - \frac{1}{4}F^{\rho\sigma}\cdot F_{\sigma\rho}\cdot F_{\mu\nu}\bigg)\bigg),$$

$$V^{\Phi^c}(A,\psi) = \mathrm{Str}e^{ik\cdot A}\bigg(\frac{1}{8\cdot 8!}(\bar\psi\cdot\Gamma^{\alpha\gamma}\not{k}\psi)\cdot(\bar\psi\cdot\Gamma_{\gamma\delta}\not{k}\psi)\cdot(\bar\psi\cdot\Gamma_{\delta\beta}\not{k}\psi)\cdot(\bar\psi\cdot\Gamma_{\alpha\beta}\not{k}\psi) + \cdots\cdots + \frac{i}{48}(\bar\psi\cdot\Gamma_{\mu\nu\rho\sigma\lambda\tau}\not{k}\psi)\cdot F^{\mu\nu}$$
$$\cdot F^{\rho\sigma}\cdot F^{\lambda\tau} + [A_\mu,\bar\psi]\cdot\Gamma_{\rho\sigma}\Gamma_\nu\psi\cdot F^{\mu\nu}\cdot F^{\rho\sigma} + \frac{i}{2}(\bar\psi\cdot\Gamma^{\mu\nu}\not{k}\psi)\cdot\bigg(F_{\mu\rho}\cdot F^{\rho\sigma}\cdot F_{\sigma\nu} - \frac{1}{4}F^{\rho\sigma}\cdot F_{\sigma\rho}\cdot F_{\mu\nu}\bigg)$$
$$-\bigg(F_{\mu\nu}\cdot F^{\nu\rho}\cdot F_{\rho\sigma}\cdot F^{\sigma\mu} - \frac{1}{4}F_{\mu\nu}\cdot F^{\nu\mu}\cdot F_{\rho\sigma}\cdot F^{\sigma\rho}\bigg)\bigg).$$

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