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Local bulk operators in AdS/CFT correspondence: A holographic description of the black hole interior

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To gain insight into how bulk locality emerges from the holographic conformal field theory (CFT), we reformulate the bulk-to-boundary map in as local a way as possible. In previous work, we carried out this program for Lorentzian anti-de Sitter (AdS), and showed the support on the boundary could always be reduced to a compact region spacelike separated from the bulk point. In the present work the idea is extended to a complexified boundary, where spatial coordinates are continued to imaginary values. This continuation enables us to represent a local bulk operator as a CFT operator with support on a finite disc on the complexified boundary. We treat general AdS in Poincaré coordinates and AdS_3 in Rindler coordinates. We represent bulk operators inside the horizon of a Banados-Teitelboim-Zanelli (BTZ) black hole and we verify that the correct bulk two-point functions are reproduced, including the divergence when one point hits the BTZ singularity. We comment on the holographic description of black holes formed by collapse and discuss locality and holographic entropy counting at finite N.

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I. INTRODUCTION

The AdS/CFT correspondence relates string theory in an asymptotically anti-de Sitter (AdS) background to a conformal field theory (CFT) living on the boundary of AdS [1–4]. The main observable of interest in the original work on AdS/CFT was the boundary S-matrix. In the present work we will focus instead on how one might recover approximate local bulk quantities from the CFT. Even with interactions included, one can hope to recover quasi-local observables in the gravitational theory [5]. In this paper we study this in detail, generalizing our earlier work [6,7]. We develop the map at leading order in $1/N$ where we can treat the gravity theory semiclassically and work with free scalar fields.

The original bulk-to-boundary map of Euclidean AdS [2,3] and its Lorentzian generalization [8,9] has been reformulated and studied further in [10–14]. In these works, one can view the construction of a local bulk operator as an integral over the entire boundary of AdS. Vanishing of commutators of local bulk operators at spacelike separation relies on delicate cancellations in this approach [12]. The strategy we will adopt, following our earlier work [6,7], is to reformulate the bulk-to-boundary map so that the support on the boundary is as small as possible.

It is worth emphasizing the physical relevance of our approach. By representing bulk operators as operators on the boundary with compact support—in fact with support that is as small as possible—we can have bulk operators whose dual boundary operators are spacelike separated. Such bulk operators will manifestly commute with each other just by the locality of the boundary theory. This statement will continue to hold at finite N. Moreover we will find interesting applications of this basis of operators to the study of black hole interiors and singularities, as well as holographic entropy counting.

We will use the following framework developed in [6,7]. The first of these works [6] mainly considered two-dimensional AdS space, and showed the boundary support of a local bulk operator could be reduced to points spacelike separated from the bulk point. This was generalized to the higher-dimensional case in [7]. In Lorentzian AdS, a free bulk scalar field ϕ is dual to a nonlocal operator in the CFT, via a correspondence

$$\phi(x, Z) \leftrightarrow \int dx' K(x'|x, Z)\mathcal{O}(x'). \tag{1}$$

Here Z is a radial coordinate in AdS which vanishes at the boundary, x represents coordinates along the boundary, and $\mathcal{O}(x')$ is a local operator in the CFT. A similar approach has been considered previously in [10–12,14]. We refer to the kernel K as a smearing function. This correspondence can be used inside correlation functions, for example

$$\begin{aligned}\langle\phi(x_1, Z_1)\phi(x_2, Z_2)\rangle_{\mathrm{SUGRA}} = \int dx_1' dx_2' K(x_1'|x_1, Z_1)\\ \times K(x_2'|x_2, Z_2)\\ \times \langle\mathcal{O}(x_1')\mathcal{O}(x_2')\rangle_{\mathrm{CFT}}.\end{aligned}$$

To construct smearing functions one begins with a field in Lorentzian AdS that satisfies the free wave equation and

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has normalizable falloff near the boundary of AdS,[1]

$$\phi(x, Z) \sim Z^{\Delta}\phi_0(x) \quad \text{as } Z \to 0.$$

The parameter Δ is related to the mass of the field. The boundary field $\phi_0(x)$ is dual to a local operator in the CFT[2]

$$\phi_0(x) \leftrightarrow \mathcal{O}(x). \tag{2}$$

We will construct smearing functions that let us solve for the bulk field in terms of the boundary field

$$\phi(x, Z) = \int dx' K(x'|x, Z)\phi_0(x'). \tag{3}$$

K should not be confused with the standard bulk-to-boundary propagator [3], since our smearing functions generate *normalizable* solutions to the *Lorentzian* equations of motion. Using the duality (2), we obtain the relation between bulk and boundary operators given in (1).

Solving for the bulk field in terms of the boundary field is not a standard Cauchy problem: since the "initial conditions" are specified on a timelike hypersurface we have neither existence nor uniqueness theorems. In global AdS it was shown that, although the smearing function is not unique, one can always construct a smearing function which has support on the boundary at points which are spacelike separated from the bulk point [6,7]. It is then interesting to see if a stronger statement can be made. Can we further reduce the support on the boundary? This was studied in [7], where smearing functions for AdS_3 were constructed in accelerating Rindler coordinates. It was shown that smearing functions can only be constructed by analytically continuing the boundary coordinates to complex values, since the naive expression derived from mode sums was divergent. This continuation leads to a well-defined smearing function with compact support on the complexified boundary of the Rindler patch; it can be thought of as arising from a retarded Green's function in de Sitter space. Moreover the support shrinks to a point as the bulk point approaches the boundary. In this way we recover the expected relation (2).

It thus seems the most economical description of local bulk physics in AdS/CFT requires the use of complexified boundary coordinates. Complexified coordinates also appeared in [15], and have been used to study the region inside horizons in [16–20]. For other approaches to recovering bulk physics see [21–23].

An outline of this paper is as follows. In Sec. II we extend the work of [7] and use complexified boundary coordinates to construct compact smearing functions in AdS spacetimes of general dimension in two ways. First we work in Poincaré coordinates and perform a Poincaré mode sum, then we Wick rotate to de Sitter space and use a retarded Green's function. In Sec. III we translate our AdS_3 results into Rindler coordinates and show that we recover bulk correlators inside the Rindler horizon. After these preliminaries we develop applications of this new formulation of the bulk/boundary map to black holes and to holographic entropy counting. In Sec. IV we argue that the Rindler smearing functions can also be used in a Banados-Teitelboim-Zanelli (BTZ) spacetime [24] and we show how the BTZ singularity manifests itself in the conformal field theory. In Sec. V we discuss local operators inside the horizon of an AdS black hole formed by collapse, where there is only a single asymptotic AdS region. This provides evidence that our results will generalize to time-dependent situations. Finally in Sec. VI we explain how the number of degrees of freedom is reduced at finite N and how this leads to a new perspective on holographic entropy counting.

II. POINCARÉ COORDINATES

In this section we construct a compact smearing function for a general-dimensional AdS spacetime. We obtain the result in two ways: by performing the Poincaré mode sum in Sec. II B, and by Wick rotating to de Sitter space in Sec. II C.

A. Preliminaries

We will work in AdS_D in Poincaré coordinates, with metric

$$ds^2 = \frac{R^2}{Z^2}(-dT^2 + |dX|^2 + dZ^2). \tag{4}$$

Here R is the AdS radius. The coordinates range over $0 < Z < \infty$, $-\infty < T < \infty$, and $X \in \mathbb{R}^{d-1}$ where $d = D - 1$. An AdS-invariant distance function is provided by

$$\sigma(T, X, Z|T', X', Z') = \frac{1}{2ZZ'}(Z^2 + Z'^2 + |X - X'|^2 - (T - T')^2). \tag{5}$$

We consider a free scalar field of mass m in this background. Normalizable solutions to the free wave equation $(-\Box + m^2)\phi = 0$ can be expanded in a complete set of modes

$$\phi(T, X, Z) = \int_{|\omega|>|k|} d\omega d^{d-1}k a_{\omega k} e^{-i\omega T} \times e^{ik\cdot X} Z^{d/2} J_\nu(\sqrt{\omega^2 - k^2}Z). \tag{6}$$

The Bessel function has order $\nu = \Delta - d/2$ where $\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2R^2}$ is the conformal dimension of the corresponding operator.

[1]This is to be contrasted with the original formulations of the bulk-to-boundary map [8,9] that include both normalizable and non-normalizable modes on the boundary.

[2]Again this should be compared with the original formulation of the bulk-to-boundary map where the non-normalizable component at the boundary is dual to a source for the CFT operator.

In Poincaré coordinates we define the boundary field by

$$\phi_0^{\rm Poincare}(T,X) = \lim_{Z\to 0}\frac{1}{Z^\Delta}\phi(T,X,Z) = \frac{1}{2^\nu\Gamma(\nu+1)}\int_{|\omega|>|k|} d\omega d^{d-1}k a_{\omega k} \times e^{-i\omega T}e^{ik\cdot X}(\omega^2-k^2)^{\nu/2}. \quad (7)$$

Note that

$$a_{\omega k} = \frac{2^\nu\Gamma(\nu+1)}{(2\pi)^d(\omega^2-k^2)^{\nu/2}} \times \int dT d^{d-1}X e^{i\omega T}e^{-ik\cdot X}\phi_0^{\rm Poincare}(T,X). \quad (8)$$

Substituting this back into the bulk mode expansion (6), we obtain an expression for the bulk field in terms of the boundary field, namely

$$\phi(T,X,Z) = \int dT' d^{d-1}X' K(T',X'|T,X,Z)\phi_0^{\rm Poincare}(T',X'), \quad (9)$$

where

$$K(T',X'|T,X,Z) = \frac{2^\nu\Gamma(\nu+1)}{(2\pi)^d}\int_{|\omega|>|k|} d\omega d^{d-1}k \times e^{-i\omega(T-T')}e^{ik\cdot(X-X')} \times Z^{d/2}J_\nu(\sqrt{\omega^2-k^2}Z)/(\omega^2-k^2)^{\nu/2}. \quad (10)$$

One could proceed to evaluate this integral representation for K along the lines of [6,7,12]. However one generically obtains a smearing function with support on the entire boundary of the Poincaré patch.[3] In the following we will improve on this by constructing smearing functions that make manifest the property that local bulk operators go over to local boundary operators as the bulk point approaches the boundary.

Such smearing functions require complexifying the boundary spatial coordinates X. We will establish this in two ways: in Sec. II B, for fields in AdS_3, by starting with the mode sum (9) and performing a suitable analytic continuation, and again in Sec. II C, for fields in general-dimensional AdS, by Wick rotating to de Sitter space and using a retarded de Sitter Green's function.

B. Poincaré mode sum

Consider a field in AdS_3. The Poincaré mode sum (9) reads

$$\phi(T,X,Z) = \frac{2^\nu\Gamma(\nu+1)}{4\pi^2}\int_{|\omega|>|k|} d\omega dk \frac{ZJ_\nu(\sqrt{\omega^2-k^2}Z)}{(\omega^2-k^2)^{\nu/2}} \times\left(\int dT'dX' e^{-i\omega(T-T')}e^{ik(X-X')} \times \phi_0^{\rm Poincare}(T',X')\right).$$

The Poincaré boundary field has no Fourier components with $|\omega|<|k|$, so provided we perform the T' and X' integrals first we can subsequently integrate over ω and k without restriction. Thus

$$\phi(T,X,Z) = 2^\nu\Gamma(\nu+1)\int d\omega dk e^{-i\omega T}e^{ikX}\frac{ZJ_\nu(\sqrt{\omega^2-k^2}Z)}{(\omega^2-k^2)^{\nu/2}} \times \phi_0^{\rm Poincare}(\omega,k), \quad (11)$$

where $\phi_0^{\rm Poincare}(\omega,k)$ is the Fourier transform of the boundary field. We now use the two integrals

$$\int_0^{2\pi} d\theta e^{-ir\omega\sin\theta - kr\cos\theta} = 2\pi J_0(r\sqrt{\omega^2-k^2}), \quad (12)$$

$$\int_0^1 rdr(1-r^2)^{\nu-1}J_0(br) = 2^{\nu-1}\Gamma(\nu)b^{-\nu}J_\nu(b) \quad (13)$$

to obtain

$$\frac{J_\nu(\sqrt{\omega^2-k^2}Z)}{(\omega^2-k^2)^{\nu/2}} = \frac{1}{\pi(2Z)^\nu\Gamma(\nu)}\int_{T'^2+Y'^2<Z^2} dT'dY' \times (Z^2-T'^2-Y'^2)^{\nu-1}e^{-i\omega T'}e^{-kY'}. \quad (14)$$

Inserting this into (11) one gets

$$\phi(T,X,Z) = \frac{\nu}{\pi}\int_{T'^2+Y'^2<Z^2} dT'dY'\left(\frac{Z^2-T'^2-Y'^2}{Z}\right)^{\nu-1} \times \int d\omega dk e^{-i\omega(T+T')}e^{ik(X+iY')}\phi_0^{\rm Poincare}(\omega,k). \quad (15)$$

We identify the second line of (15) as $\phi_0^{\rm Poincare}(T+T',X+iY')$, so we can write (recall $\nu=\Delta-1$)

$$\phi(T,X,Z) = \frac{\Delta-1}{\pi}\int_{T'^2+Y'^2<Z^2} dT'dY' \times\left(\frac{Z^2-T'^2-Y'^2}{Z}\right)^{\Delta-2} \times \phi_0^{\rm Poincare}(T+T',X+iY'). \quad (16)$$

That is, we have succeeded in expressing the bulk field in terms of an integral over a disk of radius Z in the (real T, imaginary X) plane. We can express the result in terms of the invariant distance (5),

[3] In even-dimensional AdS one can restrict to spacelike separation in the Poincaré patch [6,7].

$$\phi(T,X,Z) = \frac{\Delta - 1}{\pi} \int_{T'^2+Y'^2<Z^2} dT' dY' \times \lim_{Z'\to 0} (2Z'\sigma(T,X,Z|T+T',X+iY',Z'))^{\Delta-2} \times \phi_0^{\mathrm{Poincare}}(T+T',X+iY'). \tag{17}$$

We will obtain the generalization of this result to higher-dimensional AdS in the next subsection.

C. de Sitter continuation

Having seen that we need to analytically continue the boundary spatial coordinates to complex values in order to obtain a smearing function with compact support, we will now begin by Wick rotating the Poincaré longitudinal spatial coordinates, setting $X = iY$. This turns the AdS metric (4) into

$$ds^2 = \frac{R^2}{Z^2}(dZ^2 - dT^2 - |dY|^2).$$

This is nothing but de Sitter space written in flat Friedmann-Robertson-Walker (FRW) coordinates, with Z playing the role of conformal time (note the flip in signature). The AdS boundary becomes the past boundary of de Sitter space. Up to a divergent conformal factor the induced metric on the past boundary is

$$ds^2_{\mathrm{bdy}} = dT^2 + |dY|^2$$

i.e. a plane $\mathbb{R}^d$ which should be thought of as compactified to a sphere S^d by adding a point at infinity. The Penrose diagram is shown in Fig. 1.

In de Sitter space it is clear that the field at any point inside the Poincaré patch can be expressed in terms of data on a compact region of the past boundary.[4] With this motivation we will construct a retarded Green's function in de Sitter space and use it to reproduce and generalize the smearing function (16) that we previously obtained from a Poincaré mode sum.

The de Sitter invariant distance function is

$$\sigma(T,Y,Z|T',Y',Z') = \frac{1}{2ZZ'}(Z^2 + Z'^2 - (T-T')^2 - |Y-Y'|^2).$$

We consider a scalar field of mass m in de Sitter space. For now we take $m^2R^2 > 1$, however later we will analytically continue $m^2 \to -m^2$. The analytically continued mass can be identified with the mass of a field in AdS (note that the Wick rotation flips the signature of the metric).

The field at some bulk point can be written in terms of the retarded Green's function. The retarded Green's function coincides with the imaginary part of the commutator inside the past light cone of the future point and vanishes outside this region. The field at some bulk point is therefore

$$\phi(T,Y,Z) = \int dT' d^{d-1}Y' \left(\frac{R}{Z'}\right)^{d-1} G_{\mathrm{ret}}(T,Y,Z|T',Y',Z') \times \overleftrightarrow{\partial}_{Z'} \phi(T',Y',Z'), \tag{18}$$

where the region of integration is over a spacelike surface of fixed Z' inside the past light cone of the bulk point. In the $Z' \to 0$ limit this becomes the disk

$$(T-T')^2 + |Y-Y'|^2 < Z^2. \tag{19}$$

As $Z' \to 0$ (with other coordinates held fixed) the retarded Green's function takes the form [25]

$$G_{\mathrm{ret}} \sim iR^{-d+1}(c(-\sigma - i\epsilon)^{-d/2+i\sqrt{m^2R^2-(d/2)^2}} + c^*(-\sigma - i\epsilon)^{-d/2-i\sqrt{m^2R^2-(d/2)^2}} - \mathrm{c.c.}),$$

where we take branch cuts along the positive real σ axis and where

$$c = \frac{\Gamma(2i\sqrt{m^2R^2-(\frac{d}{2})^2})\Gamma(\frac{d}{2} - i\sqrt{m^2R^2-(\frac{d}{2})^2})}{2^{-(d/2)+i\sqrt{m^2R^2-(d/2)^2}}(4\pi)^{(d+1)/2}\Gamma(\frac{1}{2} + i\sqrt{m^2R^2-(\frac{d}{2})^2})}.$$

The boundary field is defined as usual using (7). Choosing normalizable modes from the AdS viewpoint corresponds to taking only positive frequencies in the Z direction, which have a $Z^{d/2+i\sqrt{m^2R^2-(\frac{d}{2})^2}}$ Z-dependence.

Evaluating (18) as $Z' \to 0$ we obtain the smearing function[5]

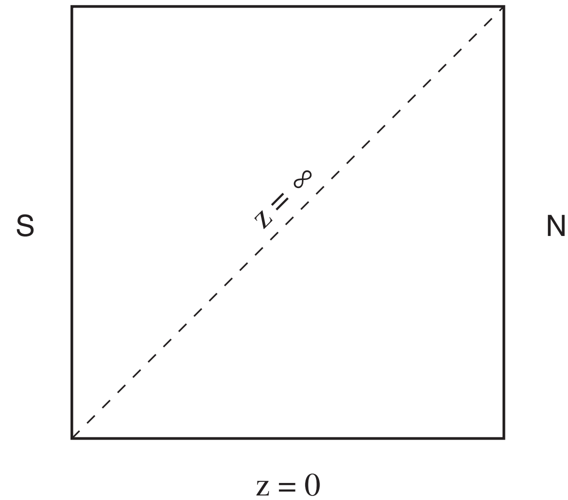

FIG. 1. The Penrose diagram for de Sitter space. Flat FRW coordinates cover the lower triangle. Horizontal slices are spheres. Each point on the diagram represents an S^{d-1} which shrinks to a point at the north and south poles (the right and left edges of the diagram).

[4] To go outside the Poincaré patch one would have to include the point at infinity in $\mathbb{R}^d$.

[5] Here we use the identities $\sin(\pi z) = \frac{\pi}{\Gamma(z)\Gamma(1-z)}$ and $\frac{\Gamma(2z)}{\Gamma(z)\Gamma(1/2+z)} = \frac{2^{2z-1}}{\sqrt{\pi}}$.

$$K(T', Y'|T, Y, Z) = \frac{\Gamma(\Delta - \frac{d}{2} + 1)}{\pi^{d/2}\Gamma(\Delta - d + 1)} \times \left(\frac{Z^2 - (T - T')^2 - |Y - Y'|^2}{Z}\right)^{\Delta - d} \times \theta(Z^2 - (T - T')^2 - |Y - Y'|^2). \quad (20)$$

However at this point we still have $\Delta = \frac{d}{2} + i\sqrt{m^2R^2 - (\frac{d}{2})^2}$. By analytically continuing $m^2 \to -m^2$ we can take Δ to coincide with the conformal dimension in AdS. Since $\sigma > 0$ in the domain of integration this analytic continuation is straightforward. Furthermore we can shift $iY \to X + iY$ and $iY' \to X + iY'$, assuming $\phi_0^{\mathrm{Poincare}}$ is analytic everywhere in the strip $|Y| < Z$; this is true for any given Poincaré mode function (6). Thus we wind up with the integral representation

$$\phi(T, X, Z) = \frac{\Gamma(\Delta - \frac{d}{2} + 1)}{\pi^{d/2}\Gamma(\Delta - d + 1)} \int_{T'^2 + |Y'|^2 < Z^2} dT' d^{d-1}Y' \times \left(\frac{Z^2 - T'^2 - |Y'|^2}{Z}\right)^{\Delta - d} \times \phi_0^{\mathrm{Poincare}}(T + T', X + iY'). \quad (21)$$

This matches (16) for $d = 2$. As a further check we can examine the limit $Z \to 0$ where we should recover (7). In this limit the region of integration becomes very small so we can bring the boundary field out of the integral and we indeed recover (7).

D. Recovering bulk correlators

In this section we show that the smearing functions we have constructed can be used to reproduce bulk correlation functions. As a corollary, this shows that the operators we have defined will commute with each other at bulk spacelike separation. For simplicity we will only treat the case of a massless field in AdS_3.

The Wightman function for a massless scalar in AdS_3 is

$$G(x|x') = \langle 0|\phi(x)\phi(x')|0\rangle_{\mathrm{SUGRA}} = \frac{1}{4\pi R} \frac{1}{\sqrt{\sigma^2 - 1}(\sigma + \sqrt{\sigma^2 - 1})}, \quad (22)$$

where σ is defined in (5), and where branch cuts are handled with a $T \to T - i\epsilon$ prescription.[6] We will consider the correlation function between an arbitrary bulk point (T, X, Z) and a point near the boundary with coordinates $(T' = 0,\ X' = 0,\ Z' \to 0)$. Taking the appropriate limit of (22) we have

$$\langle \phi(T, X, Z)\phi_0^{\mathrm{Poincare}}(0, 0)\rangle_{\mathrm{SUGRA}} = \frac{1}{2\pi R} \frac{Z^2}{(T^2 - X^2 - Z^2)^2}. \quad (23)$$

We would like to reproduce this from the CFT. To do this note that from (16) we have

$$\phi(T, X, Z) = \frac{1}{\pi} \int_{T'^2 + Y'^2 < Z^2} dT' dY' \times \phi_0^{\mathrm{Poincare}}(T + T', X + iY'). \quad (24)$$

Also by sending both points to the boundary in (22) we have the boundary correlator[7]

$$\langle \phi_0^{\mathrm{Poincare}}(T, X)\phi_0^{\mathrm{Poincare}}(0, 0)\rangle_{\mathrm{CFT}} = \frac{1}{2\pi R} \frac{1}{(T^2 - X^2)^2}. \quad (25)$$

Thus our claim is that we can reproduce (23) by computing

$$\frac{1}{\pi}\int_{T'^2 + Y'^2 < Z^2} dT' dY' \langle \phi_0^{\mathrm{Poincare}}(T + T', X + iY')\phi_0^{\mathrm{Poincare}}(0, 0)\rangle = \frac{1}{2\pi^2 R}\int_{T'^2 + Y'^2 < Z^2} dT' dY' \frac{1}{((T + T')^2 - (X + iY')^2)^2}. \quad (26)$$

Let us begin by studying this in the regime

$$|T + X| > Z \quad \text{and} \quad |T - X| > Z. \quad (27)$$

In this regime there are no poles in the range of integration, so (26) is well defined without having to give a prescription for dealing with light cone singularities in the CFT. It is convenient to work in polar coordinates, setting $T' = r\cos\theta$ and $Y' = r\sin\theta$. Defining $z = e^{i\theta}$ we have

$$\frac{1}{\pi R}\int_0^Z r dr \oint_{|z|=1} \frac{dz}{2\pi i} \frac{z}{(T + X + rz)^2(z(T - X) + r)^2}. \quad (28)$$

Evaluating the contour integral gives

$$\frac{1}{\pi R}\int_0^Z r dr \frac{T^2 - X^2 + r^2}{(T^2 - X^2 - r^2)^3} = \frac{1}{2\pi R} \frac{Z^2}{(T^2 - X^2 - Z^2)^2} \quad (29)$$

as promised.

Now let us return to the question of dealing with light cone singularities in the CFT. That is, let us ask how we can analytically continue this result outside the range (27). In general the integrand in (28) has two double poles, located at

$$z = z_1 = -\frac{r}{T - X} \quad \text{and} \quad z = z_2 = -\frac{T + X}{r}. \quad (30)$$

[6]This Wightman function identifies $|0\rangle$ as the Poincaré vacuum state.

[7]We obtained this from a boundary correlator in supergravity, but the result matches the correlator of local operators in the CFT.

In the range (27) we see that the contour always encircles the pole at z_1 and never encircles the pole at z_2. When we try to go outside this range one of the poles crosses the integration contour $|z| = 1$. So to analytically continue the calculation outside the range (27) we merely have to deform the z contour of integration so that it continues to encircle the pole at z_1 and exclude the pole at z_2.

One might ask how one can distinguish the two poles in general. Recall that the boundary CFT correlator is defined with a $T \to T - i\epsilon$ prescription. This means the poles are displaced to[8]

$$z_1 = -\frac{r}{T - X} - i\epsilon, \qquad z_2 = -\frac{T + X}{r} + i\epsilon. \tag{31}$$

We see that z_1 is always in the lower half-plane while z_2 is always in the upper half-plane. So the general prescription is to only encircle the pole in the lower half-plane. The $i\epsilon$ prescription makes the z contour integral well defined, since the poles never collide. It also makes the integral over r well defined, since the poles in r are displaced off the real axis.

This lets us see how the bulk light cone singularity emerges from the CFT. Let us perform the z integral in (28) first, followed by the r integral. The two poles pinch the z contour of integration when $r^2 = r_0^2 \equiv (T - i\epsilon)^2 - X^2$. Thus the integral over z has a pole when $r = \pm r_0$. When one of these singularities in the complex r plane hits the $r = Z$ endpoint of the contour for integrating over r, the integral over r diverges. This reproduces the bulk light cone singularity at $T^2 - X^2 = Z^2$, regulated by the appropriate $i\epsilon$ prescription.

Since our smeared operators have the correct 2-point functions, it follows that at infinite N they commute as operators in the CFT whenever the bulk points are spacelike separated. This relies on the fact that at infinite N the commutator is a c-number, and one can check that it vanishes at bulk spacelike separation by computing a correlator $\langle\psi|\mathcal{O}_1\mathcal{O}_2 - \mathcal{O}_2\mathcal{O}_1|\psi\rangle$ in any state of the CFT. However at finite N the commutator becomes an operator. The delicate cancellations which occurred at infinite N become state dependent and are no longer possible in general. Thus we do not necessarily expect the commutator to vanish at bulk spacelike separation. We discuss this point further in Sec. VI.

III. AdS_3 IN RINDLER COORDINATES

We now specialize to AdS_3. This is a particularly interesting example, since the BTZ black hole can be constructed as a quotient of AdS_3 [24]. After some preliminaries we discuss AdS_3 in accelerating Rindler-like coordinates. We show that our Poincaré results can be translated into accelerating coordinates and, with the help of an antipodal map, can be used to describe local bulk operators inside the Rindler horizon.

[8]Assuming that T and X are real and $r > 0$.

A. Preliminaries

AdS_3 can be realized as the universal cover of a hyperboloid

$$-U^2 - V^2 + X^2 + Y^2 = -R^2 \tag{32}$$

inside $\mathbb{R}^{2,2}$ with metric $ds^2 = -dU^2 - dV^2 + dX^2 + dY^2$. To describe this in Rindler coordinates we set

$$U = \frac{Rr}{r_+}\cosh\frac{r_+\phi}{R}, \qquad V = R\sqrt{\frac{r^2}{r_+^2} - 1}\,\sinh\frac{r_+ t}{R^2},$$
$$X = R\sqrt{\frac{r^2}{r_+^2} - 1}\,\cosh\frac{r_+ t}{R^2}, \qquad Y = \frac{Rr}{r_+}\sinh\frac{r_+\phi}{R} \tag{33}$$

so that the induced metric is

$$ds^2 = -\frac{r^2 - r_+^2}{R^2}dt^2 + \frac{R^2}{r^2 - r_+^2}dr^2 + r^2 d\phi^2. \tag{34}$$

Here $-\infty < t, \phi < \infty$ and $r_+ < r < \infty$. The Rindler horizon is located at $r = r_+$. These coordinates cover the right Rindler wedge of AdS_3 as shown in Fig. 2. One can continue into the future wedge by setting

$$U = \frac{Rr}{r_+}\cosh\frac{r_+\phi}{R}, \qquad V = R\sqrt{1 - \frac{r^2}{r_+^2}}\,\cosh\frac{r_+ t}{R^2},$$
$$X = R\sqrt{1 - \frac{r^2}{r_+^2}}\,\sinh\frac{r_+ t}{R^2}, \qquad Y = \frac{Rr}{r_+}\sinh\frac{r_+\phi}{R^2}$$

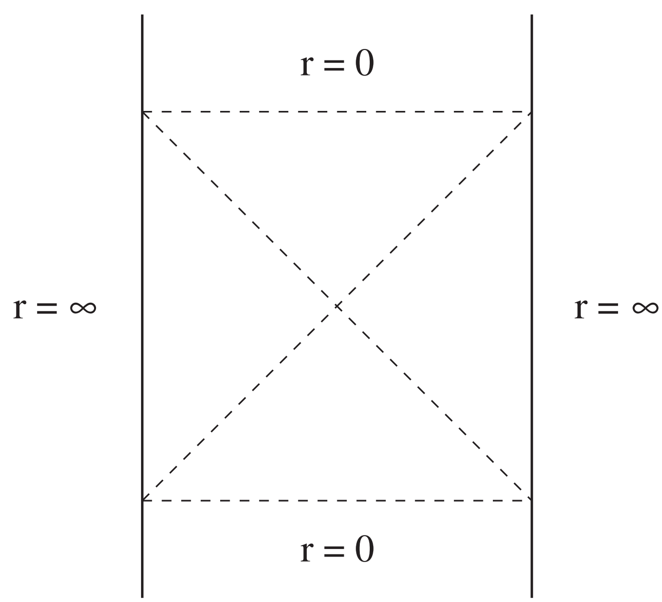

FIG. 2. A slice of constant ϕ in AdS_3, drawn as an AdS_2 Penrose diagram. The four Rindler wedges are separated by horizons at $r = r_+$.

with $0 < r < r_+$. One can extend these coordinates to the (left, past) wedges by starting from the (right, future) definitions and changing the signs of V and X. It will frequently be convenient to work with rescaled coordinates

$$\hat{t} = r_+ t/R^2, \qquad \hat{\phi} = r_+ \phi/R.$$

An AdS-invariant distance function is provided by

$$\sigma(x|x') = -\frac{1}{R^2} X_\mu X'^\mu \tag{35}$$

in terms of the embedding coordinates $X^\mu = X^\mu(x)$. For two points in the right Rindler wedge we have

$$\sigma = \frac{rr'}{r_+^2}\cosh(\hat{\phi} - \hat{\phi}') - \left(\frac{r^2}{r_+^2} - 1\right)^{1/2}\left(\frac{r'^2}{r_+^2} - 1\right)^{1/2} \times \cosh(\hat{t} - \hat{t}') \tag{36}$$

while for a point $(\hat{t}, r, \hat{\phi})$ inside the future horizon and a point $(\hat{t}', r', \hat{\phi}')$ in the $\left\{\begin{smallmatrix}\text{right}\\ \text{left}\end{smallmatrix}\right\}$ Rindler wedge we have

$$\sigma = \frac{rr'}{r_+^2}\cosh(\hat{\phi} - \hat{\phi}') \mp \left(1 - \frac{r^2}{r_+^2}\right)^{1/2}\left(\frac{r'^2}{r_+^2} - 1\right)^{1/2} \times \sinh(\hat{t} - \hat{t}'). \tag{37}$$

B. Rindler smearing functions

We could set about constructing a smearing function starting from a Rindler mode sum. For points outside the Rindler horizon this was carried out in [7], while for points inside the horizon we set up but do not evaluate the mode sum in Appendix A. However the Rindler mode sum is divergent and must be defined by analytic continuation in $\hat{t}$ and/or $\hat{\phi}$. The divergence means there is no smearing function with support on real values of the Rindler boundary coordinates.

A simpler approach to constructing the Rindler smearing function is to begin with our Poincaré result (16) and translate it into Rindler coordinates. The translation is easiest to understand in de Sitter space. Wick rotating $\hat{\phi} = iy$ turns the AdS metric (34) into

$$ds^2 = \frac{R^2}{r_+^2}\left[\frac{r_+^2}{r^2 - r_+^2}dr^2 - (r^2 - r_+^2)d\hat{t}^2 - r^2 dy^2\right].$$

This is de Sitter space in static coordinates. To avoid a conical singularity at $r = 0$ we must periodically identify $y \sim y + 2\pi$. The right Rindler wedge becomes the past wedge of de Sitter space, as shown in Fig. 3. The induced metric on the past boundary is, up to a divergent conformal factor,

$$ds_{\text{bdy}}^2 = d\hat{t}^2 + dy^2, \qquad -\infty < \hat{t} < \infty, \qquad y \sim y + 2\pi,$$

i.e. an infinite cylinder which can be compactified to a sphere by adding the north and south poles. This sphere can be identified with the past boundary that we identified working in Poincaré coordinates. However note that any observer inside the past wedge of de Sitter space can at most see one hemisphere of the past boundary, namely, the region characterized by

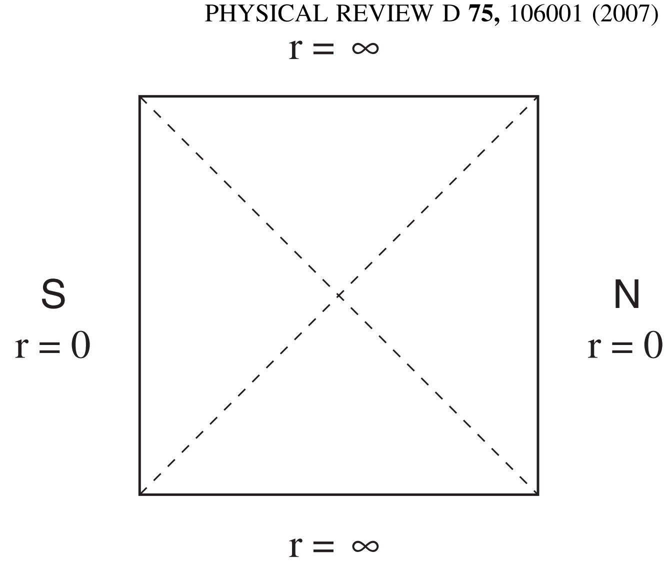

FIG. 3. de Sitter space in static coordinates.

$$-\infty < \hat{t} < \infty, \qquad -\pi/2 < y < \pi/2.$$

For a point inside the past wedge of de Sitter we can construct a retarded Green's function that lets us express the value of the field in terms of data on the past boundary. In AdS this means we can express the value of the field anywhere in the right Rindler wedge in terms of a data on the right Rindler boundary. In fact the result is a simple translation of our Poincaré result (16). We define the right boundary field in Rindler coordinates by

$$\phi_0^{\text{Rindler},R}(\hat{t}, \hat{\phi}) = \lim_{r\to\infty} r^\Delta \phi(\hat{t}, r, \hat{\phi})|_{\text{right boundary}}. \tag{38}$$

This is related to the Poincaré boundary field by

$$\phi_0^{\text{Rindler},R}(\hat{t}, \hat{\phi}) = \lim_{r\to\infty} (rZ)^\Delta \phi_0^{\text{Poincare}}(T, X). \tag{39}$$

We also have the boundary change of coordinates

$$\frac{dTdX}{Z^2} = \frac{r^2 d\hat{t} d\hat{\phi}}{r_+^2}. \tag{40}$$

Making these substitutions in (16), the value of the field at a bulk point inside the right Rindler wedge of AdS_3 is

$$\phi(\hat{t}, r, \hat{\phi}) = \frac{(\Delta - 1)2^{\Delta-2}}{\pi r_+^2}\int_{\text{spacelike}} dxdy \lim_{r'\to\infty} \times (\sigma/r')^{\Delta-2}\phi_0^{\text{Rindler},R}(\hat{t} + x, \hat{\phi} + iy), \tag{41}$$

where as $r' \to \infty$ the AdS-invariant distance (36) becomes

$$\sigma(\hat{t}, r, \hat{\phi}|\hat{t} + x, r', \hat{\phi} + iy) = \frac{rr'}{r_+^2}\left[\cos y - \left(1 - \frac{r_+^2}{r^2}\right)^{1/2}\cosh x\right] \tag{42}$$

and the integration is over "spacelike separated" points on the Wick rotated boundary, that is, over real values of (x, y) such that $\sigma > 0$.

The result (41) for bulk points in the right Rindler wedge was obtained in [7], starting from a Rindler mode sum and defining it via an analytic continuation, or alternatively from a de Sitter Green's function. Now let us ask what happens for bulk points inside the Rindler horizon. It is clear from Fig. 2 that, if we were willing to work in Poincaré coordinates, there would be no problem: we could use (16) to obtain a smearing function with compact support on the Poincaré boundary. However if we wish to work in Rindler coordinates there is a problem: the smearing function extends outside the Rindler wedge, and covers points on the (real slice of) the boundary which are to the future of the right Rindler patch.[9]

To fix this we can use the antipodal map.[10] The antipodal map acts on the embedding coordinates of Sec. III A by

$$A: X^\mu \to -X^\mu. \tag{43}$$

In terms of Rindler coordinates this can be realized by

$$A: \hat{t} \to \hat{t} + i\pi, \qquad \hat{\phi} \to \hat{\phi} + i\pi. \tag{44}$$

Note that $\sigma(x|Ax') = -\sigma(x|x')$. Fields with integer conformal dimension transform simply under the antipodal map,

$$\phi(Ax) = (-1)^\Delta \phi(x). \tag{45}$$

This is discussed in Appendix B, where we also treat the slightly more involved case of noninteger Δ.

In Rindler coordinates the antipodal map can be used to move the part of the smearing function which extends outside the right Rindler wedge over to the left boundary. To see this one starts with the Poincaré result (16) and breaks the integration region up into two pieces. One piece gives a smearing function in the right Rindler wedge, while under the antipodal map the other piece becomes a smearing function in the left Rindler wedge. Thus for a bulk point inside the Rindler horizon we have

$$\begin{aligned}\phi(\hat{t}, r, \hat{\phi}) = \frac{(\Delta - 1)2^{\Delta-2}}{\pi r_+^2}\Big[\int_{\sigma>0} dxdy \lim_{r'\to\infty} (\sigma/r')^{\Delta-2} \\ \times \phi_0^{\text{Rindler},R}(\hat{t} + x, \hat{\phi} + iy) \\ + \int_{\sigma<0} dxdy \lim_{r'\to\infty} (-\sigma/r')^{\Delta-2}(-1)^\Delta \\ \times \phi_0^{\text{Rindler},L}(\hat{t} + x, \hat{\phi} + iy)\Big].\end{aligned} \tag{46}$$

Here as $r' \to \infty$ the AdS-invariant distance (35) becomes

$$\sigma(\hat{t}, r, \hat{\phi}|\hat{t} + x, r', \hat{\phi} + iy) = \frac{rr'}{r_+^2}\left[\cos y \mp \left(\frac{r_+^2}{r^2} - 1\right)^{1/2} \sinh x\right] \tag{47}$$

when the boundary point is in the $\{{}^{\text{right}}_{\text{left}}\}$ Rindler wedge. The integration is over points with $\sigma > 0$ on the right boundary and points with $\sigma < 0$ on the left boundary, and we define

$$\phi_0^{\text{Rindler},L}(\hat{t}, \hat{\phi}) = \lim_{r\to\infty} r^\Delta \phi(\hat{t}, r, \hat{\phi})|_{\text{left boundary}}. \tag{48}$$

C. Reproducing bulk correlators

It is instructive to check that the Rindler smearing functions we have constructed let us recover the correct bulk two-point functions from the CFT,[11] especially for points inside the Rindler horizon. Clearly of special importance is the point $r = 0$, where the Rindler coordinates become singular. So in this section we show how this works for a point located at $r = 0$ and a point near the right boundary.

The AdS Wightman function is

$$\begin{aligned}G_{\text{AdS}}(x|x') &= \langle 0|\phi(x)\phi(x')|0\rangle_{\text{SUGRA}} \\ &= \frac{1}{4\pi R}\frac{1}{\sqrt{\sigma^2 - 1}}\frac{1}{(\sigma + \sqrt{\sigma^2 - 1})^{\Delta-1}}.\end{aligned} \tag{49}$$

Here $|0\rangle$ is the global or AdS-invariant vacuum state. Branch cuts are handled with a $\tau \to \tau - i\epsilon$ prescription, or equivalently $\sigma \to \sigma + i\epsilon \sin(\tau - \tau')$, where τ is the global time coordinate defined in Appendix B.[12] We consider a point near the origin of Rindler coordinates ($t = 0$, $r = r_0$, $\phi = 0$), and a point near the right boundary with coordinates (t, r, ϕ). As $r_0 \to 0$ and $r \to \infty$ the invariant distance (37) is

$$\sigma \approx \frac{r}{r_+}\left(\frac{r_0}{r_+}\cosh\hat{\phi} + \sinh\hat{t}\right).$$

Thus the AdS correlator approaches a finite, $\hat{\phi}$-independent value as $r_0 \to 0$

$$G_{\text{AdS}}(0, 0, 0|\hat{t}, r, \hat{\phi}) \approx \frac{1}{2\pi R}\left(\frac{r_+}{2r\sinh\hat{t} + i\epsilon}\right)^\Delta. \tag{50}$$

The fact that the correlator is independent of $\hat{\phi}$ reflects the fact that $r = 0$ is a fixed point of the isometry $\hat{\phi} \to \hat{\phi} +$ const.

Now let us see how this behavior is reproduced by the CFT. We will work with a field of integer conformal dimension. At $\hat{t} = r = \hat{\phi} = 0$ the smearing function (46) reduces to

[9] This is easiest to see by considering a bulk point in the future wedge of Fig. 2 and following light rays to the right boundary.

[10] Exactly the same procedure applies to AdS_2 in Rindler coordinates [6]. An alternate procedure would be to analytically continue outside the strip $-\pi/2 < y < \pi/2$.

[11] This was done in Sec. II D for Poincaré coordinates.

[12] For points inside the Poincaré patch this is equivalent to $T \to T - i\epsilon$.

$$\phi(0,0,0) = \frac{(\Delta-1)2^{\Delta-2}}{\pi r_+^\Delta}\int_0^\infty dx \sinh^{\Delta-2}x \times \int_{-\pi/2}^{\pi/2} dy(\phi_0^{\text{Rindler},R}(x,iy) + (-1)^\Delta \phi_0^{\text{Rindler},L}(x,iy)) \tag{51}$$

while as $r \to \infty$ the smearing function (41) reduces to

$$\phi(\hat{t}, r, \hat{\phi}) \approx r^{-\Delta}\phi_0^{\text{Rindler},R}(\hat{t},\hat{\phi}). \tag{52}$$

This means we should be able to recover (50) by computing

$$\frac{(\Delta-1)2^{\Delta-2}}{\pi(rr_+)^\Delta}\int_0^\infty dx \sinh^{\Delta-2}x \int_{-\pi/2}^{\pi/2} dy \langle(\phi_0^{\text{Rindler},R}(x,iy) + (-1)^\Delta \phi_0^{\text{Rindler},L}(x,iy))\phi_0^{\text{Rindler},R}(\hat{t},\hat{\phi})\rangle_{\text{CFT}}. \tag{53}$$

For convenience we will work in the regime

$$\hat{t} < 0, \qquad \hat{t} < \hat{\phi} < -\hat{t}. \tag{54}$$

In this regime the smeared CFT operators are never lightlike separated, so (53) is well defined without a prescription for dealing with light cone singularities in the CFT. The appropriate CFT correlators can be obtained from (49) by sending the bulk points to the appropriate boundary[13]

$$\begin{aligned}\langle\phi_0^{\text{Rindler},R}(\hat{t},\hat{\phi})\phi_0^{\text{Rindler},R}(\hat{t}',\hat{\phi}')\rangle_{\text{CFT}} &= \frac{(r_+^2/2)^\Delta}{2\pi R(\cosh(\hat{\phi}-\hat{\phi}') - \cosh(\hat{t}-\hat{t}'-i\epsilon))^\Delta},\\ \langle\phi_0^{\text{Rindler},R}(\hat{t},\hat{\phi})\phi_0^{\text{Rindler},L}(\hat{t}',\hat{\phi}')\rangle_{\text{CFT}} &= \frac{(r_+^2/2)^\Delta}{2\pi R(\cosh(\hat{\phi}-\hat{\phi}') + \cosh(\hat{t}-\hat{t}'))^\Delta}.\end{aligned} \tag{55}$$

Then we have

$$\begin{aligned}&\frac{(\Delta-1)r_+^\Delta}{8\pi^2 R r^\Delta}\int_0^\infty dx \sinh^{\Delta-2}x \int_{-\pi/2}^{\pi/2} dy[(\cosh(\hat{\phi}-iy) - \cosh(\hat{t}-x))^{-\Delta} + (-1)^\Delta(\cosh(\hat{\phi}-iy)+\cosh(\hat{t}-x))^{-\Delta}]\\ &= \frac{(\Delta-1)r_+^\Delta}{8\pi^2 R r^\Delta}\int_0^\infty dx \sinh^{\Delta-2}x \int_{-\pi}^{\pi} dy(\cosh(\hat{\phi}-iy)-\cosh(\hat{t}-x))^{-\Delta}\\ &= \frac{(\Delta-1)2^{\Delta-3}r_+^\Delta}{i\pi^2 R r^\Delta}\int_0^\infty dx \sinh^{\Delta-2}x \oint_{|z|=e^{\hat{\phi}}} \frac{z^{\Delta-1}dz}{(z-e^{\hat{t}-x})^\Delta(z-e^{-(\hat{t}-x)})^\Delta}.\end{aligned} \tag{56}$$

In the last line we set $z = e^{\hat{\phi}-iy}$. In the regime (58) the z contour of integration always encircles the pole at $e^{\hat{t}-x}$ and never encircles the pole at $e^{-(\hat{t}-x)}$. To analytically continue outside (54) we proceed as in Sec. II D and deform the contour of integration so that it continues to encircle the appropriate pole. This continuation gives an integral that is independent of $\hat{\phi}$, and in this way the smearing function (51) captures the fact that $r = 0$ is a fixed point of the isometry $\phi \to \phi + \text{const}$. It is entertaining to push the calculation a little further and show that the CFT exactly reproduces the bulk correlator. Just to be concrete, let us set $\Delta = 2$. Then evaluating the contour integral in (56) gives

$$-\frac{r_+^2}{\pi R r^2}\int_0^\infty dx \frac{2\cosh(\hat{t}-x)}{(2\sinh(\hat{t}-x))^3} = \frac{1}{2\pi R}\left(\frac{r_+}{2r\sinh\hat{t}}\right)^2 \tag{57}$$

in agreement with (50) for $\Delta = 2$. The result is also valid outside the range (54) using the analytic continuation described above.

IV. BTZ BLACK HOLE

To make a BTZ black hole starting from AdS_3 all we have to do is periodically identify the ϕ coordinate, $\phi \sim \phi + 2\pi$ [24,26]. Scalar fields on AdS_3 will descend to scalar fields on BTZ provided they satisfy $\phi(t, r, \phi) = \phi(t, r, \phi + 2\pi)$. The global AdS vacuum descends to the Hartle-Hawking vacuum state in BTZ.

In this construction we are identifying points separated by *real* values of the ϕ coordinate. Since the Rindler smearing functions we have constructed are translation invariant in ϕ, and since they only involve integration over the imaginary part of ϕ, we can apply our Rindler results to a BTZ black hole without modification. That is, (41) and (46) can be used to represent bulk fields in a BTZ spacetime; if the boundary field has the correct periodicity then so will the bulk field. This shows quite explicitly that we can recover local physics outside the BTZ horizon using operators that act on a single copy of the CFT, while to describe the region inside the horizon we must use operators that act on both the CFT and its thermofield double [6,16–18,27].

The BTZ black hole has a spacelike singularity at $r = 0$, which has been studied from the CFT point of view in [17,18,21,23].[14] In the semiclassical limit that we are considering this singularity should be encoded in the CFT. We are now in a position to see this directly, by studying bulk correlators with one point close to the singularity.

The BTZ Wightman function is given by an image sum [28,29]

[13]We obtained these expressions as boundary supergravity correlators, but the same result holds for correlation functions in a finite temperature CFT.

[14]The singularity is an analytic continuation of one of conical type. The curvature remains constant near the singularity.

$$G_{\rm BTZ}(x|x') = \sum_{n=-\infty}^{\infty} G_{\rm AdS}(t, r, \phi|t', r', \phi' + 2\pi n). \quad (58)$$

This diverges when $r = 0$, just because $r = 0$ is a fixed point of the isometry of shifting ϕ by a constant: when $r = 0$ the invariant distance (37) is independent of ϕ' and the image sum diverges. To estimate the divergence, note that for small r_0 the BTZ image sum is cut off at $|n| \approx \frac{1}{2\pi} \times \log(r_+/r_0)$. This means the BTZ Wightman function diverges logarithmically near the singularity

$$G_{\rm BTZ}(0, r_0, 0|\hat{t}, r, \hat{\phi}) \sim \frac{1}{2\pi^2 R}\left(\frac{r_+}{2r\sinh\hat{t}}\right)^{\Delta} \log\frac{r_+}{r_0} \quad \text{as } r_0 \to 0.$$

How does this divergence arise from the CFT viewpoint? *A priori* there are a number of possibilities:

(i) The CFT itself could be incomplete in the same sense as classical gravity.

(ii) The mapping between CFT operators and local bulk fields could become singular at this point.

(iii) The mapping could remain smooth, but the CFT operator moves outside the class of physically reasonable observables.

The boundary S-matrix in the gravity theory appears to be well defined around the BTZ background by virtue of cosmic censorship, provided one avoids processes that produce naked singularities. Hence the same will be true of the CFT correlators, so in that sense the CFT gives a complete well-defined theory at large N. Thus the first possibility is ruled out. The mapping is nonsingular, as can be seen explicitly in (51), which rules out the second possibility. It is the third possibility which is realized.

Before discussing this in more detail, let us follow through with our calculation of the bulk two-point function using the CFT. As in Sec. III C we place one point near the singularity and the other near the right boundary. Then all we have to do is replace the AdS boundary correlators with BTZ boundary correlators in (53).[15] Boundary correlators in the BTZ geometry can be obtained from (55) by performing an image sum to make them 2π periodic in ϕ [30]. However as we have seen (53) gives a result that is independent of ϕ. Therefore substituting BTZ boundary correlators in (53) leads to a divergent image sum. So the divergence is present in the CFT computation of the correlator, for the same reason it was present in the bulk.

Now let us make some comments on the interpretation of this divergence. In AdS_3 as two bulk points coincide their correlator exhibits the expected Hadamard short-distance singularity

$$G_{\rm AdS} \sim \frac{1}{4\pi R\sqrt{2(\sigma - 1)}} \quad \text{as } \sigma \to 1. \quad (59)$$

Generically as two points coincide in BTZ their correlator diverges in exactly the same way, because only one term in the image sum (58) will have a singularity. However if we place one point at the BTZ singularity then $G_{\rm BTZ}$ diverges no matter where the other point is located. This is because $r = 0$ is a fixed point of the orbifold symmetry and the symmetry operation is of infinite order.

We can use the coefficient of the singularity (59) as a definition of the norm of these operators. For generic points the norm is finite, however the norm diverges for the operator at the fixed point. One way to see this is by using a point splitting regularization and considering $\lim_{\epsilon\to 0} G_{\rm BTZ}(0, 0, 0|\epsilon_r, \epsilon_\phi, \epsilon_t)$. The invariant distance is independent of the coordinate separation in the ϕ direction if one point lies at $r = 0$, so $G_{\rm BTZ}(0, 0, 0|\epsilon_r, \epsilon_\phi, \epsilon_t)$ diverges even at finite ϵ. Thus the operator $\phi|_{r=0}$ has infinite norm.

In the CFT we interpret the operator (51) dual to $\phi|_{r=0}$ exactly as in the bulk. It is a non-normalizable operator which has divergent correlators with all operators of interest. This is how a well-behaved conformal field theory gives rise to a divergent correlation function: through the introduction of a non-normalizable operator. We will comment further in Sec. VII on how this picture generalizes when backreaction and finite N are taken into account.

V. COLLAPSE GEOMETRIES

As we have seen, it is possible to probe the region inside the horizon of a BTZ black hole using operators that act on both the left and right copies of the CFT. A similar result should hold for a general eternal AdS-Schwarzschild black hole. However in the more physical case of a black hole formed in collapse there is only a single asymptotic AdS region, and one might ask: can the region inside the horizon be described using the single copy of the CFT?

For simplicity let us work in AdS_3 and consider a large (stable) black hole formed by sending in a null shell from the boundary. The Penrose diagram is shown in Fig. 4. Consider a bulk point P inside the horizon and to the future of the shell. Can an operator inserted at that point be described in the CFT?

The answer is yes, and for fields with integer conformal dimension the explicit construction is quite simple. As can be seen from the global mode expansion given in Appendix B, fields with integer conformal dimension are single-valued on the AdS hyperboloid (periodic in global time with period 2π). Note from (33) that continuing $\hat{t} \to \hat{t} + i\pi$ has the effect of changing the sign of two of the embedding coordinates, namely

$$V \to -V \qquad X \to -X.$$

Thus for integer conformal dimension the boundary fields

[15] As is written, (53) is only valid in the range (54). To extend it outside this range we must analytically continue in ϕ, as discussed at the end of Sec. III C.

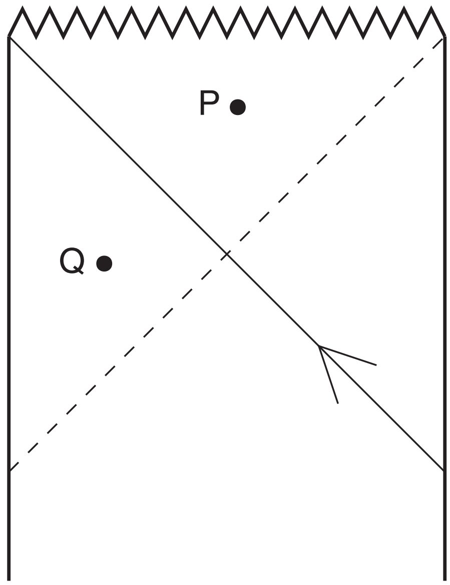

FIG. 4. An AdS black hole formed by collapse. The left edge of the diagram is the origin of AdS, the right edge is the AdS boundary. The dashed line is the black hole horizon while the solid diagonal line represents the infalling shell.

in the left and right Rindler wedges are related by

$$\phi_0^{\mathrm{Rindler},L}(\hat{t}, \hat{\phi}) = \phi_0^{\mathrm{Rindler},R}(\hat{t} + i\pi, \hat{\phi}). \qquad (60)$$

(This relation was also used in [16].) The collapse geometry can be made by taking the right and future regions of an eternal BTZ black hole and joining them across the shell to a piece of AdS_3. For points to the future of the shell we can, by analytic continuation, pretend that we are in an eternal BTZ geometry. We can therefore use the relation (60) in our BTZ smearing function (46) to represent bulk operators that are located inside the horizon.

This shows that we can represent a bulk point inside the horizon in terms of a single CFT, provided we analytically continue in both the $\hat{t}$ and $\hat{\phi}$ coordinates. Our explicit construction works for points that are to the future of the infalling shell. One could also ask about representing bulk points inside the shell, such as the point Q in Fig. 4. This is indeed possible, although the construction is more complicated since one must propagate modes across the shell [31].

VI. COMMENTS ON FINITE N

We have seen that in the semiclassical limit one can construct local operators anywhere in the bulk of AdS. However at finite N, when the Planck length is finite, holography demands that the number of independent degrees of freedom inside a volume is finite, bounded by the area of the region in Planck units. In this section we attempt to understand how this comes about.

The smeared operators we have constructed in the CFT are still well defined at finite N. For example in $\mathcal{N} = 4$ Yang-Mills we can define the operator

$$\Phi(T, X, Z) = \int dT' d^3X' K(T', X'|T, X, Z) \mathrm{Tr} F^2(T', X') \qquad (61)$$

at any N. At finite N it does not obey the correct bulk dilaton equation of motion [10]. However it is a perfectly good operator in the gauge theory, and it has the right behavior in the large-N limit to be associated with a particular point in the bulk. So as a first step, it seems reasonable to associate $\Phi(T, X, Z)$ with a point in the bulk, even at finite N. Since the bulk point was arbitrary, at first sight this means we can associate an infinite number of local operators with any given region in the bulk.

This might seem like a surprising conclusion, so let us give supporting evidence for our approach. Consider pure AdS_D, dual to a CFT_{D-1} in its ground state. The conformal symmetry of the CFT is valid at any N. This means that, even when the Planck length is finite, AdS quantum gravity has an exact $SO(D-1, 2)$ symmetry. Purely formally, we can realize this symmetry as acting on a set of coordinates (T, X, Z). The smearing functions we have constructed transform covariantly under $SO(D-1, 2)$ [7]—a property which suffices to determine them up to an overall coefficient.[16] This means that at any N, the smearing functions we have defined are the unique way to start with a primary operator in the CFT and build a representation of $SO(D-1, 2)$ which transforms as a scalar field in AdS. Since the construction we have outlined is fixed by the symmetries, the operators (61) are singled out even at finite N.

How can this continuum of operators be compatible with holography? We believe the answer is that only a few of these operators will commute with each other at finite N. At infinite N we managed to construct smeared operators in the CFT which commute with each other even though the smearing functions overlap on the boundary. We discussed this in Sec. II D. But at finite N it is implausible that all the overlapping operators will commute.[17] Let us estimate how many commuting operators we do expect. If we take a local CFT operator and smear it, it will trivially commute with another smeared operator provided the two smearings are "spacelike" to each other: that is, provided the two smearing functions have supports which only involve points on the boundary that are at spacelike separation. In this case the two smeared operators will commute with each other by locality of the boundary theory. The condition for spacelike separation was studied in [6] for AdS_3 and is easily extended to any dimension. In Poincaré coordinates, working on a hypersurface of fixed time, it boils down to

[16] The coefficient can be fixed by matching onto a properly normalized operator in the CFT in the limit that the bulk point approaches the boundary.

[17] By overlapping operators we mean the smearing functions have support at timelike relative separation.

the requirement that the separation between any two bulk operators satisfies $|\Delta X| > 2Z$. Since the necessary separation gets larger as Z increases, the maximum number of commuting operators in a given region is achieved by placing them all at the boundary of the region. For example, inside a bulk region

$$0 < X_i < L, \qquad Z_0 < Z < \infty, \qquad i = 1 \cdots d - 1$$

the maximum number of trivially commuting operators is given when they are evenly spaced along the boundary of the region, at $Z = Z_0$, with a characteristic coordinate spacing of order Z_0. Thus according to this prescription there are at most $\sim(L/Z_0)^{d-1}$ trivially commuting operators one can build in this region by smearing a single local operator in the CFT.[18] This corresponds to one commuting operator per AdS area (in units of the AdS radius of curvature R). This is far too few degrees of freedom to describe a local bulk field.

Turning back to the infinite number of operators described above, we note that—although they do not all commute—their correlation functions nonetheless look local up to $1/N$ corrections that involve mixing with other operators. The infinite set of operators can therefore be used to describe bulk physics which is approximately local, at least as far as correlation functions are concerned, as long as the $1/N$ corrections can be ignored. However note that if one tries to place operators too close together or in a state with large energy, their commutator may get a large contribution from smeared operators corresponding to bulk excitations which are outside the given spacetime volume. We should not associate such operators with independent degrees of freedom within the volume. Presumably there is a finite maximal set of operators that mutually commute up to terms that vanish as $N \to \infty$ and remain inside the given volume. Bekenstein-style arguments [33] (made on the supergravity side) support this idea. It is this set of operators which we argue counts the independent degrees of freedom inside a volume.

We obtain a natural proposal for a basis of these operators by generalizing the above construction of trivially commuting operators. Let us consider all possible degrees of freedom within a given bulk volume, rather than those associated with a particular supergravity field. For concreteness, we will consider AdS_5. We expect of order N^2 independent local operators in the boundary theory. Therefore we should be able to construct a basis of N^2 mutually commuting bulk operators as we did above for the trivially commuting operators. This implies a total of N^2 degrees of freedom per area in AdS units. This matches perfectly with the relation $l^3_{\mathrm{Planck}} = R^3/N^2$ and saturates the holographic bound.

[18] Thus the bulk region can be described by a boundary theory with a lattice spacing $\sim Z_0$. This is clearly closely related to the cutoff procedure introduced in [32].

VII. CONCLUSIONS

In this paper we developed the representation of local operators in the bulk of AdS in terms of nonlocal operators on the complexified boundary. We showed that these nonlocal operators reproduce the correct bulk-to-bulk correlation functions. In particular they reproduce the divergent correlators of an operator inserted at the BTZ singularity. We commented on black holes formed by collapse, and discussed the way in which bulk locality arises in the large-N limit but breaks down at finite N.

Local bulk operators thus provide a powerful tool for understanding the AdS/CFT correspondence. They give new insights into the way in which light cone singularities and spacelike commutativity arise in the bulk. They enable us to probe nontrivial geometries, including regions inside horizons which are naively hidden from the boundary, and they show very explicitly how a bulk singularity can manifest itself in a well-behaved CFT. Our results were all obtained in the infinite N limit. However we argued that in some contexts (two-point functions in pure AdS) our results carry over exactly to any value of N. And based on consideration of the operators at infinite N we were able to give a qualitative picture of the independent bulk degrees of freedom at finite N.

There are a number of directions for future work. We begin with a few further remarks on the nature of the BTZ singularity from the CFT viewpoint. At leading order in a large N expansion, we found that a bulk field probe of the singularity is represented by a non-normalizable operator in the CFT. Note that backreaction/finite N effects play a crucial role in understanding the physics near the singularity, even in the case of BTZ, as discussed in [16,28,34] (and references therein). Therefore we certainly expect large corrections to the smearing function within a Planck length of the singularity. It would be interesting to know whether these corrections render operators at the singularity normalizable, or whether one should simply abandon a bulk spacetime description of the physics in this region. Nevertheless it seems the operators defined by (46) have smooth analytic continuations through complex values of r from region 2_{++} (in the notation of [16]) to the past of the singularity to region 2_{-+} to the future of the singularity, avoiding the Planck scale region near the singularity. This raises the question of whether the CFT also gives a smooth description of regions to the future of the singularity. An important criterion in deciding whether certain combinations of CFT correlators reproduce sensible bulk spacetime physics, is whether the set of amplitudes can be reproduced by a unitary local bulk Lorentzian spacetime effective action. This seems to be true for regions outside the horizon, and regions to the past of the singularity, but it is unlikely this will be true if one also includes operators to the future of the singularity. It would be very interesting to show this explicitly. Moreover the resolution of the black hole information problem via AdS/

CFT suggests [35,36] that nonlocal terms appear in a bulk effective action connecting the region near the singularity with the region outside the horizon. The local operators constructed in the present work are an important first step in trying to reconstruct these new quantum gravity features of the bulk effective action.

In Sec. VI we commented on the way in which the number of commuting degrees of freedom is reduced at finite N. We also showed how bulk locality is recovered in correlation functions in the large-N limit, despite the seemingly low number of degrees of freedom (corresponding to a theory with a cutoff $\Delta X > Z$):, namely, through the presence of a continuum of bulk operators whose commutators are $\mathcal{O}(1/N)$. Constructing a precise analog of smearing functions at finite N and better understanding the analog of bulk spacetime geometry is an important open problem.

For eternal black holes we found that local operators inside the horizon are dual to operators which act on both copies of the CFT. In Sec. V we showed that, at least in some cases, one could represent an operator inside the horizon of a black hole formed by collapse in terms of a single CFT, by using an operator which is analytically continued both in the spatial and temporal coordinates of the CFT. These ideas will be further explored in [31].

This leads to an interesting question, namely, whether there is an algorithm for constructing smearing functions with compact support in a general asymptotically AdS background. The smearing functions we have constructed in this paper can all be thought of as arising from a Wick rotation of the boundary spatial coordinates. This should certainly be a well-defined operation on the analytic correlators that arise from the CFT. However a general bulk geometry will typically not have an interpretation with a real metric after performing such a continuation. One could still try to represent the smearing function as a mode sum, but it is not clear that the smearing function will have compact support on the (complexified) boundary. One way to address this issue would be to attempt to find a procedure, purely within the CFT, for identifying a set of well-behaved smearing functions. The only obvious condition to impose is that in the semiclassical limit the smeared operators should commute at bulk spacelike separation. Is that enough to uniquely determine the smearing functions?

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APPENDIX A: RINDLER MODE SUM

In this appendix we set up the Rindler mode sum for a bulk point inside the horizon. It is convenient to introduce Kruskal coordinates on AdS_3 in which

$$ds^2 = -\frac{4R^2}{(1+uv)^2}dudv + r^2 d\phi^2.$$

These coordinates are defined by

$$u = \left(\frac{r - r_+}{r + r_+}\right)^{1/2} e^{\hat{t}}, \qquad v = -\left(\frac{r - r_+}{r + r_+}\right)^{1/2} e^{-\hat{t}}$$

in the right Rindler wedge and

$$u = \left(\frac{r_+ - r}{r_+ + r}\right)^{1/2} e^{\hat{t}}, \qquad u = \left(\frac{r_+ - r}{r_+ + r}\right)^{1/2} e^{-\hat{t}}$$

in the future Rindler wedge; to cover the left and past wedges just change the signs of both u and v. A complete set of normalizable modes in the right Rindler wedge is given by

$$\phi_R(t, r, \phi) = e^{-i\omega t} e^{ik\phi} r^{-\Delta}\left(1 - \frac{r_+^2}{r^2}\right)^{-i\hat{\omega}/2} F\left(\frac{\Delta}{2} - i\hat{\omega}^+, \frac{\Delta}{2} - i\hat{\omega}^-, \Delta, \frac{r_+^2}{r^2}\right), \tag{A1}$$

where ω, $k \in \mathbb{R}$, $\hat{\omega}^{\pm} = \frac{1}{2}(\hat{\omega} \pm \hat{k})$, $\hat{\omega} = \omega R^2/r_+$, $\hat{k} = kR/r_+$. We can extend this mode to the entire Kruskal diagram by analytically continuing across the Rindler horizons. If we continue through the lower half of the complex u and v planes we get a mode which is positive frequency with respect to Kruskal time, while continuing through the upper half of the complex u and v planes gives a negative frequency Kruskal mode.[19] The analytic continuation is straightforward, with the help of a $z \to 1 - z$ transformation of the hypergeometric function. Define

$$f_{\omega k}(r) = \frac{1}{r^{\Delta}}\left(1 - \frac{r_+^2}{r^2}\right)^{-i\hat{\omega}/2} F\left(\frac{\Delta}{2} - i\hat{\omega}^+, \frac{\Delta}{2} - i\hat{\omega}^-, \Delta, \frac{r_+^2}{r^2}\right),$$

$$g_{\omega k}(r) = \frac{1}{r^{\Delta}}\left(\frac{r_+^2}{r^2} - 1\right)^{-i\hat{\omega}/2} \times \frac{\Gamma(\Delta)\Gamma(i\hat{\omega})}{\Gamma((\Delta/2) + i\hat{\omega}^+)\Gamma((\Delta/2) + i\hat{\omega}^-)} \times F\left(\frac{\Delta}{2} - i\hat{\omega}^+, \frac{\Delta}{2} - i\hat{\omega}^-, 1 - i\hat{\omega}, 1 - \frac{r_+^2}{r^2}\right).$$

Then a complete set of $\left\{\begin{smallmatrix}\text{positive}\\ \text{negative}\end{smallmatrix}\right\}$ frequency Kruskal modes is

[19] Positive and negative frequency in the sense of multiplying annihilation and creation operators in the expansion of the field. This prescription for selecting positive frequency Kruskal modes picks out the AdS-invariant vacuum state.

given by

$$\begin{aligned}\phi_R^{\pm}(t,r,\phi) &= e^{-i\omega t}e^{ik\phi}f_{\omega k}(r),\\ \phi_F^{\pm}(t,r,\phi) &= e^{-i\omega t}e^{ik\phi}(g_{\omega k}(r)+e^{\mp\pi\hat{\omega}}g_{-\omega,k}(r)),\\ \phi_L^{\pm}(t,r,\phi) &= e^{\mp\pi\hat{\omega}}e^{-i\omega t}e^{ik\phi}f_{\omega k}(r)\end{aligned} \tag{A2}$$

in the (right, future, left) Rindler wedges. This means we can express the value of the field in the future wedge in terms of data on the right and left boundaries, via

$$\begin{aligned}\phi_F(t,r,\phi) = &\int d\omega dk \frac{1}{4\pi^2}g_{\omega k}(r)\\ &\times\Big[\int dt'd\phi'(e^{-i\omega(t-t')}e^{ik(\phi-\phi')}\phi_0^{\mathrm{Rindler},R}(t',\phi')\\ &+e^{-i\omega(-t+t')}e^{ik(\phi-\phi')}\phi_0^{\mathrm{Rindler},L}(t',\phi'))\Big].\end{aligned}$$

(Recall that time is oriented oppositely on the two boundaries, so for $t=0$ this expression is in fact symmetric between the right and left boundaries.) Switching the order of integration and performing the ω and k integrals first gives a formal representation of the Rindler smearing function, essentially as the Fourier transform of $g_{\omega k}$. However it is easy to check that $g_{\omega k}$ grows exponentially as $k\to\pm\infty$. So we are not justified in switching the order of integration and the Fourier transform does not exist. One can presumably make sense of the Rindler smearing function in this approach by deforming the contours of integration as in [7]. For points inside the horizon this should reproduce the result (46) we obtained from Poincaré coordinates.

APPENDIX B: NONINTEGER Δ

In this appendix we work out the generalization of the Rindler smearing function (46) appropriate for arbitrary conformal dimension.

We first need to discuss the generalization of the antipodal map. This is easiest to understand in global coordinates, where the embedding coordinates of Sec. III A are given by

$$U = R\cos\tau/\cos\rho, \qquad V = R\sin\tau/\cos\rho,$$
$$X = R\cos\theta\tan\rho, \qquad Y = R\sin\theta\tan\rho$$

for $-\infty<\tau<\infty$, $0\le\rho<\pi/2$, $\theta\sim\theta+2\pi$. The induced metric is

$$ds^2 = \frac{R^2}{\cos^2\rho}(-d\tau^2+d\rho^2+\sin^2\rho d\theta^2).$$

The antipodal map acts by

$$A\colon (\tau,\rho,\theta)\to(\tau-\pi,\rho,\theta+\pi).$$

The global mode expansion is

$$\phi(\tau,\rho,\theta) = \sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty}a_{nl}e^{-i\omega_{nl}\tau}e^{il\theta}\sin^{|l|}\rho\cos^{\Delta}\rho \times P_n^{(|l|,\Delta-1)}(\cos 2\rho)+\text{c.c.},$$

where $\omega_{nl}=2n+|l|+\Delta$ and P_n is a Jacobi polynomial. So fields which are $\{\substack{\text{positive}\\ \text{negative}}\}$ frequency with respect to global time satisfy

$$\phi^{\pm}(x) = e^{\mp i\pi\Delta}\phi^{\pm}(Ax).$$

This means the generalization of (46) to arbitrary conformal dimension is

$$\begin{aligned}\phi = \frac{(\Delta-1)2^{\Delta-2}}{\pi r_+^2}\Big[&\int_{\sigma>0}dxdy\lim_{r'\to\infty}(\sigma/r')^{\Delta-2}\\ &\times\phi_0^{\mathrm{Rindler},R}(\hat{t}+x,\hat{\phi}+iy)\\ &+\int_{\sigma<0}dxdy\lim_{r'\to\infty}(-\sigma/r')^{\Delta-2}(e^{-i\pi\Delta}\phi_{0+}^{\mathrm{Rindler},L}\\ &\times(\hat{t}+x,\hat{\phi}+iy)+e^{i\pi\Delta}\phi_{0-}^{\mathrm{Rindler},L}(\hat{t}+x,\hat{\phi}+iy))\Big],\end{aligned} \tag{B1}$$

where we have decomposed the left boundary field into pieces $\phi_{0\pm}^{\mathrm{Rindler},L}$ that are $\{\substack{\text{positive}\\ \text{negative}}\}$ frequency with respect to global (equivalently, Kruskal) time. These may in turn be expressed in terms of integrals involving $\phi_0^{\mathrm{Rindler},L}$ and $\phi_0^{\mathrm{Rindler},R}$ over all time, as in Appendix 2 of [6].

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