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Full twisted Poincaré symmetry and quantum field theory on Moyal-Weyl spaces

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We explore some general consequences of a proper, *full* enforcement of the "twisted Poincaré" covariance of Chaichian *et al.*, Wess, Koch *et al.*, and Oeckl upon many-particle quantum mechanics and field quantization on a Moyal-Weyl noncommutative space(time). This entails the associated braided tensor product with an involutive braiding (or $\star$-tensor product in the parlance of Aschieri *et al.*) prescription for any coordinate pair of x, y generating two different copies of the space(time); the associated nontrivial commutation relations between them imply that $x - y$ is central and its Poincaré transformation properties remain undeformed. As a consequence, in quantum field theory (QFT) (even with space-time noncommutativity) one can reproduce notions (like spacelike separation, time- and normal-ordering, Wightman or Green's functions, etc.), impose constraints (Wightman axioms), and construct free or interacting theories which essentially coincide with the undeformed ones, since the only observable quantities involve coordinate differences. In other words, one may thus well realize quantum mechanics (QM) and QFT's where the effect of space(time) noncommutativity amounts to a practically unobservable common noncommutative translation of all reference frames.

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I. INTRODUCTION: MOYAL-WEYL SPACES, TWISTED POINCARÉ "GROUP," AND QFT

In the last decade a broad attention has been devoted to the construction of quantum field theories (QFT) on the perhaps simplest examples of noncommutative spaces, the so-called Moyal-Weyl spaces. These are characterized by coordinates $\hat{x}^\mu$ fulfilling the commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{1}$$

where $\theta^{\mu\nu}$ is a constant real antisymmetric matrix. The $\theta^{\mu\nu} = 0$ limit is the undeformed algebra $\mathcal{A}$ generated by commuting coordinates x^μ. For the sake of definiteness we shall suppose (with the exception of Sec. III) $\mu = 0, 1, 2, 3$ and endow the space with the ordinary Minkowski metric $\eta_{\mu\nu}$, to obtain a deformation of the $3 + 1$-dimensional Minkowski space-time. As $\theta^{\mu\nu}$ is not an isotropic tensor, relations (1) are not covariant (i.e. not form invariant) under Lorentz transformations of the reference frame (although they are invariant under translations).

The unital algebra $\hat{\mathcal{A}}$ generated by these $\hat{x}^\mu$ is isomorphic to the one $\mathcal{A}_\theta$ which is obtained by endowing the vector space underlying $\mathcal{A}$ (extended over the formal power series in $\theta^{\mu\nu}$) with a deformed product, the $\star$-product, which can be formally defined by

$$a \star b := (\bar{\mathcal{F}}^{(1)} \triangleright a)(\bar{\mathcal{F}}^{(2)} \triangleright b). \tag{2}$$

For typographical convenience we have denoted by $\bar{\mathcal{F}} \equiv \mathcal{F}^{-1}$ the inverse of the so-called twist $\mathcal{F}$. It (and therefore also the associated isomorphism ϕ: $\hat{\mathcal{A}} \to \mathcal{A}_\theta$) is not uniquely determined, but what follows does not depend on the specific choice of $\bar{\mathcal{F}}$. The simplest is

$$\bar{\mathcal{F}} \equiv \bar{\mathcal{F}}^{(1)} \otimes \bar{\mathcal{F}}^{(2)} := \exp\left(-\frac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu\right). \tag{3}$$

P_μ denote the generators of translations, and $\triangleright$ in general denotes the action of the universal enveloping algebra (UEA) $U\mathcal{P}$ of the Poincaré Lie algebra $\mathcal{P}$ (on $\mathcal{A}$ this amounts to the action of the corresponding algebra of differential operators, e.g. P_μ can be identified with $i\partial_\mu := i\partial/\partial x^\mu$). In the second expression and in (2) we have used a Sweedler notation with suppressed summation index: $\bar{\mathcal{F}}^{(1)} \otimes \bar{\mathcal{F}}^{(2)}$ stands in fact for a (infinite) sum $\sum_I \bar{\mathcal{F}}^{(1)}_I \otimes \bar{\mathcal{F}}^{(2)}_I$. Relation (2) with the specific choice (3) of the twist gives in particular

$$\hat{x}^\mu \hat{x}^\nu \xrightarrow{\phi} x^\mu \star x^\nu = x^\mu x^\nu + i\theta^{\mu\nu}/2.$$

As a result, $x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}$, i.e. again (1), as claimed. One advantage of working with $\mathcal{A}_\theta$ instead of $\hat{\mathcal{A}}$ is that integration over the original commutative space can be used also on the noncommutative one without losing its properties (in particular Stokes's theorem). In addition,

$$\int d^4x a \star b = \int d^4x ab \tag{4}$$

for any regular a, b functions in the vector space underlying $\mathcal{A}$ vanishing sufficiently fast at infinity. The definition (2) and (3) involves a power series in $\theta^{\mu\nu}$ and for the moment should be regarded as formal: it can be applied to a much larger domain if $\bar{\mathcal{F}}$ is rather realized as an integral operator, as we shall explain in (13).

Different (obviously not Lorentz-covariant) approaches to the quantization of field theory on Moyal-Weyl spaces have been proposed (see [1–4] and references therein). New complications appear, like nonunitarity [5], violation of causality [6,7], UV-IR mixing of divergences [8] and subsequent nonrenormalizability, alleged change of statistics, etc. Some of these problems, like nonunitarity [9], or the very occurrence of divergences [10], may be due simply to naive (and unjustified) applications of commutative QFT rules (path-integral methods, Feynman diagrams, etc.) and could disappear adopting a sounder field-operator approach.

In Refs. [11–13] it has been recognized that the commutation relations of $\hat{\mathcal{A}} \sim \mathcal{A}_\theta$ are in fact covariant under a deformed version of the Poincaré group, namely, the triangular noncocommutative Hopf $*$-algebra H obtained from $U\mathcal{P}$ by "twisting" [14] with $\mathcal{F}$ (this result had been in fact anticipated in terms of corepresentations of the dual Hopf algebra in Sec. 4.4.1 of [15]. For a general introduction to the twist, see e.g. [16]). This means that (up to possible isomorphisms) the algebra structure and the counit ε of $U\mathcal{P}$, H (extended over the formal power series in $\theta^{\mu\nu}$) are the same, but the coproduct is changed through the similarity transformation

$$\Delta(g) \equiv g_{(1)} \otimes g_{(2)} \to \hat{\Delta}(g) = \mathcal{F}\Delta(g)\mathcal{F}^{-1} \equiv g_{(\hat{1})} \otimes g_{(\hat{2})}, \quad g \in H = U\mathcal{P} \tag{5}$$

(at the right-hand side's (rhs's) we have again used Sweedler notation with suppressed summation indices), and the antipode S accordingly. A straightforward computation gives

$$\hat{\Delta}(P_\mu) = P_\mu \otimes \mathbf{1} + \mathbf{1} \otimes P_\mu = \Delta(P_\mu),$$
$$\hat{\Delta}(M_\omega) = M_\omega \otimes \mathbf{1} + \mathbf{1} \otimes M_\omega + P[\omega, \theta] \otimes P \neq \Delta(M_\omega),$$

where we have set $M_\omega := \omega^{\mu\nu} M_{\mu\nu}$ and used a row-by-column matrix product on the right. The left identity shows that the Hopf P subalgebra remains undeformed and equivalent to the Abelian translation group $\mathcal{T} \sim \mathbb{R}^4$. Denoting by $\triangleright$, $\hat{\triangleright}$ the (say, left) actions of $U\mathcal{P}$, H, they coincide on first degree polynomials in x^ν, $\hat{x}^\nu$,

$$P_\mu \triangleright x^\rho = i\delta^\rho_\mu = P_\mu \hat{\triangleright} \hat{x}^\rho, \qquad M_\omega \triangleright x^\rho = 2i(x\omega)^\rho, \qquad M_\omega \hat{\triangleright} \hat{x}^\rho = 2i(\hat{x}\omega)^\rho, \tag{6}$$

and more generally on irreducible representations (irreps); as noted in [11], this yields the same classification of elementary particles as unitary irreps of $\mathcal{P}$. But $\triangleright$, $\hat{\triangleright}$ differ on products of coordinates, and more generally on tensor products of representations, as $\triangleright$ is extended by the rule $g\triangleright(ab) = (g_{(1)}\triangleright a)(g_{(2)}\triangleright b)$ involving $\Delta(g)$ (the rule reduces to the usual Leibniz rule for $g = P_\mu, M_{\mu\nu}$), whereas $\hat{\triangleright}$ is extended on products of elements in both $\hat{\mathcal{A}}$, $\mathcal{A}_\theta$ by the rule

$$g\hat{\triangleright}(\hat{a}\,\hat{b}) = (g_{(\hat{1})}\hat{\triangleright}\,\hat{a})(g_{(\hat{2})}\hat{\triangleright}\,\hat{b}) \Leftrightarrow g\hat{\triangleright}(a \star b) = (g_{(\hat{1})}\hat{\triangleright} a) \star (g_{(\hat{2})}\hat{\triangleright} b), \tag{7}$$

which respects the commutation relations (1), making $\hat{\mathcal{A}}$, $\mathcal{A}_\theta$ isomorphic H-module algebras; this deforms, in particular, the Leibniz rule of $M_{\mu\nu}$ (but not of P_μ).

How to implement this twisted Poincaré covariance in QFT is the subject of debate and different proposals [17–26], two main issues being whether one should: (a) take the $\star$-product of fields at different space-time points; (b) deform the canonical commutation relations (CCR) of creation and annihilation operators a, $a^\dagger$ for free fields.

The aim of this work is to point out that a proper enforcement of twisted Poincaré covariance answers affirmatively to (a) and brings a radical simplification to the framework, in that all coordinate differences become $\star$-central, i.e. central w.r.t. the $\star$-product (Sec. II). We first explore (Sec. III) some consequences of the latter fact in n-particle quantum mechanics (QM): we find that twisted Galilei covariance is compatible with Bose or Fermi statistics and that the dynamics of an isolated system of n particles is the same as its counterpart on commutative space. As for QFT, which we treat in field-operator approach, we sketch the general consequences of (slightly adapted) Wightman axioms in Sec. IV, show in Sec. V that the latter can be satisfied by free (for simplicity scalar) fields if we also suitably deform the CCR of the a, $a^\dagger$'s so that the $\star$-commutator of the fields is equal to the undeformed counterpart, show in Sec. VI that then the time-ordered perturbative computation of Green functions of a scalar $\varphi^{\star n}$ interacting theory gives the same results as the undeformed theory. In other words, we end up in this way with twisted Poincaré covariant QFT's which are physically equivalent to their counterparts on commutative Minkowski space, with the obvious consequence that the above-mentioned complications will disappear. In Sec. VII we draw some conclusions and briefly comment on the alternatives implying violation of the cluster property by the Wightman functions.

II. THE ACTION OF THE TWISTED POINCARÉ GROUP ON SEVERAL SPACE-TIME VARIABLES

Dealing with n-point (Green's, Wightman's, etc.) functions in QFT requires n sets of noncommutative Minkowski space-time coordinates $\hat{x}^\mu_i$, $i = 1, \ldots, n$, of type (1). Similarly, dealing with n-particle QM requires n sets of noncommutative Euclidean space coordinates $\hat{x}^\mu_i$, (one for each particle) of type (1).

Our starting, basic observation is that to consistently adopt the viewpoint of twisted Poincaré covariance *one should require that also the larger algebra* $\hat{\mathcal{A}}^n$ *generated by them is a H-module algebra*, meaning, in particular, that within the latter (7) still holds. This is also the philosophy adopted in Ref. [27]. To this end one *cannot* adopt as $\hat{\mathcal{A}}^n$

the tensor product algebra of n copies of $\hat{\mathcal{A}}$, or equivalently assume trivial commutation relations

$$[\hat{x}_i^\mu, \hat{x}_j^\nu] = 0 \qquad i \neq j,$$

as done e.g. in [20,28], because the latter are incompatible with (7) by the noncocommutativity of $\hat{\Delta}$ (this can be checked e.g. by letting the Lorentz generators $M^{\rho\sigma}$ act on both sides). In fact it is a basic property of quasitriangular Hopf algebra theory (see e.g. [29]) that one has to adopt as $\hat{\mathcal{A}}^n$ rather the deformation of the tensor product algebra, usually called *braided tensor product algebra*, dictated by the quasitriangular structure $\mathcal{R}$ of H. Given two left H-module algebras $\hat{M}$, $\hat{M}'$ the braided tensor product algebra $\hat{M} \underline{\otimes} \hat{M}'$ is still $\hat{M} \otimes \hat{M}'$ as a vector space, but is characterized by the product

$$(\hat{m} \underline{\otimes} \hat{m}') \cdot (\hat{n} \underline{\otimes} \hat{n}') = \hat{m}(\mathcal{R}^{(2)} \hat{\triangleright} \hat{n}) \underline{\otimes} (\mathcal{R}^{(1)} \hat{\triangleright} \hat{m}')\hat{n}', \qquad (8)$$

where we have again used a Sweedler notation with suppressed summation index: $\mathcal{R} \equiv \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$ stands in fact for a (infinite) sum $\sum_I \mathcal{R}_I^{(1)} \otimes \mathcal{R}_I^{(2)}$. In the present case $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1} = (\mathcal{F})^{-2}$ is even triangular, i.e. $\mathcal{R}\mathcal{R}_{21} = \mathbf{1} \otimes \mathbf{1}$, implying that these rules are symmetric w.r.t. to the exchange of $\hat{M}$, $\hat{M}'$, or equivalently the braiding coincides with the ordinary flip up to a similarity transformation. If $\hat{M}$, $\hat{M}'$ are H-module algebras, deformations of two $U\mathcal{P}$-module algebras M, M', so that the isomorphisms $\hat{M} \sim M_\theta$, $\hat{M}' \sim M'_\theta$ hold, the braided tensor product (8) is isomorphic to the $\star$-tensor product $\otimes_\star$ of [27], which is defined by setting for any $m \in M_\theta$, $m' \in M'_\theta$

$$m \otimes_\star m' = (\bar{\mathcal{F}}^{(1)} \triangleright m) \otimes (\bar{\mathcal{F}}^{(2)} \triangleright m'). \qquad (9)$$

That this is the "right" deformation of the tensor product follows also from the observation that *this is nothing but the extension of the $\star$-product law* (2) *to the whole tensor product algebra $M \otimes M'$*, in the sense

$$m \otimes_\star m' = (m \otimes \mathbf{1}) \star (\mathbf{1} \otimes m'). \qquad (10)$$

If $\hat{M}$, $\hat{M}'$ are unital (8) reduces to the ordinary tensor algebra rule if either $\hat{m}' = \mathbf{1}$ or $\hat{n} = \mathbf{1}$, as $\varepsilon(\mathcal{R}^{(1)})\mathcal{R}^{(2)} = \varepsilon(\mathcal{R}^{(2)})\mathcal{R}^{(1)} = \mathbf{1}$. As for ordinary tensor product algebras, because of the trivial algebra isomorphisms $\mathbf{1} \underline{\otimes} \hat{M}' \sim \hat{M}'$, $\hat{M} \underline{\otimes} \mathbf{1} \sim \hat{M}$, one can simplify the notation by dropping the units, i.e. denote $\hat{m} \underline{\otimes} \mathbf{1}$ and $\mathbf{1} \underline{\otimes} \hat{m}'$, respectively by $\hat{m}$, $\hat{m}'$, whereby the only novelty of (8) remains concentrated in the nontrivial "cross" commutation relation

$$\hat{m}'\hat{n} = (\mathcal{R}^{(2)} \hat{\triangleright} \hat{n})(\mathcal{R}^{(1)} \hat{\triangleright} \hat{m}').$$

Similarly, we can simplify the notation denoting the sides of (10) as $m \star m'$ and replacing the previous relation by $m' \star n = (\mathcal{R}^{(2)} \hat{\triangleright} n) \star (\mathcal{R}^{(1)} \hat{\triangleright} m')$.

Choosing as $\hat{M}$, $\hat{M}'$ two copies of the $*$-algebra of functions $\hat{\mathcal{A}}$ on the Moyal-Weyl noncommutative space, calling $\hat{x}$, $\hat{y}$ the respective sets of coordinates, and noting that the action of the translation generators on the coordinates is given by

$$P_\mu \hat{\triangleright} \hat{x}^\nu = P_\mu \hat{\triangleright} \hat{y}^\nu = i\delta_\mu^\nu,$$

we find

$$\hat{x}^\mu \hat{y}^\nu = (\mathcal{R}^{(2)} \hat{\triangleright} \hat{y}^\nu)(\mathcal{R}^{(1)} \hat{\triangleright} \hat{x}^\mu) = \hat{y}^\nu \hat{x}^\mu + i\theta^{\mu\nu}.$$

These are also automatically compatible with the $*$-structure (a straightforward check, beside a consequence of $\mathcal{R}^{*\otimes *} = \mathcal{R}_{21} = \mathcal{R}^{-1}$), and with setting $\hat{x} = \hat{y}$. More generally, applying the above rule iteratively, the braided tensor product of n copies of $\hat{\mathcal{A}}$ and the $\star$-tensor product of n copies of $\mathcal{A}_\theta$ will be isomorphic H-module $*$-algebras $\hat{\mathcal{A}}^n$, $\mathcal{A}_\theta^n$, respectively, generated by real variables $\hat{x}_i^\mu$ and x_i^μ, $i = 1, 2, \ldots, n$, fulfilling the commutation relations

$$[\hat{x}_i^\mu, \hat{x}_j^\nu] = i\theta^{\mu\nu} \Leftrightarrow [x_i^\mu \stackrel{\star}{,} x_j^\nu] = i\theta^{\mu\nu}. \qquad (11)$$

This formula summarizes all the commutation relations defining $\hat{\mathcal{A}}^n \sim \mathcal{A}_\theta^n$: for $i = j$ these are the defining commutation relations of the ith copy, for $i \neq j$ these are consequences of the braided tensor (or $\star$-tensor) product between the ith and the jth copy. Summing up, the algebra $\mathcal{A}_\theta^n$ is obtained by endowing the vector space underlying the n-fold tensor product $\mathcal{A}^n$ of $\mathcal{A}$ with a new product, the $\star$-product, related to the product in $\mathcal{A}^n$ by formula (2) for any $a, b \in \mathcal{A}^n$. This encodes both the usual $\star$-product within each copy of $\mathcal{A}$, and the $\star$-tensor product of [27,30]. More explicitly, on analytic functions a, b (2) reads

$$a(x_i) \star b(x_j) = \exp\left(\frac{i}{2} \partial_{x_i} \theta \partial_{x_j}\right) a(x_i) b(x_j), \qquad (12)$$

and must be followed by the identification $x_i = x_j$ *after* the action of the bi-pseudodifferential operator $\exp[\frac{i}{2} \partial_{x_i} \theta \partial_{x_j}]$ if $i = j$.

Strictly speaking, the definitions (2) and (3) or (12) make sense if we choose a, b in a suitable subspace $\mathcal{A}' \subset \mathcal{A}$ ensuring that the involved power series in $\theta^{\mu\nu}$ is termwise well defined and converges. One such subspace can be looked for within the space of (analytic) functions that are the Fourier transforms $\check{g}$ of functions g with compact support. The determination of the largest possible $\mathcal{A}'$ is a delicate issue, about which little is known (see [31] and references therein). Anyway for field-theoretic purposes it would not be enough to work with $\mathcal{A}'$, and it is much better to *define* the $\star$-product as the integral with a nonlocal kernel

$$a(x_i) \star b(x_j) = \int d^4h \int d^4k e^{i(h \cdot x_i + k \cdot x_j - (h\theta k/2))} \check{a}(h) \check{b}(k), \qquad (13)$$

where $\check{}$ denotes the anti-Fourier transform. This is well defined if $a, b \in L^1(\mathbb{R}^4) \cap \widehat{L^1(\mathbb{R}^4)}$, can be defined even if a, b are distributions, and is designed so as to have the series (2) as a formal power expansion; see [31] for the conditions under which the latter is in fact an asymptotic expansion. More generally, one should adopt as proper definition of the action of $\mathcal{F}$, $\bar{\mathcal{F}}$ and of derived operators like $(\Delta \otimes \mathrm{id})\mathcal{F}$ the corresponding nonlocal integral operators. They also fulfill the cocycle condition $\mathcal{F}_{12}(\Delta \otimes \mathrm{id})\mathcal{F} = \mathcal{F}_{23}(\mathrm{id} \otimes \Delta)\mathcal{F}$, ensuring the associativity of the $\star$-product.

We now define an alternative set of real generators of $\mathcal{A}_\theta^n$ (or, correspondingly, of $\hat{\mathcal{A}}^n$):

$$\xi_i^\mu := x_{i+1}^\mu - x_i^\mu, \qquad i = 1, \ldots, n-1, \qquad X^\mu := \sum_{i=1}^n a_i x_i^\mu, \tag{14}$$

where a_i are real numbers such that $\sum_i a_i = 1$ (in particular one could choose $X^\mu = x_j^\mu$, for some special j). It is immediate to verify that:

(1) All ξ_i^μ are invariant under translations, (whereas X^μ is not):

$$P_\mu \hat{\triangleright} \xi_i^\nu = 0, \qquad P_\mu \hat{\triangleright} X^\nu = i\delta_\mu^\nu. \tag{15}$$

(2) X^μ generate a copy $\mathcal{A}_{\theta,X}$ of Moyal-Weyl noncommutative space, whereas the $\star$-product with ξ_i^μ (or any function thereof) reduces to the ordinary product

$$\xi_i^\mu \star b = \xi_i^\mu b = b \star \xi_i^\mu, \qquad b \in \mathcal{A}_\theta^n, \tag{16}$$

implying that the ξ_i^μ are $\star$-central in $\mathcal{A}_\theta^n$ (i.e. $\star$-commute with everything),

$$[\xi_i^\nu \overset{\star}{,} \mathcal{A}_\theta^n] = 0. \tag{17}$$

Thus the central $*$-subalgebra $\mathcal{A}_{\theta,\xi}^{n-1}$ generated by the ξ_i^μ reduces to the ordinary tensor product algebra of $n-1$ copies of the undeformed $\mathcal{A}$ [because of the trivial action (15) of the P_μ contained in the twist $\mathcal{F} = \exp(\frac{i}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu)$ and in $\mathcal{R} = \mathcal{F}^{-2}$ on the tensor factors], whereas $\mathcal{A}_\theta^n$ reduces to the tensor product algebra $\mathcal{A}_\theta^n = \mathcal{A}_{\theta,\xi}^{n-1} \otimes \mathcal{A}_{\theta,X}$. Moreover, the ξ_i^μ have the same spectral decomposition on the whole $\mathbb{R}$ as classical variables ξ^μ; in particular, 0 is in their spectrum.

(3) $\mathcal{A}_{\theta,\xi}^{n-1}$, $\mathcal{A}_{\theta,X}$ are actually H-module subalgebras, and

$$g \hat{\triangleright}(a \star b) = (g_{(1)} \triangleright a) \star (g_{(2)} \hat{\triangleright} b), \qquad a \in \mathcal{A}_{\theta,\xi}^{n-1}, \quad b \in \mathcal{A}_\theta^n, \quad g \in H, \tag{18}$$

implying in particular $g \hat{\triangleright} a = g \triangleright a$, i.e. *on* $\mathcal{A}_{\theta,\xi}^{n-1}$ *the H-action is undeformed*. In fact the Leibniz rule reduces to the undeformed one whenever a twist leg acts on a, again because of the trivial action $(15)_1$ of the P_μ's contained in $\bar{\mathcal{F}}$. The previous relation holds also without the two $\star$-products, by (16).

Summing up, any coordinate difference like ξ_i^μ can be treated as a classical, commutative variable. Any x_i^μ is a combination of $n-1$ $\star$-commutative variables ξ_i^μ and 1 $\star$-noncommutative one X^μ; or equivalently can be obtained from the zero 4-vector and $n-1$ $\star$-commutative 4-vectors by the global "noncommutative translation" X, e.g. if $X := x_1$ then

$$x_i = \sum_{j=1}^{i-1} \xi_j + X.$$

Of course, all the previous statements [with the exception of (16)] can be formulated in the isomorphic setting removing all $\star$'s, putting a $\hat{}$ over any coordinate and replacing $\mathcal{A}_\theta$, $\triangleright_\star$, $\mathcal{A}_\theta^n$, $\mathcal{A}_{\theta,\xi}^{n-1}$, $\mathcal{A}_{\theta,X}$ with the isomorphic objects $\hat{\mathcal{A}}$, $\hat{\triangleright}$, $\hat{\mathcal{A}}^n$, $\hat{\mathcal{A}}_\xi^{n-1}$, $\hat{\mathcal{A}}_X$. The result for $\hat{X}$ is like the "quantum shift operator" of [22].

Remark 1.— One immediate consequence is that on any irreducible representation $\star$-multiplication by a space-time coordinate difference $x - y$ equals multiplication by $x - y$, which is either a spacelike, a null, or a timelike 4-vector, in the usual sense.

Remark 2.— Relation (18) holds also for an infinitesimal general coordinate transformation, i.e. if g is an element of the (deformed) UEA $U\Xi_\star$ [27,30] of the Lie algebra of general vector fields on the Moyal-Weyl NC space.

We recall that the differential calculus over $\mathbb{R}^n$ remains unchanged under deformation of this space into a Moyal-Weyl NC space. This is true also if we consider the differential calculus on the larger algebra $\mathcal{A}_\theta^n$ (or the isomorphic $\hat{\mathcal{A}}^n$), and follows again from (2) and (3) and the fact that P_μ have trivial action on the derivatives. Explicitly,

$$\partial_{x_i^\mu} \star x_j^\nu = \delta_\mu^\nu \delta_j^i + x_j^\nu \star \partial_{x_i^\mu}, \qquad [\partial_{x_i^\mu} \overset{\star}{,} \partial_{x_j^\nu}] = 0 \tag{19}$$

with self-explaining notation. Since the presence of the $\star$ product has no effect on the action of the derivatives on $\mathcal{A}_\theta^n$, in the sequel we shall drop it.

Given two sets x, y of coordinates, integrating over some x^μ both sides of the identity

$$g(y) \star f(x) = (\mathcal{R}^{(2)} \hat{\triangleright} f(x)) \star (\mathcal{R}^{(1)} \hat{\triangleright} g(y)) = \exp(-i\theta^{\mu\nu}\partial_{x^\mu}\partial_{y^\nu}) f(x) \star g(y),$$

we see that any integration $\int dx^\mu$ commutes with $g(y)\star$ if f rapidly decreases at infinity; in fact, if we define the $\star$-product by the integral (13) we realize that

$$\int dx^\mu g(y) \star f(x) = g(y) \star \int dx^\mu f(x) \tag{20}$$

is true also for $f, g \in L^1(\mathbb{R}^4) \cap \widehat{L^1(\mathbb{R}^4)}$ or even some distributions, as on commutative space [of course, since the rhs (20) is independent of x^μ, terms with $\theta^{\mu\nu}\partial_{y^\nu}$ will be ineffective and disappear, as if it were $\theta^{\mu\nu} = 0$ for all ν]. Therefore, for our purposes we can consider integration over any set of coordinates as an operation commuting with $\star$-products.

III. GENERAL CONSEQUENCES FOR MANY-PARTICLE QM

In configuration space the Hamiltonian of an isolated system of n nonrelativistic (for simplicity spinless) particles

$$\mathsf{H} = \mathsf{H}_0 + \sum_{i<j} V_{ij}(|\mathbf{x}_i - \mathbf{x}_j|), \quad \mathsf{H}_0 := -\sum_{i=1}^{n} \frac{\hbar^2}{2m_i}\nabla^2_{\mathbf{x}_i} \tag{21}$$

involves only derivatives and relative coordinates ξ. Denoting as $\mathbf{X}$ the coordinates of the center of mass, as M the total mass of the system, the kinetic part H_0 can be written as the sum of $-\hbar^2\nabla^2_{\mathbf{X}}/2M$ and a second order differential operator in the ξ-derivatives only. As a consequence, the dynamics of the center of mass is free. This means that an orthogonal basis of eigenfunctions of H is $\{\exp(i\mathbf{k}\cdot\mathbf{X})\psi_j(\xi)\}$, where ψ_j are eigenfunctions of the rest Hamiltonian $\mathsf{H}_\xi := \mathsf{H} + \hbar^2\nabla^2_{\mathbf{X}}/2M$, depending on the ξ and ξ derivatives only.

Going to the noncommutative Euclidean space (the time remaining commutative) brings no change: the deformed Hamiltonian $\mathsf{H}_\star \equiv \mathsf{H}\star$ can be still split into a free part $-\hbar^2\nabla^2_{\mathbf{X}}/2M\star$ for the center-of-mass degrees of freedom and an interacting part $\mathsf{H}_\xi\star$ depending only on the relative coordinates, and both parts act on the vector space underlying both $\mathcal{A}^n_\theta$ and $\mathcal{A}^n$ (and therefore also on the subspace consisting of square-integrable wave functions) exactly as their undeformed counterparts, implying that $\{\exp(i\mathbf{k}\cdot\mathbf{X})\psi_J(\xi)\}$ is also an orthonormal basis of eigenfunctions of $\mathsf{H}_\star$ with the same eigenvalues. As a result, the deformed dynamics coincides with the undeformed one.

Assume now that the particles are identical. If the space is commutative, a wave function $\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ completely (anti)symmetric under particles' permutations can be decomposed as $\Psi = \sum_{IJ}\Psi_{IJ}\phi_I\chi_J$ in any tensor product basis $\{\phi_I(\mathbf{X})\chi_J(\xi)\}$, where $\chi_J(\xi)$ are completely (anti)-symmetric [$\phi_I(\mathbf{X})$ are automatically completely symmetric]. These symmetries are preserved by the dynamical evolution, since this is ruled by the completely symmetric evolution operator $U(t-t_0) = \exp[-\frac{i}{\hbar}\mathsf{H}'(t-t_0)]$, where $\mathsf{H}' = \mathsf{H} + \sum_i V_e(\mathbf{x}_i)$ is the total Hamiltonian with $V_{ij} \equiv V$ and V_e the external potential (if the system is not isolated). For the same reason this is true both in the Schrödinger and in the Heisenberg picture, which are related by the unitary transformation $U(t-t_0)$, and also in the interaction picture, which is related to the Schrödinger by the completely symmetric evolution operator $U_0(t-t_0) = \exp[-\frac{i}{\hbar}\mathsf{H}_0(t-t_0)]$. All the corresponding deformed statements remain true, as $\mathsf{H}'_\star \equiv \mathsf{H}'_\star$ and $\mathsf{H}_{0\star} \equiv \mathsf{H}_0\star$ are also completely symmetric.

The action of the Galilei Lie algebra $\mathcal{G}$,[1] and therefore also of its universal enveloping algebra $U\mathcal{G}$, maps $\mathcal{A}_X \to \mathcal{A}_X$, $\mathcal{A}^{n-1}_\xi \to \mathcal{A}^{n-1}_\xi$ preserving these complete (anti)-symmetries, hence amounts to a change of the coefficients Ψ_{IJ}. Interpreting Ψ, $\phi_I(\mathbf{X})$, $\chi_J(\xi)$ as elements, respectively, of $\mathcal{A}^n_\theta$, $\mathcal{A}_{\theta,X}$, $A^{n-1}_{\theta,\xi}$, the same will be true of the action of the twisted Galilei UEA H, as the latter maps $\mathcal{A}_{\theta,X} \to \mathcal{A}_{\theta,X}$, $A^{n-1}_{\theta,\xi} \to A^{n-1}_{\theta,\xi}$, by (18). Therefore there is no incompatibility between the standard complete (anti)-symmetry conditions on a wave function $\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ and the action of H. Consequently, the standard Bose, Fermi (and similarly anyon, in 2 space dimensions) statistics are compatible with twisted Galilei symmetry (in first quantization). This agrees with the general (and physically reassuring) results of Ref. [32], where it was shown (by a unitary equivalence in a n particle, abstract Hilbert space formalism) that covariance under a noncocommutative Hopf algebra obtained by twisting from a cocommutative one is compatible with usual statistics.

IV. GENERAL CONSEQUENCES FOR QFT

In the field-operator approach quantization of fields on Minkowski space obeys a set of general conditions, the Wightman axioms [33], which (as done e.g. in Ref. [34]) can be divided into a subset (in the sequel labeled by **QM**) encoding the quantum mechanical interpretation of the theory, its symmetry under space-time translations and stability, and a subset (in the sequel labeled by **R**) encoding the relativistic properties (full Lorentz covariance and locality). We now try to translate this into a field quantization procedure on a Moyal-Weyl noncommutative space

[1]We recall that $\mathcal{G}$ is generated by H_0 (kinetic term in the Hamiltonian: generates time translations of a free system), m (mass: is central), P^a (momentum components: generate space translations), L^a (angular momentum components: generate rotations), K^a (generate boosts), with $a = 1, 2, 3$, where the only nontrivial commutation relations are

$$\begin{gathered} [K^a, P^b] = im\hbar\delta^{ab}, \qquad [K^a, H_0] = i\hbar P^a, \\ [L^a, L^b] = i\epsilon^{abc}\hbar L^c, \qquad [L^a, P^b] = i\epsilon^{abc}\hbar P^c, \\ [L^a, K^b] = i\epsilon^{abc}\hbar K^c. \end{gathered} \tag{22}$$

The generators are realized as the differential operators $H_0 = -\hbar\nabla^2/2m$, $P^a = -i\hbar\partial^a$, $L^a = -i\hbar\epsilon^{abc}x^b\partial^c$, $K^a = mx^a + i\hbar t\partial^a$ in the configuration space of each single particle. Hence the observable $K^a + tP^a$ gives the mass times the space coordinate x^a of the particle. The coproducts are defined by the fact that these generators are primitive. The coproducts of m, H_0, P^a, L^a, respectively, give the addition laws for the total mass, the total kinetic energy, the total momentum, and the total angular momentum of the system, whereas the coproduct of $K^a + tP^a$ gives the total mass times the space coordinate X^a of the center of mass of the system.

keeping the QM conditions, "fully" twisting Poincaré covariance and being ready to weaken locality if necessary.

QM1 (Hilbert space structure).—The states are described by vectors of a (separable) Hilbert space $\mathcal{H}$.

QM2 (Energy-momentum spectral condition).—The group of space-time translations $\mathcal{T} \sim \mathbb{R}^4$ is a symmetry of the theory and is represented on $\mathcal{H}$ by strongly continuous unitary operators $U(a)$, $a \in \mathbb{R}^4$: the fields transform according to (30) with unit A, $U(A)$, $\Lambda(A)$. The spectrum of the generators P_μ is contained in the closed *forward cone* $\bar{V}_+ = \{p_\mu: p^2 \geq 0, p_0 \geq 0\}$. There is a *vacuum state* Ψ_0, with the property of being the unique Poincaré invariant state (*uniqueness of the vacuum*).

QM3 (Field operators).—The theory is formulated in terms of fields (in the Heisenberg representation) $\varphi^\alpha(x)$, $\alpha = 1, \ldots, N$, that are operator (on $\mathcal{H}$) valued tempered distributions on Minkowski space, with Ψ_0 a *cyclic* vector for the fields, i.e. by applying polynomials of the (smeared) fields to Ψ_0 one gets a set $\mathcal{D}_0$ dense in $\mathcal{H}$.

By taking vacuum expectation values (v.e.v.) of $\star$-products of fields one can introduce different kinds of n-point functions, that will be (mere) distributions: Wightman functions

$$\mathcal{W}^{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n) = (\Psi_0, \varphi^{\alpha_1}(x_1) \star \ldots \star \varphi^{\alpha_n}(x_n)\Psi_0), \tag{23}$$

where $\alpha_1, \ldots, \alpha_n$ enumerate possible different field species and/or $SL(2,\mathbb{C})$-tensor (spinor, vector,...) components, or (their linear combinations) Green's functions

$$G^{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n) = (\Psi_0, T[\varphi^{\alpha_1}(x_1) \star \ldots \star \varphi^{\alpha_n}(x_n)]\Psi_0), \tag{24}$$

or retarded functions, etc. In the second definition there appears the time ordering T, but there is in fact no ambiguity in defining T as on commutative Minkowski space,[2]

$$\begin{aligned} T[\varphi^{\alpha_1}(x_1) \star \varphi^{\alpha_2}(x_2) \star \ldots \star \varphi^{\alpha_n}(x_n)] &= \varphi^{\alpha_1}(x_1) \star \varphi^{\alpha_2}(x_2) \star \ldots \star \varphi^{\alpha_n}(x_n) \\ &\quad \times \vartheta(x_1^0 - x_2^0) \ldots \vartheta(x_{n-1}^0 - x_n^0) \\ &\quad + \varphi^{\alpha_2}(x_2) \star \varphi^{\alpha_1}(x_1) \star \varphi^{\alpha_3}(x_3) \ldots \\ &\quad \times \varphi^{\alpha_n}(x_n)\vartheta(x_2^0 - x_1^0) \ldots \times \vartheta(x_{n-1}^0 - x_n^0) + \ldots, \end{aligned} \tag{25}$$

as this definition involves multiplication by the $\star$-central $\vartheta(x_i^0 - x_j^0)$ (ϑ denotes the Heaviside function). [The $\star$'s preceding all ϑ can be and have been dropped, by (16).]

Arguing as in [33] for ordinary QFT, exactly the same properties follow from QM1-3 (alone). Applying a pure translation, from QM2 we find that *Wightman and Green's functions are translation invariant* and therefore may depend only on the commutative space-time variables ξ_i^μ:

$$\mathcal{W}^{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n) = W^{\alpha_1,\ldots,\alpha_n}(\xi_1,\ldots,\xi_{n-1}), \tag{26}$$

$$\mathcal{G}^{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n) = G^{\alpha_1,\ldots,\alpha_n}(\xi_1,\ldots,\xi_{n-1}). \tag{27}$$

Moreover, from QM3, QM2, QM1 it, respectively, follows

W1.—$\mathcal{W}^{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n)$ are tempered distributions depending only on the ξ_i.

W2 (Spectral condition).—The support of the Fourier transform $\tilde{W}$ of W is contained in the product of forward cones, i.e.

$$\tilde{W}^{\alpha_1,\ldots,\alpha_n}(q_1,\ldots,q_{n-1}) = 0, \qquad \text{if } \exists j: q_j \notin \bar{V}_+. \tag{28}$$

W3 (Hermiticity and Positivity).—The transformation properties of Wightman functions under complex conjugation follow from

$$\begin{aligned} &\overline{(\Psi_0, \varphi^{\alpha_1}(x_1) \star \ldots \star \varphi^{\alpha_n}(x_n)\Psi_0)} \\ &\quad = (\Psi_0, \varphi^{\alpha_n \dagger}(x_1) \star \ldots \star \varphi^{\alpha_1 \dagger}(x_n)\Psi_0). \end{aligned}$$

In particular, if all fields are Hermitian scalar then $\overline{\mathcal{W}(x_1,\ldots,x_n)} = \mathcal{W}(x_n,\ldots,x_1)$. For any terminating sequence $\underline{f} = (f_0, f_1, \ldots f_N)$, $f_j \in \mathcal{S}(\mathbb{R}^4)^j$ one has[3]

$$\sum_{j,k} \int dx dy \bar{f}_j(x_j,\ldots,x_1) f_k(y_1,\ldots,y_k) \times \mathcal{W}(x_1,\ldots,x_j; y_1,\ldots,y_k)) \geq 0. \tag{29}$$

We now recall the ordinary relativistic conditions on QFT:

R1 (Lorentz Covariance).—The universal covering group $SL(2,\mathbb{C})$ of the restricted Lorentz group is a symmetry of the theory and is represented on $\mathcal{H}$ by (strongly continuous) unitary operators $U(A)$. The fields transform covariantly under the inhomogeneous $SL(2,\mathbb{C})$ (i.e. generalized Poincaré) transformations $U(a,A) = U(a)U(A)$:

$$U(a,A)\varphi^\alpha(x)U(a,A)^{-1} = S^\alpha_\beta(A^{-1})\varphi^\beta(\Lambda(A)x + a), \tag{30}$$

with S a finite dimensional representation of $SL(2,\mathbb{C})$ and $\Lambda(A)$ the Lorentz transformation associated to $A \in SL(2,\mathbb{C})$.

R2 (Microcausality or locality).—The fields either commute or anticommute at spacelike separated points

$$[\varphi^\alpha(x), \varphi^\beta(y)]_\mp = 0, \quad \text{for } (x-y)^2 < 0. \tag{31}$$

As a consequence of QM2, R1 in QFT on commutative Minkowski space one finds

[2]In the standard approach [6,35,36] this was found to be safe and unambiguous only in the case of space-time commutativity ($\theta^{0i} = 0$), which gives commuting time variables x_i^0, so that time ordering commutes with the $\star$-product.

[3]This is the transcription of positivity of the norm of any state of the form

$$\Psi_{\underline{f}} = f_0\Psi_0 + \varphi(f_1)\Psi_0 + \varphi(f_2^{(1)})\varphi(f_2^{(2)})\Psi_0 + \ldots,$$

where $\underline{f} = (f_0, f_1, \ldots f_N)$, $f_j = \prod_{k=1}^j f_j^{(k)}(x_k)$.

W4 (Lorentz Covariance of Wightman functions).—

$$\mathcal{W}^{\alpha_1\ldots\alpha_n}(\Lambda(A)x_1,\ldots,\Lambda(A)x_n) = S^{\alpha_1}_{\beta_1}(A)\ldots S^{\alpha_n}_{\beta_n}(A)\mathcal{W}^{\beta_1\ldots\beta_n}(x_1,\ldots,x_n). \tag{32}$$

Wightman functions of scalar fields are Lorentz invariant. (Similarly for Green functions).

In order to translate R1 into a corresponding condition $R1_\star$ in the twisted Hopf algebra setting we could go either to the infinitesimal formulation (i.e. first to $\mathcal{P}$, and then deform to H), or to the dual functions-on-the-group Hopf algebra. We do not attempt this here, because it would be rather technical (especially translating the strong continuity requirement), and moreover some subtlety might be hidden in the interplay of active (or system) and passive (or coordinates) twisted Poincaré transformations appearing at the two sides of (30). We content ourselves with requiring the deformed analog of W4, which should follow from $R1_\star$ however this will look like, namely, that Wightman (and Green) functions transform under a twisted version of (32), in particular, are invariant if all involved fields are scalar. On the other hand, as these functions should be built only in terms of the ξ_i^μ and of ordinary $SL(2,\mathbb{C})$ tensors, like $\partial_{x_i^\mu}$, the isotropic tensor δ^μ_ν, spinors, γ matrices, etc., which are all annihilated by the action of P_μ, the action of the twist "legs" $\mathcal{F}^{(1)}$, $\mathcal{F}^{(2)}$ should be trivial and the transformation properties under the Lorentz generators should remain undeformed: so *these functions should admit exactly the same decomposition in Lorentz tensors as in the undeformed case (in particular should be invariant if all fields are scalar fields)*. Therefore, deferring the formulation of $R1_\star$ to possible future works, here *we shall require W4 also in the deformed case* as a temporary substitute of R1.

As for R2, it is natural to ask whether in the deformed theory one can adopt the twisted version

$R2_\star$ *(Microcausality or locality).—*The fields either $\star$-commute or $\star$-anticommute at spacelike separated points[4]

$$[\varphi^\alpha(x) \overset{\star}{,} \varphi^\beta(y)]_\mp = 0, \quad \text{for } (x-y)^2 < 0 \tag{33}$$

and whether there are also viable alternatives. That the conditions QM1-3, W4, R2 are independent and compatible can be proved arguing along the lines [33]; in particular, compatibility is proved by showing that they can be fulfilled by free fields (see next section). We thus find, in particular, that *the noncommutativity structure of a Moyal-Weyl space is compatible with locality* $R2_\star$. Whether reasonable weakenings of $R2_\star$ exist is in fact an open question also in the ordinary theory. Phenomenology suggests that the rhs of (31) should at least rapidly decrease with increasing spacelike distances, if it is not zero. On the other hand, the same results as in [37,38] should apply, namely, requiring that the rhs of (31) is zero only in some spacelike separated open subsets (see [38], or Theorem 4-1 in [33]), or is a c number decreasing faster than an exponential with spacelike distances [37], are actually only apparent weakenings, in that they imply again R2.

As consequences of $R2_\star$ one again finds [33]

*W5 (Local commutativity conditions).—*If $(x_j - x_{j+1})^2 < 0$ then

$$\mathcal{W}^{\alpha_1\ldots\alpha_n}(x_1,\ldots,x_j,x_{j+1},\ldots,x_n) = \pm\mathcal{W}^{\alpha_1\ldots\alpha_{j+1}\alpha_j\ldots\alpha_n}(x_1,\ldots,x_{j+1},x_j,\ldots,x_n); \tag{34}$$

the sign is negative if φ^{α_j}, $\varphi^{\alpha_{j+1}}$ $\star$-anticommute, and is positive otherwise.

*W6 (Cluster property).—*For any spacelike vector a and for $\lambda \to \infty$

$$\mathcal{W}^{\alpha_1\ldots\alpha_n}(x_1,\ldots,x_j,x_{j+1}+\lambda a,\ldots,x_n+\lambda a) \to \mathcal{W}^{\alpha_1\ldots\alpha_j}(x_1,\ldots,x_j)\mathcal{W}^{\alpha_{j+1}\ldots\alpha_n}(x_{j+1},\ldots,x_n) \tag{35}$$

(convergence is in the distributional sense); this is true also with permuted coordinates.

In the proof of W6 the uniqueness of the invariant state Ψ_0 plays an essential role.

Summarizing, we end up with a QFT framework on these NC spaces with QM1-3, W4, $R2_\star$ or alternatively with exactly the same constraints W1-6 on Wightman functions as in ordinary QFT on Minkowski space. Reasoning as described in [33,39,40], one should be able to prove the same, other well-known fundamental results in ordinary QFT:

(1) That Wightman functions are boundary values

$$W^{\alpha_1\ldots\alpha_n}(\xi_1,\ldots,\xi_{n-1}) = \lim_{\eta_1,\ldots,\eta_{n-1}\to 0} W^{\alpha_1\ldots\alpha_n}(\zeta_1,\ldots,\zeta_{n-1})$$

of holomorphic functions $W(\zeta_1,\ldots,\zeta_{n-1})$ in the complex variables $\zeta_i = \xi_i - i\eta_i$, the domain of holomorphy being $\{\zeta_1,\ldots,\zeta_{n-1} | \eta_j \in V^+\}$.

(2) The analogs of the spin-statistics and *CPT* theorems.

(3) That the cluster property W6 implies (Haag-Ruelle theory) the existence of asymptotic (free) fields and, under the assumption of asymptotic completeness ($\mathcal{H} = \mathcal{H}^{\rm in} = \mathcal{H}^{\rm out}$), of a unitary S matrix. This allows to derive [41] the Lehmann-Symanzik-Zimmermann (LSZ) [42] formulation of QFT, and subsequently dispersion relations for scattering amplitudes, etc.

(4) That the Wightman functions have an analytic continuation to the so-called Euclidean points, thus allowing to derive the existence and the general properties of Euclidean QFT with the analog of Schwinger functions.

[4]As already noted, spacelike separation is well defined, so that the latter condition makes sense.

We stress that these results should hold for all $\theta^{\mu\nu}$, and not only if $\theta^{0i} = 0$ as in the approach e.g. of [23,43].

V. FREE FIELDS

As in ordinary QFT, things become much more definite for *free* fields. By (19), the kinetic differential operators (D'Alambertian, Dirac operator, etc.) remain undeformed, therefore so will remain the free field equations and the consequent constraints on Wightman, Green's functions, and on the field commutation relations. For simplicity we stick to the case of a free Hermitian scalar field $\varphi_0(x)$ of mass m:

$$(\Box_x + m^2)\varphi_0 = 0. \tag{36}$$

In momentum space this becomes $(p^2 - m^2)\tilde{\varphi}(p) = 0$, so the spectrum is contained in (the two sheets of) the hyperboloid $p^2 = m^2$. We can therefore Fourier decompose $\varphi_0(x)$ into a positive and a negative frequency part in a (twisted) Lorentz invariant way,

$$\begin{aligned} \varphi_0(x) &= \varphi_0^+(x) + \varphi_0^-(x), \\ \varphi_0^+(x) &:= \int d\mu(p) e^{-ip\cdot x} a^p, \\ \varphi_0^-(x) &:= \int d\mu(p) a_p^\dagger e^{ip\cdot x} = (\varphi_0^+(x))^\dagger, \end{aligned} \tag{37}$$

where $d\mu(p) = \delta(p^2 - m^2)\vartheta(p^0)d^4p = dp^0\delta(p^0 - \omega_{\mathbf{p}})d^3\mathbf{p}/2\omega_{\mathbf{p}}$ is the invariant measure ($\omega_{\mathbf{p}} := \sqrt{\mathbf{p}^2 + m^2}$). From (36) it immediately follows $(\Box_\xi + m^2)W(\xi) = 0$ or equivalently $(p^2 - m^2)\tilde{W}(p) = 0$ in momentum space, whence the Fourier decomposition

$$W(x-y) = \int d\mu(p)[w^+(p)e^{-ip\cdot(x-y)} + w^-(p)e^{ip\cdot(x-y)}].$$

On the other hand, using QM1-3 one finds first $\varphi_0^+(x)\Psi_0 = 0$, i.e. $a^p\Psi_0 = 0$, then

$$\begin{aligned} W(x-y) &= (\Psi_0, \varphi_0(x) \star \varphi_0(y)\Psi_0) \\ &= (\Psi_0, \varphi_0^+(x) \star \varphi_0^-(y)\Psi_0), \end{aligned}$$

showing that x (respectively y) is associated only to the positive (respectively negative) frequencies, i.e. $w^-(p)$ has to vanish, and $w^+(p)$ has to be positive. But by W4 $w^+(p)$ has to be Lorentz invariant, i.e. constant, so we conclude that up to a positive factor W is given by

$$\begin{aligned} W(x-y) &= -iF^+(x-y), \\ F^+(\xi) &:= i\int \frac{d\mu(p)}{(2\pi)^3} e^{-ip\cdot\xi}, \end{aligned} \tag{38}$$

and therefore coincides with the undeformed counterpart. Moreover,

$$(\Psi_0, a^p a_q^\dagger \Psi_0) = 2\omega_{\mathbf{p}}\delta^3(\mathbf{p} - \mathbf{q}) \tag{39}$$

as in the undeformed case. The 2-point Green's function is now immediately obtained as the time-ordered combination of $W(x-y)$ and $W(y-x)$:

$$\begin{aligned} \mathcal{G}(x,y) &:= (\Psi_0, T[\varphi_0(x) \star \varphi_0(y)]\Psi_0) = G(x-y), \\ G(\xi) &:= -i[\vartheta(\xi^0)F^+(\xi) + \vartheta(-\xi^0)F^+(-\xi)] \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot\xi}}{p^2 - m^2 + i\epsilon}, \end{aligned} \tag{40}$$

and therefore coincides with (the undeformed) Feynman's propagator. Note that (38)–(40) are independent of $\mathbf{R2}_\star$ or any other assumption about the field commutation relations, which we have not used in the proof.

On the other hand, if one postulates all the axioms of the preceding section (including $\mathbf{R2}_\star$) and reasons as in the proof of the Jost-Schroer theorem, Thm 4-15[5] in [33], one proves up to a positive factor the *free field commutation relation*

$$\begin{aligned} [\varphi_0(x) \overset{\star}{,} \varphi_0(y)] &= iF(x-y), \\ F(\xi) &:= F^+(-\xi) - F^+(\xi), \end{aligned} \tag{41}$$

which coincides with the undeformed one. Incidentally, this can be proved also from just the free field equation and the assumption that the commutator is a (twisted, and therefore also untwisted) Poincaré invariant c number (see e.g. [44], page 178–179). Applying ∂_{y^0} to (41) and then setting $y^0 = x^0$ [as already noted, this is compatible with (11)] one finds *the canonical commutation relation*

$$[\varphi_0(x^0, \mathbf{x}) \overset{\star}{,} \dot{\varphi}_0(x^0, \mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y}). \tag{42}$$

As a consequence of (41), also the n-point Wightman functions coincide with the undeformed ones, i.e. vanish if n is odd and are the sum of products of two point functions (this is the so-called factorization) if n is even. This of course agrees with the cluster property W6.

Free fields fulfilling (41) can be obtained from (37) plugging creation, annihilation operators fulfilling commutation relations deformed so as to compensate the space-time noncommutativity. The first possibility[6] is to require

[5] More precisely, as it is done after the proof that (36) follows from (38).

[6] In this and other proofs one has to use the following properties of exponentials. Recalling the Baker-Campbell-Hausdorff formula $e^Ae^B = e^{A+B+C}$ (with $C := [A, B]/2$ commuting with A, B) one finds

$$\begin{aligned} e^{ip\cdot x} \star e^{iq\cdot y} &= e^{ip\cdot x + iq\cdot y - (i/2)p\theta q} \Rightarrow e^{ip\cdot x} \star e^{iq\cdot y} \\ &= e^{iq\cdot y} \star e^{ip\cdot x} e^{-ip\theta q}, \\ e^{ip\cdot x} \star e^{-ip\cdot y} &= e^{ip\cdot(x-y)}; \end{aligned}$$

the last follows from the first and $p\theta p = 0$. These relations hold, in particular, for $y = x$. More generally, by iteration of the previous result one finds

$$e^{ip_1\cdot x_1} \star e^{ip_2\cdot x_2} \star \ldots \star e^{ip_n\cdot x_n} = \exp\left(i\sum_{j=1}^n p_j \cdot x_j - \frac{i}{2}\sum_{j<k} p_j\theta p_k\right), \tag{43}$$

which holds also if $x_j = x_k$ for some j, k.

$$a_p^\dagger a_q^\dagger = e^{ip\theta' q} a_q^\dagger a_p^\dagger, \qquad a^p a^q = e^{ip\theta' q} a^q a^p,$$
$$a^p a_q^\dagger = e^{-ip\theta' q} a_q^\dagger a^p + 2\omega_{\mathbf{p}} \delta^3(\mathbf{p}-\mathbf{q}), \tag{44}$$
$$[a^p, f(x)] = [a_p^\dagger, f(x)] = 0, \qquad \theta' = \theta$$

(where $p\theta q := p_\mu \theta^{\mu\nu} q_\nu$ for any 4-vectors p, q, and any $f \in \mathcal{A}$), as adopted e.g. in [17,19,20,24] [see also the bibliographical notes after (47)]. In the sequel we wish to consider and compare also other choices of the parameters $\theta'^{\mu\nu}$. The choice $\theta' = 0$ gives the canonical commutation relations (CCR), assumed in most of the literature, explicitly [1] or implicitly, either in the operator (e.g. apparently in [22,23,25,43]), or in the path-integral approach to quantization (see e.g. [2–4] and most references therein). Note that the last term in the third equation is fixed by (39). Correspondingly, one finds the field $\star$-commutation relations

$$\varphi_0(x) \star \varphi_0(y) = e^{i\partial_x(\theta-\theta')\partial_y} \varphi_0(x) \star \varphi_0(y) + iF(x-y), \tag{45}$$

which are nonlocal unless $\theta' = \theta$. As said, the authors of [17,19,20,24] adopt $\theta' = \theta$. In [24] commutation relations of the form (41) are proposed in a $1+1$-dimensional model in order to close the chiral current algebra; in [17] (41) are proposed in any dimension, although only for scalar fields and for $\theta^{0i} = 0$; whereas the authors of [19,20] find nonlocal relations [see formula (3.23) in [20]] similar to (45), because they do not perform the $\star$-product between functions of different sets x, y of coordinates.

Let us consider two typical contributions to the 4-point Wightman function:

$$\mathcal{W}(x_1, x_2, x_3, x_4) = W(x_1 - x_2) W(x_3 - x_4) + e^{i\partial_{x_2}(\theta-\theta')\partial_{x_3}} W(x_1 - x_3) \times W(x_2 - x_4) + \ldots$$

The first term on the rhs comes from the v.e.v.'s of $\varphi_0(x_1) \star \varphi_0(x_2)$ and $\varphi_0(x_3) \star \varphi_0(x_4)$; it is Lorentz invariant by (38) and factorized. The second, nonlocal term comes from the v.e.v.'s of $\varphi_0(x_1) \star \varphi_0(x_3)$ and of $\varphi_0(x_2) \star \varphi_0(x_4)$, after using (45) to commute $\varphi_0(x_2)$, $\varphi_0(x_3)$. Only if $\theta' = \theta$ it is Lorentz invariant and factorizes into $W(x_1 - x_3) W(x_2 - x_4)$. As it depends only on $x_1 - x_3$, $x_2 - x_4$, it is invariant under the replacements $(x_1, x_2, x_3, x_4) \to (x_1, x_2 + \lambda a, x_3, x_4 + \lambda a)$, even in the limit $\lambda \to \infty$ By taking a spacelike, we conclude that if $\theta' \neq \theta$ $\mathcal{W}$ violates W4 and W6, as expected.

We present a second way to realize (41), which at first sight might appear "exotic," but we are going to theoretically motivate elsewhere. It follows from assuming nontrivial transformation laws $P_\mu \triangleright a_p^\dagger = p_\mu a_p^\dagger$, $P_\mu \triangleright a^p = -p_\mu a^p$ and extending the $\star$-product law (2) also to a^p, $a_p^\dagger$. It amounts to choosing $\theta' = -\theta$ in (44) [inserting for uniformity of notation a $\star$ symbol in each product, also in (37)] and to adding nontrivial commutation relations between the a^p, $a_p^\dagger$, and functions, in particular, exponentials, of the form

$$a_p^\dagger \star a_q^\dagger = e^{-ip\theta q} a_q^\dagger \star a_p^\dagger,$$
$$a^p \star a^q = e^{-ip\theta q} a^q \star a^p,$$
$$a^p \star a_q^\dagger = e^{ip\theta q} a_q^\dagger \star a^p + 2\omega_{\mathbf{p}} \delta^3(\mathbf{p}-\mathbf{q}), \tag{46}$$
$$a^p \star e^{iq\cdot x} = e^{-ip\theta q} e^{iq\cdot x} \star a^p,$$
$$a_p^\dagger \star e^{iq\cdot x} = e^{ip\theta q} e^{iq\cdot x} \star a_p^\dagger.$$

As a consequence, $[\varphi_0(x) \stackrel{\star}{,} f(y)] = 0$. In other words, the first three relations in (46) define an example of a general deformed Heisenberg algebra [45]

$$a^q \star a^p = R^{qp}_{rs} a^s \star a^r, \qquad a_p^\dagger \star a_q^\dagger = R^{sr}_{pq} a_r^\dagger \star a_s^\dagger,$$
$$a^p \star a_q^\dagger = \delta^p_q + R^{rp}_{qs} a_r^\dagger \star a^s \tag{47}$$

covariant under a triangular Hopf algebra H. Here the R matrix is the universal $\mathcal{R}$ in the infinite-dimensional representation of H spanned by the basis of vectors $a_p^\dagger$, the (on-shell) 4-momenta p, q, r, s playing the role of (continuous) indices, and $\delta^p_q = 2\omega_{\mathbf{p}} \delta^3(\mathbf{p}-\mathbf{q})$ is Dirac's delta (up to normalization). [The first three relations in (44) also can be considered of the form (47) after a replacement $\theta \to -\theta'$]. Such a^p, $a_p^\dagger$ can be realized [45] as the composite of operators c^p, $c_p^\dagger$ fulfilling the ordinary CCR (for the case at hand of the θ-deformed Poincaré this has been done also in [18]), so that they act on the same (undeformed) Fock space. In doing so one finds that the action of P_μ can be realized as a commutator with the operator $\tilde{P}_\mu = \int d\mu(p) c_p^\dagger c^p = \int d\mu(p) a_p^\dagger a^p$.

As historical remarks we add that, up to normalization of R, and with $p, q, r, s \in \{1, \ldots, N\}$, relations (47) are also identical to the ones defining the older q-deformed Heisenberg algebras of [46,47], based on a quasitriangular $\mathcal{R}$ in (only) the *fundamental* representation of $H = U_q su(N)$; allowing a different (possibly infinite-dimensional) representation has been considered in [48] for the $U_q su(2)$-covariant quantization of fields on the q-deformed fuzzy sphere. Going further back in the past, (46) and (47) are reminiscent of the Zamoldchikov-Faddeev [49,50] algebra, generated by deformed creation/annihilation operators of scattering states of some completely integrable $1+1$-dimensional QFT; there again the indices are discrete, but the R matrix depends on a (continuous) spectral parameter, the rapidity of the particles. In [51] the Zamoldchikov-Faddeev creation/annihilation operators have been realized as acting on the (undeformed) Fock space.

VI. THEORETICAL DEVELOPMENTS—PERTURBATIVE EXPANSION FOR INTERACTING QFT

Normal ordering should be a $\mathcal{A}^n_\theta$-bilinear map of the field algebra into itself, such that any normal-ordered expression has a trivial v.e.v., in particular $:\mathbf{1}: = 0$. Applying it to the sides of (44) we find that it is consistent to define

$$:a^p a^q: = a^p a^q, \qquad :a^\dagger_p a^q: = a^\dagger_p a^q,$$
$$:a^\dagger_p a^\dagger_q: = a^\dagger_p a^\dagger_q, \qquad :a^p a^\dagger_q: = a^\dagger_q a^p e^{-ip\theta' q}. \tag{48}$$

The phase[7] in the last relation is to account for $(44)_3$ and $:\mathbf{1}: = 0$. More generally, it is consistent to *define normal ordering* on any monomial in a^p, $a^\dagger_q$ as a map which reorders all a^p to the right of all $a^\dagger_q$ introducing a factor $e^{-ip\theta' q}$ for each flip $a^p \leftrightarrow a^\dagger_q$. For $\theta' = 0$ one finds the undeformed definition. Using $\mathcal{A}^n_\theta$-bilinearity normal ordering is extended to fields.

We first consider the assumptions leading to (41), namely (44) or (46). One finds that exactly as in the undeformed case it maps each monomial M in the fields (and their derivatives) into itself minus all lower degree monomials obtained by taking v.e.v.'s of pairs of fields appearing in M, e.g.

$$\begin{aligned} :\varphi_0(x): &= \varphi_0(x),\\ :\varphi_0(x)\star\varphi_0(y): &= \varphi_0(x)\star\varphi_0(y) - (\Psi_0, \varphi_0(x)\\ &\quad\times\star\varphi_0(y)\Psi_0),\\ :\varphi_0(x)\star\varphi_0(y)\star\varphi_0(z): &= \varphi_0(x)\star\varphi_0(y)\star\varphi_0(z)\\ &\quad-(\Psi_0, \varphi_0(x)\star\varphi_0(y)\Psi_0)\varphi_0(z)\\ &\quad-(\Psi_0, \varphi_0(x)\star\varphi_0(z)\Psi_0)\varphi_0(y)\\ &\quad-\varphi_0(x)(\Psi_0, \varphi_0(y)\\ &\quad\times\star\varphi_0(z)\Psi_0)\ldots \end{aligned}\tag{49}$$

By construction $(\Psi_0, :M:\Psi_0) = 0$. These are well-defined operators also in the limit of coinciding coordinates (e.g. $y \to x$). The above substractions amount to flipping step by step each $\varphi_0^+(x)$ to the right of each $\varphi_0^-(y)$. For instance on $\varphi_0^\varepsilon(x)\varphi_0^\eta(y)$ ($\varepsilon, \eta \in \{-, +\}$) normal ordering acts as the identity unless $\varepsilon = +$ and $\eta = -$, whereas

$$:\varphi_0^+(x)\star\varphi_0^-(y): = \varphi_0^-(y)\star\varphi_0^+(x).$$

As a consequence we find that for any monomial M' obtained from M by permutation of the field factors $:M: = :M':$, for instance

$$:\varphi_0(x)\star\varphi_0(y): = :\varphi_0(y)\star\varphi_0(x):\ldots \tag{50}$$

Moreover, as $\varphi_0(x)$ is Hermitian, any normal-ordered monomial $:\varphi_0(x_1)\star\ldots\star\varphi_0(x_n):$ is (*a fortiori* for coinciding coordinates). Summing up, under these assumptions normal ordering (49) and its properties are written in terms of the fields exactly as in the undeformed setting (apart from the occurrence of the $\star$-product symbol). Since the same occurs with time ordering (25), another straightforward consequence is that *the same Wick theorem will hold*:

$$\begin{aligned} T[\varphi_0(x)\star\varphi_0(y)] &= :\varphi_0(x)\star\varphi_0(y):\\ &\quad+(\Psi_0, T[\varphi_0(x)\star\varphi_0(y)]\Psi_0),\\ T[\varphi_0(x)\star\varphi_0(y)\star\varphi_0(z)] &= :\varphi_0(x)\star\varphi_0(y)\star\varphi_0(z):\\ &\quad+(\Psi_0, T[\varphi_0(x)\star\varphi_0(y)]\Psi_0):\\ &\quad\times\varphi_0(z): + (\Psi_0, T[\varphi_0(x)\\ &\quad\times\star\varphi_0(z)]\Psi_0):\varphi_0(y):\\ &\quad+(\Psi_0, T[\varphi_0(y)\star\varphi_0(z)]\Psi_0):\\ &\quad\times\varphi_0(z):\ldots \end{aligned}$$

Let us apply now *time-ordered perturbation theory* to an interacting field. We use the Gell-Mann-Low formula (rigorously valid under the assumption of asymptotic completeness, $\mathcal{H} = \mathcal{H}^{\rm in} = \mathcal{H}^{\rm out}$)

$$G(x_1,\ldots,x_n) = \frac{(\Psi_0, T\{\varphi_0(x_1)\star\ldots\star\varphi_0(x_n)\star\exp[-i\lambda\int dy^0 H_I(y^0)]\}\Psi_0)}{(\Psi_0, T\exp[-i\int dy^0 H_I(y^0)]\Psi_0)}. \tag{51}$$

Here φ_0 denotes the free "in" field, i.e. the incoming field in the interaction representation, and $H_I(x^0)$ is the interaction Hamiltonian in the interaction representation. The derivation of (51) is heuristic and goes as on commutative space. It involves unitary evolution operators of the form

$$\begin{aligned} U(x^0, y^0) &= \lim_{N\to\infty}\overleftarrow{\prod_{m=0}^{N-1}}\exp\left[-\frac{i}{\hbar}\frac{\Delta}{N}H_I\left(y^0+\Delta\frac{m}{N}\right)\right]\\ &= T\exp\left[-\frac{i}{\hbar}\int_{y^0}^{x^0}dtH_I(t)\right], \end{aligned}$$

where $\Delta = x^0 - y^0 > 0$ and again T always uses $\star$-central time coordinate differences as arguments of the Heaviside function. For the sake of definiteness we choose

[7]The authors of [19] omit this phase. However, their conclusions about the S matrix being undeformed remain valid, as the effect of this phase disappears when exploiting global energy-momentum conservation.

$$H_I(x^0) = \lambda \int d^3x{:}\varphi_0^{\star m}(x){:}\star,$$
$$\varphi_0^{\star m}(x) \equiv \underbrace{\varphi_0(x) \star \ldots \star \varphi_0(x)}_{m \text{ times}}. \tag{52}$$

This is a well-defined, Hermitian [by (50)] operator, with zero v.e.v. Expanding the exponential the generic term of order $O(\lambda^h)$ in the numerator of (51) will be the v.e.v.

$$\frac{\lambda^h}{i^h} \int d^4y_1 \ldots d^4y_h(\Psi_0, T[\varphi_0(x_1) \star \ldots \star \varphi_0(x_n) \star {:}\varphi_0^{\star m}(y_1){:} \star \ldots \star {:}\varphi_0^{\star m}(y_h){:}]\Psi_0), \tag{53}$$

where we have also used the property (20) that integration over any space-time variable commutes with taking $\star$-products. We proceed to evaluate this expression as in the undeformed case: applying Wick theorem to the field monomial and the fact that all normal-ordered field monomials have trivial v.e.v. we end up with exactly the *same* sum of terms given by integrals over y-variables, as in the undeformed case, of products of free propagators (40) having coordinate differences as arguments. Each of these terms is represented by a Feynman diagram. So the result for the generic term (53) will be the *same* as the undeformed one. On the other hand, the generic term of order $O(\lambda^h)$ in the denominator of (51) will be a special case of (53), the one with $n = 0$. Summing up, the numerator and denominator of (51), and therefore also the *Green functions* (51) *coincide with the undeformed ones* (at least perturbatively). They can be computed by Feynman diagrams with the undeformed Feynman rules.

In other words, at least perturbatively, *this QFT is completely equivalent to the undeformed counterpart*, and therefore also pathologies like UV-IR mixing disappear. Thus, also for the interacting theory the a_p, $a_p^\dagger$ and the space-time noncommutativities somehow compensate each other.

We now sketch how perturbation theory changes if $\theta' \neq \theta$, starting from normal-ordered field monomials. Relations (44) lead to a nonlocal (pseudodifferential) operator for each flip of a $\varphi_0^+(x)$ to the right of a $\varphi_0^-(y)$, e.g.

$${:}\,\varphi_0^+(x) \star \varphi_0^-(y){:} = e^{i\partial_x(\theta-\theta')\partial_y}\varphi_0^-(y) \star \varphi_0^+(x),$$

whereas on $\varphi_0^\varepsilon(x)\varphi_0^\eta(y)$ with $(\varepsilon, \eta) \neq (+, -)$ normal ordering still acts as the identity. As a consequence, property (50) and Wick theorem are modified, so are the Feynman rules, and UV/IR mixing for nonplanar Feynman diagrams reappears. Just to get a feeling one can consider the $\lambda\varphi^{\star 4}$ theory without normal ordering and, as in [8], one finds UV/IR mixing already in several contributions (of nonplanar tadpole diagram type) to the $O(\lambda)$ correction to the propagator.

VII. FINAL REMARKS AND CONCLUSIONS

There is still no convincing and generally accepted formulation of QFT on noncommutative spaces, even on the simplest one, the Moyal-Weyl space. One crucial aspect under debate is the form of its covariance under space(time) symmetry transformations. In this work we have argued that a Moyal-Weyl deformation of Minkowski space is compatible with the Wightman axioms (including locality) and time-ordered perturbation theory, provided one replaces products of fields by $\star$-products (also at different space-time points) and the Lorentz-covariance axiom (R1) by the appropriate twisted version $R1_\star$ (which we have not formulated yet). Both for free and interacting fields the resulting QFT's appear physically equivalent to the undeformed counterparts on commutative Minkowski space, in that their Wightman and Green's functions coincide. This can be understood as a sort of compensation of the effects of the a_p, $a_p^\dagger$ and of the space-time noncommutativities, if these are matched to each other. (To keep the size of this work contained we have not developed other important aspects, like those mentioned at the end of Sec. IV, which we hope to treat soon elsewhere. For the moment, regarding the question whether QFT on noncommutative spaces violate standard Bose or Fermi statistics, as claimed e.g. in [18,20,52], we content with drawing the reader's attention to Refs. [32,53].)

The main positive aspect of this outcome is a way to avoid all the additional complications (nonunitarity, macroscopic violation of causality, UV-IR mixing and subsequent nonrenormalizability, change of statistics,...) appeared in other approaches and to end up with a theoretically and phenomenologically satisfactory QFT, the undeformed theory (to the extent the latter can be considered satisfactory). For the free field this is achieved by matching the commutation relations of the deformed creation/annihilation operators to the space-time noncommutativity (however, we have found even two different ways to realize such a matching).

The related, obvious disappointing aspect is that in this resulting QFT there appears neither new physics nor a more satisfactory formulation of the old one (e.g. by an intrinsic UV regularization), in that all effects of space-time noncommutativity are confined in an "unobservable common noncommutative translation of all reference frames." This may indicate that Moyal-Weyl deformations considered in the framework of twisted Poincaré covariance are too trivial for this scope.

As a general remark, we would like to emphasize that the cluster property W6 is an important test for QFT on noncommutative as well as on commutative spaces: its violation implies a macroscopic (and therefore contrasting with experiments) violation of causality. It is also an easy theoretical test to carry out on free fields. For the noncommutative space at hand, our two possible prescriptions for free fields fulfill the cluster property whereas other prescriptions proposed in the literature ($\theta' \neq \theta$, see the end of Sec. V) lead to its violation.

As already noted in the paper, our results have some overlap or links with those of other works. To this regard we add some further remarks. At the 21st Nishinomiya-Yukawa Memorial Symposium on Theoretical Physics (11–15 Nov. 2006) "Noncommutative Geometry and Quantum Space-time in Physics," after presenting our results, the author of Ref. [17] pointed out his work. We have realized that there he proposes a quantization procedure for scalar fields and $\theta^{0i} = 0$ which finally coincides with the first of our two proposals and arrives at very similar conclusions, although the derivation is different and various steps of it appear to us not completely clear or justified. In [54] field quantization on the h-deformed Lobachevsky plane was performed adopting a braided tensor product among coordinates of different space-time points, as done here; by a proper treatment the authors found that the result for the 2-point function also coincides with the undeformed one. Finally, already in [15,55] Oeckl used the relation between the deformed and undeformed covariance to determine a mapping between deformed and undeformed theories (in the Euclidean formulation of QFT); in the Moyal-Weyl case this mapping allows [15] to immediately compute the deformed Green functions in terms of the undeformed ones (however they do not coincide).

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