

Multiparticle states in deformed special relativity

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We investigate the properties of multiparticle states in deformed special relativity (DSR). Starting from the Lagrangian formalism with an energy dependent metric, the conserved Noether current can be derived which is additive in the usual way. The integrated Noether current had previously been discarded as a conserved quantity, because it was correctly realized that it does no longer obey the DSR transformations. We identify the reason for this mismatch in the fact that DSR depends only on the extensive quantity of total four momentum instead of the energy-momentum densities as would be appropriate for a field theory. We argue that the reason for the failure of DSR to reproduce the standard transformation behavior in the well established limits is due to the missing sensitivity to the volume inside which energy is accumulated. We show that the soccer-ball problem is absent if one formulates DSR instead for the field densities. As a consequence, estimates for predicted effects have to be corrected by many orders of magnitude. Further, we derive that the modified quantum field theory implies a locality bound.

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I. INTRODUCTION

The phenomenology of quantum gravity has received increased attention during the last years. In the absence of a testable theory quantum gravity, predictions based on effective models have been studied which use only some few well motivated assumptions. One such assumption is the presence of a regulator in the ultraviolet, or the existence of a maximal energy scale, respectively. The requirement that Lorentz transformations in momentum space have this scale as a second invariant leads to a class of deformations of special relativity (DSR). As one of the most general expectations arising from a theory of quantum gravity, these modified Lorentz transformations have been studied extensively [1–14].

However, despite the fact that it is possible to use kinematic arguments to predict threshold corrections, a fully consistent quantum field theory with DSR is still not available. Though there are notable attempts [15–19], one has faced serious conceptual problems in the formulation of a field theory, such as the proper definition of conserved quantities in interactions, and the transformation of multiparticle states, also known as the soccer-ball problem.

In this paper, we argue that the reason for this mismatch lies in the investigation of extensive quantities like total energy and momentum, rather than intensive quantities like energy- and momentum densities which would be appropriate for a field theory. Originally, DSR was formulated [1,2] as a (classical) theory for a point particle, and it has been shown [20,21] that DSR can be understood as a deformation of the momentum space that belongs to the point particle. However, DSR—through the very introduction of a minimal length—implies a generalized uncertainty principle [22–26], which forbids it to localize a

particle to a point. Therefore, already this formulation must be interpreted as a theory for an energy distribution with maximally possible localization, and the momentum space properties for space-time points inside this space-time volume.

If one wants to construct a field theory that consistently incorporates DSR, the transformation behavior for a classical particle with four momentum \mathbf{p} can not independently be transferred to each of the single field's modes, since superposition of these modes implies that the properties at a point in space-time—and therefore the momentum space at this point—depend not only on the single mode but on the energy density of all the present excitations.

Once one realizes that the quantity to be bounded by the Planck scale should not be the total energy of a system, but rather its energy density, the soccer-ball problem vanishes and multiparticle states transform appropriately. Unfortunately, the energy density of all experimentally accessible objects is far too small to make any quantum gravitational effects of this kind important. Thus, if one formulates the quantum field theory with DSR, the so far proposed predictions are unobservable.

This paper is organized as follows. In the next section we introduce the notation. In Sec. III we briefly recall the problem of multiparticle states and examine its cause. Section IV summarizes the quantum field theory formalism previously used in Refs. [15,16,22]. In the following Sec. V, we investigate the soccer-ball problem and show how it can be resolved. Predictions are revisited in Sec. VI. The discussion and the conclusions can be found in Sec. VII.

Throughout this paper we use the convention $\hbar = c = 1$, such that the Planck mass is the inverse of the Planck length $m_p = 1/l_p$. Bold faced quantities \mathbf{p} , \mathbf{q} are four vectors. Capital Latin indices label particles. Small Greek indices, and small Latin indices from the beginning of the alphabet run from 0 to 3 and label space-time coordinates.

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Quantities with small Greek indices transform under standard Lorentz transformation; quantities with small Latin indices from the beginning of the alphabet transform under the deformed transformations. The indices k and p refer to the wave vector and momentum, respectively.

II. DEFORMED SPECIAL RELATIVITY

With use of the notation introduced in Ref. [15,16,22], the quantity $\mathbf{p} = (E, p)$ transforms as a standard Lorentz vector, and is distinguished from $\mathbf{k} = (\omega, k)$, which obeys the modified transformation that is nonlinear in \mathbf{k} . The former quantity \mathbf{p} can always be introduced, the important step is eventually its physical interpretation. In the standard DSR formalism, \mathbf{p} is often referred to as the pseudomomentum.

A general relation between \mathbf{p} and \mathbf{k} can be of the form

$$\mathbf{k} = F(\mathbf{p}) = (Ef(\mathbf{p}), pg(\mathbf{p})), \quad (1)$$

with the inverse $\mathbf{p} = F^{-1}(\mathbf{k})$, that we will denote for better readability as $F^{-1}(\mathbf{k}) \equiv G(\mathbf{k})$. As examined in Ref. [22] these theories can, but need not necessarily have an energy dependent speed of light. An obvious requirement is that the function F reduce to multiplication with \hbar in the limit of energies being small with respect to the Planck scale. In order to implement a maximum energy scale, either one or all components of \mathbf{k} should be bounded by m_p . In these theories, one has a modified dispersion relation (MDR) of the form $G(\mathbf{k})^2 = m^2$.

It is now straightforward to derive the transformation that maps $\mathbf{k} \rightarrow \mathbf{k}'$ when applying a Lorentz-boost, and which respects the invariance of the modified dispersion relation. One just keeps in mind that the relation for \mathbf{p} is the standard relation $p^2 - E^2 = p'^2 - E'^2$, from which one finds the standard Lorentz transformation in momentum space. We will denote these standard transformations with $\mathbf{p}' = L(\mathbf{p})$. Then one gets the modified Lorentz transformations acting on \mathbf{k} by requiring

$$\mathbf{k}' = F(\mathbf{p}') = F(L(\mathbf{p})) = F(L(G(\mathbf{k}))). \quad (2)$$

We will denote these transformations as $\mathbf{k}' = \tilde{L}(\mathbf{k})$. The transformations Eqs. (2) will be nonlinear in (ω, k) since F is. By construction, implemented upper bounds on one or all components of \mathbf{k} are respected. For special choices of F one finds the DSR transformations used in the literature. An explicit example [10] is $f(E) = g(E) = 1/(1 + E/m_p)$, for which one has the transformations

$$\begin{aligned} \omega' &= \frac{\gamma(\omega - vk)}{1 - \omega/m_p + \gamma(\omega - vk)/m_p}, \\ k' &= \frac{\gamma(k - \frac{v}{c^2}\omega)}{1 - \omega/m_p + \gamma(\omega - vk)/m_p}. \end{aligned} \quad (3)$$

One can understand DSR as a theory with a curved momentum space. In fact, if one integrates over all possible values of \mathbf{k} , and rewrites the integration into momentum space one finds

$$\int d^4k = \int d^4p \left| \frac{\partial F}{\partial p} \right|, \quad (4)$$

where the quantity under the right integral is the Jacobian determinant. This can be read as a curved momentum space with an energy dependent metric [27–29] g and $|\partial F/\partial p| = \sqrt{-g}$. The most extensively investigated geometry is that of de Sitter space [20,21]. For the cases investigated in Refs. [15,16,22], the geometry is conformally flat [30].

However, one should keep in mind that here we have considered single particles, and the momentum space we were referring to was the momentum space of that particle, not a global property. In fact, if one considers a field theory, every point of our space-time manifold should have a corresponding momentum space, and its properties can in principle be a function of the space-time coordinates [31]. In the limit where quantum gravitational effects are negligible, one would expect to a flat momentum space to be a very good approximation, and to recover the standard transformation laws of special relativity.

III. MULTIPARTICLE STATES

So far we have considered only one particle. The question how to generalize the formalism of DSR to multiparticle states is essential if one wants to formulate a quantum field theory. The missing description of multiparticle systems is a huge obstacle on the way to formulate the principles of the theory, and to recover the limiting cases of the standard model and special relativity. Though large progress has been made regarding the solution of this problem [18,27,32–34], the issue is still not completely settled and open questions remain [12,35,36].

In particular, one wants to construct a conserved quantity for bound states and interactions. Let us consider a two particle system with $\mathbf{p}_A, \mathbf{p}_B$, or $\mathbf{k}_A, \mathbf{k}_B$ respectively, and ask for the conserved quantity \mathbf{q} . The most obvious choice is

$$\mathbf{q} = \mathbf{p}_A + \mathbf{p}_B, \quad (5)$$

which transforms as a usual Lorentz vector, and is the way pursued in Ref. [15,16,22]. However, this option is admittedly not very exciting, and it has been pointed out [37] that in fact within DSR the construction of a conserved quantity seems to be not uniquely defined. The next obvious choice that one would take is

$$\mathbf{q} = \mathbf{k}_A + \mathbf{k}_B. \quad (6)$$

However, if one requires \mathbf{q} to obey the same deformed transformations as \mathbf{k}_A and \mathbf{k}_B , then this quantity does not transform properly. Since the transformations \tilde{L} are not

linear, one has

$$\begin{aligned} \mathbf{q}' &= \tilde{L}(\mathbf{q}) = \tilde{L}(\mathbf{k}_A + \mathbf{k}_B) \neq \tilde{L}(\mathbf{k}_A) + \tilde{L}(\mathbf{k}_B) \\ &= \mathbf{k}'_A + \mathbf{k}'_B. \end{aligned} \quad (7)$$

Note that this arises from the fact that the transformation acting on the \mathbf{k}_A (\mathbf{k}_B) is a function of \mathbf{k}_A (\mathbf{k}_B) only and not of the total conserved quantity. Since the transformation behavior reflects the properties of the curved momentum space of the particle with energy \mathbf{k} , this means that these momentum spaces are independent of each other. This is a justified expectation if the particle's energies are sufficiently localized, as not to influence each other. This is appropriate for point particles, but will definitely not be the case for plane waves.

One could of course just define the transformation behavior of \mathbf{q} to be equal to that of $\mathbf{k}_A + \mathbf{k}_B$. But then, the transformation of the conserved charge \mathbf{q} would depend on the decomposition into the added quantities and not be unique. That is, another decomposition into $\mathbf{q} = \mathbf{k}_C + \mathbf{k}_D$ would lead to a different transformation behavior. Though this seems unintuitive, and we will not further examine this transformation law, we would like to point out that this possibility remains an option.

When one discards the addition law (6), one is then lead to the conclusion that the quantity \mathbf{k} has to obey a modified addition law, which we will denote with \oplus , and which is given by

$$\mathbf{k}_A \oplus \mathbf{k}_B = F(\mathbf{p}_A + \mathbf{p}_B). \quad (8)$$

In such a way, one can define

$$\mathbf{q} = \mathbf{k}_A \oplus \mathbf{k}_B, \quad (9)$$

which transforms appropriately under applying the transformation (2)

$$\mathbf{q}' = F(L(\mathbf{p}_A + \mathbf{p}_B)) = F(\mathbf{p}'_A + \mathbf{p}'_B). \quad (10)$$

If one considers an interaction of the type $A + B \rightarrow C$, and identifies the energy of the particle c with the above defined quantity \mathbf{q} one obtains the conservation law

$$0 = \mathbf{q} - \mathbf{k}_A \oplus \mathbf{k}_B. \quad (11)$$

This conservation law deviates from the standard prescription due to the modified addition law. This gives rise to the predicted threshold modifications. In case one considers more than three particles, one has more choices for the nonlinear addition [37]. Note that \mathbf{k}_A is an element of particle A 's phase space, whereas \mathbf{k}_B belongs to particle B 's space. The addition therefore is not performed inside the single particle phase space, but instead defines a structure on the multiparticle phase space.

However, with the prescription Eq. (8) one runs into another problem. By construction, the function F creates

an upper bound on \mathbf{k} . Unfortunately, we know that bound systems of elementary particles can very well exceed the Planck mass, and this DSR formalism therefore can not apply for them. The reason for this mismatch, also known as the soccer-ball problem, is the nonlinearity of the transformations, which should be suppressed when the number of constituents grows.

One should also note that the addition law has been chosen and not been derived, which means it is an additional assumption of DSR.

IV. TOWARDS A FIELD THEORY

One way or the other, if DSR is a well defined symmetry principle, it should be possible to just derive the conserved quantity for multiparticle states, and resolve the soccer-ball problem. Indeed, this is straightforward to do, as has been shown e.g. in Ref. [16].

In the following we will use the formalism with an energy dependent metric $g_{\mu\nu}(\mathbf{k})$ that has been introduced and worked out in Ref. [16,29]. The momentum is denoted by p^i and transforms under the usual Lorentz transformation. The wave vector is obtained by converting the index with an energy dependent field that we will denote with h . Since the relation between both momentum and wave vector depends on the energy, the transformation of the wave vector will no longer be the standard Lorentz transformation. One also notices that the volume element in momentum space becomes energy dependent as previously mentioned [compare to Eq. (4)].

Under quantization, the metric becomes an operator $g_{\mu\nu}(\partial)$. The energy dependence of the metric can be interpreted as a backreaction effect on the propagating particle: If the energy density in a space-time region reaches the Planckian regime, then the particle will significantly disturb the background it propagates in. In the limit where the metric approaches that of flat Minkowski space one recovers standard special relativity.

The relation between the formerly introduced quantities of the particle is given by

$$p^i = h^i{}_\nu(\mathbf{k})k^\nu, \quad (12)$$

$$g_{\nu\kappa}(\mathbf{k}) = \eta_{ij}h^i{}_\nu(\mathbf{k})h^j{}_\kappa(\mathbf{k}), \quad (13)$$

where η is the Minkowski metric. The dispersion relation reads simply $\eta_{ij}p^i p^j = 0$, or, more intuitively

$$k_\nu g^{\nu\kappa}(\mathbf{k})k_\kappa = 0. \quad (14)$$

We will in the following refer to the dispersion relation being a modified dispersion relation (MDR) if

$$\eta^{\kappa\nu}k_\kappa k_\nu \neq 0. \quad (15)$$

Note, that this need not necessarily be the case for all

equations of the form (14). For example, when the energy dependent metric is conformally flat and of the form $g^{\kappa\nu} = a(k)\eta^{\kappa\nu}$ with some scaling function a , then the dispersion relation (14) implies the standard dispersion relation.

We can write the relation in the general form $p^i = G^i(k)$ with

$$G^i(k) = \delta^i_{\nu} k^{\nu} + \sum_{l=1}^{\infty} \frac{A_{(2l+1)}^{i\nu_1\nu_2\dots\nu_{2l+1}}}{m_p^{2l}} k_{\nu_1} k_{\nu_2} \dots k_{\nu_{2l+1}},$$

where it is taken into account that p is odd in k . A is a rank- $2l + 1$ tensor with dimensionless coefficients that are constant with respect to space-time coordinates. Here, it was assumed that m_p sets the scale for the higher order terms.

Under quantization, the local quantity k will be translated into a partial derivate. One now wants to proceed from a single- k mode

$$v_k \sim e^{ik_{\nu}x^{\nu}} \quad (16)$$

to a field and to the operator $\hat{k}_{\nu} = -i\partial_{\nu}$. The corresponding momentum operator \hat{p} should have the property

$$\hat{p}^i v_k = p^i v_k = G^i(k) v_k, \quad (17)$$

which is fulfilled by

$$\hat{p}^i = G^i(-i\partial), \quad (18)$$

since every derivation results in just another factor k . It is therefore convenient to define the higher order operator

$$\delta^i = iG^i(-i\partial). \quad (19)$$

Since G is even in k , this operator's expansion has only real coefficients that are up to signs those of G^i . Note that δ^i commutes with ∂_{κ} . A theory of this structure will usually involve higher order derivatives in the spacelike as well as in the timelike coordinates that require initial conditions. One thus expects the theory to have a rather complicated canonical structure, and to display inherently nonlocal features. In particular the equal time commutation relations will be examined in Sec. V.

From the above one can further define the operator $\tilde{\square}$ which generates the wave function that corresponds to the MDR Eq. (14)

$$\tilde{\square} = g^{\mu\nu}(\partial_{\alpha})\partial_{\mu}\partial_{\nu} = \eta_{ij}\delta^i\delta^j. \quad (20)$$

This modified D'Alembert operator plays the role of the propagator in the quantized theory. Normalized solutions to the wave equation Eq. (30) can be found in the set of modes

$$v_p(x) = \frac{1}{\sqrt{(2\pi)^3 2E}} \exp(ik_{\nu}x^{\nu}), \quad (21)$$

which solve the equation of motion when p fulfills the usual dispersion relation, or k fulfills the MDR, respectively [38]. Alternatively, one can consider an expansion in k space with $v_k = \sqrt{E/\omega} v_p$. The solutions Eq. (21) form an orthonormal set with respect to the new derivative

$$\int d^3x v_p^*(x) \overleftrightarrow{\delta^0} v_{p'}(x) = \delta(k - k') = \delta(p - p') \left| \frac{\partial G}{\partial k} \right|. \quad (22)$$

It is important to note that this complete set of orthonormal eigenfunctions of the momentum operator in the coordinate representation are not also a complete set of eigenfunctions of the coordinate operator in the momentum representation, as it usually is the case. In k space, the modes are normalized with respect to the usual scalar product. Both descriptions are equivalent. The use of which is more suitable depends on the quantity one wants to investigate. In case the standard momentum is p , it is more appropriate to express everything in p space. In the standard DSR, one would instead want to formulate everything in the modified quantity k .

The field expansion in terms of the set of solutions reads

$$\phi(x) = \int d^3p \left| \frac{\partial F}{\partial p} \right| [v_p(x) a_p + v_p^*(x) a_p^{\dagger}], \quad (23)$$

which yields the operators through forming the scalar product

$$a_p = \int dx^3 v_p^*(x) \overleftrightarrow{\delta^0} \phi(x), \quad (24)$$

$$a_p^{\dagger} = \int dx^3 \phi(x) \overleftrightarrow{\delta^0} v_p(x). \quad (25)$$

These fulfill the commutation relation [15]

$$[a_p, a_{p'}^{\dagger}] = \delta(p - p') \left| \frac{\partial G}{\partial k} \right|. \quad (26)$$

It is convenient to use the higher order operator δ^i in the setup of a field theory, instead of having to deal with an explicit infinite sum. Note, that this sum actually has to be infinite when the relation $p^i = G^i(k)$ has an asymptotic limit as one needs for an UV-regulator. Such asymptotic behavior can never be achieved with a finite power series.

For the following analysis it is important to note that the higher order operator δ^i fulfills the property

$$\phi_i(\delta^i \psi) = -(\delta^i \phi_i) \psi + \text{total divergence}, \quad (27)$$

which has been derived in Ref. [16]. As a simple example we will work with a massless scalar field. The action for the scalar field [39] takes the form

$$S = \int d^4x \sqrt{g} \mathcal{L}. \quad (28)$$

with

$$\mathcal{L} = \frac{1}{2}(\partial_\nu \phi)(g^{\nu\kappa} \partial_\kappa \phi). \quad (29)$$

Using Eq. (27), one then derives the equations from the usual variational principle to the correct form

$$g^{\nu\kappa} \partial_\kappa \partial_\nu \phi = 0. \quad (30)$$

The calculus with the higher order operator δ^i effectively summarizes the explicit dealing with the infinite series. These higher order derivative theories have been examined in Ref. [18], where also the conserved Noether currents have been derived and an explicit expression for the energy-momentum tensor can be found. For our purposes it is sufficient to note that the Noether current is a bilinear form in the fields derivatives. If one inserts the field expansion, integrates it over space and takes the vacuum expectation value, one obtains (after normal ordering) the conserved quantity

$$q^\nu = \int d^3x \langle 0| : T^\nu_0 : |0\rangle, \quad (31)$$

which is conserved with respect to the time derivative, and whose 0-component can be identified as the total energy \mathcal{E} . If one inserts a superposition of two plane waves with \mathbf{k}_1 and \mathbf{k}_2 , one finds that it is additive, since the mixing terms in the bilinear form do not contribute when the volume integration is performed. With this result from Ref. [18] the soccer-ball problem is absent. Because of the standard additivity, this expression for the total energy reduces to the usual expression already when the energy of each constituent is $\ll m_p$. In this case however, one has not only solved the multiparticle problem, but also removed the threshold modifications. In fact, this result is incompatible with the DSR interpretation in which the physically relevant and conserved quantity is obtained through a modified addition law.

As we had noticed before, this conserved quantity can then no longer be subject to the DSR transformation, the reason for which we can now identify. The above examination shows us very nicely where the problem stems from. It arises from the fact that the relation between the usual and the deformed quantity h is a function only of the one mode it acts on, and so is the metric. If we apply it to the field's expansion, each term under the integral acquires a different transformation law, and we are back to the problem (7).

In general relativity however, the metric is a function not of the energy of a single mode, but of the energy-momentum tensor of the whole quantum field. In a theory of quantum gravity, the metric g would become an operator, and the action would be a functional of g coupled to the quantum field ϕ . When applying the variational principle, both are treated as independent variables. Variation with

respect to the field ϕ results in the field's equations of motion; variation with respect to the metric should result in a quantum version of Einstein's field equations. From dimensional arguments, and to recover the classical limit, the source term in the latter equations should be the field's density, and not a global charge.

In lack of the full theory of quantum gravity, the here investigated approach can be understood as an educated guess for the arising metric. Instead of deriving it, we required it to reproduce the existence of a minimal length which captures one of the best known, and most widely examined, properties of gravitational effects in the Planckian regime. This metric then can be inserted in the field equations for ϕ which makes them nonlinear. Nevertheless, the conjectured metric operator should be a function of the field's densities, and instead of it being a function of k only, it should be of the form $g_{\mu\nu}(\partial\phi\partial\phi)$. Moreover, it follows from this that the relation between the momentum p and the wave vector k of a single mode therefore does *not* only depend on the mode's properties, but on the energy density of the whole field and Eq. (12) should correctly read

$$p^i = h^i_\nu(\partial\phi\partial\phi)k^\nu. \quad (32)$$

In contrast to the single particles that were considered for the construction of the original DSR transformations, plane waves do overlap each other. The transformations acting on one wave will therefore be sensitive to the energy content of the other waves, all of which taken together determine the structure of the momentum space bundle over the space-time. Up to dimensional factors, the standard DSR approach remains applicable for a single mode, in which case the energy density is proportional to the mode's frequency.

V. THE SOCCER-BALL PROBLEM

One of the truly surprising features of DSR is that a particle of a very tiny mass compared to the Planck scale can perceive DSR effects that are argued to be caused by quantum gravity. Naively, one would expect quantum gravitational effects to become important only when the curvature of the background is non-negligible. This is usually not the case for particles we observe.

This reflects in the above finding that the relation between the quantity with the standard properties \mathbf{p} and the modified one \mathbf{k} should be a function of the energy-momentum density rather than the total energy. One should also keep in mind that under quantization, modifications of the type $\mathbf{k} = F(\mathbf{p})$ lead to a modified commutation relation [22–26] which results in a generalized uncertainty relation. This generalized uncertainty makes it impossible to localize a particle to better precision than a Planck length, which is what one would expect.

One reproduces the equivalent of the generalized uncertainty for a quantum field theory by considering the com-

mutator of the field and its conjugated variable. With the help of the previously defined higher order operator δ^ν , one can define a conjugated momentum of the field to

$$\pi^\nu = \delta^\nu \phi = \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi)}, \quad (33)$$

with the identification $\pi^0 \equiv \pi$.

From $\pi = \delta^0 \phi(y)$ with use of the field expansion Eq. (23) one then finds in the usual way

$$[\phi(x), \pi^0(y)] = i \int d^3 p \left| \frac{\partial F}{\partial p} \right| \int d^3 p' \left| \frac{\partial F'}{\partial p'} \right| \times 2E' v_p(x) v_{p'}^*(y) [a_p, a_{p'}^\dagger]. \quad (34)$$

Using Eq. (26) this reduces to

$$[\phi(x), \pi(y)] = i \int d^3 p \left| \frac{\partial F}{\partial p} \right| 2E v_p(x) v_p^*(y) = i \int \frac{d^3 p}{(2\pi)^3} \left| \frac{\partial F}{\partial p} \right| e^{ik(x-y)}. \quad (35)$$

A specifically useful relation from Ref. [40] for $k(p)$ is

$$k_\mu(p) = \hat{e}_\mu \int_0^p e^{-\epsilon p'^2} dp', \quad (36)$$

where \hat{e}_μ is the unit vector in μ direction, $p^2 = \vec{p} \cdot \vec{p}$ and $\epsilon = l_p^2 \pi/4$ (the factor $\pi/4$ is included to assure, that the limiting value is indeed $1/l_p$). Using this example one finds

$$[\phi(x), \pi(y)] = i \int \frac{d^3 p}{(2\pi)^3} e^{ik(x-y) - \epsilon p^2}. \quad (37)$$

One sees that a nontrivial dispersion relation with a lower bound on the wavelength therefore implies a locality bound similar to that proposed in Refs. [41,42]

$$[\phi(x), \pi(y)] \neq i\delta(x-y), \quad (38)$$

which is due to the nontrivial volume element in momentum space. Rewriting the expression into k space, one realizes that this arises through the finite boundaries. Such a modification will become important, when $x-y \sim l_p$.

It therefore seems inappropriate to consider an integrated quantity that can not be localized to a point particle by using superpositions, since this is disabled by the very postulate of a minimal length. Instead, one should consider the local density of the field, and impose a bound on it. This is also a more appropriate choice simply because we want to construct a field theory for DSR.

In the standard DSR approach there is no dependence on the volume inside which we localize a mode with a given energy. We can use box modes and shrink the box as small as possible, that is as small as a Planck volume. This does

not reflect in any way in the transformation properties of the modes. If one thinks in terms of total energy, then it is not even clear in which limit an undeformed special relativity has to be recovered. The limit of a total energy \mathcal{E} very small with respect to the Planck-mass, $\mathcal{E} \ll m_p$ (single particle), as well as very large $m_p \ll \mathcal{E}$ (multiparticle) need to reduce to the standard transformation behavior, since we have observations in both cases that show no deviation from special relativity. Instead, the limit that one would like to take is that of a small energy density with respect to the Planck scale $\mathcal{E}/\text{volume} \ll m_p/l_p^3$. This means however, that the whole formalism of DSR needs to be sensitive to the volume in which we localize the energy.

Furthermore, let us recall what we found earlier that DSR can be understood as a theory with a curved momentum space. For a single particle, we were just concerned with the particle's configuration space over the particle's world line. As long as the particles are separated from each other, it is conceivable to treat their momentum spaces as independent. In the case of superpositions of modes however, the properties of the momentum spaces over the space-time in which the field extends should depend on all of the modes that contribute to the field's composition.

We are therefore lead to the conclusion that the quantity to be bounded in DSR should not be the energy of a particle, but rather the energy density of a matter field.

One sees now easily that the soccer-ball problem arises from the fact that the DSR formalism does not forbid us to consider multiparticle systems inside a region of space-time possibly as small as l_p^3 , but with an unlimited total energy of the particles. However, when we go from a microscopic system to a macroscopic system in a sequence like quark \rightarrow proton \rightarrow nucleus \rightarrow atom \rightarrow soccer-ball, then the number of constituents grows, and so does the total energy, but the energy density usually drops. Consequently, one would expect gravitational effects to become less important. In the usual DSR approach however, it is possible to place an arbitrary amount of particles arbitrarily close to each other. This is not only inconsistent with the ansatz itself to understand DSR as an effective quantum gravitational description (since the system would inevitably undergo gravitational collapse), but it is also in conflict with every day experience.

It is also important to note that for a quantized theory the number of constituents of an object is a very ill defined quantity, due to virtual particle content, and would better be avoided.

Again, we are lead to the conclusion that the quantity to consider should be the field's energy-momentum density rather than the four momentum of a particle. Or, to construct a four vector, the projection of the energy-momentum tensor on an observer's four velocity u_κ :

$$J^\nu := T^{\kappa\nu} u_\kappa \leq m_p^4. \quad (39)$$

In the rest frame this is just $\mathbf{J} = (\rho, 0, 0, 0)$. Like for the

momentum, one derives easily a deformed transformation that respects this behavior by just replacing $\omega \rightarrow J^0$, $k \rightarrow J$, and $m_p \rightarrow m_p^4$ in Eq. (2).

Now what happened to the soccer-ball problem? Well, a system's energy density is not an extensive quantity. Thus, there is no problem with an addition law. In fact, we notice immediately that the deformations and modifications vanish as one would expect for energy densities $\rho \ll m_p^4$, and soccer balls do exist and transform like we are used to from soccer balls.

If one considers the addition of energies of single modes, one arrives via Noether's theorem at the above derived quantity Eq. (31) [43]. To be in accordance with the DSR interpretation, we again lower the index and arise at the addition law Eq. (6) for the quantity with the deformed transformation behavior. The previously encountered problem that this quantity does not transform covariantly (7) is absent because the transformation \tilde{L} now is not only a function of the mode it acts on, but a function of both modes' wave vectors since both enter \mathbf{J} . Therefore, the transformations for both modes are equal. In this way, the total energy of a system can become arbitrarily large, but its transformation properties approach the special relativistic limit for small densities.

One finds the common DSR prescription for a particle with $\mathbf{k} = (\omega, k)$ if one localizes it as good as maximally possible while still respecting the generalized uncertainty, that is one sets $\mathbf{J}m_p^3 = \mathbf{k}$.

VI. PREDICTIONS REVISITED

These were the good news. Now to the bad news. The time of flight analysis for photons with different energies from γ -ray bursts has been proposed as a possible test for DSR. If the speed of light is energy dependent, one finds a possible time delay ΔT of [44,45]

$$\Delta T \sim T \frac{E}{m_p}, \quad (40)$$

where T is the duration of travel, which for a distance as large as a Gpc can be up to $\sim 10^{17}$ s. Even though for typical energies like $E \sim \text{GeV}$, the ratio to the Planck mass is tiny $E/m_p \sim 10^{-19}$, the long distance traveled results in $\Delta T \sim 10^{-2}$ s. This time delay is comparable to the typical duration of the burst, and thus potentially measurable.

However, having come to the conclusion that not the energy of a single particle is the relevant quantity but rather its energy density that curves the space it propagates in, let us repeat this analysis. The typical peak energy of a γ -ray burst is ~ 100 keV, but let us consider one of highest peak energy $\sim \text{GeV}$, which will have a typical localization of $\sim \text{fm}$, and an energy density of roughly $\rho \sim 10^{-76} m_p/l_p^3$. Thus, the effect is about 57 orders of magnitude smaller than predicted [46].

It should be pointed out that this conclusion does not apply for theories with a modified dispersion relation that explicitly break Lorentz-invariance. All the here made investigations are based on the assumption of observer-independence without a preferred frame.

VII. DISCUSSION AND CONCLUSION

In the previous sections we have seen that the transfer of the single particle DSR prescription to a field theory needs to be formulated in the field's densities rather than in integrated quantities of total four momentum. The dependence on the volume inside which energy is accumulated is necessary to recover the standard transformation behavior in the limit when the density is small compared to the Planck scale, and quantum gravitational effects can safely be neglected. It has also been previously pointed out in Ref. [34] that the soccer-ball problem might be due to the use of quantities with inappropriate dimensionality. Another related approach, suggested in Ref. [18], is that the relevant quantity could scale with the number of constituents.

We have seen that the soccer-ball problem is naturally absent if one assumes a deformation of special relativity that saturates with the energy density approaching the Planckian limit. The total energy adds linearly, but its transformation is deformed as a function of the energy density. The total energy of macroscopic systems thus can exceed the Planck mass, while the dropping energy density assures the recovery of special relativity. As a consequence, predictions for the measurement of an energy dependent speed of light with γ -ray bursts are rendered unobservable, since the scale for the effect is set by the typical energy density rather than the total energy of the photons.

Now why am I writing such a depressing paper? The reason is that despite the excitement that DSR has understandably caused, one should not neglect the demand for consistency. A model that is claimed to potentially describe nature must reproduce the known and well established theories in the range that we have confirmed them with observations. It is not difficult to make exciting predictions if one weakens this requirement. Even though I find the possibility to experimentally test quantum gravity extremely fascinating, one should carefully investigate the known problems of the approach as to whether they are fatal.

Deformed special relativity, in the interpretation as commonly used, is not able to reproduce the standard model of particle physics because multiparticle states can not be described. For the same reason, it is not possible to reproduce the usual transformation laws of special relativity for macroscopic objects.

The here presented analysis does not aim to provide a complete quantum field theory with DSR that incorporates the suggested framework, but it presents a starting point for

further investigations. I am summarizing the difficulties with the common approach here not because I like to tell depressing stories, but because I think that it is indeed possible to formulate a quantum field theory with DSR, that does not suffer from the above mentioned problems. This theory might be less exciting but also less speculative.

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