

**Cerenkov radiation in a Lorentz-violating and birefringent vacuum**

Brett Altschul\*

*Department of Physics Indiana University, Bloomington, Indiana 47405 USA*

(Received 8 February 2007; revised manuscript received 3 April 2007; published 4 May 2007)

We calculate the emission spectrum for vacuum Cerenkov radiation in Lorentz-violating extensions of electrodynamics. We develop an approach that works equally well in the presence or the absence of birefringence. In addition to confirming earlier work, we present the first calculation relevant to Cerenkov radiation in the presence of a birefringent photon  $k_F$  term, calculating the lower-energy part of the spectrum for that case.

DOI: [10.1103/PhysRevD.75.105003](https://doi.org/10.1103/PhysRevD.75.105003)

PACS numbers: 11.30.Cp, 11.30.Er, 41.20.Jb, 41.60.Bq

**I. INTRODUCTION**

In the past decade, a great deal of interest has developed in the possibility that Lorentz and *CPT* symmetries might not be exact in nature. If any violations of these important symmetries were discovered, they would be of tremendous importance. The form of the violations could potentially tell us a great deal about the new physics of the Planck scale. In fact, a number of candidate theories of quantum gravity suggest the possibility of Lorentz symmetry breaking in certain regimes. For example, Lorentz violation could arise spontaneously in string theory [1,2] or elsewhere [3]. There could also be Lorentz-violating physics in loop quantum gravity [4,5] and noncommutative geometry [6,7] theories, or Lorentz violation through spacetime-varying couplings [8], or anomalous breaking of Lorentz and *CPT* symmetries [9] in certain spacetimes.

Over the years, there have been many sensitive experimental tests of Lorentz symmetry. Modern tests of this type have included studies of matter-antimatter asymmetries for trapped charged particles [10–13] and bound state systems [14,15], determinations of muon properties [16,17], analyses of the behavior of spin-polarized matter [18,19], frequency standard comparisons [20–23], Michelson-Morley experiments with cryogenic resonators [24–26], Doppler effect measurements [27,28], measurements of neutral meson oscillations [29–34], polarization measurements on the light from distant galaxies [35–38], analyses of the spectra of energetic astrophysical sources [39,40], and others. There is a well-developed effective field theory framework, the standard model extension (SME), which parametrizes possible Lorentz violations in a local quantum field theory [41,42] and also in the gravity sector [43].

The general SME has an infinite number of parameters, since it includes nonrenormalizable operators of arbitrarily high dimensions. Practically, it is usually more useful to restrict attention to a finite subset of these operators. The most commonly considered subset is the minimal SME. This includes operators which are superficially renormalizable (that is, of dimension two, three, or four) and invariant under the standard model's  $SU(3)_c \times SU(2)_L \times$

$U(1)_Y$  gauge group. The minimal SME describes the forms of Lorentz violation that should be most important at lower energies. We shall only consider minimal SME operators in this paper, although higher-dimension operators could still have distinct and potentially quite interesting effects on the processes that we are interested in. We shall also specialize to Lorentz violations that are entirely in the electromagnetic sector, so that the matter sector is conventional.

Lorentz-violating field theories are extremely interesting theoretically, since they possess many new features that are absent in Lorentz-invariant models. Processes that are kinematically forbidden when Lorentz symmetry is exact may become allowed when this symmetry is weakly broken. One especially interesting process is vacuum Cerenkov radiation,  $e^- \rightarrow e^- + \gamma$ . This is the analogue of ordinary Cerenkov radiation in matter, and the threshold conditions are similar. An electron (or other charged particle) can emit low-energy Cerenkov photons when the electron's velocity exceeds the photons' phase speed. (When the photon energy becomes large enough that recoil effects are important, the threshold conditions becomes more complicated. This is natural, because the crucial quantity—the electron's velocity—does not remain constant through the duration of the emission process. What must exceed the phase speed of light in this case is the electron's average velocity during the emission process—averaged over the region of momentum space between the initial and final values of the electron momentum.)

The problem of vacuum Cerenkov radiation may be approached from several angles. There are a number of different operators in the SME photon sector that could give rise to this kind of process. How Cerenkov radiation works in the presence of a birefringent Chern-Simons term has been analyzed in detail, using both macroscopic techniques [44,45] and the microscopic language of Feynman diagrams [46]. Terms that do not induce birefringence have also been considered [47]. However, there are still ten coefficients in just the minimal SME photon sector whose effects have not yet been considered in this context.

Our goal in this paper is to develop a technique that will allow us to study the spectrum of vacuum Cerenkov radiation in modified electrodynamic theories. This will enable

\*Email address: [baltschu@indiana.edu](mailto:baltschu@indiana.edu)

us to fill in some of the gaps in our knowledge of how the various minimal SME terms impact the Cerenkov process. However, there are some questions that we shall not be able to answer using this method. In order to make our calculations tractable, we must make some general simplifying assumptions. However, all these assumptions are quite reasonable physically, because we know that any deviations from conventional electrodynamics (whether Lorentz violating or otherwise) must be very small at observable photon energies. Our method will be macroscopic, relying on methods qualitatively similar to those used in the calculation of ordinary Cerenkov radiation in dielectric materials. We shall also neglect any recoil effects; because of this, and because of the particular minimal SME operators that we are considering, the threshold condition takes its simplest form.

There are three important aspects of vacuum Cerenkov radiation that distinguish it from textbook Cerenkov radiation. These are dispersion, birefringence, and direction dependence. Not all the theories we shall consider have all of these properties, but understanding each of them will be important to a complete understanding of vacuum Cerenkov radiation. Of course, dispersion exists in real materials as well as Lorentz-violating vacua; the index of refraction will always be a function of frequency. The other two effects, which involve direction- and polarization-dependent speeds of light, are also seen in certain asymmetric crystals; however, they will be much more important and generally more complicated in Lorentz-violating field theories. Physical vacuum birefringence at observable wavelengths is strongly constrained by astrophysical experiments, but the property is still of significant theoretical interest.

As an input to our Cerenkov radiation calculations, we shall require the dispersion relations for the propagating modes of the electromagnetic fields. In the birefringent case, there will be two separate dispersion relations for each wave vector, which we shall denote by  $\omega_{(i)}(\vec{k})$ , corresponding to the polarization vectors  $\hat{\epsilon}_{(i)}$ . In general, the polarization structure of the normal modes of propagation will also depend on  $\vec{k}$ , and this dependence may be either on the magnitude  $k = |\vec{k}|$  or on the direction  $\hat{k} = \vec{k}/k$  (or potentially both). In situations without birefringence, the choice of polarization basis is unimportant, because all polarizations possess the same phase speed. In all cases we are considering, Cerenkov radiation is possible if the phase speed for at least one low-frequency mode of the electromagnetic field is less than 1.

We shall restrict our attention to the most physically relevant case, that in which the deviation from 1 of the vacuum speed of light is small,  $1 - \omega(\vec{k})/k \ll 1$ . This excludes some regimes of potential theoretical interest, but it covers any region for which there is a reasonable possibility of actually observing the Cerenkov radiation. Many of the excluded regions are also bedeviled by prob-

lems with stability or causality [48], which could prevent us from deriving meaningful results in any case. We shall also assume that the deviation of the dispersion relation from its conventional form is a slowly varying function of  $\vec{k}$ —so that  $|\vec{\nabla}_{\vec{k}}[\omega(\vec{k})/k]| \ll 1/k$ . Finally, we shall only consider linear modifications of electrodynamics and theories with conventional source terms.

In general, with possible Lorentz and *CPT* violations, the electromagnetic field of a propagating wave may not be transverse. This, however, is not an important effect in the regime we are considering, in which the deviations from conventional electrodynamics are small. To see why this effect is of secondary importance, we may consider the following dichotomy: a change in the dispersion relation of the electromagnetic waves without a change in the polarization structure can lead to Cerenkov radiation; however a change in the polarization states without a change in the dispersion relation cannot. Modifications to the phase speed are therefore more important. The existence of non-transverse propagating waves will only result in higher-order corrections to the effect we are interested in. We shall therefore neglect any changes that the new physics may make to the space of physical polarizations of the radiation field, and we assume that the normal mode polarization vectors  $\hat{\epsilon}_{(1)}(\vec{k})$  and  $\hat{\epsilon}_{(2)}(\vec{k})$  span the transverse subspace with  $\hat{\epsilon}_{(i)}(\vec{k}) \cdot \vec{k} = 0$ .

Because the modified electrodynamical theories we shall be considering are linear, we may work with each polarization mode separately. Because the sources of the field are not modified, the crucial question for each mode of the field is how much Cerenkov radiation a moving charge will emit with that wave vector and polarization, and this question may be answered by relatively conventional means. We need only calculate how much radiation would be emitted in that particular mode in an ordinary Cerenkov process, in a medium with the right dielectric constant to give the mode we are interested in the correct phase speed.

Our physically motivated approximations will also allow other simplifications. The smallness of the deviation of the phase speed from 1 ensures that the (appropriately generalized) Mach cone will always be very broad. Cerenkov photons must be emitted in directions close to the direction  $\hat{v}$  of the charge's motion. Moreover, there will only be emission if the charge's speed is close to 1.

With our conventional matter sector, 1 is the maximum achievable velocity for a moving charge; however, many of the expressions we shall derive would apply equally well to theories with Lorentz violation in the matter sector and speeds  $v > 1$  allowed. However, we shall assume  $v < 1$ , because it simplifies the accounting of which modes of the electromagnetic field contain vacuum Cerenkov radiation. In fact, it is not always a well-posed question which sector actually contains a Lorentz violation; some forms of matter-sector Lorentz violation can be defined away, a

change of coordinates moving the Lorentz violations into the gauge sector without changing the physics.

Since our study of the Cerenkov spectrum will be performed on a mode by mode basis, there are some interesting features of vacuum Cerenkov radiation which it is not possible to study with these techniques. It can be difficult or impossible to calculate the backreaction on the moving charge, because this quantity depends on the total emission in all modes of the electromagnetic field. This limits our ability to evaluate the high-energy part of the spectrum, where recoil is an important effect. It also limits us to considering moving charges with velocities significantly above the Cerenkov threshold; if the velocity were too close to the threshold, the recoil accompanying the first emitted photons could push the charge's velocity back below threshold, fundamentally altering the character of the process. Furthermore, for some modes, the condition that the deviation of the phase speed from 1 be small may not be met, and for higher-energy modes, new physics may come into play. Questions related to the overall stability of the process may also be difficult to answer, although they have already been considered, using a different technique, for the case which poses the most interesting questions in this regard; in the presence of a Lorentz-violating Chern-Simons term, there is no vacuum Cerenkov emission if and only if in the rest frame of the moving charge the energetic stability of the electromagnetic sector is manifest [44]. We shall also neglect consideration of any finite-duration effects, treating the Cerenkov radiation as a completely steady state process. Thus, some interesting phenomena that arise in real, finite period Cerenkov processes (such as diffraction) will not have any analogues in our analysis.

This paper is organized as follows. In Sec. II, we introduce a number of interesting theories, most of them Lorentz violating, in which vacuum Cerenkov radiation could be possible. In Sec. III, we show how to generalize the usual equations governing Cerenkov radiation, and in Sec. IV, we apply these generalizations to the specific models introduced in Sec. II. Section V presents our conclusions and outlook.

## II. MODIFIED ELECTRODYNAMIC MODELS

The free Lagrange density for the electromagnetic sector, without any of the modifications that would make vacuum Cerenkov radiation possible, is

$$\mathcal{L}_0 = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - j^\mu A_\mu. \quad (1)$$

For our purposes here, the source term may be taken to be externally specified, corresponding to a point charge  $e$  moving with velocity  $\vec{v}$  well above the Cerenkov threshold.

In this section, we shall only be concerned with the free propagation of electromagnetic waves in vacuum. This restricts us to considering modifications of the Lagrangian which are bilinear in the electromagnetic field. We shall also consider only those operators which are

superficially renormalizable—that is, operators of dimensions two, three, and four. We shall not insist on gauge invariance for the dimension two operators, since there are potentially interesting physics associated with photon mass terms. However, we shall only consider higher-dimension operators that are gauge invariant, at least at the level of the action.

We shall consider modifications to  $\mathcal{L}_0$  one at a time. Within the minimal SME, there are two types of gauge-invariant, renormalizable Lorentz-violating coefficients in the purely electromagnetic sector. (Since we are considering the Cerenkov response to an externally prescribed charge density, we shall not consider Lorentz violation in the matter sector, although Lorentz violations can have important effects on how real particles move.) These are the *CPT*-odd Chern-Simons term

$$\mathcal{L}_{AF} = \frac{1}{2}k_{AF}^\mu \epsilon_{\mu\nu\rho\sigma} F^{\nu\rho} A^\sigma \quad (2)$$

and the *CPT*-even term

$$\mathcal{L}_F = -\frac{1}{4}k_F^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \quad (3)$$

The four-index tensor  $k_F$  has the symmetries of the Riemann tensor and a vanishing double trace, leaving it with 19 independent coefficients. We shall consider the  $k_{AF}$  and  $k_F$  separately, because doing so will make the analysis much more elegant and intuitive. However, there would be no impediment in principle to doing our calculations in the presence of both  $k_{AF}$  and  $k_F$ , for which case the dispersion relations and the correct techniques for identifying the elliptically polarized normal modes of propagation are known [42].

The Chern-Simons term  $\mathcal{L}_{AF}$  (which is gauge invariant up to a total derivative) gives different dispersion relations for right- and left-handed electromagnetic waves. The positive- and negative-helicity waves have frequencies [35]

$$\omega_\pm^2 = k^2 \pm \frac{k_{AF}^0 k - |\vec{k}_{AF}| \omega_\pm \cos\theta_{AF}}{\sqrt{\frac{1}{4} - \frac{\vec{k}_{AF}^2 \sin^2\theta_{AF}}{\omega_\pm^2 - k^2}}}, \quad (4)$$

where  $\theta_{AF}$  is the angle between  $\vec{k}$  and  $\vec{k}_{AF}$ . Equation (4) is not a closed form expression for  $\omega_\pm(\vec{k})$ ; however, expanding it to leading order in the Lorentz violation, we get

$$\omega_\pm \approx k \pm (k_{AF}^0 - |\vec{k}_{AF}| \cos\theta_{AF}). \quad (5)$$

This covers the range of  $k$  that we are interested in. However, it should be noted that, at small wave numbers, the dispersion relation becomes problematical. If only  $k_{AF}^0$  is nonzero, then we have  $\omega_\pm^2 = k(k \pm 2k_{AF}^0)$ , so that  $\omega$  may become imaginary. There are then runaway solutions, which can be fixed only by allowing for acausal signal propagation. It is not clear whether the theory can be physically meaningful in this regime, especially when considering an unconventional radiation process like

Cerenkov radiation. The previous macroscopic analyses of vacuum Cerenkov radiation in the presence of  $k_{AF}$  have focused more on the spacelike case, in which such instabilities do not occur.

With  $k_F$  only, the dispersion relations are determined by the matrix equation

$$[\delta_{jk}p^2 + p_j p_k + 2(k_F)_{j\alpha\beta k} p^\alpha p^\beta][\hat{\epsilon}_{(i)}]_k = 0, \quad (6)$$

where  $p^\mu = (\omega, \vec{k})$  is the photon four-momentum. In this case, the phase speed and polarization structure are independent of the wave number, although they do depend on the propagation direction. To leading order in  $k_F$ , the polarizations are transverse and orthogonal, and the frequencies are [49]

$$\omega_\pm \approx [1 + \rho(\hat{k}) \pm \sigma(\hat{k})]k, \quad (7)$$

where  $\rho(\hat{k}) = -\frac{1}{2}\tilde{k}^\alpha{}_\alpha$ , and  $\sigma^2(\hat{k}) = \frac{1}{2}\tilde{k}^{\alpha\beta}\tilde{k}_{\alpha\beta} - \rho^2(\hat{k})$ , with  $\tilde{k}^{\alpha\beta} = k_F^{\alpha\mu\beta\nu}\hat{p}_\mu\hat{p}_\nu$  and  $\hat{p}^\mu = (1, \hat{k})$ . Since the  $k_F$  term is dimensionless, there is no dispersion in the photon spectrum; the phase speed depends only on  $\hat{k}$ . Moreover, there is no birefringence if  $\sigma = 0$ , which was the case considered in [47].

Using the explicit leading-order expression for the dispersion relation, it is possible to recast the eigenvector condition (6) as

$$[2(\rho \pm \sigma)\delta_{jk} - \hat{k}_j\hat{k}_k + 2\tilde{k}_{jk}][\hat{\epsilon}_{(\pm)}]_k = 0. \quad (8)$$

In a primed frame where a photon's energy-momentum is  $\hat{p}'^\mu = (1, \hat{e}_3)$ , the polarization vectors are

$$\hat{\epsilon}'_{(\pm)} \propto (\sin\xi, \pm 1 - \cos\xi, 0), \quad (9)$$

where  $\sigma \sin\xi = \tilde{k}'_{12}$  and  $\sigma \cos\xi = \frac{1}{2}(\tilde{k}'_{11} - \tilde{k}'_{22})$ . In these coordinates,  $\hat{\epsilon}'_{(+)}$  makes an angle  $\xi/2$  with the  $x'$  axis.

Another class of possibly Lorentz-violating models with photon speeds less than 1 may also be considered. These are models which break gauge invariance. The Lagrange density

$$\mathcal{L}_M = M^{\mu\nu}A_\mu A_\nu \quad (10)$$

has a generalized photon mass term. In the presence of  $\mathcal{L}_M$ , there are generally three propagating modes of the electromagnetic field. However, we shall neglect the novel longitudinal mode, because if the breaking of gauge invariance is weak, this mode will be correspondingly weakly coupled to charges.

Of interest is obviously the Lorentz-invariant Proca theory, with  $M^{\mu\nu} = \frac{1}{2}g^{\mu\nu}m^2$ . Other Lorentz-violating versions have also aroused some recent interest [50–52]. In all the cases that have been considered, there exists a frame in which  $M^{\mu\nu}$  is diagonal, with non-negative eigenvalues, at most one of which is different from the others. These models are never birefringent. If  $M^0_0$  vanishes, but the  $M^j_k = \frac{1}{2}m_1^2\delta^j_k$  are nonvanishing, then the theory contains

only two propagating modes; while if  $M^0_0$  is finite as well, then there is also a propagating longitudinal mode. However, the dispersion relation for the transverse modes is always

$$\omega^2 = k^2 + m_1^2. \quad (11)$$

If one of the spatial elements on the diagonal of  $M^{\mu\nu}$  differs from the others, then the situation is more complex. The basis of propagation states is not orthogonal; however, the only mode with an unconventional dispersion relation is again essentially longitudinally polarized. (As the Lorentz violation gets smaller, the associated polarization vector moves closer to  $\hat{k}$ .) The net result is that the transverse modes propagate at the same rate, and this rate is again independent of the propagation direction.

So for all the  $M^{\mu\nu}$  of interest, the transverse modes have the same type of dispersion relation. Unfortunately, while the group velocity for this dispersion relation is always less than 1, the phase velocity is the reciprocal of the group velocity and is hence always greater than 1. There is thus no vacuum Cerenkov radiation in these theories, and we shall not consider them any further.

### III. CALCULATIONAL TECHNIQUES

We must now generalize the usual techniques used to calculate rates of Cerenkov emission to cover the vacuum cases we are interested in. For a charge  $e$  moving with velocity  $\vec{v}$ , subject to  $v < 1$ , there may be Cerenkov emission if  $v$  is greater than the phase speed of light in some direction. In a conventional dielectric material, where the index of refraction is direction and polarization independent and constant (or only slowly varying) as a function of wave number, there is a sharp Mach cone, with a discontinuity (or near discontinuity) across it. The moving charge emits photons, and the cone represents the signal front for their propagation. The Cerenkov angle  $\theta_C$  is the angle between the direction  $\hat{v}$  and the propagation of the emitted photons. With dispersion, this angle becomes frequency dependent. In an anisotropic vacuum, it will also generally depend on the azimuthal angle around  $\hat{v}$ , and with birefringence, it will depend on the polarization as well. In the birefringent case, there may exist Cerenkov emission for only one of the polarizations corresponding to a given  $\vec{k}$ , since the phase speeds for the two polarizations are not the same. The opening angle of the Mach cone is  $\frac{\pi}{2} - \theta_C$ .

If there is significant dispersion, there will generally not be a Mach cone defined by a sharp shock front. However, the radiation at a given fixed frequency will all be located on a cone, although with Lorentz violation that cone need not be right angled or circular.

In the ordinary case,  $\theta_C$  is determined by the coherence condition,

$$\cos\theta_C = \frac{1}{vn}. \quad (12)$$



This can be thought of as simply the angle at which the radiation in the far field does not interfere with itself destructively, so that there is a net nonzero Poynting flux. (Since interference between waves emitted by the moving charge at different points in time is the crucial effect, the derivation of the coherence condition requires that the velocity not change appreciably during the time period under consideration.) The same constructive interference is still necessary in the Lorentz-violating case if there is to be a net outflow of radiation. So an analogous condition will govern the direction in which radiation is emitted in the more complicated cases under consideration here; however, the analog of (12) is no longer a straightforward expression for  $\theta_C$  as a function of the other parameters. The effective index of refraction can be a function of direction and hence of  $\theta_C$ . To generalize (12), we note (omitting any dependences on polarization) that  $\cos\theta_C = \hat{v} \cdot \hat{k}$ , and  $n$  generalizes to  $k/\omega(\vec{k})$ . A simple rearrangement then yields the generalized coherence condition

$$\vec{v} \cdot \vec{k} = \omega(\vec{k}). \quad (13)$$

For arbitrary anisotropic dispersion relations, (13) may be difficult to solve. However, when the effects of new physics are small, we may solve (13) perturbatively. When  $1 - \omega(\vec{k})/k \ll 1$ , the angle  $\theta_C$  will be small. The photons are all emitted in directions  $\hat{k}$  very close to  $\hat{v}$ , so to leading order we may calculate  $\theta_C$  by approximating  $\vec{k}$  by  $k\hat{v}$  on the right-hand side of (13). This gives

$$\cos\theta_C \approx 1 - \frac{\theta_C^2}{2} \approx \frac{\omega(k\hat{v})}{kv}, \quad (14)$$

$$\theta_C^2 \approx 2[1 - \omega(k\hat{v})/kv]. \quad (15)$$

Cerenkov radiation is emitted if  $\theta_C^2 > 0$ . At this level of approximation, the Mach cone is right angled and circular. Any obliquities are higher-order effects. However, the opening angle of the cone will vary with the direction  $\hat{v}$  of the charge's movement. We shall make extensive further use of the approximation  $\vec{k} \approx k\hat{v}$  in obtaining other leading-order results; this amounts to approximating the Mach cone as being a flat planar wave front.

In the absence of birefringence, (15) is all that is needed to determine the leading-order character of the radiation. Because the theory is linear, we may look at the electromagnetic field one mode at a time. In a momentum-space neighborhood of any given mode, the theory looks like conventional electrodynamics but with a speed of light  $\omega(\vec{k})/k \equiv 1/n(\vec{k})$  different from 1. The effects of dispersion are suppressed, because  $\omega(\vec{k})/k$  is slowly varying.

However, we must still deal with the birefringent case. Ordinary Cerenkov radiation is linearly polarized in the plane defined by  $\hat{v}$  and  $\hat{k}$ ; we shall denote the corresponding polarization vector by  $\hat{\epsilon}_{(0)}(\vec{k})$ . Obviously, birefringence will change the polarization of the emitted radiation.

However, the changes are really quite simple, again because we can look at the theory one mode of the field at a time. If the coherence condition is satisfied for a given mode of the field, with wave vector  $\vec{k}$  and polarization  $\hat{\epsilon}_{(i)}$ , what happens in all the orthogonal modes is unimportant. In particular, the emission in the mode we are interested in is exactly the same as in any other theory with the same  $\vec{v}$  and  $n[\vec{k}, \hat{\epsilon}_{(i)}]$ . So we can imagine replacing the theory by one with a constant, polarization-independent index of refraction  $\tilde{n} = n[\vec{k}, \hat{\epsilon}_{(i)}]$ . Calculating the emitted power in the mode of interest is then trivial. It is just the total power emitted in the conventional theory with  $\tilde{n}$ , times the squared overlap between the conventional linear polarization mode and the mode we are studying, or just  $|\hat{\epsilon}_{(i)} \cdot \hat{\epsilon}_{(0)}|^2$ . This technique for determining the radiated power is valid even beyond the leading-order approximation.

The power emitted per unit frequency in ordinary Cerenkov radiation is  $P(\omega) = \frac{e^2}{4\pi} \sin^2\theta_C \omega$ . The generalizations required by the presence of Lorentz violations and dispersion are minor, at least at leading order.  $\theta_C^2$  must be determined from (15), and we must include the polarization overlap factor, which adds the only real complication. At leading order, the Mach cone is right and circular, so  $\theta_C$  does not depend on the azimuthal angle  $\phi$ . Nor, at leading order, do the normal mode polarization vectors  $\hat{\epsilon}_{(i)}$  depend on  $\phi$ —and for the same reason, since we can approximate the  $\hat{\epsilon}_{(i)}(\vec{k})$  by  $\hat{\epsilon}_{(i)}(k\hat{v})$ . However, the polarization of ordinary Cerenkov radiation does depend strongly on  $\phi$ . In the same leading-order approximation we have been using, the polarization vector  $\hat{\epsilon}_{(0)}$  is  $\hat{\rho} \equiv \hat{\phi} \times \hat{v}$ . So the emitted power may depend on  $\phi$ . We must therefore express the power per unit frequency per unit of azimuthal angle for a given polarization mode; this is

$$P_{(i)}(\omega, \phi) \approx \frac{e^2}{(2\pi)^2} |\hat{\epsilon}_{(i)} \cdot \hat{\rho}|^2 [1 - \omega_{(i)}(k\hat{v})/kv] \omega_{(i)}(k\hat{v}). \quad (16)$$

(At this level of approximation, we may also replace the terminal  $\omega_{(i)}$  by  $k$ .) The average of  $|\hat{\epsilon}_{(i)} \cdot \hat{\rho}|^2$  over all  $\phi$  is always  $\frac{1}{2}$ . In a theory without birefringence, all the factors in (16) except  $|\hat{\epsilon}_{(i)} \cdot \hat{\rho}|^2$  are equal for the two polarizations, and the total emitted power, summed over both polarizations, is independent of  $\phi$ , as we might expect.

To leading order, we need not distinguish whether our expression for the power is the power per unit frequency or per unit wave number, nor need we worry about how the energy-momentum tensor is modified by the new physics; these effects are only important beyond leading order. So in (16), we have a simple expression for the leading-order Cerenkov spectrum in any normal mode of the radiation field that satisfies our assumptions.

### IV. APPLICATION TO SPECIFIC MODELS

We may now apply our method to the models discussed in Sec. II. Considering the  $k_{AF}$  model first, we note that, according to the leading-order expression (5) for the dispersion relation, exactly one mode of the field is superluminal and one subluminal for each value of  $\vec{k}$ . Furthermore, since the normal modes of propagation are always circularly polarized waves, the overlap expression  $|\hat{\epsilon}_{(i)} \cdot \hat{\epsilon}_{(0)}|^2$  is always exactly equal to  $\frac{1}{2}$ . The Cerenkov angle is also easy to calculate at leading order. For whichever polarization is moving more slowly than  $v$ , it is

$$\theta_C^2 \approx 2 \left( 1 - \frac{1}{v} + \frac{|k_{AF}^0 - \vec{k}_{AF} \cdot \hat{v}|}{kv} \right). \quad (17)$$

Since this depends on  $k$ , there is not a sharply defined Mach cone. The sign of the expression inside the absolute value determines which polarization this represents. The helicity of the emitted photons is  $-\text{sgn}(k_{AF}^0 - \vec{k}_{AF} \cdot \hat{v})$ ; this agrees with the results in [45] for emission close to the direction of  $\vec{v}$ .

The total power emitted per unit frequency is

$$P(\omega) \approx \frac{e^2}{4\pi} \left( 1 - \frac{1}{v} + \frac{|k_{AF}^0 - \vec{k}_{AF} \cdot \hat{v}|}{kv} \right) k, \quad (18)$$

and this is emitted in an azimuthally symmetric pattern around  $\hat{v}$ . It is clear in this case that the dispersion will cut off the Cerenkov spectrum at high energies, because the absolute value term in (17) is divided by  $k$ . For large wave numbers  $k > |k_{AF}^0 - \vec{k}_{AF} \cdot \hat{v}|/(1 - v)$ , there is no emission, and this ensures that there is no ultraviolet divergence in the total power.

For the theory with  $k_F$ , the results are equally straightforward. In this case, if  $\rho(\hat{v}) < -|\sigma(\hat{v})|$ , there can be Cerenkov radiation in both polarization modes. However, unless  $\sigma(\hat{v}) = 0$ , the two polarizations will have Mach cones of different width and different rates of emission. Indeed, the Cerenkov angles are

$$\theta_C^2 \approx 2 \left[ 1 - \frac{1}{v} - \frac{\rho(\hat{v}) \pm \sigma(\hat{v})}{v} \right]. \quad (19)$$

Figure 1 shows the possible shapes of the Mach cones for two different particle velocities  $\vec{v}$ . In one direction, there are two broad cones, corresponding to two different polarization states, but in the second direction, only a single Mach cone is possible.

If we choose coordinates so that  $\hat{v} = \hat{e}_3$ , the polarization vectors corresponding to the two cones are given by (9). The angular overlap factors are then, using  $\hat{\phi} \times \hat{v} = (\cos\phi, \sin\phi, 0)$ ,

$$|\hat{\epsilon}_{(+)} \cdot \hat{\epsilon}_{(0)}|^2 \approx \cos^2(\phi - \xi/2), \quad (20)$$

$$|\hat{\epsilon}_{(-)} \cdot \hat{\epsilon}_{(0)}|^2 \approx \sin^2(\phi - \xi/2). \quad (21)$$

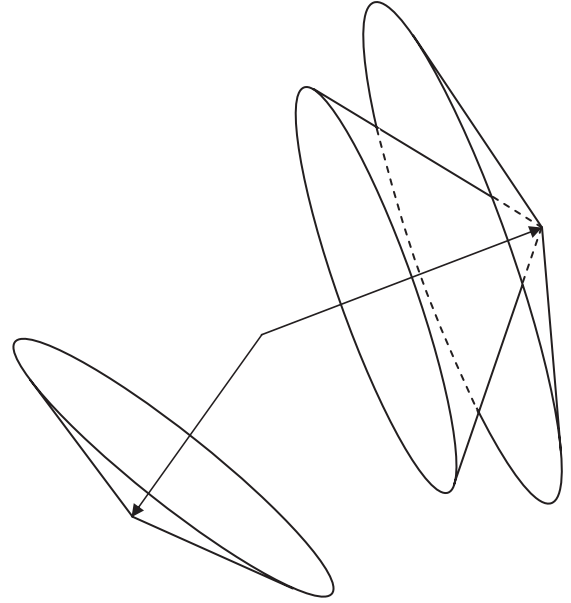


FIG. 1. Possible shapes of the Mach cones corresponding to charges moving in two different directions. In one direction, two Mach cones are possible, but in another direction only one. In the leading-order approximation, the cones are right angled and circular, although the cones shown here are exaggeratedly narrow.

If  $\sigma(\hat{v}) = 0$ , then the radiation intensity is independent of  $\phi$ ; moreover, the Cerenkov angle—and thus the power emitted—agrees in this case with the results calculated in [47] by somewhat different methods. However, because there more generally is birefringence, there can be an angular dependence in the total power. The two polarizations are emitted in two perpendicularly oriented dipole-like patterns. (These are not, however, dipole radiation patterns in the usual sense, since the waves are all directed into a narrow angular range around the direction  $\hat{v}$ .) Figure 2 shows the intensity and polarization on the surface of one of the two possible Mach cones.

The expressions for  $\theta_C^2$  and the polarization overlap factors give us the power emitted and its angular distribution,

$$P_{(+)}(\omega, \phi) \approx \frac{e^2}{(2\pi)^2} \left[ 1 - \frac{1}{v} - \frac{\rho(\hat{v}) + \sigma(\hat{v})}{v} \right] \times \omega \cos^2(\phi - \xi/2), \quad (22)$$

$$P_{(-)}(\omega, \phi) \approx \frac{e^2}{(2\pi)^2} \left[ 1 - \frac{1}{v} - \frac{\rho(\hat{v}) - \sigma(\hat{v})}{v} \right] \times \omega \sin^2(\phi - \xi/2). \quad (23)$$

These results should be accurate whenever our approximations are valid. However, with the  $k_F$  term, there is an obvious problem with the spectrum at large  $k$ . Since  $\theta_C^2$

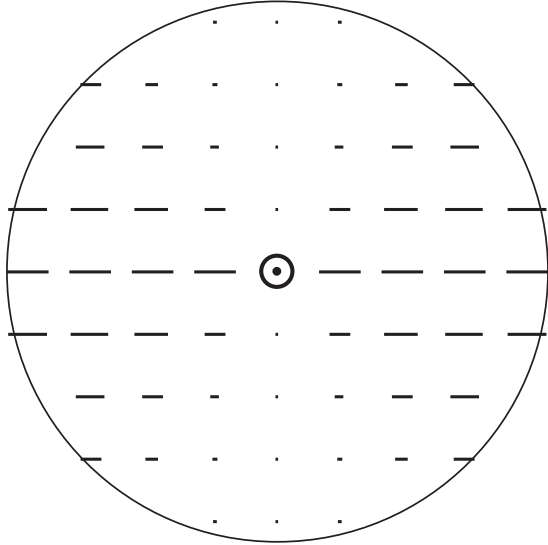


FIG. 2. Polarization of the radiation on a single Mach cone in the birefringent  $k_F$  theory. The cone is seen from above, with the charge moving out of the plane toward the viewer. The lines indicate the polarization direction of the radiation at various points on the cone's surface, with their lengths denoting the relative intensities.

and the polarization factors are independent of  $k$ , the spectrum appears to diverge at high energies.

Some new effect, not considered here, must enter to cut off the spectrum. We have neglected the recoil of the emitting particle. Including it ought to render the whole expression finite; the charge will not radiate away more energy than it possesses [53]. So there is a natural cutoff at the energy scale of the radiating particle itself. However, this is not necessarily the relevant cutoff. New physics may enter at a scale lower than the energy scale of the particle, and the scale of the new physics may represent the physically meaningful cutoff scale.

These questions were previously discussed in [47] and a partial solution put forward for the special case considered there. The part of  $k_F$  that does not cause birefringence mixes with Lorentz-violating coefficients in the matter sector under renormalization [54]. Although the pure gauge sector does not make reference to any mass scale, the matter sector will contain massive charged particles, which will set a definite scale for the theory. This implies that new physics must enter at a scale  $\Lambda \sim mk_F^{-1/2}$ , where  $m$  is the lightest charged particle mass; without the appearance of new physics, the theory will exhibit pathological properties at high energies [48].  $\Lambda$  is essentially the largest scale at which new physics can enter, and it is comparable to the threshold energy for the Cerenkov process.

The appearance of a high-energy cutoff comparable to the energy threshold for vacuum Cerenkov is actually desirable, since it provides a uniform ultraviolet regulator for the total power emitted (although the Cerenkov spectrum, even with the cutoff, still has a number of counter-

intuitive properties [47]). In this special case, any charge emitting vacuum Cerenkov radiation must have an energy comparable to or larger than  $\Lambda$ , so we expect  $\Lambda$ —not the charge's energy—to be the most relevant cutoff scale. A charge whose energy is far above the threshold level can emit radiation at a rate limited by  $\Lambda$  for an extended period, with the velocity decaying only comparatively slowly over this time, so that recoil effects are unimportant.

However, the birefringent part of  $k_F$  does not mix with any other Lorentz-violating coefficients at leading order, and so an electromagnetic theory containing only this form of Lorentz violation is equally valid at all energy scales. It is obviously possible that new physics may cut off this theory as well, but there is no indication of at what scale that cutoff should come. Or it may be impossible to consider this theory without taking into account the backreaction on the charge, which loses momentum as it radiates. Unlike the previous case, it is possible that recoil effects might provide the only cutoff for the Cerenkov spectrum. In any case, the spectrum we have calculated should be a perfectly valid first approximation at sufficiently low energies, but to understand the totality of the vacuum Cerenkov process in the presence of this kind of Lorentz violation, a different approach is required.

The new physics which should cut off the theory with a nonbirefringent  $k_F$  enter the photon sector through radiative corrections, and it would be interesting to consider how other loop effects might impact our results. One type of effect may be of particular interest—photon splitting,  $\gamma \rightarrow N\gamma$ , which is forbidden on shell in a gauge- and Lorentz-invariant theory. Photon splitting amplitudes are generally nonzero in the presence of Lorentz violation, however [55]. An entirely speculative yet interesting possibility in the context of vacuum Cerenkov radiation is that the emitted photons may split into multiple collinear photons, and the amplitude for this process may interfere destructively with the primary Cerenkov amplitude. To determine whether this could actually occur would require evaluation of the amplitude for many-photon Cerenkov emission, including its phase, as well as the photon splitting amplitude in the presence of a general  $k_F$ ; and neither of these amplitudes is known at present. A tricky balancing between terms at different orders in  $e^2$  would also be required in this scenario. So whether photon splitting or other radiative corrections have important impacts on the Cerenkov process is unknown.

## V. CONCLUSION

In this paper, we have worked out a method for determining the spectrum of vacuum Cerenkov radiation, working with the electromagnetic field mode by mode. For our leading-order results to be useful for a given mode, Lorentz violations must affect the energy-momentum relation for the mode in question only slightly. However, apart from that, the method is rather general, allowing us to treat the

$k_{AF}$  and  $k_F$  terms, birefringent and not, on equal footing. We have rederived previous results for a number of cases, and we have also provided the first calculation of the lower-energy part of the Cerenkov spectrum in the presence of a birefringent  $k_F$ .

However, there are many interesting questions that are still unanswered. There are regions of the spectrum where our approximations are simply not valid. The small  $k$  region in the  $k_{AF}$  theory is outside the realm of our approximations' validity, although it has been examined by other means. The large  $k$  domain in the presence of  $k_F$  (where new physics may come into play to cut off the Cerenkov process) raises further questions. Much more could also be said about the backreaction on the moving charge. One technique that might make it easier to take this effect into account would be to calculate the Cerenkov

spectrum using Feynman diagrams, as was done for the  $k_{AF}$  in [46]. Without understanding recoil effects, it may be impossible to calculate the highest-energy portions of the Cerenkov spectrum or even the low-energy spectrum for particles with energies very close to the threshold. Because these kinds of deep questions still exist, vacuum Cerenkov radiation remains a very interesting area in the study of Lorentz violation.

## ACKNOWLEDGMENTS

The author is grateful to V. A. Kostelecký for helpful discussions. This work is supported in part by funds provided by the U. S. Department of Energy (D.O.E.) under Cooperative Research Agreement No. DE-FG02-91ER40661.

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- [1] V. A. Kostelecký and S. Samuel, Phys. Rev. D **39**, 683 (1989).
  - [2] V. A. Kostelecký and R. Potting, Nucl. Phys. **B359**, 545 (1991).
  - [3] B. Altschul and V. A. Kostelecký, Phys. Lett. B **628**, 106 (2005).
  - [4] R. Gambini and J. Pullin, Phys. Rev. D **59**, 124021 (1999).
  - [5] J. Alfaro, H. A. Morales-Técotl, and L. F. Urrutia, Phys. Rev. D **65**, 103509 (2002).
  - [6] I. Mocioiu, M. Pospelov, and R. Roiban, Phys. Lett. B **489**, 390 (2000).
  - [7] S. M. Carroll, J. A. Harvey, V. A. Kostelecký, C. D. Lane, and T. Okamoto, Phys. Rev. Lett. **87**, 141601 (2001).
  - [8] V. A. Kostelecký, R. Lehnert, and M. J. Perry, Phys. Rev. D **68**, 123511 (2003).
  - [9] F. R. Klinkhamer and C. Rupp, Phys. Rev. D **70**, 045020 (2004).
  - [10] R. Bluhm, V. A. Kostelecký, and N. Russell, Phys. Rev. Lett. **79**, 1432 (1997).
  - [11] R. Bluhm, V. A. Kostelecký, and N. Russell, Phys. Rev. D **57**, 3932 (1998).
  - [12] G. Gabrielse, A. Khabbaz, D. S. Hall, C. Heimann, H. Kalinowsky, and W. Jhe, Phys. Rev. Lett. **82**, 3198 (1999).
  - [13] H. Dehmelt, R. Mittleman, R. S. Van Dyck, Jr., and P. Schwinberg, Phys. Rev. Lett. **83**, 4694 (1999).
  - [14] R. Bluhm, V. A. Kostelecký, and N. Russell, Phys. Rev. Lett. **82**, 2254 (1999).
  - [15] D. F. Phillips, M. A. Humphrey, E. M. Mattison, R. E. Stoner, R. F. C. Vessot, and R. L. Walsworth, Phys. Rev. D **63**, 111101(R) (2001).
  - [16] R. Bluhm, V. A. Kostelecký, C. D. Lane, Phys. Rev. Lett. **84**, 1098 (2000).
  - [17] V. W. Hughes *et al.*, Phys. Rev. Lett. **87**, 111804 (2001).
  - [18] R. Bluhm and V. A. Kostelecký, Phys. Rev. Lett. **84**, 1381 (2000).
  - [19] B. R. Heckel, C. E. Cramer, T. S. Cook, S. Schlamminger, E. G. Adelberger, and U. Schmidt, Phys. Rev. Lett. **97**, 021603 (2006).
  - [20] C. J. Berglund, L. R. Hunter, D. Krause, Jr., E. O. Prigge, M. S. Ronfeldt, and S. K. Lamoreaux, Phys. Rev. Lett. **75**, 1879 (1995).
  - [21] V. A. Kostelecký and C. D. Lane, Phys. Rev. D **60**, 116010 (1999).
  - [22] D. Bear, R. E. Stoner, R. L. Walsworth, V. A. Kostelecký, and C. D. Lane, Phys. Rev. Lett. **85**, 5038 (2000).
  - [23] P. Wolf, F. Chapelet, S. Bize, and A. Clairon, Phys. Rev. Lett. **96**, 060801 (2006).
  - [24] P. Antonini, M. Okhapkin, E. Göklü, and S. Schiller, Phys. Rev. A **71**, 050101 (2005).
  - [25] P. L. Stanwix, M. E. Tobar, P. Wolf, M. Susli, C. R. Locke, E. N. Ivanov, J. Winterflood, and F. van Kann, Phys. Rev. Lett. **95**, 040404 (2005).
  - [26] S. Herrmann, A. Senger, E. Kovalchuk, H. Müller, and A. Peters, Phys. Rev. Lett. **95**, 150401 (2005).
  - [27] G. Saathoff, S. Karpuk, U. Eisenbarth, G. Huber, S. Krohn, R. Muñoz Horta, S. Reinhardt, D. Schwalm, A. Wolf, and G. Gwinner, Phys. Rev. Lett. **91**, 190403 (2003).
  - [28] C. D. Lane, Phys. Rev. D **72**, 016005 (2005).
  - [29] V. A. Kostelecký, Phys. Rev. Lett. **80**, 1818 (1998).
  - [30] V. A. Kostelecký, Phys. Rev. D **61**, 016002 (1999).
  - [31] Y. B. Hsiung, Nucl. Phys. B, Proc. Suppl. **86**, 312 (2000).
  - [32] K. Abe *et al.*, Phys. Rev. Lett. **86**, 3228 (2001).
  - [33] J. M. Link *et al.*, Phys. Lett. B **556**, 7 (2003).
  - [34] B. Aubert *et al.*, Phys. Rev. Lett. **96**, 251802 (2006).
  - [35] S. M. Carroll, G. B. Field, and R. Jackiw, Phys. Rev. D **41**, 1231 (1990).
  - [36] S. M. Carroll and G. B. Field, Phys. Rev. Lett. **79**, 2394 (1997).
  - [37] V. A. Kostelecký and M. Mewes, Phys. Rev. Lett. **87**, 251304 (2001).
  - [38] V. A. Kostelecký and M. Mewes, Phys. Rev. Lett. **97**, 140401 (2006).



- [39] T. Jacobson, S. Liberati, and D. Mattingly, *Nature* (London) **424**, 1019 (2003).
- [40] B. Altschul, *Phys. Rev. Lett.* **96**, 201101 (2006).
- [41] D. Colladay and V. A. Kostelecký, *Phys. Rev. D* **55**, 6760 (1997).
- [42] D. Colladay and V. A. Kostelecký, *Phys. Rev. D* **58**, 116002 (1998).
- [43] V. A. Kostelecký, *Phys. Rev. D* **69**, 105009 (2004).
- [44] R. Lehnert and R. Potting, *Phys. Rev. Lett.* **93**, 110402 (2004).
- [45] R. Lehnert and R. Potting, *Phys. Rev. D* **70**, 125010 (2004); **70**, 129906 (2004).
- [46] C. Kaufhold and F. R. Klinkhamer, *Nucl. Phys.* **B734**, 1 (2006).
- [47] B. Altschul, *Phys. Rev. Lett.* **98**, 041603 (2007).
- [48] V. A. Kostelecký and R. Lehnert, *Phys. Rev. D* **63**, 065008 (2001).
- [49] V. A. Kostelecký and M. Mewes, *Phys. Rev. D* **66**, 056005 (2002).
- [50] G. Gabadadze and L. Grisa, *Phys. Lett. B* **617**, 124 (2005).
- [51] G. Dvali, M. Papucci, and M. D. Schwartz, *Phys. Rev. Lett.* **94**, 191602 (2005).
- [52] B. Altschul, *Phys. Rev. D* **73**, 036005 (2006).
- [53] T. Jacobson, S. Liberati, and D. Mattingly, *Ann. Phys.* (N.Y.) **321**, 150 (2006).
- [54] V. A. Kostelecký, C. D. Lane, and A. G. M. Pickering, *Phys. Rev. D* **65**, 056006 (2002).
- [55] V. A. Kostelecký and A. G. M. Pickering, *Phys. Rev. Lett.* **91**, 031801 (2003).