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Extra force in $f(R)$ modified theories of gravity

Orfeu Bertolami*

Instituto Superior Técnico, Departamento de Física and Centro de Física dos Plasmas, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal

Christian G. Böhmer†

Institute of Cosmology & Gravitation, University of Portsmouth, Portsmouth PO1 2EG, United Kingdom

Tiberiu Harko‡

Department of Physics and Center for Theoretical and Computational Physics, The University of Hong Kong, Pok Fu Lam Road, Hong Kong

Francisco S. N. Lobo§

Institute of Cosmology & Gravitation, University of Portsmouth, Portsmouth PO1 2EG, United Kingdom and Centro de Astronomia e Astrofísica da Universidade de Lisboa, Campo Grande, Ed. C8 1749-016 Lisboa, Portugal

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The equation of motion for massive particles in $f(R)$ modified theories of gravity is derived. By considering an explicit coupling between an arbitrary function of the scalar curvature, R, and the Lagrangian density of matter, it is shown that an extra force arises. This extra force is orthogonal to the four-velocity and the corresponding acceleration law is obtained in the weak-field limit. Connections with MOND and with the Pioneer anomaly are further discussed.

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I. INTRODUCTION

Higher-order curvature theories of gravity have recently received a great deal of attention in connection with the possibility of giving rise to cosmological models in the context of which one can address the issue of the accelerated expansion of the Universe without the need of *ad hoc* scalar fields [1–6]. These theories involve corrections to the Einstein-Hilbert action by considering a nonlinear function of the curvature scalar, $f(R)$. Earlier interest in $f(R)$ theories was motivated by inflationary scenarios as for instance, in the Starobinsky model, where $f(R) = R - \Lambda + \alpha R^2$ was considered [7]. Other motivations include the search for wormhole-type solutions [8]. In these studies, different approaches are used throughout the literature. These include the metric formalism, where the action is varied with respect to the metric; the Palatini formalism, where the metric and the connections are treated as separate variables; and the metric-affine formalism, which generalizes the Palatini variation, where the matter part of the action depends and is varied with respect to the connection [9].

Recently, it has been argued that most models proposed so far in the metric formalism violate weak-field solar system constraints [10], although viable models do exist [11]. Furthermore, it has been argued that higher-order gravity may explain the flatness of the rotation curves of galaxies [12–14]. For instance, in the context of $f(R) = f_0 R^n$ theories, the obtained gravitational potential is shown to differ from the Newtonian one due to the appearance of a repulsive term that increases with the distance from the center. The rotation curves of our Galaxy were studied, and compared with the observed data, so to assess the viability of these theories and to estimate the typical length scale of the correction. It was shown at first approximation, where spherically symmetric and thin disk mass distributions were considered, that a good agreement with data can be obtained with just the stellar disk and the interstellar gas.

In this paper we aim to derive the equation of motion for massive particles in a class of generalized gravitational models in which the Lagrangian of the gravitational field is an arbitrary function of the curvature scalar. The study of the equation of motion is of fundamental importance for the understanding of the structure and properties of gravitational theories. One of the most effective ways to test gravitational theories is by matching their predictions with the motion of real objects. For this purpose, we point out that the covariant conservation equation for a symmetric energy-momentum tensor, corresponding to matter is not, in general, conformally invariant [15]. One is led to relax the covariant conservation of the matter energy-momentum by considering a coupling between the matter Lagrangian and an arbitrary function of the curvature scalar. It is interesting to note that nonlinear couplings of matter with gravity were analyzed in the context of the accelerated expansion of the Universe [16], and in the study of the cosmological constant problem [17]. This is reminiscent of the situation in scalar-tensor theories of gravity and also arises in string theory. Nonminimal couplings have also been extensively considered in the litera-

*Electronic address: orfeu@cosmos.ist.utl.pt
†Electronic address: christian.boehmer@port.ac.uk
‡Electronic address: harko@hkucc.hku.hk
§Electronic address: francisco.lobo@port.ac.uk

ture, namely, between a scalar field and matter, including baryons and dark matter [14,18–23]. These couplings imply the violation of the equivalence principle, which is highly constrained by solar system experimental tests [24,25]. However, it has been recently reported, from data of the Abell Cluster A586, that interaction of dark matter and dark energy does imply the violation of the equivalence principle [26]. Notice that the violation of the equivalence principle is also found as a low-energy feature of some compactified version of higher-dimensional theories.

In what follows, by considering a coupling between a function of the curvature scalar and the matter Lagrangian, we show that in higher-order curvature theories of gravity the equation of motion of massive particles is nongeodesic. Thus, as shall be shown in Sec. II, the equation describing the trajectory of the particle has an extra-force term, which is orthogonal to its four-velocity. The nongeodesic nature of motion is a distinct feature of these $f(R)$ theories, found also in the context of a scalar field model with a suitable potential and proposed [27] as a solution for the Pioneer anomaly problem [28]. We shall discuss this question in Sec. III. Furthermore, it will also be addressed how our approach can be regarded as a covariant realization of a Modified Newtonian Dynamics (MOND) [29], even though free from the arbitrariness of its phenomenological version and somewhat simpler than its more recent realization which involves besides gravity, a vector and two scalar fields, the so-called TeVeS approach [30].

II. THE EQUATION OF MOTION IN $f(R)$ GRAVITATIONAL THEORIES

The action for the modified theories of gravity considered in this work takes the following form

$$S = \int \left\{ \frac{1}{2} f_1(R) + [1 + \lambda f_2(R)] \mathcal{L}_m \right\} \sqrt{-g} d^4x, \quad (1)$$

where $f_i(R)$ (with $i = 1, 2$) are arbitrary functions of the Ricci scalar R and $\mathcal{L}_m$ is the Lagrangian density corresponding to matter. Note that the strength of the interaction between $f_2(R)$ and the matter Lagrangian is characterized by a coupling constant λ. Analogous nonlinear gravitational couplings with a matter Lagrangian were also considered in the context of proposals to address the cosmic accelerated expansion [16], and in the analysis of the cosmological constant problem [17].

Varying the action with respect to the metric $g_{\mu\nu}$ yields the field equations, given by

$$\begin{aligned} F_1(R) R_{\mu\nu} - \tfrac{1}{2} f_1(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu F_1(R) + g_{\mu\nu} \Box F_1(R) \\ = -2\lambda F_2(R) \mathcal{L}_m R_{\mu\nu} + 2\lambda (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) \\ \times \mathcal{L}_m F_2(R) + [1 + \lambda f_2(R)] T_{\mu\nu}^{(m)}, \end{aligned} \quad (2)$$

where we have denoted $F_i(R) = f_i'(R)$, and the prime represents the derivative with respect to the scalar curvature. The matter energy-momentum tensor is defined as

$$T_{\mu\nu}^{(m)} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta(g^{\mu\nu})}. \quad (3)$$

Now, taking into account the covariant derivative of the field equations (2), the Bianchi identities, $\nabla^\mu G_{\mu\nu} = 0$, and the identity

$$(\Box \nabla_\nu - \nabla_\nu \Box) F_i = R_{\mu\nu} \nabla^\mu F_i, \quad (4)$$

one finally deduces the relationship

$$\nabla^\mu T_{\mu\nu}^{(m)} = \frac{\lambda F_2}{1 + \lambda f_2} [g_{\mu\nu} \mathcal{L}_m - T_{\mu\nu}^{(m)}] \nabla^\mu R. \quad (5)$$

Thus, the coupling between the matter and the higher derivative curvature terms describes an exchange of energy and momentum between both. Analogous couplings arise after a conformal transformation in the context of scalar-tensor theories of gravity, and also in string theory. In the absence of the coupling, one verifies the conservation of the energy-momentum tensor [31], which can also be verified from the diffeomorphism invariance of the matter part of the action [24,32,33]. Note that from Eq. (5), the conservation of the energy-momentum tensor is also verified if $f_2(R)$ is a constant or the matter Lagrangian is not an explicit function of the metric.

In order to test the motion in our model, we consider for the energy-momentum tensor of matter a perfect fluid

$$T_{\mu\nu}^{(m)} = (\epsilon + p) u_\mu u_\nu - p g_{\mu\nu}, \quad (6)$$

where ϵ is the overall energy density and p, the pressure, respectively. The four-velocity, u_μ, satisfies the conditions $u_\mu u^\mu = 1$ and $u^\mu u_{\mu;\nu} = 0$. We also introduce the projection operator $h_{\mu\lambda} = g_{\mu\lambda} - u_\mu u_\lambda$ from which one obtains $h_{\mu\lambda} u^\mu = 0$.

By contracting Eq. (5) with the projection operator $h_{\mu\lambda}$, one deduces the following expression

$$\begin{aligned} (\epsilon + p) g_{\mu\lambda} u^\nu \nabla_\nu u^\mu - (\nabla_\nu p)(\delta_\lambda^\nu - u^\nu u_\lambda) \\ - \frac{\lambda F_2}{1 + \lambda f_2} (\mathcal{L}_m + p)(\nabla_\nu R)(\delta_\lambda^\nu - u^\nu u_\lambda) = 0. \end{aligned} \quad (7)$$

Finally, contraction with $g^{\alpha\lambda}$ gives rise to the equation of motion for a fluid element

$$\frac{Du^\alpha}{ds} \equiv \frac{du^\alpha}{ds} + \Gamma_{\mu\nu}^\alpha u^\mu u^\nu = f^\alpha, \quad (8)$$

where we have introduced the space-time connection $\Gamma_{\mu\nu}^\alpha$, which is expressed in terms of the Christoffel symbols constructed from the metric, and where

$$f^\alpha = \frac{1}{\epsilon + p} \left[\frac{\lambda F_2}{1 + \lambda f_2} (\mathcal{L}_m + p) \nabla_\nu R + \nabla_\nu p \right] h^{\alpha\nu}. \quad (9)$$

As one can immediately verify, the extra force f^α is orthogonal to the four-velocity of the particle,

$$f^\alpha u_\alpha = 0, \tag{10}$$

which can be seen directly from the properties of the projection operator. This is consistent with the usual interpretation of the force, according to which only the component of the four-force that is orthogonal to the particle's four-velocity can influence its trajectory.

Notice that massless particles do follow geodesics and therefore for them $f^\alpha = 0$.

III. THE ACCELERATION LAW IN $f(R)$ GRAVITY

To derive the acceleration law in the $f(R)$ gravity, we start by assuming that the motion of a particle in a spacetime with metric $g_{\mu\nu}$ is given by Eq. (8). The presence of the extra force f^α implies that the motion of the particle is nongeodesic. For $f^\alpha = 0$ we recover the geodesic equation of motion. The usual gravitational effects, due to the presence of an arbitrary mass distribution, are assumed to be contained in the term $a_N^\alpha = \Gamma^\alpha_{\mu\nu} u^\mu u^\nu$. In three dimensions and in the Newtonian limit, Eq. (8) can be formally represented as a three-vector equation of the form

$$\vec{a} = \vec{a}_N + \vec{f}, \tag{11}$$

where $\vec{a}$ is the total acceleration of the particle, $\vec{a}_N$ is the gravitational acceleration, and $\vec{f}$ is the acceleration (per unit mass) due to the presence of the extra force. If $\vec{f} = 0$, the equation of motion is the usual Newtonian one, $\vec{a} = \vec{a}_N$, which for a pointlike mass distribution is given by $\vec{a} = -GM\vec{r}/r^3$.

Taking the square of Eq. (11) one obtains

$$\vec{f} \cdot \vec{a}_N = \tfrac{1}{2}(a^2 - a_N^2 - f^2), \tag{12}$$

where the dot stands for the three-dimensional scalar product. Equation (12) can be interpreted as a general relation which expresses the unknown vector $\vec{a}_N$ as a function of the total acceleration $\vec{a}$, the extra force $\vec{f}$, and the magnitudes a^2, a_N^2, and f^2. From Eq. (12) one can express the vector $\vec{a}_N$, as one can easily verify, in the form

$$\vec{a}_N = \frac{1}{2}(a^2 - a_N^2 - f^2)\frac{\vec{a}}{\vec{f} \cdot \vec{a}} + \vec{C} \times \vec{f}, \tag{13}$$

where $\vec{C}$ is an arbitrary vector perpendicular to the vector $\vec{f}$. In the following, we assume for simplicity, that $\vec{C} = 0$.

The mathematical consistency of Eq. (13) requires that $\vec{f} \cdot \vec{a} \neq 0$, that is, vectors $\vec{f}$ and $\vec{a}$ cannot be orthogonal to each other. We consider that both vectors are parallel. Therefore, we can represent the gravitational acceleration of a particle in the presence of an extra force as

$$\vec{a}_N = \frac{1}{2}(a^2 - a_N^2 - f^2)\frac{\vec{a}}{fa}. \tag{14}$$

In the limit of very small gravitational accelerations $a_N \ll a$, we obtain the relation

$$\vec{a}_N \approx \frac{1}{2}a\left(1 - \frac{f^2}{a^2}\right)\frac{1}{f}\vec{a}. \tag{15}$$

If one denotes

$$\frac{1}{a_E} \equiv \frac{1}{2f}\left(1 - \frac{f^2}{a^2}\right), \tag{16}$$

then Eq. (15) can be immediately written as

$$\vec{a}_N \approx \frac{a}{a_E}\vec{a}, \tag{17}$$

which has a striking resemblance with the equation put forward phenomenologically in the so-called MOND approach [29]. It then follows that $a \approx \sqrt{a_E a_N}$, and since $a_N = GM/r^2$, then $a \approx \sqrt{a_E GM}/r = v_{tg}^2/r$, where v_{tg} is the rotation velocity of the particle under the influence of a central force. Therefore, it follows that $v_{tg}^2 \to v_\infty^2 = \sqrt{a_E GM}$, from which arises the Tully-Fisher relation $L \sim v_\infty^4$ as $v_\infty^4 = a_E GM$, where L is the luminosity that is assumed to be proportional to the mass [29].

Notice however, that in the framework of $f(R)$ gravity, a_E is not a universal constant as it depends on local curvature features. This might explain why it is somewhat difficult to match the whole galactic phenomenology within the framework of MOND (see e.g. [34] for a critical assessment). Nevertheless, this feature of our model opens up quite interesting possibilities as we will see next.

Indeed, in general, the definition of a_E, Eq. (16), allows one to formally represent the extra force as a function of a and a_E, that is

$$\frac{f}{a_E} = -\left(\frac{a}{a_E}\right)^2 \pm \left(\frac{a}{a_E}\right)\sqrt{1 + \left(\frac{a}{a_E}\right)^2}. \tag{18}$$

Hence, through Eq. (16), Eq. (14) can be rewritten as

$$\vec{a}_N = \frac{a}{a_E}\left[h\left(\frac{a}{a_E}\right)\left(\frac{a_N}{a}\right)^2 + 1\right]\vec{a}, \tag{19}$$

where

$$h\left(\frac{a}{a_E}\right) \equiv \frac{1}{2}\left(\frac{a}{a_E}\right)^{-1}\left(\frac{a}{a_E} \mp \sqrt{1 + \frac{a^2}{a_E^2}}\right)^{-1}. \tag{20}$$

Upon substitution into Eq. (16), we verify $a^2 = v_\infty^4/r^2 = a_E GM/r^2$, which yields for a_E:

$$a_E = \frac{f^2 r^2}{GM} + 2f. \tag{21}$$

Suppose now that $f \sim GM\alpha/r$, where α is a constant, then in the large r limit, when $f \to 0$, $a_E \approx \alpha^2$ is a constant, whose numerical value is determined by the physical properties of the extra force.

Given the environmental nature of this extra force, only phenomenology can guide us in its identification. In the galactic context, it seems natural to identify a_E with the $a_0 = 10^{-10}$ m/s^2, the threshold acceleration of MOND.

On the other hand, in the solar system and its neighborhood, the Pioneer anomaly, whether a real effect unrelated with systematic effects (see e.g. [35] and references therein), suggests that $a_E = a_{\rm Pio} = (8.5 \pm 1.3) \times 10^{-10}\ \mathrm{m/s^2}$. Interestingly, our approach allows for a unified explanation for these two problems and can account for the fact that the characteristic acceleration of each class of observations is somewhat different.

IV. DISCUSSION AND CONCLUSIONS

In this work we have studied a class of generalized gravitational models, in which the Lagrangian density of the gravitational sector is an arbitrary function of the scalar curvature and an explicit coupling between the scalar curvature term and the matter Lagrangian density. Interestingly, we have found that the equation of motion of massive particles is nongeodesic. Therefore, the equation describing the trajectory of particles exhibits a term representing an extra force, which is orthogonal to its four-velocity. We have also shown that our models have similar features with the phenomenological approach of MOND, providing an alternative formulation to this proposal without the need of introducing, such as in the TeVeS proposal, extra fields besides the metric and the scalar curvature, which now plays the role of an additional scalar field. Furthermore, we have shown that the extra force is consistent with the so-called Pioneer anomaly. A distinct feature of the formalism outlined in this work is that it allows to establish a connection between the problem of the rotation curve of galaxies, via a solution somewhat similar to the one put forward in the context of MOND, and the Pioneer anomaly, even though the characteristic acceleration of these two classes of observation is somewhat different, about $10^{-10}\ \mathrm{m/s^2}$. Certainly, a more detailed study of the solar system implications of our model via the parametrized post-Newtonian analysis remains still to be performed, but it will be considered elsewhere.

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