Possible constraints on the duration of inflationary expansion from quantum stress tensor fluctuations

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We discuss the effect of quantum stress tensor fluctuations in de Sitter spacetime upon the expansion of a congruence of timelike geodesics. We treat a model in which the expansion fluctuations begin on a given hypersurface in de Sitter spacetime, and find that this effect tends to grow, in contrast to the situation in flat spacetime. This growth potentially leads to observable consequences in inflationary cosmology in the form of density perturbations which depend upon the duration of the inflationary period. In the context of our model, the effect may be used to place upper bounds on this duration.

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I. INTRODUCTION

Quantum fluctuations of the stress tensor operator have been studied in numerous recent papers [1–19]. Fluctuations of the stress tensor drive passive fluctuations of the gravitational field, which are to be distinguished from the active fluctuations due to the quantization of the gravitational degrees of freedom. Stress tensor fluctuations play a crucial role in stochastic gravity, and their role in the early universe has been discussed by several authors.

In the present paper, we will be concerned with the effects of quantum stress tensor fluctuations in inflationary cosmology. Since the pioneering paper by Guth [20], the inflationary paradigm has been extensively developed and now seems to be in good agreement with observations [21,22]. One of the successes of inflation is a natural solution of the horizon problem; inflationary expansion allows the entire observable universe today to have arisen from a region within which all parts were once in causal contact. The rapid expansion smooths any initial classical perturbations, and leads to a subsequent universe which is relatively independent of the duration of inflation, so long as there is inflation by at least a factor of 10^{23}. However, it can be shown that inflation could not have had an infinite duration in the past [23]. A key prediction of inflationary cosmology is a nearly flat spectrum of initial density perturbations, which arise from the intrinsic quantum fluctuations of an inflaton field. However, quantum stress tensor fluctuations of all quantum fields, not just the inflaton, should also contribute to density perturbations by means of passive metric fluctuations. Unlike the fluctuations of a nearly free inflaton field, quantum stress tensor fluctuations are expected to have a non-Gaussian probability distribution. Evidence for this comes from calculations in simple cases which reveal that in general the third moment is nonzero, and hence the probability distribution cannot be symmetric [24,25].

Here we will consider the effects of stress tensor fluctuations of the conformally invariant scalar field and the electromagnetic field in de Sitter spacetime upon geodesics of test particles. We will employ the Raychaudhuri equation to calculate fluctuations in the expansion θ of a congruence of comoving timelike geodesics, and then use the results to draw inferences about density perturbations in the post-inflationary period. The outline of this paper is as follows: Some basic formalism will be developed in Sec. II. This formalism will be applied to inflationary cosmology in Sec. III. We show that θ-fluctuations build up during inflation and influence the redshifting of radiation after inflation. In this model, we will make no specific references to the mechanism by which inflation ends, but will show that the resulting density perturbations grow as the length of the inflationary epoch increases. This will lead to an upper bound on the duration of inflation. In Sec. IV, we will examine the effects of θ-fluctuations on the dynamics of an inflaton field, and show that there is a further mechanism by which stress tensor fluctuations create density perturbations sensitive to the length of inflation. This model leads to a stronger bound on the duration of inflation. The results will be discussed in Sec. V.

II. BASIC FORMALISM

A. The Raychaudhuri equation and the conservation law

The key tool which we will employ for studying the effects of passive metric fluctuations in de Sitter spacetime will be the Raychaudhuri equation, which can be written for a congruence of timelike geodesics with four-velocity u^μ as

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$$\frac{d\theta}{d\lambda} = -R_{\mu\nu}u^\mu u^\nu - \frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu}. \quad (1)$$

Here $\theta = u^\mu_{;\mu}$ is the expansion of the congruence and λ is an affine parameter. In addition, $R_{\mu\nu}$ is the Ricci tensor, $\sigma^{\mu\nu}$ is the shear, and $\omega^{\mu\nu}$ is the vorticity of the congruence. The vorticity may be set to be zero, and we will assume that the shear is negligible. In this case, the equation reduces to

$$\frac{d\theta}{d\lambda} = -R_{\mu\nu}u^\mu u^\nu - \frac{1}{3}\theta^2. \quad (2)$$

Consider the case of a Robertson-Walker spacetime, where the metric can be written in terms of the scale factor $a(t)$ as

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \quad (3)$$

In the case of comoving geodesics which remain at rest in these coordinates, $u^\mu = \delta^\mu_t$ and the expansion is given by

$$\theta = \theta_0 = 3\frac{\dot{a}}{a}, \quad (4)$$

where $\dot{a} = da/dt$. With a perfect fluid source, the matter stress tensor is

$$T_{\mu\nu} = (p + \rho)u_\mu u_\nu + p g_{\mu\nu}, \quad (5)$$

where ρ is the energy density and p is the pressure. The Ricci tensor is given in terms of the stress tensor by Einstein's equations

$$R_{\mu\nu} = 8\pi(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T), \quad (6)$$

and for the perfect fluid we have

$$R_{\mu\nu}u^\mu u^\nu = 4\pi(\rho + 3p). \quad (7)$$

Thus θ_0 satisfies

$$\frac{d\theta_0}{dt} = -4\pi(\rho + 3p) - \frac{1}{3}\theta_0^2. \quad (8)$$

The conservation law for a perfect fluid can be expressed in terms of the expansion as [26,27]

$$\dot{\rho} + (\rho + p)\theta = 0. \quad (9)$$

Write the equation of state as $p = w\rho$ and assume w is constant. The density can be expressed as

$$\rho(t) = \rho(0)e^{-(1+w)\int_0^t \theta(t_1)dt_1}. \quad (10)$$

For the case of unperturbed Robertson-Walker spacetime, this is equivalent to

$$\rho(t) = \rho_0(t) = \rho(0)\left[\frac{a(t)}{a(0)}\right]^{-3(1+w)}. \quad (11)$$

B. Fluctuations of the expansion

Now we wish to consider perturbations of the background spacetime produced by stress tensor fluctuations. Let $\theta = \theta_0 + \theta_1$, $\rho = \rho_0 + \delta\rho$, and $p = p_0 + \delta p$. For now, we treat $\delta\rho$, and δp as independent variables. Let the Ricci tensor term in the Raychaudhuri equation be expressed as a sum of a classical part, given by Eq. (7), and a smaller fluctuating quantum part, denoted by $(R_{\mu\nu}u^\mu u^\nu)_q$. Note that the quantum field responsible for the fluctuations is distinct from the classical perfect fluid. Because of the possibility of pressure gradients, the fluid elements will not in general move along geodesics and the Raychaudhuri equation acquires an additional term on the right-hand side [28,29] of

$$-\frac{\nabla^2 \delta p}{\rho + p}, \quad (12)$$

where ∇^2 is the Laplacian operator in a constant t hypersurface. If we expand the Raychaudhuri equation to first order in θ_1, $\delta\rho$, δp, and $(R_{\mu\nu}u^\mu u^\nu)_q$, we find

$$\frac{d\theta_1}{dt} = -4\pi(\delta\rho + 3\delta p) - \frac{\nabla^2\delta p}{\rho_0 + p_0} - (R_{\mu\nu}u^\mu u^\nu)_q - \frac{2}{3}\theta_0\theta_1. \quad (13)$$

This equation may be integrated to find

$$\theta_1(t) = -a^{-2}(t)\int_{t_0}^t dt' a^2(t')\left[4\pi(\delta\rho(t') + 3\delta p(t')) - \frac{\nabla^2\delta p}{\rho_0 + p_0} - (R_{\mu\nu}u^\mu u^\nu)_q\right], \quad (14)$$

with the initial condition $\theta_1(t_0) = 0$.

To leading order, we may regard $\delta\rho$ and δp as perturbations which have some source other than quantum stress tensor fluctuations. In most inflationary models, these perturbations are driven by quantum fluctuations of the inflaton field, but for our purposes they can be treated as being either classical, or at least uncorrelated with the stress tensor fluctuations. Thus it is convenient to split θ_1 into a "classical" part θ_{1c} which depends upon $\delta\rho$ and δp, and a quantum part, θ_{1q}, driven by the stress tensor fluctuations. Then $\theta_1 = \theta_{1c} + \theta_{1q}$, and

$$\theta_{1q}(t) = -a^{-2}(t)\int_{t_0}^t dt' a^2(t')(R_{\mu\nu}u^\mu u^\nu)_q. \quad (15)$$

Thus θ_{1q} is given by the fluctuating part of the Ricci tensor, which is in turn given by the quantum stress tensor through Eq. (6). In this case, we can construct a correlation function for the expansion as an integral of the Ricci tensor correlation function:

$$K_{\mu\nu\alpha\beta} = \langle R_{\mu\nu}(x)R_{\alpha\beta}(x')\rangle - \langle R_{\mu\nu}(x)\rangle\langle R_{\alpha\beta}(x')\rangle, \quad (16)$$

and write

$$\langle\theta(t_1)\theta(t_2)\rangle - \langle\theta(t_1)\rangle\langle\theta(t_2)\rangle = a^{-2}(t_1)a^{-2}(t_2)\int_{t_0}^{t_1} dt\, a^2(t) \times \int_{t_0}^{t_2} dt' a^2(t') \times K_{\mu\nu\alpha\beta}u^\mu u^\nu u^\alpha u^\beta. \quad (17)$$

Note that if θ_{1c} is truly classical and nonfluctuating, we do not need to make a distinction between θ and θ_{1q} in the above expression, as the correlation function for both is the same: only the fluctuating part contributes to the correlation function. If $\delta\rho$ and δp do indeed fluctuate, then we assume that their fluctuations are uncorrelated with those of the stress tensor, so their effect would be to add another term in the expansion correlation function. This assumption will be discussed in more detail later. In writing Eq. (17), we are essentially assuming that the θ-fluctuations vanish before the $t = t_0$ hypersurface, which amounts to a sudden switching assumption.

In this paper, we will restrict our attention to passive metric fluctuations caused by conformally invariant quantum fields. In this case, the classical stress tensor in Robertson-Walker spacetime is related to that in flat spacetime by a conformal transformation:

$$T^{\rm RW}_{\mu\nu}(x) = a^{-4}(t)T^{\rm flat}_{\mu\nu}(x). \quad (18)$$

The quantum stress tensor operator in curved spacetime has an anomalous trace, so curved spacetime expectation values cannot be obtained directly by a conformal transformation of the corresponding flat spacetime expectation value. However, the contribution of the anomalous trace to the stress tensor operator is a c-number, and hence will cancel in a stress tensor correlation function [30]. Thus we can express the correlation function for the stress tensor in Robertson-Walker spacetime as the conformal transform of the corresponding flat space correlation function,

$$C^{\rm RW}_{\mu\nu\alpha\beta}(x, x') = a^{-4}(t)a^{-4}(t')C^{\rm flat}_{\mu\nu\alpha\beta}(x, x'), \quad (19)$$

where

$$C_{\mu\nu\alpha\beta}(x, x') = \langle T_{\mu\nu}(x)T_{\alpha\beta}(x')\rangle - \langle T_{\mu\nu}(x)\rangle\langle T_{\alpha\beta}(x')\rangle. \quad (20)$$

Because the anomalous trace of the stress tensor does not contribute to correlation functions, we can use the Einstein equation, Eq. (6), to relate the Ricci and stress tensor correlation functions:

$$K_{\mu\nu\alpha\beta}(x, x') = (8\pi)^2 C_{\mu\nu\alpha\beta}(x, x'). \quad (21)$$

Here we will assume that the quantum field in Robertson-Walker spacetime is in the conformal vacuum state, so the corresponding flat space correlation function will be that for the Minkowski vacuum state. In this case, we have

$$K_{\mu\nu\alpha\beta}(x, x')u^\mu u^\nu u^\alpha u^\beta = (8\pi)^2 C_{tt t' t'}(x, x') = (8\pi)^2 a^{-4}(t)a^{-4}(t')\mathcal{E}, \quad (22)$$

where $\mathcal{E}$ is the flat space vacuum energy density correlation function.

C. Treatment of singular integrands

Observable quantities are expressed as integrals of stress tensor correlation functions, as in Eq. (17). However, the integrands in these integrals contain singularities which appear not to be integrable, specifically higher order poles on the real axis. Nonetheless, such integrals can be given an unambiguous, finite value. One approach which could be used is dimensional regularization. This approach has been studied in flat spacetime in Ref. [17], where it was shown that spacetime integrals of stress tensor correlation functions such as $C_{\mu\nu\alpha\beta}(x, x')$ are actually finite in dimensional regularization. This means that if we were to evaluate an integral of $C_{\mu\nu\alpha\beta}(x, x')$ in spacetime dimension $4 + \varepsilon$, and then take the limit that $\varepsilon \to 0$, the result will be finite. (This is not true for integrals of correlation functions involving a time-ordered product of stress tensor operators. In this case, the singularity as $\varepsilon \to 0$ is proportional to counterterms in the gravitational action quadratic in the curvature.)

An alternative approach, which is easier to use in practice, is to define the integrals by an integration by parts procedure. This is the generalized principal value discussed in Ref. [31], and used in Refs. [12,15,16,18]. The basic idea is to reexpress an integral of the form

$$\int_a^b \frac{f(x)}{(x-c)^n}dx \quad (23)$$

by use of the identity

$$\frac{1}{(x-c)^n} = (-1)^{n-1}(n-1)!\frac{d^n}{dx^n}\ln(x-c), \quad (24)$$

and then perform successive integrations by parts to express the original integral as a sum of finite boundary terms and an integral whose integrand has only a logarithmic singularity. An equivalent approach is to seek an antiderivative, $G(x)$ of the function $F(x) = f(x)/(x-c)^n$, that is, $G'(x) = F(x)$, and write

$$\int_a^b F(x)dx = G(b) - G(a). \quad (25)$$

This result would be trivial if $F(x)$ had no singularities. In the present case, the only nontrivial aspect arises from whether the contour of integration goes above or below the higher order pole at $x = c$. However, the residue of this pole is pure imaginary if $f(x)$ and its derivatives are real. This is the case in the present problem, where one needs to integrate a real valued correlation function to get a real answer. Consequently, the residue of the pole will not contribute to the final result.

III. FIRST MODEL: θ-FLUCTUATIONS AND REDSHIFTS AFTER REHEATING

In this section, we will examine the fluctuations of the expansion during an inflationary period, and their subsequent effects in creating density fluctuations by differential redshifts after the end of inflation. It will be convenient to use the conformal time η rather than the proper time t of the comoving observers. The two time coordinates are related by

$$d\eta = a^{-1}(t)dt. \tag{26}$$

Let the inflationary phase begin at $\eta = \eta_0$ and end at $\eta = 0$. During this interval, the spacetime will be taken to be de Sitter space, which may be represented as a spatially flat Robertson-Walker metric with

$$a(\eta) = \frac{1}{1 - H\eta}, \qquad \eta_0 \le \eta \le 0. \tag{27}$$

For $\eta \ge 0$, we take the scale factor to be that of a radiation-dominated universe, for which

$$a(\eta) = 1 + H\eta. \tag{28}$$

These forms are chosen so that both $a(\eta)$ and its first derivative are continuous at $\eta = 0$. In terms of comoving time, the scale factors are

$$a(t) = e^{H(t-t_R)}, \qquad t \le t_R, \tag{29}$$

and

$$a(t) = \sqrt{1 + 2H(t - t_R)}, \qquad t \ge t_R, \tag{30}$$

where $t = t_R$ is the comoving time at which inflation ends.

The expansion correlation function, both during and after inflation, may be written as

$$\langle\theta(\eta_1)\theta(\eta_2)\rangle - \langle\theta(\eta_1)\rangle\langle\theta(\eta_2)\rangle = (8\pi)^2 a^{-2}(\eta_1)a^{-2}(\eta_2)\int_{\eta_0}^{\eta_1}\frac{d\eta}{a(\eta)}\int_{\eta_0}^{\eta_2}\frac{d\eta'}{a(\eta')}\mathcal{E}(\Delta\eta, r), \tag{31}$$

where $\Delta\eta = \eta - \eta'$ and $r = |\mathbf{x} - \mathbf{x}'|$ is the coordinate space separation of the pair of points at which θ is measured. Here we will assume that the θ-fluctuations vanish at the beginning of inflation, $\eta = \eta_0$. During inflation, there is no classical matter present. However, after reheating at $\eta = 0$, variations in θ cause the matter in different spatial regions to redshift at different rates, leading to variations in the density of the classical matter. Once reheating has occurred, $\delta p = w\delta\rho$, where $w = 1/3$ in our model. Equation (10) for the energy density can be written as

$$\rho(t) = \rho(0)e^{-(1+w)\int_0^t \theta_0 dt_1}e^{-(1+w)\int_0^t \theta_1(t_1)dt_1} \approx \rho_0(t)\left[1 - (1+w)\int_0^t \theta_1(t_1)dt_1 + O(\theta_1^2)\right], \tag{32}$$

so that

$$\frac{\delta\rho}{\rho_0} = -(1+w)\int_{t_R}^t \theta_1(t_1)dt_1 = -(1+w)\int_0^\eta \frac{d\eta_1}{a(\eta_1)}\theta_1. \tag{33}$$

The expansion fluctuations lead to density fluctuations given by

$$\left\langle\left(\frac{\delta\rho}{\rho}\right)^2\right\rangle = (8\pi)^2(1+w)^2\int_0^{\eta_s}\frac{d\eta_1}{a(\eta_1)}\int_0^{\eta_s}\frac{d\eta_2}{a(\eta_2)} \times \int_{\eta_0}^{\eta_1}\frac{d\eta}{a(\eta)}\int_{\eta_0}^{\eta_2}\frac{d\eta'}{a(\eta')}\mathcal{E}(\Delta\eta, r). \tag{34}$$

In the above expression, the integrals on η_1 and η_2 represent the differential redshifting, and hence have a lower limit at $\eta = 0$, the reheating time. The integrals on η and η' describe the effects of quantum stress tensor fluctuations on the expansion θ. The integration range is the beginning of inflation at $\eta = \eta_0$ to a time $\eta = \eta_s$ when the density variations are measured. We can take η_s to be the time of last scattering, when the density fluctuations of the cosmic background radiation were established.

The expression Eq. (34) contains contributions from all length scales, whereas observations are sensitive only to a finite range of scales. Thus, in order to compare the results of our calculations with observation, we should look at the power spectrum $P_k(\eta_s)$ defined by

$$\left\langle\left(\frac{\delta\rho}{\rho}\right)^2\right\rangle = \int d^3k e^{i\mathbf{k}\cdot\Delta\mathbf{x}}P_k(\eta_s). \tag{35}$$

We will first compute the density fluctuations as a function of r, and then Fourier transform the result to obtain $P_k(\eta_s)$.

Let

$$F(\eta_1, \eta_2) = \int_{\eta_0}^{\eta_1}\frac{d\eta}{a(\eta)}\int_{\eta_0}^{\eta_2}\frac{d\eta'}{a(\eta')}\mathcal{E}(\Delta\eta, r) = F_0 + F_1(\eta_1) + F_1(\eta_2) + F_2(\eta_1, \eta_2), \tag{36}$$

where

$$\begin{aligned} F_0 &= \int_{\eta_0}^{0}\frac{d\eta}{a(\eta)}\int_{\eta_0}^{0}\frac{d\eta'}{a(\eta')}\mathcal{E}(\Delta\eta, r),\\ F_1(\eta_1) &= \int_{0}^{\eta_1}\frac{d\eta}{a(\eta)}\int_{\eta_0}^{0}\frac{d\eta'}{a(\eta')}\mathcal{E}(\Delta\eta, r),\\ F_1(\eta_2) &= \int_{\eta_0}^{0}\frac{d\eta}{a(\eta)}\int_{0}^{\eta_2}\frac{d\eta'}{a(\eta')}\mathcal{E}(\Delta\eta, r),\\ F_2(\eta_1, \eta_2) &= \int_{0}^{\eta_1}\frac{d\eta}{a(\eta)}\int_{0}^{\eta_2}\frac{d\eta'}{a(\eta')}\mathcal{E}(\Delta\eta, r). \end{aligned} \tag{37}$$

Here F_0 describes correlated stress tensor fluctuations entirely within the de Sitter phase, F_2 similarly describes fluctuations entirely in the radiation-dominated phase, and F_1 describes the correlation of fluctuations between the de Sitter and radiation-dominated phases. Note that

$$F_0 = \langle\theta(0)\theta(0)\rangle - \langle\theta(0)\rangle\langle\theta(0)\rangle = \langle(\Delta\theta)^2\rangle \tag{38}$$

is the variance of the expansion at the end of inflation.

To go further, we need the explicit form for the flat space energy density correlation function, $\mathcal{E}(r, \Delta\eta)$. For the case of the electromagnetic field, it is

$$\mathcal{E}_{\rm em} = \frac{(r^2 + 3\Delta\eta^2)^2}{4\pi^4(r^2 - \Delta\eta^2)^6}. \tag{39}$$

For the conformal scalar field,

$$\mathcal{E}_{\rm scalar} = \frac{(\Delta\eta^2 + 3r^2)(r^2 + 3\Delta\eta^2)}{12\pi^4(r^2 - \Delta\eta^2)^6}. \tag{40}$$

Let us first consider F_0, which may be written as

$$F_0 = \int_0^{|\eta_0|} d\eta(1 + H\eta) \int_0^{|\eta_0|} d\eta'(1 + H\eta')\mathcal{E}(\Delta\eta, r). \tag{41}$$

This integral may be evaluated using algebraic symbol manipulation programs. We have used both *Mathematica* and *Maxima* with equivalent results. The program is asked to find antiderivatives of the integrand, which are then evaluated at the appropriate limits. (See the discussion in Sec. II C.) The result is rather complicated, but simplifies greatly in the limit that $H|\eta_0| \gg 1$. For the electromagnetic field case, we find the asymptotic form

$$F_0 \approx \frac{8H^2|\eta_0|^2}{5\pi^2 r^6} \tag{42}$$

for $H|\eta_0| \gg 1$. For the scalar case in the same limit,

$$F_0 \approx \frac{8H^2|\eta_0|^2}{15\pi^2 r^6}. \tag{43}$$

The remarkable feature of this result is that it grows with increasing $|\eta_0|$, and hence depends upon the length of the inflationary period. Note that F_2 is independent of $|\eta_0|$, and F_1 is found to go to a finite limit for large $|\eta_0|$. Thus in this limit,

$$F(\eta_1, \eta_2) \approx F_0. \tag{44}$$

Note that

$$H|\eta_0| \approx e^{H(t_R - t_0)} \tag{45}$$

is the net expansion factor during inflation, which needs to be greater than about 10^{23} to solve the horizon problem. Thus the large $H|\eta_0|$ approximation is an extremely good one. Note that there are no real particles moving along the comoving geodesics during inflation. Nonetheless, the expansion θ at the end of inflation has observable consequences. If we consider the reheating to occur very quickly, then we are effectively matching de Sitter spacetime and the radiation-dominated Robertson-Walker spacetime across the $\eta = 0$ hypersurface. Any such matching must satisfy the Israel junction conditions, that the extrinsic curvature of this hypersurface be continuous. (Note that θ is the trace of the extrinsic curvature tensor of this surface.) This implies that θ must be continuous, even in the case where there are spatial variations in θ. As a result, the expansion fluctuations generated by stress tensor fluctuations in the de Sitter phase persist in the radiation-dominated phase and cause density fluctuations. Recall that the geodesics in de Sitter space with which we are concerned become the comoving geodesics in the post-inflationary universe. This choice breaks the de Sitter invariance.

In order to compute the power spectrum of these fluctuations, we must find the Fourier transform of $1/r^6$, which may be done by integration by parts as follows:

$$\begin{aligned}\frac{1}{(2\pi)^3}\int d^3x \frac{1}{r^6} e^{-i\mathbf{k}\cdot\Delta\mathbf{x}} &= \frac{1}{2\pi^2 k}\int_0^\infty dr \frac{1}{r^5}\sin(kr) \\ &= \frac{1}{4\pi^2 k 4!}\int_{-\infty}^\infty dr \sin(kr)\frac{d^4}{dr^4}\left(\frac{1}{r}\right) \\ &= \frac{1}{4\pi^2 k 4!}\int_{-\infty}^\infty dr \left(\frac{1}{r}\right)\frac{d^4}{dr^4}\sin(kr) \\ &= \frac{k^3}{48\pi^2}\int_0^\infty dr \frac{\sin(kr)}{r} = \frac{k^3}{96\pi}.\end{aligned} \tag{46}$$

For the electromagnetic case, this leads to

$$P_k(\eta_s) \approx \frac{32H^2|\eta_0|^2 k^3}{15\pi}\left(\int_0^{\eta_s}\frac{d\eta_1}{a(\eta_1)}\right)^2(1 + w)^2. \tag{47}$$

We may evaluate the integral in the above expression using Eqs. (28) and (30) to find

$$P_k(\eta_s) \approx \frac{32|\eta_0|^2 k^3}{15\pi}\ln^2[a(\eta_s)](1 + w)^2 \ell_p^4, \tag{48}$$

where we have explicitly written the powers of ℓ_p, the Planck length. If we are interested in the effects of fluctuations within a finite bandwidth, $(k, k + \Delta k)$, then we can write

$$\left\langle\left(\frac{\delta\rho}{\rho}\right)^2\right\rangle = \int d^3k e^{i\vec{k}\cdot\Delta\vec{x}} P_k(\eta_s) \approx \Delta k k^2 e^{i\vec{k}\cdot\Delta\vec{x}} P_k. \tag{49}$$

For the purpose of a rough estimate, let us take $\Delta k \approx k$ and $e^{i\vec{k}\cdot\Delta\vec{x}} \approx 1$. Then the corresponding density perturbation is given by

$$\left(\frac{\delta\rho}{\rho}\right)_{\rm rms} = \sqrt{\left\langle\left(\frac{\delta\rho}{\rho}\right)^2\right\rangle} \approx \sqrt{k^3 P_k} \approx \ell_p^2|\eta_0|k^3\ln[a(\eta_s)]. \tag{50}$$

Note that we have taken $a = 1$ at the end of inflation. As a result, $1/a(\eta_s)$ is the redshift factor between reheating and the last scattering surface,

$$a(\eta_s) \approx \frac{E_R}{1\ \mathrm{eV}}, \tag{51}$$

where E_R is the reheating energy scale. We should have

$$\left(\frac{\delta\rho}{\rho}\right)_{\mathrm{rms}} \lesssim 10^{-4}, \tag{52}$$

which leads to an upper bound on the duration of inflation

$$H|\eta_0| \lesssim 10^{-4} \frac{H}{\ell_p^2 k^3 \ln(\frac{E_R}{1\ \mathrm{eV}})}. \tag{53}$$

If E_R is close to the scale of inflation, then the vacuum energy density during inflation is $V_0 \approx E_R^4$, and

$$H^2 = \frac{8\pi}{3}\ell_p^2 V_0 \approx \frac{8\pi}{3}\ell_p^2 E_R^4. \tag{54}$$

Because we have chosen $a = 1$ at the end of inflation, the scale factor today is

$$a_{\mathrm{now}} \approx 10^3 a(\eta_s) \approx 10^3 \frac{E_R}{1\ \mathrm{eV}}, \tag{55}$$

and k is related to the physical wave number today, k_P, by $k = a_{\mathrm{now}} k_P$. Let $k_P = 2\pi/\lambda$ correspond to the typical intergalactic separation today, $\lambda \approx 2$ Mpc, or $k_P \approx 10^{-24}\ \mathrm{cm}^{-1}$. Then we may combine the above relations to write the bound on the expansion factor during inflation as

$$H|\eta_0| \lesssim 10^{79}\left(\frac{10^{12}\ \mathrm{GeV}}{E_R}\right), \tag{56}$$

ignoring the weak logarithmic dependence upon E_R.

The k^3 dependence found in Eq. (48) indicates a non-scale invariant spectrum of fluctuations which rises at shorter wavelengths. The same dependence upon k was found recently in a somewhat different context by Lombardo and Nacir [32]. (See Eq. (68) of their paper.) If observational data for $k_P > 10^{-24}\ \mathrm{cm}^{-1}$ were available, then one might be able to obtain tighter bounds on the duration of inflation. The smallest scales on which the cosmic microwave background has been observed is about 5 arcminutes [33], which corresponds to $k_P \approx 10^{-24}\ \mathrm{cm}^{-1}$. Similarly, the role of higher values of k_P in large scale structure formation is unclear because of non-linear classical effects.

The effect we are considering depends upon trans-planckian modes in the sense that the modes of the quantized electromagnetic or scalar fields which give the dominant contribution have wavelengths much shorter than the Planck length. Let L_i be a given proper length at the beginning of inflation, and let L_f be the corresponding scale today stretched by the cosmological expansion. These two scales are approximately related by

$$L_f = 10^3\left(\frac{E_R}{1\ \mathrm{eV}}\right)H|\eta_0|L_i, \tag{57}$$

as $H|\eta_0|$ is the expansion during inflation, $E_R/1$ eV is that between reheating and last scattering, and there has been an additional expansion by a factor of about 10^3 after last scattering. If we take $L_f \approx 10^{24}$ cm, $E_R \approx 10^{12}$ GeV, and $H|\eta_0| \approx 10^{79}$, then $L_i \approx 10^{-46}\ell_p$. It is well known that transplanckian modes play a crucial role in the conventional approach to black hole evaporation [34]. It is possible to obtain black hole evaporation without transplanckian modes, but only at the price of introducing a Lorentz noninvariant dispersion relation [35].

IV. SECOND MODEL: SINGLE FIELD SLOW-ROLL INFLATION

Let ϕ be an inflaton field which obeys the equation of motion in a Robertson-Walker metric

$$\Box\phi = \frac{1}{a^2}\nabla^2\phi - \frac{1}{a^3}\frac{\partial}{\partial t}\left(a^3\frac{\partial\phi}{\partial t}\right) = V'(\phi), \tag{58}$$

where $V(\phi)$ is a relatively flat potential, and $\nabla^2\phi$ is the flat space Laplacian operator. During a slow-roll phase, the second time derivative of ϕ is assumed to be small. If, in addition, the spatial gradient terms are small, then

$$3\frac{\dot a}{a}\dot\phi \approx V'(\phi). \tag{59}$$

Note that $\theta_0 = 3\dot a/a = 3H$ is the expansion of unperturbed de Sitter spacetime. Now we wish to generalize this description to include small spatial variations of the expansion. If the vorticity of the comoving geodesics vanishes and the shear remains small, then the spacetime metric can be written as

$$ds^2 = -dt^2 + a^2(t,\mathbf{x})(dx^2 + dy^2 + dz^2). \tag{60}$$

The expansion of the comoving geodesics, those with four-velocity $u^\mu = \delta^\mu_t$ in these coordinates, is again

$$\theta = u^\mu_{;\mu} = 3\frac{\dot a}{a}, \tag{61}$$

and the shear and vorticity vanish. The equation of motion for an inflaton field with self-coupling $V(\phi)$ now becomes

$$\Box\phi = \frac{1}{a^3}\nabla\cdot(a\nabla\phi) - \frac{1}{a^3}\frac{\partial}{\partial t}\left(a^3\frac{\partial\phi}{\partial t}\right) = V'(\phi). \tag{62}$$

Let each of the quantities a, ϕ, and θ consist of a homogeneous part and a small inhomogeneous perturbation:

$$a = a_0(t) + a_1(t,\mathbf{x}),$$
$$\phi = \phi_0(t) + \phi_1(t,\mathbf{x}), \quad \text{and} \quad \theta = \theta_0(t) + \theta_1(t,\mathbf{x}). \tag{63}$$

We make the slow-roll approximation for the homogeneous part, $\phi_0(t)$, which satisfies

$$\dot{\phi}_0 = -\frac{V'(\phi_0)}{\theta_0}, \qquad (64)$$

where $\theta_0 = 3\dot{a}_0/a_0$. We retain the second time derivative of $\phi_1(t, \mathbf{x})$, which satisfies

$$\ddot{\phi}_1 + \theta_0\dot{\phi}_1 + \theta_1\dot{\phi}_0 - \frac{1}{a_0^2}\nabla^2\phi_1 = -V''(\phi_0)\phi_1, \qquad (65)$$

where we have expanded Eq. (62) to first order in all of the inhomogeneous perturbations and used

$$V'(\phi_0 + \phi_1) \approx V'(\phi_0) + V''(\phi_0)\phi_1. \qquad (66)$$

Let us consider the case where the potential is approximately linear during the period of interest, so we may set $V''(\phi_0) \approx 0$. If we use Eq. (64) and $\theta_0 = 3H$, we may write Eq. (65) as

$$\ddot{\phi}_1 + 3H\dot{\phi}_1 - \frac{1}{a_0^2}\nabla^2\phi_1 = \frac{V'(\phi_0)}{3H}\theta_1. \qquad (67)$$

Let us Fourier transform this equation and define

$$\delta\phi_k(t) = \int d^3k e^{i\mathbf{k}\cdot\Delta\mathbf{x}}\phi_1(t, \mathbf{x}), \qquad (68)$$

and

$$\delta\theta_k(t) = \int d^3k e^{i\mathbf{k}\cdot\Delta\mathbf{x}}\theta_1(t, \mathbf{x}). \qquad (69)$$

We also change from comoving time t to conformal time η. Then $\delta\phi_k(\eta)$ satisfies

$$\frac{d^2\delta\phi_k}{d\eta^2} + 2Ha_0\frac{d\delta\phi_k}{d\eta} + k^2\delta\phi_k = \frac{V'(\phi_0)}{3H}a_0^2\delta\theta_k. \qquad (70)$$

Let $G(\eta, \eta')$ be a retarded Green's function for this equation which satisfies

$$\frac{d^2G}{d\eta^2} + 2Ha_0\frac{dG}{d\eta} + k^2G = \delta(\eta - \eta'), \qquad (71)$$

and $G(\eta, \eta') = 0$ if $\eta < \eta'$. It can be expressed as

$$G(\eta, \eta') = \frac{1}{W[\varphi_1(\eta')\varphi_2(\eta')]}[\varphi_1(\eta_<)\varphi_2(\eta_>) - \varphi_1(\eta)\varphi_2(\eta')]. \qquad (72)$$

Here φ_1 and φ_2 are two linearly independent solutions of Eq. (71) without the delta-function source term, and W is their Wronskian. Here $\eta_>$ and $\eta_<$ are, respectively, the greater and lesser of η and η'. These solutions may be taken to be

$$\varphi_1(\eta) = \tfrac{1}{2}\sqrt{\pi}H|\eta|^{3/2}H_{3/2}^{(1)}(\eta) \qquad (73)$$

and

$$\varphi_2(\eta) = \tfrac{1}{2}\sqrt{\pi}H|\eta|^{3/2}H_{3/2}^{(2)}(\eta), \qquad (74)$$

where $H_{3/2}^{(1)}$ and $H_{3/2}^{(2)}$ are Hankel functions. The Wronskian becomes

$$W[\varphi_1(\eta')\varphi_2(\eta')] = \frac{i}{a_0^2(\eta')}. \qquad (75)$$

If we set

$$F = \frac{V'(\phi_0)}{3H}a_0^2\theta_1, \qquad (76)$$

then we can write the solution of Eq. (70) as

$$\delta\phi_k(\eta) = \int_{\eta_0}^{0} d\eta' F(\eta')G(\eta, \eta') = \int_{\eta_0}^{\eta} d\eta' F(\eta')G(\eta, \eta'), \qquad (77)$$

where we use the fact that $G(\eta, \eta') = 0$ for $\eta' > \eta$. In the expression for $G(\eta, \eta')$, we may now set $\eta_> = \eta$ and $\eta_< = \eta'$. Let us now split the integration range into two parts as

$$\delta\phi_k(\eta) = \int_{\eta_0}^{\eta_c} d\eta' F(\eta')G(\eta, \eta') + \int_{\eta_c}^{\eta} d\eta' F(\eta')G(\eta, \eta'). \qquad (78)$$

Here $\eta_c = -1/k$ is the horizon-crossing time for mode k in conformal time. Note that in the first integral in the above expression, we have $k|\eta| < 1 < k|\eta'|$, and in the second integral we have $k|\eta| < k|\eta'| < 1$. If we assume that $k|\eta| \ll 1 \ll k|\eta'|$, and use the limiting forms for the Hankel functions, we find

$$G \approx \frac{\cos(k\eta')}{k^2\eta'}. \qquad (79)$$

Similarly, if we assume $k|\eta| \ll k|\eta'| \ll 1$, then we have

$$G \approx -\tfrac{1}{3}\eta'. \qquad (80)$$

Let us now make the approximation that we may use Eq. (79) in the first integral in Eq. (78), and Eq. (80) in the second. The result is

$$\delta\phi_k(\eta) \approx -\frac{V_0'}{9H^3}\left[3\eta_c^2\int_{\eta_0}^{\eta_c} d\eta' \frac{\cos(\eta'/\eta_c)}{|\eta'|^3} - \int_{\eta_c}^{\eta} d\eta' \frac{1}{|\eta'|}\right]\delta\theta_k, \qquad (81)$$

where we have used $k = 1/|\eta_c|$ and our assumption that $V_0' = V'(\phi_0)$ is approximately constant.

We now consider the effects of expansion fluctuations upon the evolution of the inflaton field. The variance of $\delta\phi_k(\eta)$ is

$$\langle(\Delta\phi_k)^2\rangle = \left(\frac{V_0'}{9H^2}\right)^2\left[3(H\eta_c)^2\int_{\eta_0}^{\eta_c} d\eta_1 a_0^2(\eta_1)\cos(\eta_1/\eta_c) - \int_{\eta_c}^{\eta} d\eta_1\right]\left[3(H\eta_c)^2\int_{\eta_0}^{\eta_c} d\eta_2 a_0^2(\eta_2) \times \cos(\eta_2/\eta_c) - \int_{\eta_c}^{\eta} d\eta_2\right]a_0(\eta_1)a_0(\eta_2) \times[\langle\delta\theta_k(\eta_1)\delta\theta_k(\eta_2)\rangle - \langle\delta\theta_k(\eta_1)\rangle\langle\delta\theta_k(\eta_2)\rangle]. \quad (82)$$

The correlation function in the above integral is just the Fourier transform of the coordinate space expansion correlation function given in Eq. (31). It is convenient to evaluate the integrals in coordinate space and to write variance of $\delta\phi(\eta)$ as

$$\langle(\Delta\phi)^2\rangle = \left(\frac{8\pi V_0'}{9H^2}\right)^2\left[3(H\eta_c)^2\int_{\eta_0}^{\eta_c} d\eta_1 a_0^2(\eta_1) \times\cos(\eta_1/\eta_c) - \int_{\eta_c}^{\eta} d\eta_1\right]\left[3(H\eta_c)^2 \times\int_{\eta_0}^{\eta_c} d\eta_2 a_0^2(\eta_2)\cos(\eta_2/\eta_c) - \int_{\eta_c}^{\eta} d\eta_2\right]a_0^{-1}(\eta_1)a_0^{-1}(\eta_2)I(\eta_1,\eta_2), \quad (83)$$

where

$$I(\eta_1,\eta_2) = \int_{\eta_0}^{\eta_1} d\eta a_0^{-1}(\eta)\int_{\eta_0}^{\eta_2} d\eta' a_0^{-1}(\eta')\mathcal{E}(\Delta\eta, r). \quad (84)$$

Here we have evaluated the variance of ϕ at the end of inflation, $\eta = 0$. This quantity is directly related to the density perturbations in the post-inflationary era. This issue has been discussed extensively by previous authors [36–40] in a context where $\Delta\phi$ arises from the intrinsic quantum fluctuations of the inflaton field. However, the relation between $\Delta\phi$ and the density perturbation is the same in the present context. The basic idea is that spatial variations in ϕ cause different regions to reheat at different times, with a typical time variation of order $\Delta t \approx \Delta\phi/\dot{\phi}_0$. After reheating, the energy density decreases as $\rho = \rho_R(t_R/t)^2$, where ρ_R is the density at reheating. This leads to density variations whose magnitude is of order

$$\frac{\delta\rho}{\rho} \approx \frac{2\Delta t}{t_R} \approx \frac{2\Delta\phi}{t_R\dot{\phi}_0} \approx \frac{6H\Delta\phi}{t_R V_0'}. \quad (85)$$

In the last step, we have used Eq. (64) with $V_0' = V'(\phi_0)$. Thus we have

$$\left\langle\left(\frac{\delta\rho}{\rho}\right)^2\right\rangle = \left(\frac{6H}{t_R V_0'}\right)^2\langle(\Delta\phi)^2\rangle, \quad (86)$$

with $\langle(\Delta\phi)^2\rangle$ given by Eq. (83). The physical picture for the conversion of ϕ-fluctuations into density variations described above was first given by Guth and Pi [37]. It is possible to give a more rigorous, gauge-invariant discussion [29,41]. However, for a single inflaton model, the results are essentially equivalent to Eq. (86).

Consider first the case of an electromagnetic field. It is convenient to express Eq. (39) as

$$\mathcal{E}_{\rm em} = -\frac{1}{480\pi^4}\left[\frac{d^5}{ds^5}\frac{(r^2+3\Delta\eta^2)^2}{(s-\Delta\eta^2)}\right]_{s=r^2}, \quad (87)$$

and to interchange the order of the s-differentiations and the integrations in Eq. (84). In the limit of large $|\eta_0|$, the result is

$$I \approx \frac{H^2\eta_0^2}{40\pi^4 r^6}, \quad (88)$$

which is independent of η_1 and η_2 to leading order. Now the integrations on these variables in Eq. (83) can be performed in terms of the cosine integral function, $ci(x)$, with the result

$$\langle(\Delta\phi)^2\rangle \approx \frac{2\eta_0^2\eta_c^4(V_0')^2\ell_p^4}{405\pi^2 r^6}[1+6ci(1)]^2, \quad (89)$$

where $ci(1) \approx 0.337$. Here we assume that $H|\eta_c| \gg 1$. The corresponding expression for the conformal scalar field is smaller by a factor of $1/3$. In this model, the fluctuations grow at the same rate with increasing $|\eta_0|$ as in the case of the model in Sec. III. This arises despite the fact that $\Delta\phi_k$ for an individual mode only begins to grow after the mode leaves the horizon. The growth with increasing $|\eta_0|$ comes from the fact that the expansion fluctuations grow from the beginning of inflation, as described by Eq. (31). If it were not for this growth, the factor of ℓ_p^4 would tend to make the effects of stress tensor fluctuations small compared to those of the intrinsic quantum fluctuations of ϕ. The corresponding power spectrum of density perturbations is

$$P_k \approx \frac{\ell_p^4 H^2\eta_0^2\eta_c^4 k^3}{540\pi^3 t_R^2}[1+6ci(1)]^2 \approx \frac{\ell_p^4 H^2\eta_0^2}{540\pi^3 t_R^2 k}[1+6ci(1)]^2, \quad (90)$$

where we set $|\eta_c| \approx 1/k$ only at the last step. If we make the same estimates as were made in Eq. (50), we find

$$\left(\frac{\delta\rho}{\rho}\right)_{\rm rms} \approx \sqrt{k^3 P_k} \approx 10^{-2}\frac{\ell_p^2 H|\eta_0|k}{t_R}. \quad (91)$$

Note that both here and in the model of Sec. III, we find a spectrum which is not scale invariant. Here $(\delta\rho/\rho)_{\rm rms} \propto k$. If we again set $k_P \approx 10^{-24}\ {\rm cm}^{-1}$ and assume that $E_R \approx V_0^{1/4}$, we obtain the following upper bound on the duration of inflation:

$$H|\eta_0| \lesssim 10^{45}\left(\frac{10^{12}\ \mathrm{GeV}}{E_R}\right)^3. \tag{92}$$

This is considerably more restrictive than Eq. (56), but is still compatible with adequate inflation to solve the horizon problem.

V. SUMMARY AND DISCUSSION

We have analyzed the effects of stress tensor fluctuations of conformally invariant quantum fields in de Sitter spacetime. One unexpected result of this analysis is that expansion fluctuations grow during a de Sitter phase. This might be interpreted as due to the background spacetime altering the anticorrelated fluctuations. In Minkowski spacetime, quantum fluctuations tend to have strong correlations and anticorrelations. If one were to evaluate the variance of the expansion in Minkowski spacetime, that is, compute Eq. (31) with $a = 1$, the result would be independent of η_0 in the limit that $|\eta_0| \to \infty$. This is closely related to the anticorrelations found in sampled energy density measurements in flat spacetime. Similar anticorrelations are present when a charged particle or a mirror is coupled to vacuum fluctuations in flat spacetime, causing the mean squared velocity to approach a constant even in the absence of dissipation [42–44]. It is the presence of nonconstant $a(\eta)$ functions in the integrand of Eq. (31) which upsets the flat spacetime cancellations and leads to a result which grows with increasing $|\eta_0|$.

It is this growth which allowed us to infer the constraints, Eqs. (56) and (92) on the duration of the inflationary phase. These constraints may come as a surprise, as one usually expects inflation to erase the memory of the past history of the universe. It is certainly true that the exponential expansion quickly suppresses classical perturbations. However, the effect discussed here amounts to a type of quantum instability of de Sitter spacetime: the cumulative effects of passive metric fluctuations eventually lead to a spatially inhomogeneous spacetime. The direct effect on the spacetime geometry grows rather slowly, as reflected in the constraint equation (56). However, when the θ-fluctuations couple to the inflaton field, the result can be a stronger constraint, Eq. (92). Both of these constraints are consistent with adequate inflation to solve the horizon problem. The possibility of effects which grow during inflation and react against the expansion has been discussed by several authors as a possible solution for the cosmological constant problem. Among the effects considered are the growth of long wavelength classical perturbations [45,46] and backreaction due to quantum gravity effects [47]. The effect discussed in the present paper is distinct from either of these effects, especially as it produces an increasingly inhomogeneous universe rather than a backreaction against the cosmological constant.

Here is it worth revisiting three of the assumptions which we made in our analysis. One was the assumption that the $\delta\rho$ and δp terms in Eq. (14) are uncorrelated with the term produced by stress tensor fluctuations. This assumption should hold so long as the dominant source of density perturbations is other than the stress tensor fluctuations. Of course, if inflation were to last sufficiently long, this would no longer be true; $\delta\rho$ and δp would be predominately due to these fluctuations. However, because the contributions of stress tensor fluctuations are highly non-Gaussian and nonscale invariant, they can give at best a very small contribution to the total primordial fluctuation spectrum in our universe. Hence our assumption seems to be justified. Another assumption which we have made is ignoring the shear term in the Raychaudhuri equation. The comoving geodesics in de Sitter spacetime are certainly shear-free, so it is reasonable to start in a state where $\sigma_{\mu\nu} = 0$. Shear could develop only if there were large fluctuations of the Weyl tensor. This seems unlikely, but Weyl tensor fluctuations from passive metric fluctuations need to be better understood. The third assumption is the sudden switching assumption first made in writing Eq. (17). This amounts to assuming that the expansion fluctuations vanish before the $t = t_0$ hypersurface, and that the effects of the stress tensor fluctuations appear suddenly after that time. In future work, we plan to examine more general initial conditions to test the dependence of our results upon the initial conditions.

A particularly interesting possibility is that inflation lasted for a time only slightly less than the constraints derived above. In this case, one would predict a small, but potentially observable effect from the quantum stress tensor fluctuations. This effect is expected to manifest itself in a non-Gaussian and nonscale invariant component in the density perturbations. It would also offer a possible probe of transplanckian physics.

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