

**Kaon semileptonic decay ( $K_{l3}$ ) form factors from the instanton vacuum**Seung-il Nam<sup>\*</sup> and Hyun-Chul Kim<sup>†</sup>*Department of Physics and Nuclear Physics & Radiation Technology Institute (NuRI),**Pusan National University, Busan 609-735, Republic of Korea*

(Received 8 March 2007; published 11 May 2007)

We investigate the kaon semileptonic decay ( $K_{l3}$ ) form factors within the framework of the nonlocal chiral quark model ( $\chi$ QM) from the instanton vacuum, taking into account the effects of flavor SU(3) symmetry breaking. We also consider the problem of gauge invariance arising from the momentum-dependent quark mass in the present work. All theoretical calculations are carried out without any adjustable parameter, the average instanton size ( $\rho \sim 1/3$  fm) and the interinstanton distance ( $R \sim 1$  fm) having been fixed. We also show that the present results satisfy the Callan-Treiman low-energy theorem as well as the Ademollo-Gatto theorem. Using the  $K_{l3}$  form factors, we evaluate relevant physical quantities. It turns out that the effects of flavor SU(3) symmetry breaking are essential in reproducing the kaon semileptonic form factors. The present results are in good agreement with experiments, and are compatible with other model calculations.

DOI: [10.1103/PhysRevD.75.094011](https://doi.org/10.1103/PhysRevD.75.094011)

PACS numbers: 12.39.Ki, 13.20.Eb

**I. INTRODUCTION**

It is of great importance to understand semileptonic decays of kaons ( $K_{l3}$ ), since it plays a significant role in determining the Cabibbo-Kobayashi-Maskawa matrix element  $|V_{us}|$  precisely [1,2]. Furthermore, it provides a simple phenomenological basis for testing the breaking of flavor SU(3) symmetry: In exact flavor SU(3) symmetry, the kaon semileptonic form factor  $f_+(0)$  becomes unity. The Ademollo-Gatto [3] theorem asserts that of second order are the corrections of flavor SU(3) symmetry breaking to the form factors of vector currents at zero momentum transfer ( $q^2 = 0$ ). However, when Goldstone bosons are involved, the Ademollo-Gatto theorem must be modified: While the singlet part of flavor SU(3) symmetry breaking preserves flavor SU(3) symmetry, it breaks chiral SU(3)  $\times$  SU(3) symmetry. Langacker and Pagels showed that the corrections of flavor SU(3) symmetry breaking appear to first order due to the presence of that singlet part [4–6]. The effect of flavor SU(3) symmetry breaking on the kaon semileptonic decay form factor is known to be around 3%–5%, which is rather small.

The well-known soft-pion Callan-Treiman [7] theorem connects the ratio of the pion and kaon decay constants to the semileptonic form factors of the kaon at  $q^2 = m_K^2 - m_\pi^2$  (Callan-Treiman point). Any chiral quark model should satisfy the Callan-Treiman theorem with flavor SU(3) symmetry breaking. Experimentally, there are a certain amount of data to judge theoretical calculations [8,9]. Thus, the kaon semileptonic decay form factor provides a basis to examine the validity and reliability of any theoretical theory and model for hadrons.

There have been a great number of theoretical works: chiral perturbation theory ( $\chi$ PT) [5,10], lattice QCD

(LQCD) [11,12], a Dyson-Schwinger method [13,14], constituent quark models [15–17], and so on. In the present work, we will investigate the  $K_{l3}$  form factor within the framework of the nonlocal chiral quark model ( $\chi$ QM) derived from the instanton vacuum. We will consider the leading order in the large  $N_c$  expansion and flavor SU(3) symmetry breaking explicitly. The meson-loop corrections, which are of  $1/N_c$  order, are neglected. The model has several virtues: All relevant QCD symmetries are satisfied within the model, and there are only two parameters: The average size of instantons ( $\rho \sim 1/3$  fm) and average interinstanton distance ( $R \sim 1$  fm), which can be determined by the internal constraint such as the saddle-point equation [18–20]. These values for  $\rho$  and  $R$  have been supported in various LQCD simulations recently [21–23]. There is no further adjustable parameter in the model.

As being discussed previously, since the effects of flavor SU(3) symmetry breaking are essential in the present work, we employ the modified low-energy effective partition function with flavor SU(3) symmetry breaking [24–26]. This partition function extends the former one derived in the chiral limit [19,20]. It has been proven that the partition function with flavor SU(3) symmetry breaking is very successful in describing the low-energy hadronic properties such as various QCD condensates, magnetic susceptibilities, meson distribution amplitudes, and so on [27–31]. However, the presence of the nonlocal interaction between quarks and pseudo-Goldstone bosons breaks the Ward-Takahashi identity for Nöther currents. Since the kaon semileptonic decay form factors involve the vector current, we need to deal with this problem. While Ref. [32] proposed a systematic way as to how the conservation of the Nöther current is restored, one has to handle the integral equation. References [29,31] derived the light-quark partition function in the presence of the external gauge fields. With this gauged partition function, it was shown that the low-energy theorem for the transition from two-photon

<sup>\*</sup>Electronic address: [sinam@pusan.ac.kr](mailto:sinam@pusan.ac.kr)<sup>†</sup>Electronic address: [hchkim@pusan.ac.kr](mailto:hchkim@pusan.ac.kr)

state to the vacuum via the axial anomaly was satisfied [29]. Moreover, the magnetic susceptibility of the QCD vacuum and the meson distribution amplitudes were obtained successfully [30,31]. Thus, in the present work, we will investigate the kaon semileptonic decay ( $K_{l3}$ ) form factors, using the gauged low-energy effective partition function from the instanton vacuum with flavor SU(3) symmetry breaking explicitly taken into account.

We sketch the present work as follows: In Sec. II, we briefly explain the general formalism relevant for studying the  $K_{l3}$  form factor. In Sec. III, we introduce the nonlocal chiral quark model from the instanton vacuum. In Sec. IV, the numerical results are discussed, and are compared with those of other works. The final section is devoted to summarize the present work and to draw conclusions.

## II. SEMILEPTONIC KAON DECAY

In the present work, we are interested in the following kaon semileptonic decays ( $K_{l3}$ ) in two different isospin channels:

$$\begin{aligned} K^+(p_K) &\rightarrow \pi^0(p_\pi)l^+(p_l)\nu_l(p_\nu): K_{l3}^+, \\ K^0(p_K) &\rightarrow \pi^-(p_\pi)l^+(p_l)\nu_l(p_\nu): K_{l3}^0, \end{aligned} \quad (1)$$

where  $l$  and  $\nu_l$  stand for the leptons (either the electron or the muon) and neutrinos. The relevant diagrams for the  $K_{l3}$  form factor are depicted in Fig. 1 in which we define the momenta for the particles involved. The nonlocal contributions in Figs. 1(b) and 1(c) arise from the gauged effective chiral action that will be discussed in Sec. III.

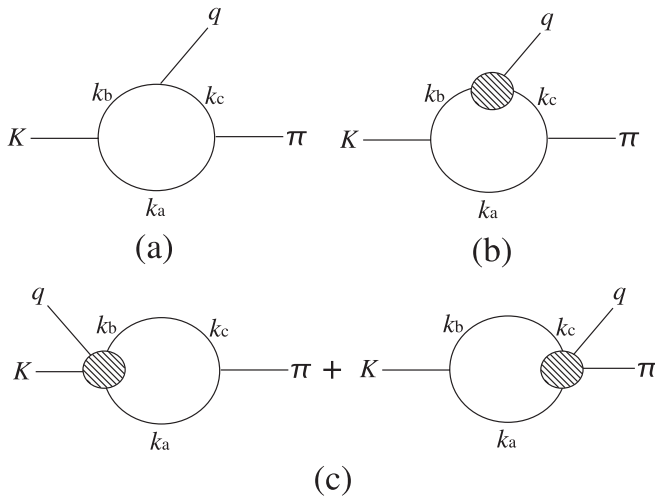


FIG. 1. Schematic diagrams for the kaon semileptonic decay form factor. We consider the contributions from the local (a) and nonlocal vector-quark vertices (b) and nonlocal vector-quark-meson vertices (c). Here, we define the relevant momenta as follows:  $k_a = k - p/2 - q/2$ ,  $k_b = k + p/2 - q/2$ , and  $k_c = k + p/2 + q/2$ , where  $k$ ,  $p$ , and  $q$  stand for the loop-integral variable, initial kaon, and vector-field momenta, respectively.

The decay amplitude ( $T_{K \rightarrow l\nu\pi}$ ) can be expressed as follows [10]:

$$T_{K \rightarrow l\nu\pi} = \frac{G_F}{\sqrt{2}} \sin\theta_c [w^\mu(p_l, p_\nu) F_\mu(p_K, p_\pi)], \quad (2)$$

where  $G_F$  is the well-known Fermi constant:  $G_F = 1.166 \times 10^{-5} \text{ GeV}^{-2}$ .  $\theta_c$  denotes the Cabibbo angle. We define, respectively, the weak leptonic and hadronic matrix elements ( $w^\mu$  and  $F_\mu$ ) with the  $\Delta S = 1$  vector current ( $j_\mu^{S\mu}$ ) as

$$w^\mu(p_l, p_\nu) = \bar{u}(p_\nu)\gamma^\mu(1 - \gamma_5)v(p_l), \quad (3)$$

$$\begin{aligned} F_\mu(p_K, p_\pi) &= c \langle \pi(p_\pi) | j_\mu^{S\mu} | K(p_K) \rangle \\ &= c \langle \pi(p_\pi) | \bar{\psi}\gamma_\mu \lambda^{4+i5} \psi | K(p_K) \rangle \\ &= (p_K + p_\pi)_\mu f_{l+}(t) + (p_K - p_\pi)_\mu f_{l-}(t), \end{aligned} \quad (4)$$

where  $c$  is the isospin factor, and set to be unity and  $1/\sqrt{2}$  for  $K_{l3}^0$  and  $K_{l3}^+$ , respectively. The matrix  $\lambda^{4+i5}$  denotes the combination of the two Gell-Mann matrices,  $(\lambda^4 + i\lambda^5)/2$ , for the relevant flavor in the present problem. The  $\psi$  denotes the quark field. The momentum transfer is defined as  $Q^2 = (p_K - p_\pi)^2 \equiv -t$ .

$f_{l\pm}$  represent the vector form factors with the corresponding lepton  $l$  ( $P$ -wave projection). Alternatively, the form factor  $F_\mu(p_K, p_\pi)$  can be expressed in terms of the scalar form factor ( $f_{l0}$ ,  $S$ -wave projection) and the vector form factor  $f_{l+}$  defined as follows:

$$\begin{aligned} F_\mu(p_K, p_\pi) &= f_{l+}(t)(p_K + p_\pi)_\mu \\ &\quad + \frac{(m_\pi^2 - m_K^2)(p_K - p_\pi)_\mu}{t} \\ &\quad \times [f_{l+}(t) - f_{l0}(t)]. \end{aligned} \quad (5)$$

Hence, the  $f_{l0}$  can be written as the linear combination of  $f_{l+}$  and  $f_{l-}$ :

$$f_{l0}(t) = f_{l+}(t) + \left[ \frac{t}{m_K^2 - m_\pi^2} \right] f_{l-}(t). \quad (6)$$

Since the isospin breaking effects are almost negligible, we will consider only the  $K^0 \rightarrow \pi^- \nu l^+$  decay channel. Input values for the numerical calculations are given as follows:  $m_K \simeq 495 \text{ MeV}$  and  $m_\pi \simeq 140 \text{ MeV}$ , respectively. The up- and down-quark masses are taken as their average value:  $m_q \equiv (m_u + m_d)/2 \simeq 5 \text{ MeV}$ , while the strange-quark mass  $m_s$  as around  $150 \text{ MeV}$ .

It has been well known that the experimental data for  $f_{l+,0}$  can be reproduced qualitatively well by the linear and quadratic fits [8]:

$$\text{linear: } f_{l+,0}(t) = f_{l+,0}(0) \left[ 1 + \frac{\lambda_{l+,0}}{m_\pi^2} (t - m_l^2) \right],$$

$$\text{quadratic: } f_{l+,0}(t) = f_{l+,0}(0) \left[ 1 + \frac{\lambda'_{l+,0}}{m_\pi^2} (t - m_l^2) + \frac{\lambda''_{l+,0}}{2m_\pi^4} (t - m_l^2)^2 \right], \quad (7)$$

where  $m_l$  is the lepton mass. The slope parameter  $\lambda_{l+}$  is deeply related to the  $K \rightarrow \pi$  decay radius ( $\langle r^2 \rangle^{K\pi}$ ) as follows [10]:

$$\lambda_+ \simeq \frac{1}{6} \langle r^2 \rangle^{K\pi} m_\pi^2. \quad (8)$$

Moreover, this radius can be expressed in terms of the Gasser-Leutwyler low-energy constant  $L_9$  in the large  $N_c$  limit [5]:

$$L_9 = \frac{1}{12} F_\pi^2 \langle r^2 \rangle^{K\pi}. \quad (9)$$

To obtain the decay rate  $d\Gamma_{K \rightarrow l\nu\pi}$ , we use the convention defined in Ref. [10]:

$$d\Gamma_{K \rightarrow l\nu\pi} = \frac{1}{16m_K(2\pi)^5} \sum_{\text{spins}} \frac{d^3 p_l}{E_l} \frac{d^3 p_\nu}{E_\nu} \frac{d^3 p_\pi}{E_\pi} \times \delta^4(p_K - p_l - p_\nu - p_\pi) |T_{K \rightarrow l\nu\pi}|^2. \quad (10)$$

This expression can be further simplified as a function of  $t$  [13]:

$$\begin{aligned} \frac{d\Gamma_{K \rightarrow l\nu\pi}}{dt} &= \frac{G_F^2 |V_{us}|^2}{24\pi^3} \left(1 - \frac{m_l^2}{t}\right)^2 \left[ |\vec{p}_\pi|^3 \left(1 + \frac{m_l^2}{2t}\right) f_{l+}^2(t) \right. \\ &\quad \left. + m_K^2 |\vec{p}_\pi| \left(1 - \frac{m_\pi^2}{m_K^2}\right)^2 \frac{3m_l^2}{8t} f_{l0}^2(t) \right] \\ &\simeq \frac{G_F^2 |V_{us}|^2}{24\pi^3} |\vec{p}_\pi|^3 f_{e+}^2(t) \quad \text{for } m_l = m_e \simeq 0, \quad (11) \end{aligned}$$

where the three momentum of the pion  $|\vec{p}_\pi|$  is defined by

$$|\vec{p}_\pi| = \left[ \left( \frac{m_K^2 + m_\pi^2 - t}{2m_K} \right)^2 - m_\pi^2 \right]^{1/2}. \quad (12)$$

This three momentum of the pion constrains the physically accessible region for the decay, i.e.,

$$m_l^2 \leq t \leq (m_K - m_\pi)^2. \quad (13)$$

### III. NONLOCAL CHIRAL QUARK MODEL FROM THE INSTANTON VACUUM

In this section, we show how to derive the hadronic matrix element given in Eq. (4) within the framework of the nonlocal  $\chi$ QM from the instanton vacuum. We begin by the low-energy effective QCD partition function derived from the instanton vacuum [20,24–26]:

$$\begin{aligned} Z_{\text{eff}} &= \int \mathcal{D}\psi \mathcal{D}\psi^\dagger \mathcal{D}\mathcal{M} \exp \int d^4x \left[ \psi_f^\dagger(x) (i\not{\partial} \right. \\ &\quad \left. + im_f) \psi_f(x) + i \int \frac{d^4k d^4p}{(2\pi)^8} e^{-i(k-p)\cdot x} \psi_f^\dagger(k) \right. \\ &\quad \left. \times \sqrt{M_f(k_\mu)} U_{fg}^{\gamma_5} \sqrt{M_g(p_\mu)} \psi_g(p) \right]. \quad (14) \end{aligned}$$

$M_f(k)$  is the dynamically generated quark mass being momentum dependent, whereas  $m_f$  stands for the current-quark mass with flavor  $f$ . The nonlinear background pseudo-Goldstone field  $U^{\gamma_5}$  is given by

$$\begin{aligned} U^{\gamma_5} &= U(x) \frac{1 + \gamma_5}{2} + U^\dagger(x) \frac{1 - \gamma_5}{2} \\ &= 1 + \frac{i}{F_{\mathcal{M}}} \gamma_5 \mathcal{M} \cdot \lambda - \frac{1}{2F_{\mathcal{M}}^2} (\mathcal{M} \cdot \lambda)^2 \dots \quad (15) \end{aligned}$$

with the meson decay constants  $F_\pi = 93$  MeV and  $F_K = 113$  MeV fixed to the experimental data. The meson field  $U$  is defined as  $U = \exp[i\lambda \cdot \mathcal{M}/F_{\mathcal{M}}]$ . The octet pseudo-scalar meson field  $\mathcal{M}$  is defined as follows:

$$\begin{aligned} \mathcal{M} \cdot \lambda &= \sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}} \eta \end{pmatrix}, \quad (16) \end{aligned}$$

References [24,25] showed how to improve the low-energy effective QCD partition function in Eq. (14) by taking into account effects of flavor SU(3) symmetry breaking, so that the dynamical quark mass acquires the contribution of the  $m_f$  corrections:

$$M_f(k) = M_0 F^2(k) \left[ \sqrt{1 + \frac{m_f^2}{d^2}} - \frac{m_f}{d} \right], \quad (17)$$

where  $M_0$  is the dynamical quark mass with zero momentum transfer in the chiral limit. Its value is determined by the saddle-point equation:  $M_0 \simeq 350$  MeV.  $F(k)$  is the momentum-dependent part which arises from the Fourier transform of the fermionic zero-mode solutions in the instantons. However, we will employ the simple-pole type parametrization for  $F(k)$ :

$$F(k) = \frac{2\Lambda^2}{2\Lambda^2 + k^2}, \quad \Lambda = \rho^{-1} \simeq 600 \text{ MeV} \quad (18)$$

which shows a very similar behavior to the original expression of  $F(k)$ . The value of  $d$  in the bracket of Eq. (17) can be computed within the model [25,32]:

$$d = \sqrt{\frac{0.08385}{2N_c} \frac{8\pi\bar{\rho}}{R^2}} \simeq 0.193 \text{ GeV}. \quad (19)$$

As mentioned previously, the momentum-dependent dynamical quark mass  $M_f(k)$  breaks the conservation of the Nöther (vector) currents. References [29,31] derived the

light-quark partition function in the presence of the external vector field:

$$\begin{aligned} \tilde{Z}_{\text{eff}} = & \int \mathcal{D}\psi \mathcal{D}\psi^\dagger \mathcal{D}\mathcal{M} \exp \int d^4x \left[ \psi_f^\dagger(x) (i\not{\partial} + \mathcal{V} \right. \\ & + im_f) \psi_f(x) + i \int \frac{d^4k d^4p}{(2\pi)^8} e^{-i(k-p)\cdot x} \psi_f^\dagger(k) \\ & \left. \times \sqrt{M_f(k_\mu + V_\mu)} U_{fg}^{\gamma_5} \sqrt{M_g(p_\mu + V_\mu)} \psi_g(p) \right]. \end{aligned} \quad (20)$$

$$\begin{aligned} & = \left[ \frac{\delta \tilde{\mathcal{S}}_{\text{eff}}}{\delta V_\mu} \right]_{V=0} \\ & = -N_c \text{Tr} \left[ \frac{1}{i\not{\partial} + im_f + i\sqrt{M_f(i\partial_\mu)} U_{fg}^{\gamma_5} \sqrt{M_g(i\partial_\mu)}} \gamma_\mu \lambda^{4-i5} \right]_{V=0} \\ & + N_c \text{Tr} \left[ \frac{\left[ \frac{\partial}{\partial V_\mu} (\sqrt{M_f(i\partial_\mu)}) U_{fg}^{\gamma_5} \sqrt{M_g(i\partial_\mu)} - i\sqrt{M_g(i\partial_\mu)} U_{fg}^{\gamma_5} \frac{\partial}{\partial V_\mu} (\sqrt{M_g(i\partial_\mu)}) \right] \lambda^{4-i5}}{i\not{\partial} + im_f + i\sqrt{M_f(i\partial_\mu)} U_{fg}^{\gamma_5} \sqrt{M_g(i\partial_\mu)}} \right]_{V=0}. \end{aligned} \quad (22)$$

The first term in Eq. (22) is usually called a *local* contribution, and the other two terms the *nonlocal* ones. Since we are interested in the decay process with two on-mass shell pseudoscalar mesons, that is, the pion and kaon as shown in Eq. (4), the local contribution can be written as follows:

$$\begin{aligned} \left[ \frac{\delta \tilde{\mathcal{S}}_{\text{eff}}}{\delta V_\mu} \right]_{\text{local}, V=0}^{K\pi} = & \frac{2N_c}{F_\pi F_K} \text{Tr} \left[ \frac{\sqrt{M_f(i\partial_\mu)} \gamma_5 \mathcal{M}^a \lambda^a \sqrt{M_g(i\partial_\mu)}}{\mathcal{D}(i\partial_\mu)} \right. \\ & \times \frac{\gamma^\mu \lambda^{4-i5}}{\mathcal{D}(i\partial_\mu)} \\ & \left. \times \frac{\sqrt{M_f(i\partial_\mu)} \gamma_5 \mathcal{M}^b \lambda^b \sqrt{M_g(i\partial_\mu)}}{\mathcal{D}(i\partial_\mu)} \right]. \end{aligned} \quad (23)$$

The pseudoscalar meson field  $\mathcal{M}^a$  can be either the kaon or the pion, depending on flavor.  $\mathcal{D}$  denotes the abbreviation for the quark-propagator:

$$\mathcal{D}_f(i\partial_\mu) = i\not{\partial} - i[m_f + M_f(i\partial_\mu)]. \quad (24)$$

Then, the corresponding matrix element can be obtained as follows:

$$\begin{aligned} F_\mu^{\text{nonlocal(b)}} = & \frac{8N_c}{F_\pi F_K} \int \frac{d^4k}{(2\pi)^4} \frac{\sqrt{M_q(k_c)} \sqrt{M_q(k_c)} M_q(k_a) M_s(k_b) \mu}{[k_a^2 + \bar{M}_q^2(k_a)][k_b^2 + \bar{M}_s^2(k_b)][k_c^2 + \bar{M}_q^2(k_c)]} [\bar{M}_q(k_c) k_a \cdot k_b + \bar{M}_s(k_b) k_a \cdot k_c \\ & - \bar{M}_q(k_a) k_b \cdot k_c + \bar{M}_q(k_a) \bar{M}_s(k_b) \bar{M}_q(k_c)] - (b \leftrightarrow c), \\ F_\mu^{\text{nonlocal(c)}} = & -\frac{4N_c}{F_\pi F_K} \int \frac{d^4k}{(2\pi)^4} \frac{\sqrt{M_q(k_a)} \sqrt{M_s(k_b)} \sqrt{M_q(k_c)} \sqrt{M_q(k_a)} [k_a \cdot k_b + \bar{M}_q(k_a) \bar{M}_s(k_b)] \mu}{[k_a^2 + \bar{M}_q^2(k_a)][k_b^2 + \bar{M}_s^2(k_b)]} \\ & + \frac{4N_c}{F_\pi F_K} \int \frac{d^4k}{(2\pi)^4} \frac{\sqrt{M_q(k_a)} \sqrt{M_s(k_b)} \sqrt{M_q(k_c)} \sqrt{M_q(k_a)} [k_a \cdot k_b + \bar{M}_q(k_a) \bar{M}_s(k_b)] \mu}{[k_a^2 + \bar{M}_q^2(k_a)][k_b^2 + \bar{M}_s^2(k_b)]} + (b \leftrightarrow c), \end{aligned} \quad (27)$$

The effective chiral action then becomes

$$\begin{aligned} \tilde{\mathcal{S}}_{\text{eff}} = & -N_c \text{Tr} \ln [i\not{\partial} + \mathcal{V} + im_f \\ & + i\sqrt{M_f(i\partial_\mu + V_\mu)} U_{fg}^{\gamma_5} \sqrt{M_g(i\partial_\mu + V_\mu)}], \end{aligned} \quad (21)$$

where Tr denotes the trace over space-time, flavor, and spin spaces. Calculating the functional derivative of  $\tilde{\mathcal{S}}_{\text{eff}}$  with the external vector field  $V$ , we obtain the relevant operator expression for the  $K \rightarrow \pi$  semileptonic decay form factors:

$$\left\langle K(p_K) \left| \left[ \frac{\delta \tilde{\mathcal{S}}_{\text{eff}}}{\delta V_\mu} \right]_{\text{local}, V=0}^{K\pi} \right| \pi(p_\pi) \right\rangle. \quad (25)$$

The matrix element for the local contribution can be immediately expressed as

$$\begin{aligned} F_\mu^{\text{local(a)}} = & \frac{8N_c}{F_\pi F_K} \int \frac{d^4k}{(2\pi)^4} \\ & \times \frac{M_q(k_a) \sqrt{M_s(k_b)} M_q(k_c)}{[k_a^2 + \bar{M}_q^2(k_a)][k_b^2 + \bar{M}_s^2(k_b)][k_c^2 + \bar{M}_q^2(k_c)]} \\ & \times [[k_a \cdot k_b + \bar{M}_q(k_a) \bar{M}_s(k_b)] k_{c\mu} \\ & - [k_b \cdot k_c + \bar{M}_s(k_b) \bar{M}_q(k_c)] k_{a\mu} \\ & + [k_a \cdot k_c + \bar{M}_q(k_a) \bar{M}_q(k_c)] k_{b\mu}], \end{aligned} \quad (26)$$

where  $\bar{M}_f(k) = m_f + M_f(k)$ . The relevant momenta are defined as  $k_a = k - p/2 - q/2$ ,  $k_b = k + p/2 - q/2$ , and  $k_c = k + p/2 + q/2$ , in which  $k$ ,  $p$ , and  $q$  denote the internal quark, initial kaon, and transferred momenta, respectively, as depicted in Fig. 1. The trace  $\text{tr}_\gamma$  runs over Dirac spin space. Similarly, we can evaluate the nonlocal contributions as follows [28,30]:

where  $\sqrt{M_f(k)} = \partial\sqrt{M_f(k)}/\partial k_\mu\mu$ . The local (a), nonlocal (b), and nonlocal (c) contributions correspond to the diagrams (a), (b), and (c) in Fig. 1, respectively.

#### IV. RESULTS AND DISCUSSIONS

We now discuss various numerical results for the kaon semileptonic decay ( $K_{l3}$ ) form factors in the present work. We facilitate the Breit-momentum framework for the calculation, since we are free to choose an arbitrary momentum framework because of the Lorentz invariance. The relevant momenta for the calculation are defined as follows ( $-Q^2 \equiv t > 0$ ):

$$p = \left(0, 0, i\frac{\sqrt{t}}{2}, \sqrt{\frac{m_K^2 - m_\pi^2 + t}{4\sqrt{t}}}\right), \quad q = (0, 0, -i\sqrt{t}, 0),$$

$$k = (k_r \sin\phi \sin\psi \cos\theta, k_r \sin\phi \sin\psi \sin\theta, k_r \sin\phi \cos\psi, k_r \cos\phi, ). \quad (28)$$

We first consider the case of  $K_{e3}$ . Since the electron mass is negligible in comparison to those of the pion and the kaon, it can be set to be zero. In the left panel of Fig. 2, we draw the numerical results for  $f_{e^+}(t)$  (solid),  $f_{e^-}(t)$  (dotted), and  $f_{e0}(t)$  (dashed) within the physically accessible regions constrained by Eq. (13). Note that the scalar form factor  $f_{e0}(t)$  is derived by using Eq. (6). We observe that the  $f_{e^+}(t)$  and  $f_{e0}(t)$  are almost linearly increasing functions of  $t$ , whereas  $f_{e^-}(t)$  decreases. At  $t = 0$ , our results demonstrate that  $f_{e^+}(0) = f_{e0}(0) = 0.947$  and  $f_{e^-}(0) = -0.137$ . In the chiral limit,  $f_{e^+}(0)$  and  $f_{e^-}(0)$  should be unity and zero, respectively, which is related to the Ademollo-Gatto theorem in the case of pseudo-Goldstone bosons [3–5]:

$$\lim_{q \rightarrow 0} F_\mu^{\text{local(a)}} \simeq 2p_\mu + \mathcal{O}(m_q). \quad (29)$$

The Ademollo-Gatto theorem in Eq. (29) can be easily tested in the nonlocal  $\chi$ QM. Considering  $q \rightarrow 0$  and ignoring the terms being proportional to  $k \cdot p$ , we can rewrite the leading contribution of Eq. (26) to order  $\mathcal{O}(m_q)$  as follows:

$$\lim_{q \rightarrow 0} F_\mu^{\text{local(a)}} \simeq 2[1 + R(m_s)]p_\mu, \quad (30)$$

where

$$R(m_s) = \frac{1}{2} \left[ \int \frac{d^4k}{(2\pi)^4} \frac{M^2(k)m_s[m_s + 2M(k)]}{[k^2 + M^2(k)]^3} \right] \times \left[ \int \frac{d^4k}{(2\pi)^4} \frac{M^2(k)}{[k^2 + M^2(k)]^2} \right]^{-1}. \quad (31)$$

To evaluate Eq. (30), we employ the ratio  $F_K/F_\pi$  computed within the same framework and expanded in terms of the strange quark mass ( $m_s$ ) (see Refs. [20,33–38] for more details):

$$\frac{F_K}{F_\pi} \simeq 1 + R(m_s). \quad (32)$$

We also use that  $k_b = k_c \rightarrow k + p/4$  since these two momenta share  $p/2$  as  $q \rightarrow 0$ . Note that we consider only the local contribution for  $F_{\mathcal{M}}$  in Eq. (32). We, however, verified that the nonlocal contributions in Eq. (27) also satisfy the Ademollo-Gatto theorem analytically.

The effect of flavor SU(3) symmetry breaking is found to be rather small in the  $K_{e3}$  form factor, i.e. its effect is around 5%. In other approaches, for example, in  $\chi$ PT, the  $K_{e3}$  form factor is known to be  $f_{e^+}(t) = 0.961 \pm 0.008$  [2], in LQCD,  $f_{e^+}(0) = 0.960 \pm 0.009$  [12], and  $0.952 \pm 0.006$  [11].

In the right panel of Fig. 2 we draw the ratio of  $f_{e^+}(t)$  and  $f_{e^+}(0)$  with respect to the CPLEAR experimental data [39], and linear (dashed) and quadratic (dotted) fits for the ratio using the PDG data [8]:  $\lambda_{e^+} = (2.960 \pm 0.05) \times 10^{-2}$ ,  $\lambda'_{e^+} = (2.485 \pm 0.163) \times 10^{-2}$ , and  $\lambda''_{e^+} = (1.920 \pm 0.062) \times 10^{-3}$ . In the present calculation, we obtain  $\lambda_{e^+} = 3.028 \times 10^{-2}$  for the linear fit, which is very close to the experimental one,  $2.960 \times 10^{-2}$ . Since our result for  $f_{e^+}$  is almost linear as shown in Fig. 1, we get almost a negligible value for the slope parameter  $\lambda''$  when the quadratic fit is taken into account. Being compared with other model calculations, the present results are com-

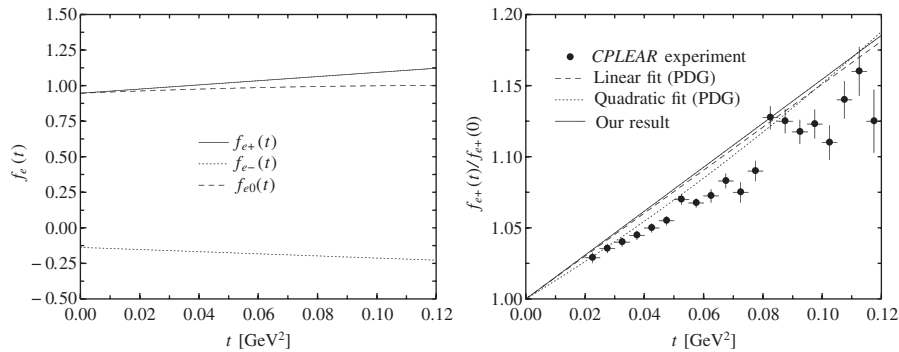


FIG. 2.  $K_{e3}$  form factors,  $f_{e^+}(t)$  (solid),  $f_{e^-}(t)$  (dotted), and  $f_{e0}(t)$  (dashed) are shown in the left panel, while in the right panel the ratio of  $f_{e^+}(t)$  and  $f_{e^+}(0)$  is given (solid). We also draw the CPLEAR experimental data [39], and linear (dashed) and quadratic (dotted) fits using the PDG data [8].



parable to those from  $\chi$ PT [10,40], and other models [11,13,14,41,42]. We compare explicitly the present results with those from other approaches in Table I.

Using Eqs. (8) and (9), we can easily estimate the  $K_{e3}$  decay radius and low-energy constant  $L_9$ , respectively. As for the  $K_{e3}$  decay radius, we obtain  $\langle r^2 \rangle^{K\pi} = 0.366 \text{ fm}^2$ . This value is slightly larger than that in  $\chi$ PT [5] (see Table I). The low-energy constant  $L_9$  turns out to be  $6.78 \times 10^{-3}$ , which is comparable to  $7.1 \sim 7.4 \times 10^{-3}$  [5] and  $6.9 \times 10^{-3}$  [10,48].

The ratio of the pion and kaon weak decay constants  $F_K/F_\pi$  can be deduced from the scalar form factor  $f_0$  via the Callan-Treiman soft-pion theorem [7]. In the soft-pion limit ( $p_\pi \rightarrow 0$ ), the  $K_{e3}$  form factor can be written as [49]

$$\lim_{p_\pi \rightarrow 0} F_\mu(p_\pi, p_K) = p_{K\mu} \frac{F_K}{F_\pi}. \quad (33)$$

Using Eqs. (4) and (6), we obtain the following expression:

$$\begin{aligned} \lim_{p_\pi \rightarrow 0} F_\mu(p_\pi, p_K) &= \lim_{p_\pi \rightarrow 0} (p_\pi + p_K)_\mu [f_{l+}(\Delta_{\text{CT}}) \\ &\quad + f_{l-}(\Delta_{\text{CT}})] \\ &\simeq p_{K\mu} f_{l0}(\Delta_{\text{CT}}), \end{aligned} \quad (34)$$

where the value of  $\Delta_{\text{CT}} = m_K^2 - m_\pi^2$  is called the Callan-Treiman point which cannot be accessible physically. Combining Eq. (33) with Eq. (34), we finally arrive at the final expression of the  $K_{l3}$  form factor for the Callan-Treiman theorem in terms of the scalar form factor and the ratio,  $F_K/F_\pi$ :

$$f_{e0}(\Delta_{\text{CT}}) = \frac{F_K}{F_\pi}. \quad (35)$$

From our numerical calculation using Eq. (35), we find that  $F_K/F_\pi = 1.08$ , which is around 10% smaller than the empirical value (1.22). The reason is due to the fact that the kaon weak decay constant turns out to be underestimated if we ignore the meson-loop  $1/N_c$  corrections [50]. In the large  $N_c$  limit, the ratio can be expressed in terms of the low-energy constant  $L_5$ :

$$\frac{F_K}{F_\pi} = 1 + \frac{4}{F_\pi^2} (m_K^2 - m_\pi^2) L_5. \quad (36)$$

Using the value of  $F_K/F_\pi = 1.08$ , we obtain  $L_5 = 7.67 \times 10^{-4}$  which is quite underestimated by about a factor 2, compared with the phenomenological value  $1.4 \times 10^{-3}$  [48]. It is already well known that in order to reproduce the  $L_5$  within the  $\chi$ QM the meson-loop  $1/N_c$  corrections are essential.

In the soft-pion limit, the model should satisfy the Callan-Treiman theorem given in Eq. (35). Taking the limit  $p_\pi \rightarrow 0$  for Eq. (26), we can show that Eq. (26) satisfies the Callan-Treiman theorem, using Eq. (32) as follows:

$$\lim_{p_\pi \rightarrow 0} F_\mu^{\text{local(a)}} \simeq [1 + R(m_s)] p_\mu, \quad (37)$$

where  $k_a = k_c \rightarrow k$  as  $p_\pi \rightarrow 0$ . Inserting Eq. (32) into Eq. (37), we can show that the present result satisfies the Callan-Treiman theorem in Eq. (33) [Eq. (35)] in the case of the local contribution. The nonlocal ones also fulfill the theorem.

The decay width of  $K \rightarrow \pi \nu e$  can be easily computed by using the result of  $f_{l+0}$  and Eq. (11). It turns out that  $\Gamma_{e3} = 6.840 \times 10^6/\text{s}$  and  $\Gamma_{\mu3} = 4.469 \times 10^6/\text{s}$  with  $|V_{us}| = 0.22$  taken into account [8,51]. The results are slightly smaller than the experimental data [ $\Gamma_{e3} = (7.920 \pm 0.040) \times 10^6/\text{s}$  and  $\Gamma_{\mu3} = (5.285 \pm 0.024) \times 10^6/\text{s}$ ] [8].

TABLE I. Various numerical results for  $K^0 \rightarrow \pi^- \nu e^+$  ( $K^0 \rightarrow \pi^- \nu \mu^+$ ). The results from other model calculations and the experiments are listed as well.

$l = e(\mu)$	$f_{l+}(m_l^2)$	$-f_{l-}(m_l^2)$	$\lambda_{l+} \times 10^2$	$\lambda_{l0}$	$\xi_l =  f_{l+}/f_{l-} $	$\Gamma_{l3} [10^6/\text{s}]$	$\langle r^2 \rangle^{K\pi} [\text{fm}^2]$
Present	0.947(0.963)	0.137(0.145)	3.03	0.0136	0.147(0.152)	6.840(4.469)	0.366
[9]	...	...	2.45	...	0.28	...	0.292
[5]	1.022	...	...	...	...	...	0.36
[2]	0.972	...	...	...	...	...	...
[11]	0.952	...	2.12	...	...	...	0.376
[13]	0.964	0.100	2.70	0.018	0.11	7.38(4.90)	0.322
[14]	0.980(1.11)	0.24(0.27)	2.80	0.0026	0.35	...	0.334
[16]	0.962	...	2.60	0.025	0.01	7.3(4.92)	0.310
[42]	...	...	2.80	...	0.28	...	0.334
[43]	0.93	0.26	1.90	...	0.28	...	...
[44]	0.9874	...	...	...	...	...	...
[45]	0.981	...	...	...	...	...	...
[46]	0.960	...	2.60	0.0089	...	...	0.310
[47]	0.7	0.068	1.52	...	0.097	...	0.181
[8] (Experiment)	...	...	$2.96 \pm 0.05$	...	...	$7.920 \pm 0.040$ $(5.285 \pm 0.024)$	...
[39] (Experiment)	...	...	$2.45 \pm 0.12$	...	...	...	...

All numerical results are summarized in Table I with the experimental data and those of other approaches for comparison.

## V. SUMMARY AND CONCLUSION

In the present work, we have investigated the kaon semileptonic decay ( $K_{l3}$ ) form factors within the framework of the gauged nonlocal chiral quark model from the instanton vacuum. The effect of flavor SU(3) symmetry breaking were taken into account. We calculated the vector form factors ( $f_{\pm}$ ), scalar form factor ( $f_0$ ), slope parameters ( $\lambda_{+,0}$ ), decay width ( $\Gamma_{l3}$ ), etc. as demonstrated in Table I. We found that the present results of the kaon semileptonic decay form factors are in a qualitatively good agreement with experiments. We emphasize that there were no adjustable free parameters in the present investigation. All results were obtained with only two parameters from the instanton vacuum, i.e. the average instanton size ( $\bar{\rho} \sim 1/3$  fm) and interinstanton distance ( $R \sim 1$  fm).

In the present investigation, we have considered only the leading-order contributions in the large  $N_c$  limit. While these contributions reproduce the observables relevant for kaon semileptonic decay in general, it seems necessary to take into account the meson-loop  $1/N_c$  corrections in order to reproduce quantitatively the kaon decay constant  $f_K$  and the low-energy constant  $L_5$ . As noticed already in Refs. [50,52,53], these meson-loop corrections can play an important role in producing the kaon properties. The related works are under progress.

## ACKNOWLEDGMENTS

The present work is supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) (KRF-2006-312-C00507). The work of S.i.N. is supported by the Brain Korea 21 (BK21) project in Center of Excellency for Developing Physics Researchers of Pusan National University, Korea. S.i.N. would like to express his gratitude to M. Khlopov for the fruitful comments.

- 
- [1] N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963).
  - [2] H. Leutwyler and M. Roos, Z. Phys. C **25**, 91 (1984).
  - [3] M. Ademollo and R. Gatto, Phys. Rev. Lett. **13**, 264 (1964).
  - [4] P. Langacker and H. Pagels, Phys. Rev. Lett. **30**, 630 (1973).
  - [5] J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 517 (1985).
  - [6] M. Baće and D.T. Cornwell, Phys. Rev. D **10**, 2694 (1974).
  - [7] C.G. Callan and S.B. Treiman, Phys. Rev. Lett. **16**, 153 (1966).
  - [8] W.M. Yao *et al.* (Particle Data Group), J. Phys. G **33**, 1 (2006).
  - [9] L.M. Chounet, J.M. Gaillard, and M.K. Gillard, Phys. Rep. **4**, 199 (1972).
  - [10] J. Bijnens, G. Colangelo, G. Ecker, and J. Gasser, arXiv:hep-ph/9411311.
  - [11] N. Tsutsui *et al.* (JLQCD Collaboration), Proc. Sci., LAT2005 (**2006**) 357.
  - [12] D. Becirevic *et al.*, Nucl. Phys. **B705**, 339 (2005).
  - [13] C.R. Ji and P. Maris, Phys. Rev. D **64**, 014032 (2001).
  - [14] Yu. Kalinovsky, K.L. Mitchell, and C.D. Roberts, Phys. Lett. B **399**, 22 (1997).
  - [15] N. Isgur, Phys. Rev. D **12**, 3666 (1975).
  - [16] H.M. Choi and C.R. Ji, Phys. Rev. D **59**, 034001 (1998).
  - [17] H.M. Choi and C.R. Ji, Phys. Lett. B **460**, 461 (1999).
  - [18] E.V. Shuryak, Nucl. Phys. **B203**, 93 (1982).
  - [19] D. Diakonov and V.Y. Petrov, Nucl. Phys. **B245**, 259 (1984).
  - [20] D. Diakonov and V.Y. Petrov, Nucl. Phys. **B272**, 457 (1986).
  - [21] M.C. Chu, J.M. Grandy, S. Huang, and J.W. Negele, Phys. Rev. D **49**, 6039 (1994).
  - [22] J.W. Negele, Nucl. Phys. B, Proc. Suppl. **73**, 92 (1999).
  - [23] T. DeGrand, Phys. Rev. D **64**, 094508 (2001).
  - [24] M. Musakhanov, Eur. Phys. J. C **9**, 235 (1999).
  - [25] M. Musakhanov, arXiv:hep-ph/0104163.
  - [26] M. Musakhanov, Nucl. Phys. **A699**, 340 (2002).
  - [27] S.i. Nam and H.-Ch. Kim, Phys. Lett. B **647**, 145 (2007).
  - [28] H.Y. Ryu, S.i. Nam, and H.-Ch. Kim, arXiv:hep-ph/0610348 [Phys. Rev. D (unpublished)].
  - [29] M.M. Musakhanov and H.-Ch. Kim, Phys. Lett. B **572**, 181 (2003).
  - [30] S.i. Nam and H.-Ch. Kim, Phys. Rev. D **74**, 076005 (2006).
  - [31] H.-Ch. Kim, M. Musakhanov, and M. Siddikov, Phys. Lett. B **608**, 95 (2005).
  - [32] P.V. Pobylitsa, Phys. Lett. B **226**, 387 (1989).
  - [33] H.J. Munczek and D.W. McKay, Phys. Rev. D **39**, 888 (1989); **46**, 5209(E) (1992).
  - [34] H.J. Munczek and A.M. Nemirovsky, Phys. Rev. D **28**, 181 (1983).
  - [35] D.W. McKay and H.J. Munczek, Phys. Rev. D **28**, 187 (1983).
  - [36] D.W. McKay and H.J. Munczek, Phys. Rev. D **32**, 266 (1985).
  - [37] D. Atkinson, H. de Groot, and P.W. Johnson, Phys. Rev. D **43**, 218 (1991).
  - [38] R.D. Bowler and M.C. Birse, Nucl. Phys. **A582**, 655 (1995).
  - [39] A. Apostolakis *et al.* (CPLEAR Collaboration), Phys. Lett. B **473**, 186 (2000).
  - [40] E.P. Shabalin, Yad. Fiz. **51**, 464 (1990) [Sov. J. Nucl. Phys. **51**, 296 (1990)].
  - [41] C.R. Ji and H.M. Choi, Phys. Lett. B **513**, 330 (2001).
  - [42] A. Afanasev and W.W. Buck, Phys. Rev. D **55**, 4380

- (1997).
- [43] D. Scora and N. Isgur, Phys. Rev. D **52**, 2783 (1995).
- [44] V. Cirigliano, M. Knecht, H. Neufeld, H. Rupertsberger, and P. Talavera, Eur. Phys. J. C **23**, 121 (2002).
- [45] V. Cirigliano, H. Neufeld, and H. Pichl, Eur. Phys. J. C **35**, 53 (2004).
- [46] D. Becirevic *et al.*, Eur. Phys. J. A **24S1**, 69 (2005).
- [47] A.O. Barut and K.C. Tripathy, Phys. Rev. **178**, 2278 (1969).
- [48] G. Ecker, Prog. Part. Nucl. Phys. **35**, 1 (1995).
- [49] B.W. Lee, Phys. Rev. Lett. **20**, 617 (1968).
- [50] H.-Ch. Kim, M.M. Musakhanov, and M. Siddikov, Phys. Lett. B **633**, 701 (2006).
- [51] G. Calderon and G. Lopez Castro, Phys. Rev. D **65**, 073032 (2002).
- [52] S.i. Nam, H.-Ch. Kim, A. Hosaka, and M.M. Musakhanov, Phys. Rev. D **74**, 014019 (2006).
- [53] J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465 (1985).