Comment on "Primordial magnetic seed field amplification by gravitational waves"

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We consider the amplification of cosmological magnetic fields by gravitational waves as it was recently presented by Betschart *et al.* That study confined to infinitely conductive environments, arguing that on spatially flat Friedmann backgrounds the gravito-magnetic interaction proceeds always as if the Universe were a perfect conductor. We explain why this claim is not correct and then reexamine the Maxwell-Weyl coupling at the limit of ideal magnetohydrodynamics. We find that the scales of the main results of Betschart *et al.* were not properly assessed and that the incorrect scale assessment has compromised both the physical and the numerical results of the paper. This comment aims to clarify these issues on the one hand, while on the other it takes a closer look at the gauge invariance and the nonlinearity of the formalism proposed by Betschart *et al.*

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I. INTRODUCTION

The interaction between electromagnetic fields and gravitational waves and the possible energy transfer between the Weyl and the Maxwell fields has a long research history. A mechanism for the amplification of large-scale magnetic fields by gravity waves of similar size soon after inflation was recently proposed in [1,2]. In the poorly conductive environment of early reheating the analysis (see [2] for details) indicated a resonant magnetic amplification proportional to the square of the field's scale and also to the gravitationally induced shear anisotropy. These features meant that Weyl-curvature distortions could provide a very efficient early-universe dynamo of superhorizon-sized magnetic fields. For example, fields with a current comoving scale of approximately 10 kpc and a strength of $\sim 10^{-34}$ G, like those produced in [3], could be amplified by many orders of magnitude by the end of reheating.

The same gravito-magnetic interaction has been applied to infinitely conductive cosmologies in [4]. Central to that study is the claim that on spatially flat Friedmann-Robertson-Walker (FRW) backgrounds the Maxwell-Weyl coupling proceeds always as if the Universe were a perfect conductor. We argue that this is not the case and explain why the above mentioned paper arrived at the opposite result. In addition, we find serious problems in the scale assessments of [4]. These have prevented the authors from recognizing the main effect of the interaction, namely, the large-scale superadiabatic amplification of the *B* field, and compromised their results. With the present comment we draw attention to these issues by correcting the mathematics, where necessary, and by clarifying the physics. We also reexamine and question the gauge invariance and the nonlinearity of the formalism proposed in [4].

II. ON THE ELECTRIC CURL AND THE ROLE OF CONDUCTIVITY

Section III B of [4] argues that the Maxwell-Weyl coupling proceeds unaffected by the conductivity of the medium and as if the Universe were a perfect conductor. This follows the claim that the gravitationally induced, second order, electric field (E_a) is curl free if the zero order FRW model has flat spatial sections. This is not the case, however, because

$$(\mathrm{curl}E_a)^{\cdot} = -\Theta \mathrm{curl}E_a + \mathcal{R}_{ab}\tilde{B}^b - \mathrm{curl}\mathcal{J}_a - \mathrm{D}^2 B_a, \quad (1)$$

at second order. Here Θ is the volume expansion, $\mathcal{R}_{ab}$ is the first-order 3-Ricci tensor, $\tilde{B}_a$ is the first-order magnetic field, $\mathcal{J}_a$ is the spatial current, and $\mathrm{D}^2 = \mathrm{D}_a\mathrm{D}^a$ is the 3-dimensional Laplacian operator. Therefore, even when $\mathrm{curl}E_a$ initially vanishes, there are sources in the right-hand side of (1) that will generally lead to a nonzero electric curl. Switching the latter off is not a consistent constraint. Note that the current term depends on the conductivity of the medium, while the 3-curvature distortions and the magnetic field fluctuations are caused by the presence of gravity-wave perturbations. These set the field lines into motion causing magnetic fluctuations that produce an electric component. The latter has $\mathrm{curl}E_a \neq 0$ because of (1). This means that the gravito-magnetic interaction does not always proceed as if the conductivity of the Universe were infinite. In [4], the authors arrived at the opposite conclusion because they considered the second time-derivative of $\mathrm{curl}E_a$ instead of the first, namely, (see Eq. (29) in [4])

$$(\mathrm{curl}E_a)^{\cdot\cdot} + \tfrac{7}{3}\Theta(\mathrm{curl}E_a)^{\cdot} - \mathrm{D}^2\mathrm{curl}E_a + [\tfrac{7}{9}\Theta^2$$
$$+ \tfrac{1}{6}(\mu - 9p) + \tfrac{5}{3}\Lambda]\mathrm{curl}E_a = \mathrm{curl}K_a. \quad (2)$$

In the above, which is said to hold at the second perturbative level, the pair μ and p represents the density and the

pressure of the matter, Λ is the cosmological constant and K_a is a gravito-magnetic source term. Arguing that curlK_a vanishes when the FRW background is spatially flat (see Eq. (30) in [4]), the paper claims that the electric field will remain curl free if it was so initially. However, this *a priori* sets $(\text{curl}E_a)^{\cdot} = 0$ at all times in Eq. (2), although the latter is not necessarily the case because of expression (1).

By switching the electric curl off the authors have inadvertently confined their study to idealized perfectly conductive universes, bypassing all plasma effects and the implications of finite conductivity. Because of that [4] cannot follow the gravito-magnetic interaction in the poorly conductive stages of early reheating and therefore the comparison with [1,2] was inappropriate.

III. ON THE SCALE ASSESSMENT AND THE NATURE OF THE AMPLIFICATION

Restricting to perfectly conducting cosmological environments we have (see Sec. II C, IV B in [4])

$$\dot{B}_a + \tfrac{2}{3}\Theta B_a = \sigma_{ab}\tilde{B}^b \equiv I_a, \tag{3}$$

where B_a is the total magnetic field, $\tilde{B}_a$ is the original, and σ_{ab} is the transverse shear component. Then, the gravitationally induced B field during the radiation era is

$$B_R = \tilde{B}_0\left(\frac{a_0}{a}\right)^2\left\{1 + \frac{2}{3}\left(\frac{\sigma}{H}\right)_0\left[\left(\frac{a_0}{a}\right) - 1\right] + \frac{5}{6}\left(\frac{\sigma}{H}\right)_0\left[\left(\frac{a}{a_0}\right)^2 - 1\right]\right\}, \tag{4}$$

while for dust and late reheating we have

$$B_{D/RH} = \tilde{B}_0\left(\frac{a_0}{a}\right)^2\left\{1 + \frac{2}{3}\left(\frac{\sigma}{H}\right)_0\left[\left(\frac{a_0}{a}\right)^{3/2} - 1\right] + 2\left(\frac{\sigma}{H}\right)_0\left[\left(\frac{a}{a_0}\right) - 1\right]\right\}, \tag{5}$$

(see Eqs. (46), (47) in [4]). Above a is the scale factor, H is the Hubble parameter, and the zero suffix marks the onset of the gravito-magnetic interaction. These expressions were treated as strictly infinite wavelength solutions (see Sec. IV B 1 in [4]), when in reality they cover all finite superhorizon scales.[1] This incorrect scale assessment has

[1]Assuming dust and setting $\tau = t/t_0$ the gravito-magnetic source term in the right-hand side of (4) reads

$$I_{(\ell)}(\tau) = [D_1 J_{5/2}(2\ell\tau^{1/3}/a_0 H_0) + D_2 Y_{5/2}(2\ell\tau^{1/3}/a_0 H_0)]\tau^{-5/2}, \tag{6}$$

where ℓ is the wave number of the gravitational wave (see Eq. (48) in [4]). Finite superhorizon scales have $\ell\tau^{1/3}/a_0 H_0 \ll 1$ and the series expansion of the Bessel functions (with the initial conditions of [4]) give

$$I(\tau) = 2\sigma_0\tilde{B}_0\tau^{-5/3} - \sigma_0\tilde{B}_0\tau^{-10/3}. \tag{7}$$

Inserted into Eq. (4) the above leads to solution (5), which therefore applies to all finite super-Hubble lengths and not only to infinite wavelengths. Similarly one can show that (5) also covers all finite superhorizon lengths.

led to the sequence of problems that we will discuss below. Here we note that the domains of (4) and (5) are essential for the additional reason that both results show a deviation from the standard a^{-2} magnetic decay law and a super-adiabatic amplification of the field (i.e. $B = $ constant for radiation and $B \propto a^{-1}$ for dust/reheating).

Returning to [4] we find [see Eq. (51) there] that the final solution for the gravitationally induced magnetic field, during both the radiation and the dust eras, is

$$\frac{B}{\rho_\gamma^{1/2}} \simeq \left[1 + \frac{1}{10}\left(\frac{\lambda_{\tilde{B}}}{\lambda_H}\right)_0^2\left(\frac{\sigma}{H}\right)_0\right]\left(\frac{\tilde{B}}{\rho_\gamma^{1/2}}\right)_0 \Rightarrow B$$

$$\simeq \tilde{B}_0\left[1 + \frac{1}{10}\left(\frac{\lambda_{\tilde{B}}}{\lambda_H}\right)_0^2\left(\frac{\sigma}{H}\right)_0\right]\left(\frac{a_0}{a}\right)^2, \tag{8}$$

since $\rho_\gamma \propto a^{-4}$ is the radiation density.[2] It is also argued (see Sec. IV B 2 after Eq. (51) in [4]) that the above spans all finite wavelengths irrespective of their size relative to the horizon. This cannot be correct because (8) means $B \propto a^{-2}$ on all scales, while (4) and (5) give $B = $ constant and $B \propto a^{-1}$ on super-Hubble lengths. Consider, for example, a large-scale B mode propagating through the radiation era and ignore any effects from reheating. Then, Eq. (4) gives

$$B \simeq \tilde{B}_0\left[1 + \left(\frac{a}{a_0}\right)_0^2\left(\frac{\sigma}{H}\right)_0\right]\left(\frac{a_0}{a}\right)^2$$

$$= \tilde{B}_0\left[1 + \left(\frac{\lambda_{\tilde{B}}}{\lambda_H}\right)_0^2\left(\frac{\sigma}{H}\right)_0\left(\frac{\lambda_H}{\lambda_{\tilde{B}}}\right)^2\right]\left(\frac{a_0}{a}\right)^2. \tag{9}$$

Well outside the horizon $\lambda_H/\lambda_{\tilde{B}} \ll 1$, which means that using (8) instead of (9) on those scales will grossly overestimate the amplification of the field. The two expressions agree only at horizon crossing when $\lambda_{\tilde{B}} = \lambda_H$. Therefore, at best, Eq. (8) works for scales that have crossed the Hubble radius. However, the magnetic growth takes place earlier outside the horizon and results from the superadiabatic amplification of the field seen in (4) and (5). In [4] the reader is unaware of the nature of the magnetic growth and where it occurs. The root of the problem was assigning (4) and (5) to infinite wavelengths. As a result, these solutions were bypassed as completely negligible (see, in particular, Sec. V after Eq. (60) in [4]), the superadiabatic nature of the amplification was not recognized, the domain of (8) was incorrectly assessed, and ultimately the physics of the gravito-magnetic interaction was misrepresented.

An additional complication is that Eq. (8) alone cannot provide the total amplification of a mode that went through

[2]Expression (8) was obtained after replacing λ_{GW} with $\lambda_{\tilde{B}}$ in Eqs. (49) and (50) of [4]. However, this substitution was arbitrary because no relation between the two scales was previously established and the original magnetic field was given a zero wave number (see Sec. IV B in [4]). The latter means that $\lambda_{\tilde{B}}$ is ill defined (i.e. $\lambda_{\tilde{B}} \to \infty$) and misleadingly suggests an infinitely strong induced B field in (8). Note that assigning a nonzero wave number to $\tilde{B}$ provides a useful relation between λ_B, λ_{GW}, and $\lambda_{\tilde{B}}$ (see Eq. (10) in [2]).

different epochs (e.g. reheating and radiation), because each domain has different growth rates [compare (4) and (5)]. Consider an inflationary B mode that crosses the horizon in the radiation era (like that in Sec. V of [4]). Assuming for simplicity that the growth is always strong, the residual field according to (8) is $B = \tilde{B}_0(\sigma/H)_0 \times (\lambda_{\tilde{B}}/\lambda_H)_0^2(a_0/a)^2$. This result was used in [4] but it does not account for the reheating effects, for which we also need (5). Then,

$$B = \tilde{B}_0\left(\frac{\sigma}{H}\right)_0\left(\frac{\sigma}{H}\right)_{RH}\left(\frac{\lambda_{\tilde{B}}}{\lambda_H}\right)_0^2\left(\frac{a_0}{a}\right)^2, \qquad (10)$$

where RH marks the end of reheating. The difference in the above is made by the extra factor $(\sigma/H)_{RH}$. The latter is typically very small and reflects the fact that the magnetic growth rate during (late) reheating is considerably slower than that of the radiation era. Therefore, not appreciating the role of (4) and (5) has also compromised the numerical results of [4].

IV. ON THE GAUGE INVARIANCE AND THE NONLINEARITY

Studies of cosmological perturbations are known to suffer from the gauge problem. The aim of [4] is to provide a nonlinear treatment of the gravito-magnetic interaction free from gauge ambiguities. This meant integrating the magnetic induction equation (i.e. Eq. (3) above) with respect to B_a (see Sec. IV in [4]). However, B_a is treated as the perturbation of $\tilde{B}_a$ (see Sec. II C in [4]). The latter has nonzero linear value and this makes B_a a gauge-dependent vector at second order by known theorems [5]. In an attempt to circumvent the problem the auxiliary variable β_a, with $\beta_a \equiv \dot{B}_a + 2\Theta B_a/3$ was temporarily introduced (see Sec. II C in [4]). This has zero linear value and is therefore gauge-invariant at second order. Nevertheless, β_a has been of little practical use because the authors still had to solve for the gauge-dependent vector B_a to extract any meaningful information about the evolution of the field (see Eqs. (4), (5), and (8) here).[3] This fact makes the analysis and the results of [4] gauge dependent.

The nonlinearity of [4] is built on a set of four spacetimes (see Sec. II there). However, the new setting has not

improved our understanding of the interaction because still only the gravito-magnetic effects of [1,2] are accounted for and all other nonlinearities (including the magnetic effects on the shear) are excluded. There is no new information in the nonlinear equations of [4], relative to what is already encoded in the linear formulas of [1,2].[4] This should not have happened and the reason it does is the selective setting of [4], which restricts the study and prevents it from going beyond the linear level.

V. DISCUSSION

The Maxwell-Weyl coupling and the possible energy transfer between the two fields have a long research history. Recently this interaction was proposed as a very efficient (resonant) amplification mechanism of large-scale magnetic fields during the poorly conductive stages of early reheating [1,2]. The same mechanism was studied at the ideal MHD limit in [4], claiming that the gravito-magnetic interaction proceeds always as if the Universe were a perfect conductor when the background spatial geometry is Euclidean. We explained here why the afore-mentioned claim is not correct and how it restricts the generality of [4]. We also demonstrated that the aforementioned paper did not properly monitor the gravito-magnetic interaction. In particular, solutions which in [4] were assigned to infinite wavelengths only, were found to hold on all finite super-Hubble scales. Also solutions that work only inside the horizon were applied to all finite scales. As a result, the physical interpretation and the numerical results of [4] were seriously compromised. With this comment we have attempted to clarify these issues by correcting, where necessary, the mathematics of the analysis and by explaining the physics of the interaction. In the process we also considered and questioned the gauge-invariance and the nonlinearity of the formalism proposed in [4].

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[3]If the authors had found a way of evaluating the magnetic growth by means of β_a exclusively, their results would have been gauge invariant. This was not possible however. Moreover, at the ideal magnetohydrodynamics (MHD) limit β_a is all but redundant because $\beta_a \equiv \sigma_{ab}B^b$ (see Secs. II C, IV B in [4] or Eq. (3) here).

[4]Expressions (4) and (10), the latter with $\Lambda = 0$, of [4] are identical to Eqs. (2b) and (4) in [2]. Also, at the MHD limit, the magnetic wave equation of [2] [see Eq. (3) there] and expression (12) in [4] reduce to Eq. (3) here. Moreover, the constraints used in [4] [see Sec. II B 2 there) are the standard linear ones. These constraints ensure that the traceless tensors remain transverse at all times, a highly nontrivial issue for any truly nonlinear study.

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