Instability of classical dynamics in theories with higher derivatives

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It is shown explicitly that in classical dynamics of theories with higher derivatives there arises the exponential instability with respect to external dissipative force. For this aim, the equations of motion for the Pais-Uhlenbeck fourth order oscillator with damping are investigated and the corrections to the oscillator frequencies due to the friction force are calculated.

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I. INTRODUCTION

All fundamental physical theories are defined by differential equations at most of the second order. However, in the course of searching for new theories the models described by higher-order derivative Lagrangians are considered too [1]. It is worth mentioning here the gauge theories with higher derivatives [2,3], the gravitation models with higher-order curvature corrections to the Einstein-Hilbert action [4–13], the models of point particles with Lagrangians depending on curvature and torsion of the world trajectories [14,15], rigid strings [16]. The higher derivative theories have some appealing properties, in particular, the convergence of the Feynman diagrams is improved [17], for instance, the conformal gravity is not only power counting renormalizable quantum field theory [5], but it turns out to be asymptotically free [6,7].

Recently the interest in theories with higher derivatives revived again. For example, such models are considered when looking for the theory of everything [18,19] and when modifying the gravity to make it predict observed cosmic phenomena without the need for dark energy and maybe for dark matter [13,20].

It is generally believed that the theories with higher derivatives are intrinsically sick due to the states with negative norm (''ghosts'') which spoil the unitarity [21]. This peculiarity of theories at hand is a direct consequence of unboundness from below of their quantum spectrum [13,22,23]. All this leads to instability of higher derivative theories both at classical and quantum levels. Sometimes this instability is referred to as the Ostrogradskian instability (see, for example, the review [13]). The recent papers [18,19] have convincingly demonstrated that it is instructive to investigate the general problem of instability of theories with higher derivatives first of all in the framework of their classical dynamics.

The present Brief Report seeks to draw the attention to the instability of theories with higher derivatives with respect to external dissipative force; namely, it will be shown that in classical dynamics of such models there arises the exponential instability (runaway solutions) due to the external friction force. As far as we know, the instability of this kind has not been considered yet.

When investigating the higher derivative theories it is convenient to use the Pais-Uhlenbeck forth order oscillator [1,13,18,24], a quantum mechanical analog of a field theory containing both second and fourth order derivative terms. This model, upon introducing therein damping, will be used in the present paper also.

II. PAIS-UHLENBECK FOURTH ORDER OSCILLATOR WITH DAMPING

Without damping the coordinate $x(t)$ of the Pais-Uhlenbeck oscillator obeys the equation [1]

$$\frac{d^4x}{dt^4} + (\omega_1^2 + \omega_2^2)\frac{d^2x}{dt^2} + \omega_1^2\omega_2^2 x = 0, \tag{2.1}$$

the general solution to which has the form

$$x(t) = a_1 e^{i\omega_1 t} + a_1^* e^{-i\omega_1 t} + a_2 e^{i\omega_2 t} + a_2^* e^{-i\omega_2 t}, \tag{2.2}$$

where a_n are the complex amplitudes. In what follows, we assume that the positive frequencies ω_1 and ω_2 are different and $\omega_2 > \omega_1$. The case of equal frequencies $\omega_1 = \omega_2$ requires a special consideration [18,24].

By making use of the Ostrogradski formalism [25] one can develop the Hamiltonian description of this model (see, for example, [13,18,24]). As a result, the conserved energy acquires the following values

$$E_1 = (a_1 a_1^* + a_1^* a_1)\omega_1^2(\omega_2^2 - \omega_1^2) > 0, \tag{2.3}$$

for the oscillations with the frequency ω_1 and

$$E_2 = (a_2 a_2^* + a_2^* a_2)\omega_2^2(\omega_1^2 - \omega_2^2) < 0 \tag{2.4}$$

for the frequency ω_2.

The energy of the conservative system (2.1) is an integral of motion, and the fact that it can acquire negative values (2.4) does not lead to any physical contradictions. A different situation arises when the system under consideration experiences external action.

Let us introduce the friction force into the equation of motion (2.1)

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$$\frac{d^4x}{dt^4} + (\omega_1^2 + \omega_2^2)\frac{d^2x}{dt^2} + \omega_1^2\omega_2^2 x + 2\gamma\frac{dx}{dt} = 0, \quad (2.5)$$

where 2γ is a positive constant (friction coefficient). As before the time dependence of the solutions to Eq. (2.5) is described by the factor $e^{i\omega t}$, where the frequency ω is the root of the equation

$$\omega^4 - (\omega_1^2 + \omega_2^2)\omega^2 + \omega_1^2\omega_2^2 + 2i\gamma\omega = 0. \quad (2.6)$$

In order to simplify the calculations we assume that the damping is weak, so the perturbation theory is applicable. Upon substituting in Eq. (2.6)

$$\omega = \omega_n + i\Delta\omega_n, \qquad n = 1, 2, \quad (2.7)$$

we arrive at the equation for $\Delta\omega_n$. In the approximation linear in γ it reads

$$2\omega_n^2\Delta\omega_n - (\omega_1^2 + \omega_2^2)\Delta\omega_n + \gamma = 0, \qquad n = 1, 2. \quad (2.8)$$

Thus, the frequency shifts for ω_1 and ω_2 prove to be real and equal in the absolute value but they have the opposite signs

$$\Delta\omega_1 = \frac{\gamma}{\omega_2^2 - \omega_1^2} > 0, \quad (2.9)$$

$$\Delta\omega_2 = \frac{\gamma}{\omega_1^2 - \omega_2^2} < 0, \quad (2.10)$$

because we have assumed that $\omega_2 > \omega_1$. As a result, the solution

$$x_1(t) \sim e^{i\omega_1 t - \Delta\omega_1 t} \quad (2.11)$$

exponentially decreases in time, while the solution

$$x_2(t) \sim e^{i\omega_2 t + |\Delta\omega_2| t} \quad (2.12)$$

exponentially increases in time. The damping behavior of the solution $x_1(t)$ is physically motivated, namely, due to the external friction force the amplitudes a_1 and a_1^* in Eq. (2.3) decrease in time and the energy E_1 also decreases being positive. Unlike this, the amplitudes a_2 and a_2^* in Eq. (2.4) exponentially grow up and there are no reasons to stop this process, since the energy E_2, being negative, decreases without bound. Thus we arrive at the exponential instability of the classical dynamics of the Pais-Uhlenbeck oscillator due to the external dissipative force (runaway solutions).

We can also consider the Pais-Uhlenbeck oscillator which experiences an arbitrary external force $f(t)$

$$\frac{d^4x}{dt^4} + (\omega_1^2 + \omega_2^2)\frac{d^2x}{dt^2} + \omega_1^2\omega_2^2 x = f(t). \quad (2.13)$$

The general solution to this equation can be expressed in terms of the relevant Green function

$$x(t) = \int_{-\infty}^{\infty} G(t - t')f(t')dt' = \int_{-\infty}^{\infty} \bar{G}(\omega)\bar{f}(\omega)d\omega, \quad (2.14)$$

where

$$G(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega t}\bar{G}(\omega)d\omega,$$
$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega t}\bar{f}(\omega)d\omega. \quad (2.15)$$

From Eq. (2.13) we deduce in a straightforward way

$$\bar{G}(\omega) = \frac{1}{\omega_1^2 - \omega_2^2}\left(\frac{1}{\omega^2 - \omega_2^2} - \frac{1}{\omega^2 - \omega_1^2}\right). \quad (2.16)$$

Now we see from Eqs. (2.14) and (2.16), that the forces with spectral densities localized around ω_1^2 and ω_2^2 give rise to displacement x of opposite signs. Obviously, it implies that one of these displacements is unphysical.

III. CONCLUSION

By making use of a simple and clear example, tractable analytically, we have revealed the exponential instability of the theories with higher derivatives with respect to the external dissipative force. In view of irremovable character of dissipative processes, any theory with higher derivatives, in order to be viable, should involve the mechanism that prevents such an instability.

In closing, it is worthy to note that the instability problem in nonrelativistic mechanical models with higher derivatives has its own peculiarities [22,26,27].

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