Batalin-Fradkin-Tyutin embedding of noncommutative chiral bosons

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A two dimensional model of chiral bosons in noncommutative field space is considered in the framework of the Batalin-Fradkin-Tyutin Hamiltonian embedding method converting the second-class constrained system into the first-class one. The symmetry structure associated with the first-class constraints is explored and the propagation speed of fields is equivalent to that of the second-class constraint system.

I. INTRODUCTION

The model of chiral bosons in two dimensions is basically a constrained system. Although it is simple, the study of its structure may give us helpful insights in understanding various models with some chiral structure, like theories of self-dual objects appearing in superstring theory. One of the major issues about the chiral boson model is how to treat its constraint structure consistently in the canonical Hamiltonian formalism. Because of the interesting feature of the model itself and its implicit relevance to other models, there have been lots of studies from various viewpoints, which are based largely on three approaches, the Floreanini-Jackiw (FJ) method [1], the method with linear constraint [2], and the Batalin-Fradkin-Tyutin (BFT) embedding method [3].

The usual playground for the study of chiral bosons is assumed to be restricted to the commutative field space. On the other hand, there is some argument about the possibility that some noncommutative effects could take place in ultra high-energy physics without violating Lorentz invariance. Motivated by this, one may consider the chiral bosons in the noncommutative field space, based on achievements of the previous studies. Indeed, the construction of the corresponding model has been given, and the problems on the bosonization and Lorentz invariance have been studied in Ref. [4].

Having the model of chiral bosons in the noncommutative field space, it is natural to ask about its canonical structure and investigate, if any, its difference from the commutative model. In this article, we study the canonical structure of the model in the framework of the BFT embedding method [3]. The BFT method converts the second-class constrained system into the first-class one by introducing auxiliary fields and hence extending the phase

space, and allows one to have local symmetries associated with the first-class constraints (See Refs. [5–8] for chiral bosons, Refs. [9–15] for the Chern-Simons model, and Ref. [16] for the noncommutative D-brane system). The resulting full first-class constrained system in the extended phase space usually has many fields (infinite number of fields in our case). We consider the propagation speed of each field and investigate the consistency in Lorentz invariance.

The organization of this paper is as follows. In the next section, we introduce the model of chiral boson in noncommutative field space and take into account of its second-class constraints via the method of symplectic structure [1]. The BFT embedding of the model follows in Sec. III and the second-class constraints are fully converted into the first-class one in the extended phase space. The resulting extended system is shown to have infinite local symmetries. At the end, from the equations of motion of the fields, the propagation speed in the noncommutative field space is considered. Conclusions are given in Sec. IV.

II. NONCOMMUTATIVE CHIRAL BOSON

The noncommutativity in field space is basically represented by the nonvanishing commutator between different elementary fields. The action for a theory in noncommutative field space is constructed in a way that such noncommutativity is realized. The theory of chiral boson in noncommutative field space has been constructed in Ref. [4]. In order to study the theory of a chiral boson in a noncommutative field space, the Poisson brackets in this model have been deformed by the noncommutative parameter θ. In this case, the action is given by

$$S = \int d^2x \left[-\frac{2}{1+\theta^2} \dot{\phi}_a \Delta_{ab} \phi'_b - \phi'_a \phi'_a \right], \quad (1)$$

where the overdot and the prime denote the derivatives with respect to time and space, respectively [4]. The left (right) moving field is represented by the subscript a with positive (negative) sign. The 2×2 matrix Δ_{ab} encodes the

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noncommutativity of the field space with the noncommutative parameter θ and is defined by

$$\Delta_{ab} \equiv \frac{a}{2}(\theta\epsilon_{ab} - \delta_{ab}) = \frac{1}{2}\begin{pmatrix} -1 & \theta \\ \theta & 1 \end{pmatrix}, \qquad (2)$$

the inverse of which is

$$\Delta_{ab}^{-1} = \frac{4}{1+\theta^2}\Delta_{ab}. \qquad (3)$$

Note that for the $\theta \to 0$ limit, the action reduces to the FJ action [1].

The system (1) is basically a constrained one, since the canonical momentum Π_a of the field ϕ_a does not contain any time evolution of the field as can be easily seen by $\Pi_a = -\frac{2}{1+\theta^2}\Delta_{ab}\phi_b'$. The primary constraints are then

$$\Omega_a = \Pi_a + \frac{2}{1+\theta^2}\Delta_{ab}\phi_b' \approx 0, \qquad (4)$$

and the evaluation of the Poisson bracket between them gives $\{\Omega_a(x), \Omega_b(y)\} = \frac{4}{1+\theta^2}\Delta_{ab}\partial_x\delta(x-y)$, from which we see that the primary constraints are in the second class. The time evolution of primary constraints by using the primary Hamiltonian defined by $H_p = H_c + \int dx\lambda_a\Omega_a$, where H_c is the canonical Hamiltonian corresponding to the action (1),

$$H_c = \int dx\phi_a'\phi_a', \qquad (5)$$

results in fixing the Lagrangian multiplier fields λ_a. Therefore, the primary constraints (4) form a full set of constraints for the system (1).

All the constraints are in the second class and hence a proper procedure is required to implement them consistently. Although the usual Dirac procedure [17] may be considered, we take a smarter method based on the symplectic structure developed by Floreanini and Jackiw (FJ) [1], which is basically concerned with symplectic structure. Since our system (1) is just the first order one, the symplectic structure method is especially suitable. The symplectic structure, say C_{ab}, is read off from the first order term in time derivative, and its precise form in the present case is obtained as $C_{ab} = \frac{4}{1+\theta^2}\Delta_{ab}\partial_x\delta(x-y)$. The so-called FJ bracket between the field variables ϕ_a is simply given by the inverse of the symplectic structure, which is $C_{ab}^{-1}(x, y) = \Delta_{ab}\frac{1}{\partial_x}\delta(x-y) = \Delta_{ab}\epsilon(x-y)$, where $\epsilon(x-y)$ is the step function. The resulting nonvanishing brackets between elementary fields are then obtained as follows. $\{\phi_a(x), \phi_b(y)\}_{FJ} = \Delta_{ab}\epsilon(x-y)$, $\{\phi_a(x), \Pi_b(y)\}_{FJ} = \frac{1}{2}\delta_{ab}\delta(x-y)$, $\{\Pi_a(x), \Pi_b(y)\}_{FJ} = -\frac{1}{1+\theta^2}\Delta_{ab}\partial_x\delta(x-y)$. They are equivalent to the Dirac brackets and they recovers the conventional brackets for chiral bosons for the $\theta \to 0$ limit.

III. BFT EMBEDDING

In this section, we consider the system (1) in the framework of the BFT Hamiltonian embedding method and convert it into the first-class constrained system. For notational convenience, we replace the fields ϕ_a and Π_a with $\phi_a^{(0)}$ and $\Pi_a^{(0)}$, respectively. The second-class constraints (4) are then written as

$$\Omega_a^{(0)} = \Pi_a^{(0)} + \frac{2}{1+\theta^2}\Delta_{ab}\phi_b^{(0)\prime} \approx 0. \qquad (6)$$

In order to convert these constraints into first-class ones, we first extend the phase space by introducing auxiliary fields $\phi_a^{(1)}$ (one auxiliary field for each constraint), which satisfies

$$\{\phi_a^{(1)}(x), \phi_b^{(1)}(y)\} = \gamma_{ab}(x, y), \qquad (7)$$

with γ_{ab} determined later on.

In the extended phase space, a proper modification of constraints $\Omega_a^{(0)}$ is given by

$$\tilde{\Omega}_a^{(0)} = \Omega_a^{(0)} + \sum_{k=1}^{\infty} \omega_a^{(1,k)}, \qquad (8)$$

which has to satisfy the boundary condition $\tilde{\Omega}_a^{(0)}|_{\phi_a^{(1)}=0} = \Omega_a^{(0)}$ and the requirement of strong involution, $\{\tilde{\Omega}_a^{(0)}, \tilde{\Omega}_b^{(0)}\} = 0$, to accomplish the BFT embedding. Here we would like to note that the strong involution is valid only for the Abelian theory, which is the case at hand. As for the non-Abelian case, the weak involution should be considered. The correction $\omega_a^{(1,k)}$ at a given order k is assumed to be proportional to $(\phi_a^{(1)})^k$. To begin with, we consider the first order correction which is given by $\omega_a^{(1,1)} = \int dy X_{ab}(x, y)\phi_b^{(1)}$. It is not so difficult to show that the requirement of strong involution leads us to have the simple solution for γ_{ab} of Eq. (7) and X_{ab} as

$$\gamma_{ab}(x, y) = \Delta_{ab}\epsilon(x-y), \qquad (9)$$

$$X_{ab}(x, y) = \frac{4}{1+\theta^2}\Delta_{ab}\partial_x\delta(x-y). \qquad (10)$$

We see that the constraints (8) become the first-class one already at the level of the first correction. This means that it is not necessary to consider higher order corrections and hence we can safely set them to zero. The resulting first-class constraint in the phase space extended by introducing the fields $\phi_a^{(1)}$ is then

$$\tilde{\Omega}_a^{(0)} = \Pi_a^{(0)} + \frac{2}{1+\theta^2}\Delta_{ab}(\phi_b^{(0)})' + \frac{4}{1+\theta^2}\Delta_{ab}(\phi_b^{(1)})'. \qquad (11)$$

The canonical Hamiltonian $H_c^{(0)} \equiv H_c$ of Eq. (5) is the one only for the fields $\phi_a^{(0)}$. Similar to the modification of constraints in Eq. (8), it should also be modified properly in the extended phase space. The new canonical Hamiltonian is defined by $H_c^{(1)} = H_c^{(0)} + h^{(1)}$, where $h^{(1)}$ is determined from the involutive condition $\{\tilde{\Omega}_a^{(0)}, H_c^{(1)}\} = 0$. In the present case, what we get is

$$H_c^{(1)} = \int dx((\phi_a^{(0)})' + (\phi_a^{(1)})')((\phi_a^{(0)})' + (\phi_a^{(1)})'). \quad (12)$$

Given this Hamiltonian, we can obtain the corresponding Lagrangian by considering the partition function to explore the constraint structure in the extended phase space. The phase space partition function is given by

$$Z = \int \prod_{a=\pm} \prod_{n=0,1} \mathcal{D}\phi_a^{(n)} \mathcal{D}\Pi_a^{(0)} \delta[\tilde{\Omega}_a^{(0)}]\delta[\Gamma_a^{(0)}]$$
$$\times \det|\{\tilde{\Omega}_a^{(0)}, \Gamma_a^{(0)}\}|e^{iS^{(1)}}, \quad (13)$$

where

$$S^{(1)} = \int d^2x \left(\Pi_a^{(0)}\dot{\phi}_a^{(0)} + \frac{1}{2}\int dy \phi_a^{(1)}(x)\gamma_{ab}^{-1}(x,y)\dot{\phi}_b^{(1)}(y) \right)$$
$$- \int dt H_c^{(1)}, \quad (14)$$

and $\Gamma_a^{(0)}$ are gauge fixing conditions to make the nonvanishing determinant of $\tilde{\Omega}_a^{(0)}$ and $\Gamma_a^{(0)}$. Through the usual procedure of path integration with respect to the momenta $\Pi_a^{(0)}$ and by noticing from Eq. (9) $\gamma_{ab}^{-1}(x,y) = \frac{4}{1+\theta^2}\Delta_{ab}\partial_x\delta(x-y)$, the Lagrangian density $[S^{(1)} = \int d^2x \mathcal{L}^{(1)}]$ is obtained as

$$\mathcal{L}^{(1)} = -\frac{2}{1+\theta^2}(\dot{\phi}_a^{(0)}\Delta_{ab}(\phi_b^{(0)})' + \dot{\phi}_a^{(1)}\Delta_{ab}(\phi_b^{(1)})')$$
$$- (\phi_a^{(0)})'(\phi_a^{(0)})' - (\phi_a^{(1)})'(\phi_a^{(1)})'$$
$$- \frac{4}{1+\theta^2}\dot{\phi}_a^{(0)}\Delta_{ab}(\phi_b^{(1)})' - 2(\phi_a^{(0)})'(\phi_a^{(1)})'. \quad (15)$$

From this Lagrangian, the canonical momenta conjugate to $\phi_a^{(0)}$ and $\phi_a^{(1)}$ are derived as $\Pi_a^{(0)} = -\frac{2}{1+\theta^2}\Delta_{ab}(\phi_b^{(0)})' - \frac{4}{1+\theta^2}\Delta_{ab}(\phi_b^{(1)})'$, $\Pi_a^{(1)} = -\frac{2}{1+\theta^2}\Delta_{ab}(\phi_b^{(1)})'$, which lead to the following constraints:

$$\tilde{\Omega}_a^{(0)} = \Pi_a^{(0)} + \frac{2}{1+\theta^2}\Delta_{ab}(\phi_b^{(0)})'$$
$$- \left(\Pi_a^{(1)} - \frac{2}{1+\theta^2}\Delta_{ab}(\phi_b^{(1)})' \right) \approx 0, \quad (16)$$

$$\Omega_a^{(1)} = \Pi_a^{(1)} + \frac{2}{1+\theta^2}\Delta_{ab}(\phi_b^{(1)})' \approx 0, \quad (17)$$

where the constraint $\tilde{\Omega}_a^{(0)}$ has been rewritten by using the constraint $\Omega_a^{(1)}$. The time evolution of these constraints via the primary Hamiltonian based on $H_c^{(1)}$ gives no more constraints and thus we see that $\tilde{\Omega}_a^{(0)}$ and $\Omega_a^{(1)}$ form a full set of constraints for the system described by $\mathcal{L}^{(1)}$.

The Poisson bracket structure between the constraints, Eqs. (16) and (17), shows that $\Omega_a^{(1)}$ are in second-class, while $\tilde{\Omega}_a^{(0)}$ are the first-class constraints as expected. This means that the system in the extended phase space is not a fully first-class constrained one and the procedure of BFT embedding is not yet completed. At this point, we observe

that $\Omega_a^{(1)}$ is exactly the same as $\Omega_a^{(0)}$ in Eq. (6) if $\phi_a^{(0)}$ and $\Pi_a^{(0)}$ are substituted for $\phi_a^{(1)}$ and $\Pi_a^{(1)}$, respectively. By introducing another auxiliary field, say $\phi_a^{(2)}$, and taking the same steps from Eq. (6) to Eq. (17), we can convert $\Omega_a^{(1)}$ into the first-class constraints $\tilde{\Omega}_a^{(1)}$. However, the canonical momenta $\Pi_a^{(2)}$ of $\phi_a^{(2)}$ give new constraints $\Omega_a^{(2)}$ which are in the second-class. It is necessary to introduce the third auxiliary fields $\phi_a^{(3)}$, and the story continues forever. We note that this kind of infinite dimensional extended phase space appears also in the study of Abelian Chern-Simons theory [14].

Then, the infinite repeat of the BFT embedding method gives us finally the canonical Hamiltonian of the fully first-class constrained system, which is

$$\tilde{H}_c = \int dx \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(\phi_a^{(m)})'(\phi_a^{(n)})'. \quad (18)$$

The corresponding Lagrangian is obtained as

$$\mathcal{L} = \sum_{n=0}^{\infty}\left[-\frac{2}{1+\theta^2}\dot{\phi}_a^{(n)}\Delta_{ab}(\phi_b^{(n)})' - (\phi_a^{(n)})'(\phi_a^{(n)})' \right]$$
$$+ 2\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\left[-\frac{2}{1+\theta^2}\dot{\phi}_a^{(m)}\Delta_{ab}(\phi_b^{(n)})' \right.$$
$$\left. - (\phi_a^{(m)})'(\phi_a^{(n)})' \right]. \quad (19)$$

From the canonical momenta $\Pi_a^{(n)}$ conjugate to the fields $\phi_a^{(n)}$,

$$\Pi_a^{(n)} = -\frac{2}{1+\theta^2}\Delta_{ab}(\phi_b^{(n)})' - \frac{4}{1+\theta^2}\Delta_{ab}\sum_{m=n+1}^{\infty}(\phi_b^{(m)})', \quad (20)$$

we get the constraints

$$\tilde{\Omega}_a^{(n)} = \Pi_a^{(n)} + \frac{2}{1+\theta^2}\Delta_{ab}(\phi_b^{(n)})'$$
$$- \left(\Pi_a^{(n+1)} - \frac{2}{1+\theta^2}\Delta_{ab}(\phi_b^{(n+1)})' \right) \approx 0, \quad (21)$$

which are in first-class, as it should be. It can be easily checked that the constraints (21) satisfies $\{\tilde{\Omega}_a^{(n)}(x), \tilde{H}_c\} = 0$.

Now we are in a position to be able to investigate new local symmetries of the first-class constrained system (19). The total action is written as $S = \int d^2x\sum_{n=0}^{\infty}\Pi_a^{(n)}\dot{\phi}_a^{(n)} - \int dt\tilde{H}_c + \int d^2x\sum_{n=0}^{\infty}\lambda_a^{(n)}\tilde{\Omega}_a^{(n)}$, where $\lambda_a^{(n)}$'s are Lagrange multipliers. It can be shown that the action is invariant under the following local gauge transformations:

$$\delta\phi_a^{(n)} = -\epsilon_a^{(n)} + \epsilon_a^{(n-1)}, \quad (22)$$

$$\delta\Pi_a^{(n)} = -\frac{2}{1+\theta^2}\Delta_{ab}[(\epsilon_a^{(n)})' + (\epsilon_a^{(n-1)})'], \quad (23)$$

$$\delta\lambda_a^{(n)} = -\dot{\epsilon}_a^{(n)}, \tag{24}$$

where $\epsilon_a^{(n)}(x)$ are infinitesimal gauge parameters with $\epsilon_a^{(-1)} = 0$ and n is non-negative integer valued. As is well established, these local symmetries are generated by the first-class constraints. Since there are infinite number of first-class constraints in the present situation, the model we are considering has infinite local symmetries.

Finally, we consider the propagation of fields in the noncommutative field space. The equations of motion for the fields $\phi_a^{(n)}$ are derived from the variation of the Lagrangian (19) as $\sum_{n=0}^{\infty}[\dot{\phi}_a^{(n)} + 2\Delta_{ab}(\phi_b^{(n)})'] = 0$, where Eq. (3) has been used. In light cone coordinates $x^{\pm}$ ($\equiv (ct \pm x)/2$), these equations split into two parts $\sum_{n=0}^{\infty}\partial_-\phi_+^{(n)} = -\theta\sum_{n=0}^{\infty}(\phi_-^{(n)})'$, $\sum_{n=0}^{\infty}\partial_+\phi_-^{(n)} = -\theta\sum_{n=0}^{\infty}(\phi_+^{(n)})'$, from which we can obtain

$$\sum_{n=0}^{\infty}(\Box - \theta^2\partial_x^2)\phi_a^{(n)} = 0, \tag{25}$$

where $\Box \equiv (1/c^2)\partial_t^2 - \partial_x^2$. This means that, by the effect of the noncommutativity in the field space, the propagation speed of the fields is modified to

$$c \to c' = c\sqrt{1 + \theta^2}, \tag{26}$$

which was noticed in Ref. [4]. As was pointed out by the authors of [4], however, this modification of the propagation speed does not mean the violation of Lorentz invariance. The present formulation in the framework of the BFT embedding method shows that such modification takes place for all the fields with exactly the same manner, and thus does not lead to any inconsistency in Lorentz invariance.

IV. CONCLUSION

We have shown that the second-class constraint system for the chiral bosons in the noncommutative field space has been converted into the first-class constraint system by using the BFT method, where the resulting brackets can be implemented by the conventional Poisson algebra. The resulting equation of motion (25) is symmetric under the transformation of (22), which can be shown by the total summation of the infinitesimal transformation parameters which are canceled completely. In general, the original second-class constraint system can be interpreted as a gauge fixed version of the first-class constraint system in the context of the BFT method. Therefore, the equations of motion (25) and (26) have been derived in a gauge independent fashion. Of course, each scalar field in the first-class constraint system has the same velocity with that of the velocity in the gauge fixed system corresponding to the second-class constraint system if the auxiliary field $\phi_a^{(n)}$ has its angular frequency $w_a^{(n)}$ and the wave number $k_a^{(n)}$, respectively.

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