PHYSICAL REVIEW D **75,** 085020 (2007)

Quantum critical transport, duality, and M theory

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(Received 11 January 2007; published 27 April 2007)

We consider charge transport properties of $2 + 1$ dimensional conformal field theories at nonzero temperature. For theories with only Abelian U(1) charges, we describe the action of particle-vortex duality on the hydrodynamic-to-collisionless crossover function: this leads to powerful functional constraints for self-dual theories. For $\mathcal{N} = 8$ supersymmetric, SU(N) Yang-Mills theory at the conformal fixed point, exact hydrodynamic-to-collisionless crossover functions of the SO(8) R-currents can be obtained in the large N limit by applying the anti-de Sitter/conformal field theory (AdS/CFT) correspondence to M theory. In the gravity theory, fluctuating currents are mapped to fluctuating gauge fields in the background of a black hole in $3 + 1$ dimensional anti-de Sitter space. The electromagnetic self-duality of the $3 + 1$ dimensional theory implies that the correlators of the R-currents obey a functional constraint similar to that found from particle-vortex duality in $2 + 1$ dimensional Abelian theories. Thus the $2 + 1$ dimensional, superconformal Yang Mills theory obeys a "holographic self-duality" in the large N limit, and perhaps more generally.

DOI: 10.1103/PhysRevD.75.085020 PACS numbers: 73.43.Nq, 11.10.Wx, 11.25.Yb

I. INTRODUCTION

The quantum phase transitions of two (spatial) dimensional systems have been the focus of much study in the condensed matter community. Prominent examples include the superfluid-insulator transition in thin films [1–3], the transitions between various quantum Hall states [4,5], and magnetic ordering transitions of Mott insulators and superconductors which have applications to the cuprate compounds [6–8]. Of particular interest in this paper are the transport properties of conserved quantities such as the electrical charge or the total spin: these are characterized by a (charge or spin) conductivity σ, which can in general be a complicated function of frequency ω, wave vector k, temperature T, and various couplings characterizing the ground state.

It is often the case that the quantum critical point is described by a strongly interacting quantum field theory in $2 + 1$ spacetime dimensions D. Examples are (i) the superfluid-insulator transition in the boson Hubbard model at integer filling [9–11], which is described by the φ^4 field theory with O(2) symmetry, and so is controlled by the Wilson-Fisher fixed point in $D = 2 + 1$; (ii) the spin-gap paramagnet to Néel order transition of coupled spin dimers/ladders/layers which is described by the O(3) φ^4 field theory [12,13]; and (iii) the "deconfined" critical point of a $S = 1/2$ antiferromagnet between a Néel and a valence bond solid state [14,15], which is described by the $\mathbb{CP}^1$ model with a noncompact U(1) gauge field [16]. In all these cases the critical point is described by a relativistic conformal field theory (CFT). With an eye towards such experimentally motivated applications, our purpose here is to explore the transport properties of general interacting CFTs in $D = 2 + 1$.

A crucial property of CFTs in $D = 2 + 1$ (which actually applies more generally to any critical theory in 2 spatial dimensions which obeys hyperscaling) is that the conductivity is $1/\hbar$ times a dimensionless number. For U(1) currents, there is also a prefactor of $(e^*)^2$ where e^* is the unit of charge—we will drop this factor below. For non-Abelian Noether currents, the normalization of charge is set by a conventional normalization of the generators of the Lie algebra. We will be working with relativistic theories, and therefore set $\hbar = k_B = c = 1$.

Initial discussions [17–19] of this dimensionless conductivity at the quantum critical point were expressed in terms of ground state correlations of the CFT. Let J^a_μ represent the set of conserved currents of the theory; here $\mu = 0, 1, 2$ is a spacetime index, and a labels the generators of the global symmetry. In the CFT, $J^a_\mu(x)$ has dimension 2, and so current conservation combined with Lorentz and scale invariance imply for the Fourier transform of the retarded correlator $C^{ab}_{\mu\nu}(x)$ at zero temperature[1]:

$$C^{ab}_{\mu\nu}(p)|_{T=0} = \sqrt{p^2}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)K_{ab}, \tag{1.1}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1)$, $p_\mu = (-\omega, \boldsymbol{k})$ is spacetime momentum, and $p^2 = \boldsymbol{k}^2 - \omega^2$. We define $\sqrt{p^2}$ so that it is analytic in the upper half-plane of ω and $\mathrm{Im}\sqrt{p^2} \le 0$ for $\omega > 0$. The parameters K_{ab} are a set of universal,

[1]If needed, a "diamagnetic" or "contact" term has been subtracted to ensure current conservation. In theories with Chern-Simons terms, an additional term proportional to $\epsilon_{\mu\nu\lambda}p_\lambda$ is permitted in Eq. (1.1) and the $T > 0$ generalization in Eq. (1.4). See Appendix B.

momentum-independent dimensionless constants characterizing the CFT, which are the analog of the central charge of the Kac-Moody algebra of CFTs in $D = 1 + 1$. Application of the Kubo formula at $T = 0$ shows that [17,19] the K_{ab} are equal to the conductivities $\sigma_{ab} = K_{ab}$, thus setting up the possibility of observing these in experiments.

It was also noted [18,19] that particle-vortex duality [20–22] of theories with Abelian symmetry mapped the $T = 0$ conductivities to their inverse (we review this mapping in Sec. II). In self-dual theories, this imposes constraints on the values of the K_{ab}, possibly allowing them to be determined exactly. However, the field theories considered in these early works were not self-dual (see Appendix B). Duality, and possible self-duality, was also considered in the context of theories containing Chern-Simons terms, relevant to quantum Hall systems [23–28]. We comment on these works in Appendix B, but the body of the paper considers only theories without Chern-Simons terms. For our purposes, more relevant is the self-dual field theory proposed recently by Motrunich and Vishwanath [16], and we discuss its charge transport properties below.

It was subsequently pointed out [29–31] that the K_{ab} are *not* the d.c. conductivities observed at small but nonzero temperature. The key point [31–33] is that at nonzero T, the time $1/T$ is a characteristic "collision" or "decoherence" time of the excitations of the CFT. Consequently the transport at $\omega \ll T$ obeys "collision-dominated" hydrodynamics, while that at $\omega \gg T$ involves "collisionless" motion of excitations above the ground state. Therefore, the limits $\omega \to 0$ and $T \to 0$ do not, in general, commute, and must be taken with great care; the constants K_{ab} above are computed in the limit $\omega/T \to \infty$, while the d.c. conductivities involve $\omega/T \to 0$.

This contrast between the collisionless and collision-dominated behavior is most clearly displayed in the correlations of the conserved densities. Taking the tt component of Eq. (1.1) we obtain the response

$$C^{ab}_{tt}(\omega, k) = K_{ab}\frac{-k^2}{\sqrt{k^2 - \omega^2}}, \qquad ||\omega| - k| \gg T, \quad (1.2)$$

which characterizes the collisionless response of the CFT at $T = 0$. We have also noted above that we expect the same result to apply at $T > 0$ provided ω and $k = |\boldsymbol{k}|$ are large enough, and away from the light cone. The $T > 0$ correlations are the Fourier transform of the retarded real-time correlators. These are related by analytic continuation to the Euclidean space correlations defined at the Matsubara frequencies, which are integer multiples of $2\pi T$. The low frequency hydrodynamic regime $\omega \ll T$ is only defined in real time (Minkowski space). In this regime, the arguments of Ref. [29] imply that the 'collision-dominated' response has the structure

$$C^{ab}_{tt}(\omega, k) = \sum_\lambda \chi^\lambda_{ab}\frac{-D_\lambda k^2}{-i\omega + D_\lambda k^2}, \qquad |\omega|, k \ll T, \quad (1.3)$$

where D_λ are the diffusion constants of a set of diffusive eigenmodes labeled by λ, and χ^λ_{ab} are the corresponding susceptibilities. Scaling arguments imply that [34] $D_\lambda = \mathcal{D}_\lambda/T$ and $\chi^\lambda_{ab} = \mathcal{C}^\lambda_{ab}T$, where the $\mathcal{D}_\lambda$, $\mathcal{C}^\lambda_{ab}$ are a set of universal numbers characterizing the hydrodynamic response of the CFT. The d.c. conductivities can be obtained from the Kubo formula by

$$\sigma_{ab} = \lim_{\omega\to 0}\lim_{k\to 0}(i\omega/k^2)C^{ab}_{tt},$$

where the order of limits is significant. At any fixed $T > 0$, the limits of small k and ω imply that this Kubo formula has to be applied to Eq. (1.3), and leads to Einstein relations between the T-independent universal conductivities and the diffusivities. The distinct forms of Eqs. (1.2) and (1.3) make it clear that, in general, the universal d.c. conductivities bear no direct relationship to the K_{ab}; the latter, as we will see below in Eq. (1.7), are related to the high frequency conductivity.

It is worth noting here in passing that the structure in Eq. (1.3) does *not* apply to CFTs in $D = 1 + 1$, where a result analogous to Eq. (1.2) holds also in the low frequency and low momentum limit; see Appendix A for further discussion of this important point.

Returning to consideration of all the components of the $C^{ab}_{\mu\nu}$ in $D = 2 + 1$, an alternative presentation of the collisionless-to-hydrodynamic crossover is obtained by writing down the generalization of Eq. (1.1) to $T > 0$. Current conservation and spatial rotational invariance, without Lorentz invariance at $T > 0$, generalize Eq. (1.1) to

$$C^{ab}_{\mu\nu}(\omega, \boldsymbol{k}) = \sqrt{p^2}(P^T_{\mu\nu}K^T_{ab}(\omega, k) + P^L_{\mu\nu}K^L_{ab}(\omega, k)), \quad (1.4)$$

where $k = |\boldsymbol{k}|$, and $P^T_{\mu\nu}$ and $P^L_{\mu\nu}$ are orthogonal projectors defined by

$$P^T_{00} = P^T_{0i} = P^T_{i0} = 0, \qquad P^T_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2}, \\ P^L_{\mu\nu} = \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) - P^T_{\mu\nu}, \quad (1.5)$$

with the indices i, j running over the 2 spatial components. The constants K_{ab} have each been replaced by *two* dimensionless, universal, temperature-dependent functions $K^{L,T}_{ab}(\omega, k)$, characterizing the longitudinal and transverse response. These functions are dimensionless, and hence they can *only* depend upon the dimensionless ratios ω/T and k/T, as is also the case for the conductivities. Spatial rotational invariance, and the existence of finite correlation length at $T > 0$ which ensures analyticity at small $\boldsymbol{k}$, imply that the longitudinal and transverse response are equal to each other at $\boldsymbol{k} = 0$, and, by the Kubo formula, are both equal to the zero momentum, frequency-dependent complex conductivity, $\sigma_{ab}(\omega/T)$:

$$\sigma_{ab}(\omega/T) = K^L_{ab}(\omega, 0) = K^T_{ab}(\omega, 0). \quad (1.6)$$

Also at $T = 0$, these functions reduce to the constants in Eq. (1.1):

$$\sigma_{ab}(\infty) = K_{ab} = K^L_{ab}(\omega, k)|_{T=0} = K^T_{ab}(\omega, k)|_{T=0}. \quad (1.7)$$

The functions $K^{L,T}_{ab}(\omega, k)$ are clearly of great physical interest, and it would be useful to compute them for a variety of CFTs. A number of computations have appeared [29,30,35–38], and show interesting structure in the conductivity as a function of ω/T, encoding the hydrodynamic-to-collisionless crossover for a variety of tractable models. Here we will present some additional results which shed light on the role duality can play on the form of these functions.

In Sec. II we will consider the role of duality in Abelian systems, by examining the self-dual noncompact, easy-plane, $\mathbb{CP}^1$ field theory discussed by Motrunich and Vishwanath [16]. Closely related results apply to other Abelian CFTs whose particle-vortex duals have been described in the literature [28,39–42], some of which are supersymmetric (in which case, particle-vortex duality is known as "mirror symmetry"). The Lagrangian formulation of the $\mathbb{CP}^1$ theory involves two complex scalar fields and one gauge field A_μ, which is coupled to a gauge current $J_{1\mu}$. The theory has a global $U(1) \times Z_2$ symmetry, and we will denote by $J_{2\mu}$ the Noether current arising from the U(1) global symmetry. There is another conserved current, the topological current $J^\mu_{\rm top} = \epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda$, which is conserved by the Bianchi identity. The topological and Noether currents exchange under the self-duality. As we will see in Sec. II, the two-point correlator of $J^\mu_{\rm top}$ is the inverse of that of $J_{1\mu}$. We use the notations of Eqs. (1.1) and (1.4) with $a, b = 1, 2$.

The Z_2 symmetry ensures that the cross correlations of the $J_{1\mu}$, $J_{2\mu}$ currents vanish, and consequently there are only two constants $K_1 \equiv K_{11}$ and $K_2 \equiv K_{22}$ in Eq. (1.1), and similarly for the $T > 0$ functions in Eq. (1.4). We examine the duality transformations of these functions in Sec. II and show that the existence of a self-dual critical point leads to the functional relations[2]

$$K^L_1(\omega, k)K^T_2(\omega, k) = \frac{1}{\pi^2}, \quad (1.8a)$$

$$K^L_2(\omega, k)K^T_1(\omega, k) = \frac{1}{\pi^2}, \quad (1.8b)$$

which hold for general T, while for the constants in Eq. (1.1) this implies $K_1K_2 = 1/\pi^2$. Note that these relations are not sufficient to determine the conductivities $\sigma_{1,2}(\omega/T)$; from Eq. (1.6), only their product obeys $\sigma_1(\omega/T)\sigma_2(\omega/T) = 1/\pi^2$, at all ω/T. Thus we expect that for this self-dual model, the conductivities will remain nontrivial functions of ω/T exhibiting the hydrodynamic-collisionless crossover, and their functional form has to be determined from the solution of a quantum Boltzmann equation.

In Sec. III, we turn to a field theory with non-Abelian symmetries: the supersymmetric Yang Mills (SYM) gauge theory with a SU(N) gauge group and $\mathcal{N} = 8$ supersymmetry [43]. At long distances, the theory flows under the renormalization group to a strongly coupled $2 + 1$ dimensional $\mathcal{N} = 8$ superconformal field theory (SCFT), which is believed to describe degrees of freedom on a stack of N M2-branes [44,45]. In the limit of large N, the SCFT can be analyzed by using the AdS/CFT correspondence [46]. The gravity description of the SCFT is given by M theory on $3 + 1$ dimensional anti-de Sitter space times a seven-sphere, and in the large N limit corresponds to $10 + 1$ dimensional supergravity on $AdS_4 \times S^7$. The AdS/CFT correspondence provides a method to compute real-time response functions at finite temperature [47,48], in which case the gravity theory contains a black hole in AdS_4. In the limit of low frequency and momentum $\omega \ll T, k \ll T$ one finds hydrodynamic behavior in the SCFT [49].[3] The surprising solvability in this limit therefore demands our attention.[4]

The $2 + 1$ dimensional SCFT has a global SO(8) R-symmetry (the symmetry of the seven-sphere in the supergravity description), and therefore has a set of conserved currents J^a_μ, $a = 1, \ldots, 28$. The SO(8) symmetry implies that $K_{ab} = K\delta_{ab}$, and so there is only a single universal constant K at zero temperature. Similarly, in Eq. (1.4) there are only two independent functions $K^L(\omega, k)$ and $K^T(\omega, k)$ which characterize the CFT response at finite temperature. In Sec. III we will compute these functions in the $N \to \infty$ limit, for all values of ω/T and k/T. We also prove that these functions obey the identity

[2]We only keep the one-photon irreducible (1PI) part in $K^{L,T}_1$, as explained in Sec. II.

[3]Hydrodynamic charge transport at small ω and k is of course not specific to the $\mathcal{N} = 8$ SCFT in $2 + 1$ dimensions. Hydrodynamics from the supergravity description was first found in strongly coupled $\mathcal{N} = 4$ SYM in $3 + 1$ dimensions [50], and later in a variety of other strongly coupled field theories [51–54]. In strongly coupled $\mathcal{N} = 4$ SYM in $D = 3 + 1$, hydrodynamic-to-collisionless crossover functions $K^{L,T}(\omega, k)$ were computed in [55]. Note that in $D = 3 + 1$ the conductivity is not dimensionless [31], but is proportional to T in the hydrodynamic limit $\omega \ll T$.

[4]Of course, there are other well-known $D = 2 + 1$ CFTs which are solvable in the large N limit, such as the O(N) φ^4 field theory. However, all of these are theories of particles which are infinitely long-lived at $N = \infty$, and so do not exhibit hydrodynamic behavior in this limit. Indeed, an infinite-order resummation of the $1/N$ expansion is invariably necessary [31] (via the quantum Boltzmann equation) to obtain hydrodynamics. These solvable theories become weakly coupled as $N \to \infty$, while the $\mathcal{N} = 8$ SYM remains strongly coupled even as $N \to \infty$.

$$K^L(\omega, k)K^T(\omega, k) = \frac{N^3}{18\pi^2}, \tag{1.9}$$

at general T, which is strikingly similar to Eqs. (1.8). Now this relation and Eq. (1.6) do indeed determine $\sigma(\omega/T)$ (and K) to be the frequency-independent constant which is the square root of the right-hand side of Eq. (1.9). In other words, for this model, the hydrodynamic and high-frequency collisionless conductivities are equal to each other. Nevertheless, the theory does have a hydrodynamic-to-collisionless crossover at all nonzero k (as we will review in Sec. III), where $K^L(\omega, k) \neq K^T(\omega, k)$, and so Eq. (1.9) is not sufficient to fix the correlators at $k \neq 0$. Thus the identity Eq. (1.9) causes all signals of the hydrodynamic-collisionless crossover to disappear *only* at $k = 0$.

The similarity of Eq. (1.9) to Eq. (1.8) suggests that explanation of the frequency independence of the conductivity of the $\mathcal{N} = 8$ SYM SCFT lies in a self-duality property. Section III demonstrates that this is indeed the case. Under the AdS/CFT correspondence, the two-point correlation function of the SO(8) R-currents in $D = 2 + 1$ is holographically equivalent to the correlator of a SO(8) gauge field on an asymptotically AdS_4 background. In the large N limit, the action of the SO(8) gauge field is Gaussian, and is easily shown to possess electromagnetic (EM) self-duality under which the electric and magnetic fields are interchanged. We demonstrate in Sec. III D that it is precisely this EM self-duality of the $3 + 1$ dimensional gauge field which leads to the constraint (1.9) in the SCFT. Thus the SYM theory obeys a self-duality which is not readily detected in $2 + 1$ dimensions, but becomes explicit in the holographic theory in $3 + 1$ dimensions. The generalization of the particle-vortex duality of Abelian CFTs in $D = 2 + 1$ to non-Abelian CFTs is facilitated by the holographic extension to the theory on AdS_4.

There have been a few earlier studies connecting dualities in $D = 4$ to those in $D = 3$. Sethi [56] considered the Kaluza-Klein reduction of S-duality from $D = 4$ to $D = 3$ by compactifying the $D = 4$ theory on a circle in one dimension. This is quite different from the connection above, using a holographic extension. The work of Witten [28] makes a connection which is the same as ours above (see also the work of Leigh and Petkou [57]). He examined the connection between Abelian particle-vortex duality ("mirror symmetry") of CFTs in $D = 2 + 1$ to the action of $SL(2, Z)$ on Abelian gauge theories on AdS_4 at zero temperature. We have considered a similar connection at nonzero temperature for the $\mathcal{N} = 8$ SCFT, and shown that it is "holographically self-dual" in the large N limit; combined with the non-Abelian SO(8) symmetry (which implies a single K), the constraints for the current correlators are stronger than those for Abelian theories.

We will also consider in Appendix E other non-Abelian theories with known gravity descriptions. In particular, we will show that for a theory on a stack of D2 branes, a nontrivial dilaton profile prevents EM self-duality. In this case, we do not have the constraint (1.9), and so find a frequency dependent conductivity.

II. ABELIAN, NONCOMPACT $\mathbb{CP}^1$ MODEL

This section will consider duality properties and current correlations of the Abelian, easy-plane $\mathbb{CP}^1$ model of Ref. [16]. This is a theory of two complex scalars $z_{1,2}$ and a noncompact U(1) gauge field A_μ; the noncompactness is necessary to suppress instantons (monopoles), and we indicate below Eq. (2.13) the modifications required when monopoles are present.

More generally, one can consider dualities of the noncompact $\mathbb{CP}^{N-1}$ model where the global SU(N) flavor symmetry has been explicitly broken down to $U(1)^{N-1} \times G_N$, with G_N some subgroup of the permutation group of N objects [42]. The $N = 1$ case, which is better known as the Abelian Higgs model, will be described in Appendix B. The $N = 2$ case (with $G_2 = Z_2$) is described below. The $T > 0$ results below have a generalization to all $N > 2$, with the mappings spelled out in Ref. [42]. Only the $N = 2$ case is self-dual, and this is our reason for focusing on it.

It is interesting to note that the duality properties of the noncompact $\mathbb{CP}^{N-1}$ models have strikingly similar counterparts in $D = 2 + 1$ theories with $\mathcal{N} = 4$ supersymmetry [39–41]. In particular, the correspondence is to the theories with one U(1) vector (gauge) multiplet and N matter hypermultiplets (SQED-N). SQED-1 is dual to a theory of a single hypermultiplet, with no vector multiplet[5]; this corresponds to the duality, reviewed in Appendix B, of the Abelian Higgs model to the theory of a single complex scalar with no gauge field (also known as the XY model or the O(2) φ^4 field theory). Next, SQED-2 is self-dual, as is our $N = 2$ case. For $N > 2$, the dual of SQED-N is a quiver gauge theory, as is the case for the $\mathbb{CP}^{N-1}$ models [42].[6] Our results below for $T > 0$ should have straightforward extensions to these $\mathcal{N} = 4$ supersymmetric theories.

A. Conserved currents

Let us now begin our analysis of the nonsupersymmetric $N = 2$ case. The action of the noncompact $\mathbb{CP}^1$ theory is

[5]The theory of a single hypermultiplet is free. This is because the Gaussian fixed point is protected by $\mathcal{N} = 4$ supersymmetry [41]. In the nonsupersymmetric case, the Gaussian fixed point is unstable to the interacting Wilson-Fisher fixed point.

[6]A quiver gauge theory consists of a direct product of gauge group factors along with matter fields transforming in the bifundamental representation of pairs of group factors. The word quiver is used because the bifundamental fields are often represented as arrows.

$$\mathcal{S} = \int d^2x\, dt \Big[|(\partial_\mu - iA_\mu)z_1|^2 + |(\partial_\mu - iA_\mu)z_2|^2 + s(|z_1|^2 + |z_2|^2) + u(|z_1|^2 + |z_2|^2)^2 + v|z_1|^2|z_2|^2 + \frac{1}{2e^2}(\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda)^2 \Big], \tag{2.1}$$

with $u > 0$ and $-4u < v < 0$. For these negative values of v, the phase for s sufficiently negative has $|\langle z_1\rangle| = |\langle z_2\rangle| \neq 0$. We can also define a gauge-invariant vector order parameter $\vec{N} = z^* \vec{\sigma} z$, where $\vec{\sigma}$ are the Pauli matrices, and the constraint $v < 0$ implies that $\vec{N}$ prefers to lie in the xy plane: hence "easy-plane" (for $v > 0$, $\vec{N}$ would be oriented along the z "easy-axis," realizing an Ising order parameter). The $\mathbb{CP}^1$ model is usually defined with fixed length constraint $|z_1|^2 + |z_2|^2 = 1$, but here we have only implemented a soft constraint by the quartic term proportional to u; we expect that the models with soft and hard constraints have the same critical properties. We are interested in the nature of the quantum phase transition accessed by tuning the value of s to a critical value $s = s_c$. For $s > s_c$, we have a "Coulomb" phase $\langle \vec{N} \rangle = 0$ with a gapless photon, while for $s < s_c$ there is a "Higgs" phase with $\langle \vec{N} \rangle \neq 0$. The phase diagram [16] of the model in the s, T plane is shown in Fig. 1. Both the Higgs and Coulomb phases have phase transitions as the temperature is raised: for the former it is driven by the loss of the Higgs (quasi)-long-range order, while for the latter it is a "confinement-deconfinement" transition of the z particle-antiparticle pairs formed from the logarithmic Coulomb force. Neither of these transitions is of interest to us in this paper. Rather, we will compute $T > 0$ correlations of the CFT associated with the quantum critical point, and these describe the physical properties of the shaded quantum critical region in Fig. 1.

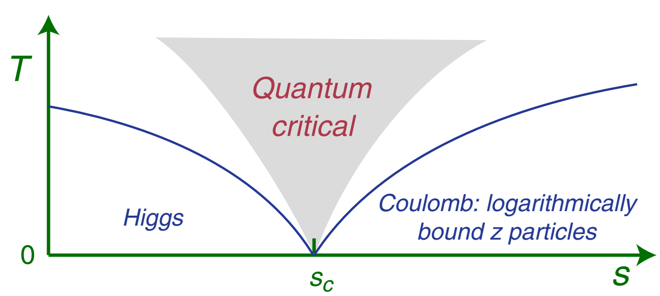

FIG. 1 (color online). Phase diagram [16] of the easy-plane noncompact $\mathbb{CP}^1$ model (Eq. (2.1)) in 2 spatial dimensions as a function of the coupling s and temperature T. The quantum critical point is at $s = s_c$, $T = 0$. The finite T correlations of the CFT describe the shaded quantum critical region; the boundary of the shaded region is a crossover into a different physical region, not a phase transition. The full lines are Kosterlitz-Thouless (KT) phase transitions. The KT line for $s < s_c$ describes the disappearance of quasi-long-range xy order of $\vec{N}$. The KT transition for $s > s_c$ describes the deconfinement of z quanta which are logarithmically bound by the Coulomb interaction in the low temperature phase into particle-antiparticle pairs. The phase diagram can also be described in terms of the dual w theory in Eq. (2.11). Duality interchanges the two sides of $s = s_c$ (T remains invariant under duality), and the z Coulomb phase is interpreted as a w Higgs phase and vice versa.

The theory has a discrete Z_2 symmetry which exchanges z_1 and z_2. The continuous symmetries are a gauge U(1) symmetry

$$z_1 \to z_1 e^{i\phi}; \qquad z_2 \to z_2 e^{i\phi}; \qquad A_\mu \to A_\mu + \partial_\mu \phi \tag{2.2}$$

and a global U(1) symmetry

$$z_1 \to z_1 e^{i\varphi}; \qquad z_2 \to z_2 e^{-i\varphi}. \tag{2.3}$$

Associated with these symmetries we can define two currents

$$J_{1\mu} = i(z_1^*(\partial_\mu - iA_\mu)z_1 - z_1(\partial_\mu + iA_\mu)z_1^*) + i(z_2^*(\partial_\mu - iA_\mu)z_2 - z_2(\partial_\mu + iA_\mu)z_2^*) \tag{2.4}$$

and

$$J_{2\mu} = i(z_1^*\partial_\mu z_1 - z_1\partial_\mu z_1^*) - i(z_2^*\partial_\mu z_2 - z_2\partial_\mu z_2^*). \tag{2.5}$$

Note that J_1 is even under the Z_2 symmetry, while J_2 is odd. Current conservation implies that at $T > 0$ these have two-point correlators of the form in Eq. (1.4), with the 4 distinct functions $K_{1,2}^{L,T}$.

Now consider the correlators of the gauge field A_μ. It is useful to write this in terms of the leading quadratic terms in the Coleman-Weinberg effective potential:

$$W = \frac{1}{2}\int_{k,\omega}\Big\{ -(k_i A_0 + \omega A_i)^2 \Big[\frac{1}{e^2} + \frac{\Pi^L(k,\omega)}{-\omega^2 + k^2}\Big] + A_i A_j\Big(\delta_{ij} - \frac{k_i k_j}{k^2}\Big)\Big[\frac{k^2}{e^2} + \Pi^T(k,\omega) + \frac{\Pi^L(k,\omega)\omega^2}{-\omega^2 + k^2}\Big]\Big\} + \ldots, \tag{2.6}$$

where $\Pi^{L,T}$ are the two components of the photon self-energy (the "polarization" operator); these are related to the current correlations by $\Pi^{L,T} = \sqrt{p^2} K_1^{L,T}$.

A key point is that at the conformal fixed point describing the phase transition at the quantum critical point $s = s_c$ we can safely take the limit $e \to \infty$ in the above. This is because $\dim[\Pi] = 1$, and so the induced polarizations are more singular than the bare Maxwell term. This is a very generic property of CFTs with gauge fields in $D = 2 + 1$. From the effective potential we can obtain the form of the gauge-invariant two-point correlators in the critical regime (it is easiest to work this out in the Coulomb gauge $k_i A_i = 0$):

$$\langle \epsilon_{ij} k_i A_j; \epsilon_{i'j'} k_{i'} A_{j'} \rangle = \frac{k^2}{\Pi^T(k,\omega)}, \qquad \langle \epsilon_{i'j'} k_{i'} A_{j'}; (k_i A_0 + \omega A_i) \rangle = \epsilon_{i'i} \frac{\omega k_{i'}}{\Pi^T(k,\omega)},$$
$$\langle (k_i A_0 + \omega A_i); (k_j A_0 + \omega A_j) \rangle = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \frac{\omega^2}{\Pi^T(k,\omega)} - \frac{k_i k_j}{k^2} \frac{(-\omega^2 + k^2)}{\Pi^L(k,\omega)}. \tag{2.7}$$

B. Vortices and duality

Here we will build a dual description of the $\mathbb{CP}^1$ model, treating the vortices of the original model as complex scalar fields in the dual description. Consider the topological vortex excitations in the Higgs state of the action (2.1). These are characterized [58] by a pair of winding numbers (n_1, n_2) associated with the phases of z_1 and z_2 out at spatial infinity. In general, such a vortex has a logarithmically diverging energy because the currents are only partially screened by the gauge field A_μ. By an extension of the Abrikosov-Nielsen-Olesen argument, it can be seen that the coefficient of the logarithmically divergent energy is proportional to

$$\left(2\pi n_1 - \int d^2x \epsilon_{ij} \partial_i A_j\right)^2 + \left(2\pi n_2 - \int d^2x \epsilon_{ij} \partial_i A_j\right)^2, \tag{2.8}$$

and this is minimized when the total A_μ flux is quantized as [16,42,58]

$$\int d^2x \epsilon_{ij} \partial_i A_j = \pi(n_1 + n_2). \tag{2.9}$$

Let us now identify the (1, 0) vortex as the worldline of a dual particle w_1, the (0, 1) vortex as the worldline of a dual particle w_2, and try to construct a dual theory by introducing complex scalar fields $w_1(x)$, $w_2(x)$. Then from Eq. (2.9), Lorentz covariance implies that the total w current is related to the A_μ flux:

$$\frac{1}{\pi} \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda = i(w_1^* \partial_\mu w_1 - w_1 \partial_\mu w_1^*) + i(w_2^* \partial_\mu w_2 - w_2 \partial_\mu w_2^*). \tag{2.10}$$

A second key property is that there are forces with a logarithmic potential between the $w_{1,2}$ particles. These are also easily seen from the structure of the classical vortex solutions of Eq. (2.1). Also, it is the *difference* of the z_1 and z_2 currents, which is not screened by the A_μ field, which contributes to an *attractive* logarithmic potential between the w_1 and w_2 particles. Another way to see this is to consider the configuration of the gauge-invariant Higgs field (N_x, N_y) around each vortex: the w_1 has an anticlockwise winding of the $\arg(N_x + iN_y)$, while the w_2 has a clockwise winding. Because there is a finite stiffness associated with this Higgs order, a w_1 particle will attract a w_2 particle, while two w_1 (or w_2) particles will repel each other.

We can now guess the form of the effective theory for the $w_{1,2}$ particles. We mediate that logarithmic potential as the Coulomb potential due to a new "dual" gauge field $\tilde{A}_\mu$. Then general symmetry arguments and the constraints above imply the dual theory [16]

$$\tilde{\mathcal{S}} = \int d^2x dt \Big[|(\partial_\mu - i\tilde{A}_\mu) w_1|^2 + |(\partial_\mu + i\tilde{A}_\mu) w_2|^2 + \tilde{s}(|w_1|^2 + |w_2|^2) + \tilde{u}(|w_1|^2 + |w_2|^2)^2 + \tilde{v}|w_1|^2|w_2|^2 + \frac{1}{2\tilde{e}^2}(\epsilon^{\mu\nu\lambda}\partial_\nu \tilde{A}_\lambda)^2 \Big]. \tag{2.11}$$

Note especially the difference in the charge assignments from (2.1)—now the $w_{1,2}$ particles have opposite charges under $\tilde{A}_\mu$. Apart from this, the theories have an identical form, and so current correlation functions $\tilde{K}_{1,2}^{L,T}$, associated with the global and gauge U(1) symmetries, will have the same dependence upon the couplings in $\tilde{\mathcal{S}}$ as the $K_{1,2}^{L,T}$ have on $\mathcal{S}$. However, the explicit expressions for the current in terms of the field operators have a sign interchanged:

$$\tilde{J}_{1\mu} = i(w_1^*(\partial_\mu - i\tilde{A}_\mu) w_1 - w_1(\partial_\mu + i\tilde{A}_\mu) w_1^*) - i(w_2^*(\partial_\mu + i\tilde{A}_\mu) w_2 - w_2(\partial_\mu - i\tilde{A}_\mu) w_2^*) \tag{2.12}$$

and

$$\tilde{J}_{2\mu} = i(w_1^* \partial_\mu w_1 - w_1 \partial_\mu w_1^*) + i(w_2^* \partial_\mu w_2 - w_2 \partial_\mu w_2^*). \tag{2.13}$$

We note in passing the extension of the above analysis to a *compact* $\mathbb{CP}^1$ theory of the z particles. Following Polyakov [59], we have to include monopoles which change the A_μ flux by 2π. This can be achieved by adding the term $-y_m(w_1 w_2 + w_1^* w_2^*)$ to the w action $\tilde{\mathcal{S}}$, where y_m is the monopole fugacity. This monopole operator is neutral under $\tilde{A}_\mu$ charge and, from Eq. (2.10), catalyzes the required change in A_μ flux. This is a relevant perturbation: the theories for the z and w particles are no longer equivalent under duality, and the universality class of the transition is changed. We will not consider the compact case further; for more details, see the review [60].

Returning to the noncompact theory, we note the duality mapping can now also be carried backwards from the w theory to the z theory, and from (2.10) we see that the theories $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are connected by the relations

$$\frac{1}{\pi} \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda = \tilde{J}_{2\mu}, \tag{2.14a}$$

$$\frac{1}{\pi} \epsilon_{\mu\nu\lambda} \partial^\nu \tilde{A}^\lambda = J_{2\mu}. \tag{2.14b}$$

From these relations, Eq. (2.7), and the definition (1.4), we immediately obtain the relation between K_1 and K_2:

$$K_1^T(\omega, k)\tilde{K}_2^L(\omega, k) = \frac{1}{\pi^2}, \qquad \tilde{K}_1^T(\omega, k)K_2^L(\omega, k) = \frac{1}{\pi^2}, \tag{2.15a}$$

$$K_1^L(\omega, k)\tilde{K}_2^T(\omega, k) = \frac{1}{\pi^2}, \qquad \tilde{K}_1^L(\omega, k)K_2^T(\omega, k) = \frac{1}{\pi^2}. \tag{2.15b}$$

Now, assuming a single second-order transition obtained by tuning the parameter s, the above reasoning implies that this critical point must be self-dual, $K_1^{T,L} = \tilde{K}_1^{T,L}$, and $K_2^{T,L} = \tilde{K}_2^{T,L}$. Self-duality thus immediately implies relation (1.8), as claimed in the introduction.

Monte Carlo simulations [61] of a current loop model related to $\mathcal{S}$ observe a weak first-order transition. This is possibly because they are using a particular lattice action which is not within the domain of attraction of the self-dual point. In any case, the duality mappings between the two phases on either side of the transition apply, and the constraints on a possible CFT remain instructive.

III. THE M2-BRANE THEORY

This section examines the transport properties of the non-Abelian SU(N) Yang Mills theory in $D = 2 + 1$ with $\mathcal{N} = 8$ supersymmetry. The weak-coupling action and field content of this theory is most directly understood by dimensional reduction of the $\mathcal{N} = 1$ SYM theory in $D = 9 + 1$ on the flat torus T^7 [62]. This reduction shows that the $D = 2 + 1$ theory has an explicit SO(7) R-charge global symmetry. The $D = 9 + 1$ SYM theory has only a single gauge coupling constant, and therefore, so does the $D = 2 + 1$ theory. The latter coupling has a positive scaling dimension, and flows to strong coupling in the infrared. It is believed [43] that the flow is to an infrared-stable fixed point that describes a SCFT. It was also argued that this SCFT has an emergent R-charge symmetry which is expanded to SO(8). We shall be interested in the transport properties of this SO(8) R-charge in the SCFT at $T > 0$ in the present section.

We are faced by a strongly coupled SCFT, and a perturbative analysis of the field theory described above is not very useful. Instead, remarkable progress is possible using the connection to string theory and the AdS/CFT correspondence. The $D = 2 + 1$ SYM theory is contained in the low energy description of type IIA string theory in the presence of a stack of N D2-branes. The flow to strong coupling of the SYM theory corresponds in string theory to the lift of ten-dimensional type IIA strings to 11-dimensional M theory [46]. So we can directly access the $D = 2 + 1$ SYM SCFT by considering M theory in the presence of a stack of N M2-branes [45]. In the large N limit, M theory can be described by the semiclassical theory of 11-dimensional supergravity, and this will be our main tool in the analysis described below. This formulation also makes the SO(8) R-charge symmetry explicit, because the M2-branes curve the spacetime of 11-dimensional supergravity to $AdS_4 \times S^7$.

Another powerful feature of the supergravity formulation is that it can be extended to $T > 0$. We have to consider supergravity in a spacetime which is asymptotically AdS_4, but which also contains a black hole. The Hawking temperature of the black hole then corresponds to the temperature of the SCFT [63] (for example, fluctuation-dissipation theorems are satisfied [48]). Hydrodynamics of the SCFT emerges from the semiclassical supergravity dynamics in the presence of the black hole.[7]

Turning to our explicit computation of dynamics in M theory, we consider the gravitational background associated with a stack of N M2-branes, with $N \gg 1$ [45,49,65],

$$ds^2 = \frac{r^4}{R^4}[-f(r)dt^2 + dx^2 + dy^2] + \frac{R^2}{r^2}\left[\frac{dr^2}{f(r)} + r^2 d\Omega_7^2\right], \tag{3.1}$$

where $f(r) = 1 - r_0^6/r^6$. It is more convenient for us to change coordinates from r to $u = (r_0/r)^2$, in terms of which

$$ds^2 = \frac{r_0^4}{R^4 u^2}[-f(u)dt^2 + dx^2 + dy^2] + \frac{R^2}{4u^2 f}du^2 + R^2 d\Omega_7^2 \tag{3.2}$$

and $f(u) = 1 - u^3$. The horizon of the black hole is located at $u = 1$, and the boundary of AdS_4 is at $u = 0$.

The relationship between the quantities in the world volume SCFT (N and temperature T) and those of the metric (R and r_0) are given by [45,49]

$$\pi^5 R^9 = \sqrt{2}N^{3/2}\kappa^2, \qquad T = \frac{3}{2\pi}\frac{r_0^2}{R^3}, \tag{3.3}$$

where κ is the gravitational coupling strength of $D = 10 + 1$ supergravity.

There is a precise correspondence between correlation functions computed in the $D = 2 + 1$ CFT and correlation functions of supergravity fields computed in the metric (3.1) [46–48]. We will use this to compute charge transport properties.

In the metric (3.1) a 7-sphere factors out: $R^2 d\Omega_7^2$. The spacetime thus has a SO(8) symmetry. This matches with the global symmetry in the M2 world volume theory: there is a R-charge which transforms under the same global symmetry. The following subsections will compute the two-point correlations of the R-charge currents, $J_{a\mu}$, with $a = 1, \ldots, 28$.

[7] Strictly speaking, the appearance of a black hole is dual to being at finite temperature *and* being in a deconfined phase; it is possible to have a finite-temperature gravitational description without a black hole [63,64].

The existence of a compact 7-sphere makes it possible to do Kaluza-Klein reduction on this space. We expand all fields in terms of spherical harmonics on the 7-sphere. The original fields of M theory are the metric tensor $g_{\mu\nu}$ and a three-index antisymmetric tensor $A_{\mu\nu\lambda}$. Upon Kaluza-Klein reduction, an SO(8) gauge field appears from the components of the metric and the threeform where only one index is in the AdS_4 directions (t, x, y, and u) and the others are in the S^7 directions (see Appendix C for details). The action for this gauge field is

$$S = -\frac{1}{4g_{4D}^2}\int d^4x\sqrt{-g}g^{MA}g^{NB}F^a_{MN}F^a_{AB}, \qquad (3.4)$$

where uppercase Latin indices A, B, M, N run four values of t, x, y, and u (in contrast to Greek indices α, β, μ, ν which run t, x, and y). The four-dimensional gauge coupling constant g_{4D} is *dimensionless*, and its large N value is computed in Appendix C

$$\frac{1}{g_{4D}^2} = \frac{\sqrt{2}}{6\pi}N^{3/2}. \qquad (3.5)$$

Although we focus on the gravity background constructed from a stack of N M2-branes in flat 11-dimensional space, there are a number of related examples which are easily understood from considering (3.4). The key observation, which we discuss further in Sec. III D, is that (3.4) exhibits classical electric-magnetic duality. In the case of our M2-brane theory, this duality is close enough to a self-duality to enforce a relation on the current-current two-point functions and result in a frequency-independent conductivity. In fact, this self-duality holds in a more general context. Consider an 11-dimensional space which factorizes into $\mathbb{R}^{2,1}$ and a Calabi-Yau fourfold which develops a local singularity. By placing a stack of M2-branes at the singularity, we should obtain a more exotic $2+1$ dimensional conformal field theory which still has at least a U(1) global R-symmetry. Kaluza-Klein reduction of the gravity theory will yield precisely (3.4) and our results on holographic self-duality will carry over to these more general cases.

There are two other interesting generalizations to consider in which holographic self-duality fails. After Kaluza-Klein reduction, the gauge fields F_{AB} will support electrically charged black holes [66]. These black holes are dual to introducing an R-charge chemical potential to the field theory. Another interesting $2+1$ dimensional field theory with a holographic description is the theory living on a stack of D2-branes in type IIA string theory. In both cases, there is generically a nontrivial scalar which appears in a modification of (3.4) as a coupling constant which depends on the holographic radial direction. The relation on the two-point functions will be between a theory with coupling $g_{4D}(u)$ and one with coupling $1/g_{4D}(u)$. For details concerning this more general perspective, see Appendix E.

A. Current-current correlators

We now proceed to the computation of the two-point correlators of the $J_{a\mu}$ in the CFT at $T>0$. Here we will work in Minkowski space (real frequencies and time), and so define the current correlation as follows:

$$C_{\mu\nu}(x-y)\delta_{ab} = -i\theta(x^0-y^0)\langle[J_{a\mu}(x), J_{b\nu}(y)]\rangle. \qquad (3.6)$$

The δ_{ab} follows from SO(8) symmetry. The expectation value is taken in a translation-invariant state, so we can Fourier transform to $C_{\mu\nu}(p)$, where $p_\mu = (-\omega, \boldsymbol{k})$. Spectral density is proportional to the imaginary part of the retarded function,

$$\rho_{\mu\nu}(p) = -2\,\mathrm{Im}C_{\mu\nu}(p). \qquad (3.7)$$

It is an odd, real function of p, whose diagonal components are positive (for positive frequency). Expectation values of all global conserved charges are assumed to vanish in the equilibrium state; in other words we consider systems without chemical potentials. Conservation of $J_{a\mu}(x)$ implies that the correlation functions may be defined so that they satisfy the Ward identity[8] $p^\mu C_{\mu\nu}(p) = 0$. Then, as in Sec. I and in Eq. (1.4), we can write $C_{\mu\nu}$ in the form

$$C_{\mu\nu}(p) = P^T_{\mu\nu}\Pi^T(\omega,k) + P^L_{\mu\nu}\Pi^L(\omega,k). \qquad (3.8)$$

(The relationship between Π and K is $\Pi^{T,L} = \sqrt{p^2}K^{T,L}$.) Without loss of generality one can take the spatial momentum oriented along the x direction, so that $p = (\omega, k, 0)$. Then the components of the retarded current-current correlation function are

$$C_{yy}(\omega,k) = \Pi^T(\omega,k), \qquad (3.9)$$

as well as

$$C_{tt} = \frac{k^2}{\omega^2-k^2}\Pi^L(\omega,k),$$
$$C_{tx} = C_{xt} = \frac{-\omega k}{\omega^2-k^2}\Pi^L(\omega,k), \qquad (3.10)$$
$$C_{xx} = \frac{\omega^2}{\omega^2-k^2}\Pi^L(\omega,k).$$

B. Correlation functions from AdS/CFT

In order to find the retarded function, one needs to study fluctuations of vector fields on the background spacetime created by a stack of M2-branes. At the linear order the fields satisfy the equations

$$\partial_M(\sqrt{-g}g^{MA}g^{NB}F_{AB}) = 0. \qquad (3.11)$$

[8] One may choose to define the correlation functions in such a way that local (in position space) counterterms appear on the right-hand side of the Ward identities. The correlation functions defined in this way will differ from $C_{\mu\nu}(p)$ by analytic functions of ω and $\boldsymbol{k}$.

These equations are to be solved with the boundary conditions

$$\lim_{u\to 0} A_\mu(u,x) = A^0_\mu(x), \tag{3.12}$$

at $u = 0$. Near $u = 1$ one imposes the outgoing wave boundary condition, which means that for u slightly less than 1 the solution is purely a wave that propagates toward the horizon. Because of translational invariance with respect to x one can solve for each Fourier mode $e^{ip\cdot x}$ separately. The result can be represented in the form

$$A_\mu(u,p) = M_\mu{}^\nu(u,p)A^0_\nu(p). \tag{3.13}$$

Then, according to the AdS/CFT prescription formulated in Ref. [47], the current-current correlator can be found from the formula[9]

$$C_{\mu\nu}(p) = -\chi \lim_{u\to 0} M'_{\mu\nu}(u,p), \tag{3.14}$$

where χ is the constant that appears in the normalization of the action,

$$S = \frac{\chi}{2}\int du\, d^3x (A_t'^2 - fA_x'^2 - fA_y'^2 + \ldots), \tag{3.15}$$

(only terms with two derivatives with respect to u are written). In our case $\chi = 4\pi T/3g^2_{4\mathrm{D}}$. It turns out that χ is precisely the charge susceptibility.[10]

The prescription given above might appear *ad hoc*. However it is a special case of a more general AdS/CFT prescription that gives real-time correlators of any number of operators [48]. For our task, however, the above prescription is technically most straightforward to implement.

We work in the radial gauge $A_u = 0$, and take all fields $A_\mu(x)$ to be proportional to $e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}$. Taking momentum $\boldsymbol{k}$ along the x direction, $\boldsymbol{k} = (k, 0)$, one finds that the fluctuating vector fields satisfy the following equations [49]

$$wA_t' + qfA_x' = 0, \tag{3.16}$$

$$A_t'' - \frac{1}{f}(wqA_x + q^2A_t) = 0, \tag{3.17}$$

$$A_x'' + \frac{f'}{f}A_x' + \frac{1}{f^2}(wqA_t + w^2A_x) = 0, \tag{3.18}$$

$$A_y'' + \frac{f'}{f}A_y' + \frac{1}{f^2}(w^2 - q^2f)A_y = 0. \tag{3.19}$$

Here prime denotes derivative with respect to u; w and q are the dimensionless frequency and momentum, $w \equiv 3\omega/(4\pi T)$, $q \equiv 3k/(4\pi T)$. Note that the equation for the transverse potential A_y decouples from the rest. Moreover, Eq. (3.18) can be shown to follow from Eqs. (3.16) and (3.17) and so is not independent. Combining Eqs. (3.16) and (3.17) one can obtain an equation that does not involve A_x,

$$A_t''' + \frac{f'}{f}A_t'' + \frac{1}{f^2}(w^2 - q^2f)A_t' = 0. \tag{3.20}$$

One can think about this equation as a second-order equation for A_t'. It was observed in [49] that *Eq.* (3.20) *has the same form as the equation for* A_y. Such degeneracy is unusual, and we now proceed to explore its implications.

1. Transverse channel

Let us start with the retarded function for transverse currents, $C_{yy}(\omega, \boldsymbol{k})$. According to the AdS/CFT prescription (3.14),

$$C_{yy}(p) = -\chi \lim_{u\to 0} M'_{yy}(u,p). \tag{3.21}$$

The function $M_{yy}(u,p)$ is the solution to Eq. (3.19) which satisfies the outgoing wave boundary condition on the horizon $u = 1$, and $M_{yy}(0,p) = 1$ at the boundary $u = 0$.

Let us denote a solution to Eq. (3.19) which satisfies the outgoing boundary condition at the horizon as $\psi(u)$. The normalization of $\psi(u)$ is left arbitrary. Near $u = 0$, Eq. (3.19) allows two asymptotic solutions, which can be expressed in terms of the Frobenius series,

$$Z_{\mathrm{I}}(u) = 1 + hZ_{\mathrm{II}}(u)\ln u + b_{\mathrm{I}}^{(1)}u + \ldots, \tag{3.22}$$

$$Z_{\mathrm{II}}(u) = u(1 + b_{\mathrm{II}}^{(1)}u + b_{\mathrm{II}}^{(2)}u^2 + \ldots). \tag{3.23}$$

The coefficient $b_I^{(1)}$ is arbitrary, and we set it to zero. All other coefficients are determined by substituting expansion (3.25) in the original Eq. (3.19). In particular, we find that $h = 0$, therefore

$$Z_{\mathrm{I}}(0) = 1, \qquad Z_{\mathrm{I}}'(0) = 0, \qquad Z_{\mathrm{II}}(0) = 0, \qquad Z_{\mathrm{II}}'(0) = 1. \tag{3.24}$$

The outgoing wave solution $\psi(u)$ can be expressed as

$$\psi(u) = \mathcal{A}Z_{\mathrm{I}}(u) + \mathcal{B}Z_{\mathrm{II}}(u), \tag{3.25}$$

where $\mathcal{A}$ and $\mathcal{B}$ depend on the parameters of the equation, in particular, on w and q. From Eq. (3.24) it follows that $\psi(0) = \mathcal{A}$ and $\psi'(0) = \mathcal{B}$. The properly normalized mode function is $M_{yy}(u,p) = \psi(u)/\psi(0)$, and therefore we find

$$C_{yy}(w,q) = -\chi\frac{\mathcal{B}(w,q)}{\mathcal{A}(w,q)}. \tag{3.26}$$

[9] Greek indices on $M_{\mu\nu}$ are raised using the flat space Minkowski metric.

[10] The hydrodynamic density-density response function found in [49] is $C_{tt} = (1/g^2_{4\mathrm{D}})k^2/(i\omega - D_ck^2)$. Comparing this to the hydrodynamic form $C_{tt} = \chi D_c k^2/(i\omega - D_ck^2)$, we find the above value for charge susceptibility χ.

2. Longitudinal channel

Let us now look at the correlators in the longitudinal channel: C_{tt}, C_{tx}, and C_{xx}. For that we need to solve Eqs. (3.16) and (3.17). First, we know that $A_t'(u)$ satisfies the same equation as $A_y(u)$. Therefore, we can write $A_t'(u) = c\psi(u)$, where c is some coefficient. This coefficient can be fixed from the boundary conditions at $u = 0$ by employing Eqs. (3.17) and $\psi'(0) = \mathcal{B}$. We find

$$A_t'(u) = \left[\frac{\mathcal{A}}{\mathcal{B}} Z_{\rm I}(u) + Z_{\rm II}(u)\right](wqA_x^0 + q^2 A_t^0). \tag{3.27}$$

From Eq. (3.16) we also find

$$A_x'(u) = -\frac{1}{f}\left[\frac{\mathcal{A}}{\mathcal{B}} Z_{\rm I}(u) + Z_{\rm II}(u)\right](w^2 A_x^0 + wqA_t^0). \tag{3.28}$$

These equations are to be compared with Eq. (3.13), from which one extracts $M'_{\mu\nu}(u, p)$. Putting $u = 0$, one finds the correlators

$$C_{tt}(w, q) = \chi q^2 \frac{\mathcal{A}(w, q)}{\mathcal{B}(w, q)}, \qquad C_{xx}(w, q) = \chi w^2 \frac{\mathcal{A}(w, q)}{\mathcal{B}(w, q)}. \tag{3.29}$$

In Appendix D we show that at zero momentum, $q = 0$, the mode equation (3.19) can be solved analytically, which allows one to determine $\Pi^T(w, 0) = \Pi^L(w, 0)$. However, one can determine the conductivity without explicitly solving the mode equation, as we now show.

C. Conductivity

We see that both C_{yy} and C_{xx} are expressed in terms of the same connection coefficients $\mathcal{A}$ and $\mathcal{B}$. Eliminating the coefficients, we find

$$C_{xx}(w, q)C_{yy}(w, q) = -\chi^2 w^2, \qquad C_{tt}(w, q)C_{yy}(w, q) = -\chi^2 q^2. \tag{3.30}$$

Expressed in terms of the self-energies Π^T, Π^L this reads

$$\Pi^T(w, q)\Pi^L(w, q) = -\chi^2(w^2 - q^2). \tag{3.31}$$

Note that this relation holds for all w and q: we have not made any small-frequency approximations anywhere. In fact, we did not even have to solve the mode equations. Combining Eqs. (1.4), (3.8), and (3.31), we obtain our main result in Eq. (1.9).

As discussed in Sec. I, at zero momentum, rotation invariance implies that $\Pi^T = \Pi^L$, therefore relation (3.31) uniquely determines the self-energy[11] $\Pi^T(\omega, 0) = \Pi^L(\omega, 0) = -i\chi w$ for all w. The conductivity is given by $\sigma(\omega/T) = i\Pi^T(\omega, 0)/\omega$, and we find

$$\sigma(\omega/T) = \chi\frac{3}{4\pi T} = \chi D_c = \frac{1}{g_{\rm 4D}^2}, \tag{3.32}$$

where $D_c = 3/(4\pi T)$ is the diffusion constant found in [49]. Note that the Einstein relation between the conductivity and the diffusion constant is satisfied. Also, as noted earlier, it is surprising that $\sigma(\omega/T)$ is actually independent of ω/T. [Dependence upon ω/T is found at all nonzero k, as is shown below.] This ω-independence is a consequence of the relation (3.31), which in turn follows from the fact that A_t' and A_y satisfy the same equation in the bulk. It can be traced back to the electromagnetic duality of the classical action (3.4), as we now show.

D. Electric-magnetic duality

Even though the origin of the relation (3.31) is puzzling from the point of view of the microscopic degrees of freedom in the $\mathcal{N} = 8$ SCFT, its origin from the bulk point of view can be traced to electric-magnetic (EM) duality of an Abelian gauge field. Indeed, current-current correlators are computed from the Maxwell equations in the four-dimensional bulk, and it is precisely in four dimensions that Maxwell equations may possess EM duality.

Although in general the R-symmetry may be non-Abelian and hence be dual to a non-Abelian gauge field in the bulk, we work in the classical supergravity limit and must keep N large. At large N, the gauge coupling $g_{\rm 4D} \propto N^{-3/4}$ is very small, and our non-Abelian gauge field factorizes into a number of effectively Abelian pieces to leading order in $1/N$.

If we write equations of motion in terms of the gauge-invariant F_{MN} (rather than the vector potential), then Maxwell equations have to be supplemented by a Bianchi identity,

$$\partial_M(\sqrt{-g}F^{MN}) = 0, \tag{3.33a}$$
$$\partial_M(\sqrt{-g}\tfrac{1}{2}\varepsilon^{MNAB}F_{AB}) = 0, \tag{3.33b}$$

where ε^{MNAB} is the totally antisymmetric tensor, with $\varepsilon^{0123} = 1/\sqrt{-g}$. Now, one can introduce G_{MN} defined as $F^{MN} = \frac{1}{2}\varepsilon^{MNAB}G_{AB}$, which can be inverted to give $G^{MN} = -\frac{1}{2}\varepsilon^{MNAB}F_{AB}$. Expressed in terms of G, the equations of motion become

$$\partial_M(\sqrt{-g}\tfrac{1}{2}\varepsilon^{MNAB}G_{AB}) = 0, \tag{3.34a}$$
$$\partial_M(\sqrt{-g}G^{MN}) = 0. \tag{3.34b}$$

Maxwell equations for F become a Bianchi identity for G, and vice versa. G_{MN} is the dual field strength tensor, and we can also define a dual vector potential B_M by $G_{MN} = \partial_M B_N - \partial_N B_M$. Note that the validity of EM duality does not depend on the background spacetime having any particular symmetries such as Lorentz symmetry, or rotational symmetry.

[11] Up to a sign, which can be fixed by requiring positivity of the spectral function $\rho_{yy} = -2\,{\rm Im}\Pi^T$.

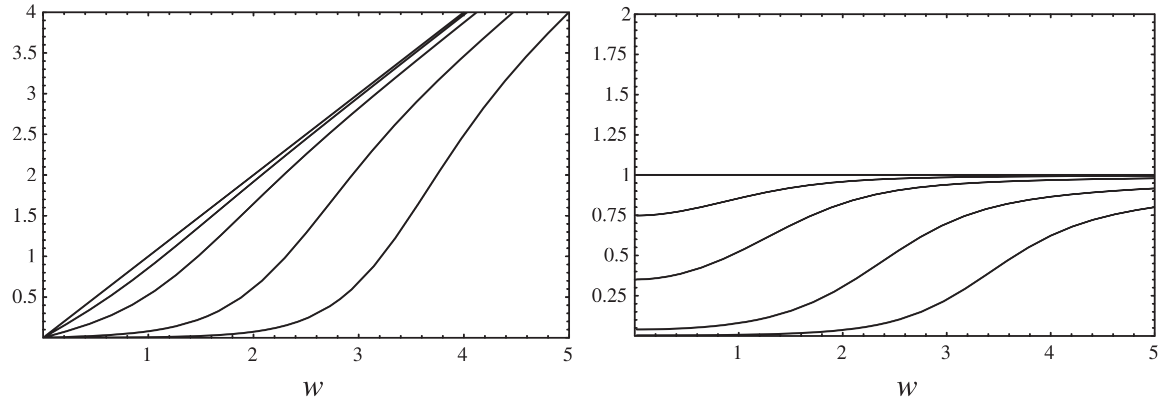

FIG. 2. Imaginary part of the retarded function $C_{yy}(\omega, k)$, plotted in units of $(-\chi)$, as a function of dimensionless frequency $w \equiv 3\omega/(4\pi T)$, for several values of dimensionless momentum $q \equiv 3k/(4\pi T)$. Curves from left to right correspond to $q = 0, 0.5, 1.0, 2.0, 3.0$. Left: $\mathrm{Im}C_{yy}(w, q)$, Right: $\mathrm{Im}C_{yy}(w, q)/w$.

From the point of view of AdS/CFT, the EM dual theory in the bulk will correspond to some theory on the boundary, which is a dual of the original SCFT. In particular, the dual vector potential B_μ will couple to the dual current $\tilde{J}_\mu$, and one can compute two-point functions $C^{\rm dual}_{\mu\nu}(\omega, k)$ in the dual theory.

In components we have $F^{tz} = G_{xy}/\sqrt{-g}$. This means that the equation for $\sqrt{-g}F^{tz}$ obtained from Eqs. (3.33) is the same as the equation for G_{xy}, obtained from the dual Eqs. (3.34). In our particular example of the nonextremal M2 background metric, we have $\sqrt{-g}F^{tu} \propto A_t'(u)$, and $G_{xy} \propto kB_y(u)$ (in the radial gauge). Thus the equation for $A_t'(u)$ is the same as the equation for $B_y(u)$. Then, by the argument in Sec. III B we find a relation between the self-energies $\Pi^{T,L}$ in the original theory, and the self-energies $\tilde{\Pi}^{T,L}$ in the dual theory:

$$\Pi^T(w, q)\tilde{\Pi}^L(w, q) = -\chi^2(w^2 - q^2), \tag{3.35a}$$

$$\tilde{\Pi}^T(w, q)\Pi^L(w, q) = -\chi^2(w^2 - q^2). \tag{3.35b}$$

For our M2-branes, EM duality is a self-duality, and the EM dual theory is the same as the original theory, as is evident from Eqs. (3.33) and (3.34). Therefore, $C_{\mu\nu} = C^{\rm dual}_{\mu\nu}$, and $\tilde{\Pi}^T = \Pi^T$, $\tilde{\Pi}^L = \Pi^L$. This gives back our main result (3.31).[12] In the case when there are nontrivial background profiles for scalar fields, the EM dual theory is not equivalent to the original theory. This is discussed in Appendix E.

[12]The present discussion assumes that the coupling constant $g^2_{\rm 4D}$ is not inverted in the dual theory, which is justified for a free, sourceless, Abelian gauge field. One could formally repeat the same steps leading to Eq. (3.31), assuming $\tilde{g}^2_{\rm 4D} = 1/g^2_{\rm 4D}$, as is standard in EM duality. However, in this case the coupling constant $\tilde{g}^2_{\rm 4D} \propto N^{3/2}$ becomes large, invalidating the bulk description in terms of a classical gauge field.

E. Full spectral functions

We will now evaluate the spectral functions numerically, for all ω and k. To do so, we find a solution $\psi(u)$ to the mode equation (3.19) with the outgoing boundary conditions at the horizon $u = 1$. Then, as described in Sec. III B, the retarded two-point function $C_{yy}(\omega, k)$ is proportional to $\psi'(0)/\psi(0)$, while $C_{tt}(\omega, k)$ is proportional to $\psi(0)/\psi'(0)$.

Figure 2 shows the imaginary part of the transverse current-current correlation function, plotted in units of $(-\chi)$. At zero momentum, $\mathrm{Im}C_{yy}$ is a linear function of $w \equiv 3\omega/(4\pi T)$ for all w, as shown in the previous subsection. At large frequency, the spectral function asymptotes to $\mathrm{Im}C_{yy} \sim (-\chi)w$, regardless of the value of $q \equiv 3k/(4\pi T)$.

The longitudinal correlators are directly related to the conserved R-charge density, and so are more direct probes of hydrodynamic behavior, and the hydrodynamic-to-collisionless crossover. Figure 3 shows the imaginary part of the density-density correlation function divided by q^2. At small momentum and frequency, one clearly sees the diffusive peak, consistent with the hydrodynamic expression in Eq. (1.3)

$$\mathrm{Im}\, C_{tt}(\omega, k) = D_c\chi \frac{-\omega k^2}{\omega^2 + (D_c k^2)^2}, \qquad |\omega| \ll T \quad \text{and} \quad k \ll T. \tag{3.36}$$

At large frequency, the asymptotic form of the spectral function is expected to be determined by the collisionless ground state correlator. The latter was presented in Eq. (1.2), and here has the form

$$\mathrm{Im}\, C_{tt}(\omega, k) = \frac{1}{g^2_{\rm 4D}} \mathrm{sgn}(\omega) \frac{(-k^2)}{\sqrt{\omega^2 - k^2}}, \qquad |\omega| - k \gg T. \tag{3.37}$$

Figure 3, right, shows that this form is indeed well obeyed. Indeed, Eqs. (3.36) and (3.37) are exactly the correlators

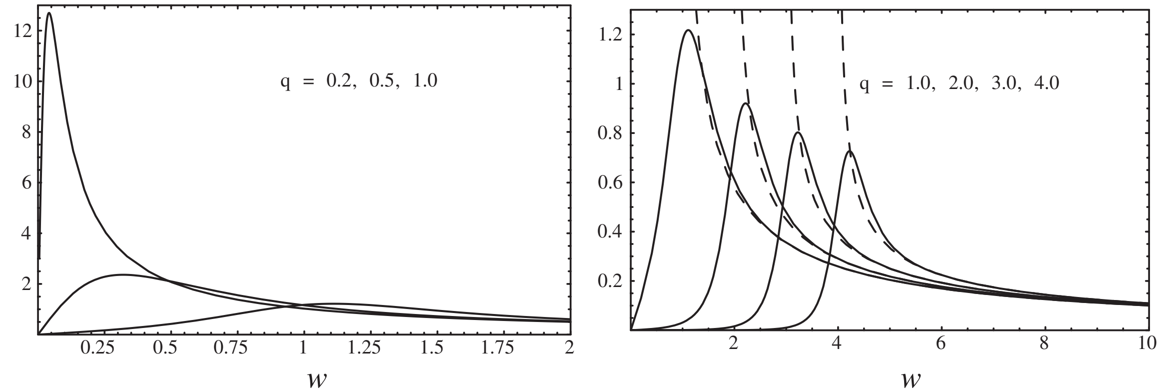

FIG. 3. Imaginary part of the retarded function $C_{tt}(w, q)/q^2$, plotted in units of $(-\chi)$, as a function of dimensionless frequency $w \equiv 3\omega/(4\pi T)$, for several values of dimensionless momentum $q \equiv 3k/(4\pi T)$. Curves from left to right correspond to $q = 0.2$, 0.5, 1.0 (left panel), and $q = 1.0$, 2.0, 3.0, 4.0 (right panel). The dashed curves are plots of Eq. (3.37) divided by k^2.

expected across a hydrodynamic-to-collisionless crossover in a generic system [67]: the prefactor of k^2 in Eq. (3.37) is required by charge conservation even at large ω, while the factor of $1/\sqrt{\omega^2 - k^2}$ is set by the CFT current scaling dimension and Lorentz invariance.

In Fig. 4, we illustrate the crossover from the hydrodynamic regime to the collisionless regime. For each value of q we find the value $w_{\max}$ where the function $\mathrm{Im}C_{tt}(w, q)$ reaches its maximal value, and plot the resulting function $w_{\max}(q)$. As we see on Fig. 4, at small q the location of the peak is $w_{\max} = q^2$, in accordance with hydrodynamics. At large q it slowly reaches the asymptotic collisionless behavior $w_{\max} = q$.

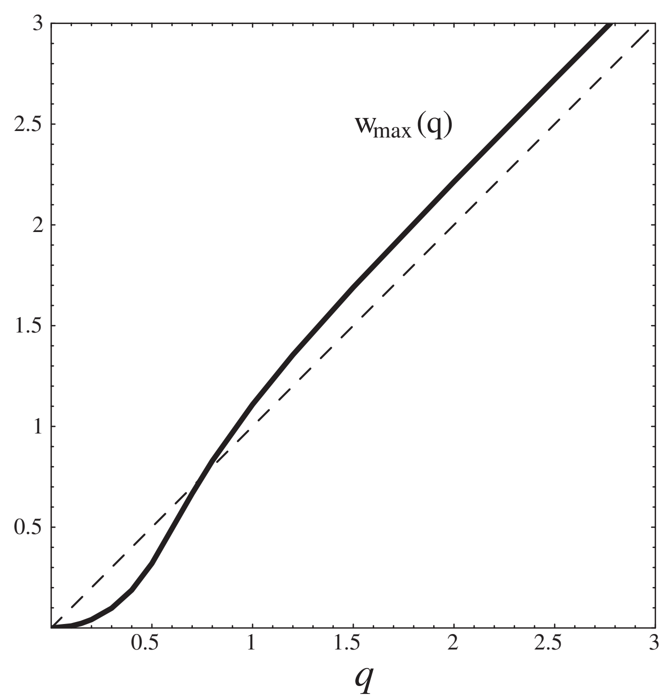

FIG. 4. The position of the peak of the spectral function in Fig. 3. The dashed line is $w = q$.

What is unexpected, is that the two prefactors in Eqs. (3.36) and (3.37), $D_c\chi$ and $g_{4\mathrm{D}}^{-2}$, happen to be equal to each other, as we saw in Eq. (3.32). We have also seen that this surprising feature is a consequence of the general functional relations in Eqs. (1.9) and (3.31). As we have discussed, such functional relations are not expected to apply to a typical $D = 2 + 1$ CFT, but only those which enjoy special self-duality symmetries. Here the self-duality of the gauge theory on AdS_4 led to the identical form of Eqs. (3.19) and (3.20) which was shown eventually to lead to Eqs. (1.9) and (3.31). In Appendix E, we consider a R-symmetry gauge field action with a nontrivial dilaton which spoils the holographic self-duality and the frequency-independent conductivity. The field theory on a D2-brane in type IIA string theory is an example with such a dilaton.

IV. CONCLUSIONS

We considered finite-temperature charge transport of quantum field theories in $D = 2 + 1$ dimensions: the easy-plane $\mathbb{CP}^1$ model, and the CFT living on a stack of N M2-branes in M theory (the $\mathcal{N} = 8$, SU(N) SYM theory). In the former theory, Abelian particle-vortex self-duality imposes a relationship (Eq. (1.8)) between different current correlators. In the latter theory, we found a strikingly similar relationship (Eq. (1.9)) between longitudinal and transverse components of the correlators of the SO(8) R-charge. This relationship led to a frequency-independent conductivity for the M2 world volume theory at zero wave vector, but hydrodynamic behavior and the hydrodynamic-collisionless crossover did appear at nonzero wave vectors. We also demonstrated that for the D2-brane theory, our argument for frequency-independent conductivity fails because of a nontrivial dilaton background.

We traced the origin of the SO(8) charge correlation constraint of the SYM theory, and its frequency-independent conductivity, to an electromagnetic self-duality of the holographic theory on AdS_4. Thus, the generalization of three-dimensional Abelian particle-vortex duality to non-Abelian theories becomes manifest only after a holographic extension to a four-dimensional theory. For Abelian theories, the AdS/CFT connection between particle-vortex duality in three dimensions and the $SL(2, Z)$ invariance of four-dimensional Abelian gauge theories was explored earlier in [28,57].

Our results for the SU(N) SYM theory were established at large N. Does holographic self-duality, and the relationship[13] Eq. (1.9), hold also for finite N? The fact that the large N theory has hydrodynamic behavior is evidence for the "generic" nature of this limit. Furthermore, Eq. (1.9) has the same structure as Eq. (1.8), and the latter is believed to be an exact relationship, obtained without a large N limit. While these facts are encouraging, establishing self-duality at finite N requires looking at the full M theory on AdS_4. Its low energy limit is $\mathcal{N} = 8$ supergravity [66,68–71] (Sec. III considered only the SO(8) gauge fields of this theory), and its "generalized $E_{7(7)}$ duality invariance" [69] (which appears to include EM duality) has remnants in M theory [72].

It would be very interesting to find an Abelian field theory which obeyed a relationship as simple as Eq. (1.9), found here for the SYM theory. An unsuccessful attempt to find such a theory is described in Appendix B. The closest we could get is Eq. (1.8), obeyed by the easy-plane $\mathbb{CP}^1$ model [16] and its expected generalization to the SQED-2 theory with $\mathcal{N} = 4$ supersymmetry [39–41]. A fundamental feature of Abelian particle-vortex duality is the exchange of U(1) "flavor" and "topological" currents, and we have not been able to construct a theory in which these currents are equivalent to each other (which would lead to a single K in Eq. (1.1)). However, non-Abelian theories can have additional symmetries which rotate different U(1) currents into each other; this was important for the simplicity of Eq. (1.9).

Finally, we would like to emphasize that the unexpected relation between the self-energies found in this paper,

$$K^L(\omega, k)K^T(\omega, k) = \text{const}, \tag{4.1}$$

holds beyond the $\mathcal{N} = 8$ SYM theory.[14] It applies to the CFTs whose electromagnetic response is described by the Maxwell action (3.4) in the 3 + 1 dimensional asymptotically AdS space. Thus the relation (4.1) should be viewed as another example of universality that characterizes finite-temperature response in the AdS/CFT correspondence. Previous examples of such universality include the universal value of the viscosity to entropy density ratio $\eta/s = 1/4\pi$ [73], and a possible universal value of the friction coefficient for a heavy particle [74]. Unlike these other examples, the universal relation (4.1) applies only to 2 + 1 dimensional CFTs at finite temperature. On the other hand, unlike these other examples, the universal relation (4.1) applies at arbitrary ω and k.

[13] Of course, the constant on the right-hand side of Eq. (1.9) would have finite N corrections. The issue is whether the right-hand side remains independent of ω and k for $T > 0$ also at finite N.

[14] As described in Appendix C, there is a whole class of 2 + 1 dimensional CFTs satisfying Eq. (4.1). For large N field theories which are dual to M theory on $AdS_4 \times X$, where X is a seven-dimensional Sasaki-Einstein manifold, with currents normalized as in Appendix C, the value of the constant in the right-hand side of Eq. (4.1) is $N^3/(2\pi^{10})\,\mathrm{Vol}(X)^2$.

ACKNOWLEDGMENTS

We thank M. Ernebjerg, J. Liu, D. Shih, M. Strassler, A. Strominger, C. Vafa, and A. Vishwanath for useful discussions. This research was supported by the NSF under grants No. PHY99-07949 (P. K. and C. H.) and No. DMR-0537077 (S. S.), and by the DOE under grants No. DE-FG02-96ER40956 (C. H.) and No. DE-FG02-00ER41132 (D. T. S.). C. H. thanks the organizers of the String Phenomenology workshop at the KITP, UCSB. P. K. thanks the organizers of the INT workshop "From RHIC to LHC: achievements and opportunities" at the University of Washington, and the Harvard University Physics Department, where part of this work was completed.

APPENDIX A: THERMAL CORRELATORS OF CFTs IN $D = 1 + 1$

First, let us consider an arbitrary Lorentz-scalar observable $\mathcal{O}$ of a CFT in $D = 1 + 1$ with scaling dimension h. Then, at $T = 0$, its two-point correlator in Euclidean space is

$$C_{\mathcal{O}}(\tau, x)|_{T=0} \sim \frac{1}{(x^2 + \tau^2)^h}, \tag{A1}$$

while the corresponding correlator in momentum and imaginary frequency is $\sim(\omega^2 + k^2)^{h-1}$. By the conformal map from the infinite plane to the cylinder with circumference $1/T$, we can obtain the form of the correlation at $T > 0$:

$$C_{\mathcal{O}}(\tau, x) \sim \left[\frac{\pi^2 T^2}{\sin(\pi T(\tau - ix))\sin(\pi T(\tau + ix))}\right]^h. \tag{A2}$$

Notice that this expression is periodic in τ, with period $1/T$. Now let us Fourier transform Eq. (A2) to momenta k and Matsubara frequencies ω_n; because of the periodicity, the ω_n must be integer multiples of $2\pi T$, and the result is

$$C_{\mathcal{O}}(i\omega_n, k) \sim T^{2h-2}\frac{\Gamma(1-h)}{\Gamma(h)} \times \frac{\Gamma(\frac{h}{2} + \frac{|\omega_n|+ik}{4\pi T})\Gamma(\frac{h}{2} + \frac{|\omega_n|-ik}{4\pi T})}{\Gamma(1 - \frac{h}{2} + \frac{|\omega_n|+ik}{4\pi T})\Gamma(1 - \frac{h}{2} + \frac{|\omega_n|-ik}{4\pi T})}. \tag{A3}$$

Finally, we analytically continue this expression to real frequencies from the upper half-frequency plane ($\omega_n > 0$) with the mapping $i\omega_n \rightarrow \omega$ to obtain the retarded two-point correlator at $T > 0$. This is a nontrivial function of ω and k, which describes relaxation of $\mathcal{O}$ correlations at $T > 0$. See Chapter 4 of Ref. [31] for more details.

Now let us consider the special case of a conserved current, and search for the collisionless-to-hydrodynamic crossover. In this case, $h = 1$. Note that Eq. (A3) has a pole at $h = 1$ with a residue which is ω and k independent; this reflects a logarithmic cutoff dependence in the Fourier transform of Eq. (A2), and the finite ω and k dependent contribution is obtained by subtracting the pole. However, the current is not a Lorentz scalar, so the above results do not directly apply anyway. The density correlator at $T = 0$ in Euclidean space is

$$C_{tt}(\tau, x)|_{T=0} \sim \frac{1}{(\tau - ix)^2} + \frac{1}{(\tau + ix)^2}. \quad \text{(A4)}$$

In momentum and real frequency space, the Fourier transform of this is cutoff independent because of the nonzero Lorentz spin:

$$C_{tt}(\omega, k)|_{T=0} \sim \frac{-k^2}{k^2 - \omega^2}. \quad \text{(A5)}$$

This is, of course, the generalization of Eq. (1.2) to $D = 1 + 1$. We can obtain the $T > 0$ density correlator by a conformal mapping of Eq. (A4), as was done earlier in Eq. (A2); here the corresponding expression is

$$C_{tt}(\tau, x) \sim \left[\frac{\pi T}{\sin(\pi T(\tau - ix))}\right]^2 + \left[\frac{\pi T}{\sin(\pi T(\tau + ix))}\right]^2. \quad \text{(A6)}$$

Finally, let us Fourier transform Eq. (A6) to momentum and Matsubara frequency space. Carrying out this transformation yields an initially surprising result. Although the real space result in Eq. (A6) depends upon temperature, the $T > 0$ result in momentum and frequency space has the same form as that at $T = 0$ in Eq. (A5):

$$C_{tt}(i\omega_n, k) \sim \frac{-k^2}{k^2 + \omega_n^2}. \quad \text{(A7)}$$

The inverse Fourier transforms of Eqs. (A5) and (A7) differ only because the frequency ω_n is discrete, while ω is continuous. So there is no hydrodynamic behavior at $T > 0$, and no analog of the result in Eq. (1.3).

The physical interpretation of the absence of hydrodynamic behavior is simple. CFTs in $D = 1 + 1$ can be holomorphically factorized, and consequently, there are no interactions or collisions between left and right movers. To obtain collisions, one has to consider the influence of formally irrelevant perturbations which can couple left and right movers. Only then will hydrodynamic behavior emerge: see Ref. [75]. In contrast, in $D = 2 + 1$, hydrodynamics emerges already in the conformal scaling limit [29].

APPENDIX B: ABELIAN DUALITY WITH ONE COMPLEX SCALAR

Here we will make some remarks on the duality properties of theories of a single complex scalar coupled to a U(1) gauge field with a Chern-Simons term in $D = 2 + 1$. Such theories have been studied extensively in the context of the quantum Hall effect [23–28,76]. A $\mathcal{N} = 3$ supersymmetric generalization of the theory below has been studied by Kapustin and Strassler [41] and their results are very similar to our $T = 0$ results below. We will also present results for the theory without the Chern-Simons term (whose supersymmetric analog, noted in Sec. II, is the $\mathcal{N} = 4$ SQED-1 theory [39–41]).

We consider a theory with the following action, which is essentially the single scalar version of Eq. (2.1), with an additional Chern-Simons term:

$$\mathcal{S}_{\rm cs} = \int d^2x\, dt \bigg[|(\partial_\mu - iA_\mu)z|^2 + s|z|^2 + u|z|^4 + \frac{1}{2e^2}(\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda)^2 + \frac{\alpha}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \bigg]. \quad \text{(B1)}$$

In general, this theory is not a CFT. However, as in Sec. II, we can imagine accessing a second-order phase transition out of a Higgs phase at a critical value of the "mass" term $s = s_c$; we are interested here in the duality properties of such a CFT.

First, standard methods [20,21] can be used to obtain a dual version of the action $\mathcal{S}_{\rm cs}$: we can either use the continuum arguments of Sec. II B, or apply Poisson summation methods to a lattice discretization [24,25,27]. From this we obtain a dual field theory, which has the same formal structure at long wavelengths:

$$\tilde{\mathcal{S}}_{\rm cs} = \int d^2x\, dt \bigg[|(\partial_\mu - i\tilde{A}_\mu)w|^2 + \tilde{s}|w|^2 + \tilde{u}|w|^4 + \frac{1}{2\tilde{e}^2}(\epsilon^{\mu\nu\lambda}\partial_\nu \tilde{A}_\lambda)^2 + \frac{\tilde{\alpha}}{4\pi}\epsilon^{\mu\nu\lambda}\tilde{A}_\mu\partial_\nu \tilde{A}_\lambda \bigg]. \quad \text{(B2)}$$

The similarity between $\mathcal{S}_{\rm cs}$ and $\tilde{\mathcal{S}}_{\rm cs}$ is encouraging and suggests that we may be able to use it to define a self-dual CFT. However, we will now argue that this is not the case.

In general, the relationship of the coupling constants in $\mathcal{S}_{\rm cs}$ and $\tilde{\mathcal{S}}_{\rm cs}$ is nonuniversal, and dependent upon the nature of the ultraviolet cutoff (with one exception, see below). However, there are a number of crucial constraints, which are readily apparent from the explicit transformations. From these constraints we find that there are 2 distinct sets of theories which are connected by duality:

Class A: Theories with no Chern-Simons terms: These theories have $\alpha = \tilde{\alpha} = 0$. Then the duality mappings show that we must have either $e = 0$ or $\tilde{e} = 0$ but *not* both

[20,21]; this is because in a theory with zero electric charge, duality maps the coefficient of the matter kinetic energy (the "stiffness") to the electric charge squared of the dual theory. Without loss of generality, let us choose $e = 0$. Then we may set $A_\mu = 0$, which then defines $\mathcal{S}_{\rm cs}$ as the theory of a single scalar with a global U(1) symmetry (the XY model). The theory $\tilde{\mathcal{S}}_{\rm cs}$ is the Abelian Higgs model which has a gauged U(1) "symmetry." Thus we have obtained the familiar duality [21] between the XY model and the Abelian Higgs model in $D = 2 + 1$. It is evident from the distinct nature of these models that they are not self-dual [20,21].

Class B: Theories with Chern-Simons terms: Now both α and $\tilde{\alpha}$ must be nonzero, and indeed the lattice duality transformations show that they satisfy

$$\alpha\tilde{\alpha} = -1, \tag{B3}$$

and this is the only relationship between the couplings of $\mathcal{S}_{\rm cs}$ and $\tilde{\mathcal{S}}_{\rm cs}$ which is universal. Furthermore, the requirement that either e or $\tilde{e}$ vanish no longer appears; in general, both are nonzero and finite. The duality also shows that it is not possible to eliminate the kinetic terms of both gauge fields, i.e. it is not possible to set both $e = \infty$ and $\tilde{e} = \infty$. Even if e.g. we eliminate the gauge kinetic term in $\mathcal{S}_{\rm cs}$ by setting $e = \infty$, then the duality yields a finite $\tilde{e}$ because, by the particle-vortex prescription, the kinetic energy of $\tilde{A}$ is related to the kinetic energy of the z particles, and the latter is finite. Because we are searching for a self-dual theory, we need to keep both e and $\tilde{e}$ finite. The implication of a finite e (or $\tilde{e}$) is that the flux-attachment transformation associated with the Chern-Simons term is "smeared out": each z particle worldline has a total of $2\pi/\alpha$ A_μ flux attached, but this flux is spread out over a finite length scale determined by e. This smearing also means that the transformation $1/\alpha \to 1/\alpha + 1$ does not map the theory onto itself. This transformation is the T operation defined by Witten [28], who also found that T did not leave the theory invariant. On the other hand, Fradkin and Kivelson [24] claimed T invariance for their model, which was defined in terms of infinitely thin particle and flux worldlines on a lattice with long-range interactions. It is unclear to us whether their model can be mapped to a local continuum action for a CFT.

Let us now consider correlators of the field theories without a Chern-Simons term, in class A. As discussed above, we choose the theory $\mathcal{S}_{\rm cs}$ to have $e = 0$ and $\alpha = 0$, so this describes the O(2) φ^4 theory (the XY model). We are interested in the CFT at some critical $s = s_c$. The two-point correlator of the U(1) current, $C_{\mu\nu}$, of $\mathcal{S}_{\rm cs}$ obeys Eqs. (1.1) and (1.4) with a single constant K, and a single set of functions $K^{L,T}(\omega, k)$. Similarly, the dual theory, $\tilde{\mathcal{S}}_{\rm cs}$ (which has $\tilde{e} \neq 0$, $\tilde{\alpha} = 0$ and is the Abelian Higgs model), has a correlator $\tilde{C}_{\mu\nu}$, and the corresponding $\tilde{K}$. Then the analog of the duality considerations of Sec. II imply that

$$K^T(\omega, k)\tilde{K}^L(\omega, k) = \frac{1}{4\pi^2}, \qquad K^L(\omega, k)\tilde{K}^T(\omega, k) = \frac{1}{4\pi^2}, \tag{B4}$$

and its $T = 0$ limit $K\tilde{K} = 1/(4\pi^2)$. This theory in class A is not self-dual, so the above relations do not allow us to determine the conductivities $\sigma(\omega/T)$ and $\tilde{\sigma}(\omega/T)$, and only constrain their product.

Next, we consider correlators of class B. The field theory $\mathcal{S}_{\rm cs}$ defines a CFT at some $s = s_c$, and we ask if this CFT can be self-dual. At $T = 0$, we have to generalize the form of the current correlator $C_{\mu\nu}$ from Eq. (1.1) to [24,27,28]

$$C_{\mu\nu}(p)|_{T=0} = K\sqrt{p^2}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + H\epsilon_{\mu\nu\lambda}p^\lambda, \tag{B5}$$

where K, H are two real constants characterizing the CFT. Note that, at the gapless conformal fixed point, there is no simple relationship[15] between the coupling constant α and the constant H, although a theory in class B is expected to have a nonzero H. At $T > 0$, the generalization of Eq. (1.4) is

$$C_{\mu\nu}(\omega, \boldsymbol{k}) = \sqrt{p^2}(P^T_{\mu\nu}K^T(\omega, k) + P^L_{\mu\nu}K^L(\omega, k)) + H(\omega, k)\epsilon_{\mu\nu\lambda}p^\lambda, \tag{B6}$$

with 3 distinct functions of ω/T and k/T on the right-hand side; note that even the Hall conductivity (equal to $H(\omega, 0)$) is a function of ω/T [30]. Similarly, we can also consider the dual-correlator $\tilde{C}_{\mu\nu}$ of the theory (B2) and define a corresponding set of parameters $\tilde{K}$ and $\tilde{H}$. The analog [23–25,27,28] of the arguments in Sec. II shows the following exact relationship between these parameters at $T = 0$:

$$(H + iK)(\tilde{H} + i\tilde{K}) = -\frac{1}{4\pi^2}. \tag{B7}$$

The real and imaginary parts of Eq. (B7) generalize the $T = 0$ limit of Eq. (B4) to class B. For $T > 0$, we have

$$\tilde{K}^T(\omega, k) = \frac{K^T(\omega, k)}{D(\omega, k)}; \qquad \tilde{K}^L(\omega, k) = \frac{K^L(\omega, k)}{D(\omega, k)}; \qquad \tilde{H}(\omega, k) = -\frac{H(\omega, k)}{D(\omega, k)}, \tag{B8}$$

with

$$D(\omega, k) \equiv 4\pi^2(K^T(\omega, k)K^L(\omega, k) + H^2(\omega, k)). \tag{B9}$$

Note that Eqs. (B8) reduce to Eqs. (B4) when $H = 0$, and to Eq. (B7) at $T = 0$.

For the class B model to be self-dual, we clearly need $\tilde{K} = K$ and $\tilde{H} = H$. From Eq. (B7) we observe that this is

[15] The one-loop expression for H obtained from $\mathcal{S}_{\rm cs}$ is exact as long as $s \neq s_c$, but the CFT at $s = s_c$ has corrections at all orders [30,76].

only possible for $K = 1/(2\pi)$ and $H = 0$. However, a model with $H = 0$, which surely requires $\alpha = 0$, is not in class B. It is in class A, and we argued earlier that a class A model could not be self-dual.

To conclude, although the model $\mathcal{S}_{\rm cs}$, and its dual $\tilde{\mathcal{S}}_{\rm cs}$, define interesting CFTs, with their correlators obeying Eqs. (B4), (B7), and (B8), we have shown that such CFTs cannot be self-dual. This conclusion is in accord with those of Kapustin and Strassler [41] and Witten [28] on related models.

APPENDIX C: NORMALIZATION OF GAUGE FIELD ACTION ON AdS_4

The R-symmetry gauge field can be thought of as arising from Kaluza-Klein reduction of an 11-dimensional supergravity solution on a regular positive curvature Sasaki-Einstein manifold X of real dimension seven. The size of the gauge group is determined by the isometry group of X. For instance, when $X = S^7$, the group is SO(8). By definition, Sasaki-Einstein manifolds have at least one U(1) isometry. In this section, we normalize the U(1) R-symmetry gauge field action in terms of the 11-dimensional gravitational coupling using results of Ref. [77]. Although the identification of this gauge field as a combination of metric and F_4 form perturbations in $D = 11$ supergravity predates Ref. [77] (see [78]), Ref. [77] provides a convenient starting point for considering issues of normalization. The normalization is not sensitive to temperature, and hence it is convenient to work here at $T = 0$.

To first order, the vector potential A perturbs the 11-dimensional metric as follows:

$$ds^2 = \frac{r^2}{L^2}\eta_{\alpha\beta}dx^\alpha dx^\beta + L^2\frac{dr^2}{r^2} + 4L^2 ds_X^2, \qquad \text{(C1)}$$

where

$$ds_X^2 = \left(\frac{q}{4}\right)^2\left(d\psi + \frac{4}{q}\sigma + \frac{2}{q}A\right)^2 + h_{a\bar{b}}dz^a d\bar{z}^b. \qquad \text{(C2)}$$

The Minkowski tensor $\eta_{\alpha\beta}$ runs over the three coordinates x^0, x^1, and x^2. Together the coordinates x^i and r give four-dimensional anti-de Sitter space with radius of curvature L.[16] Here $h_{a\bar{b}}$ is a Kähler-Einstein metric on a complex three-dimensional manifold we will call V. Setting $A = 0$, X would be a U(1) fibration over the threefold, giving rise to a real seven-dimensional Sasaki-Einstein manifold. The one form σ is constructed such that $d\sigma = 2\omega$ where ω is the Kähler form on V. With the angle ψ constrained to lie between 0 and 2π, the integer q obeys the relation $\omega = \pi q c_1/4$ where c_1 is the first Chern class of the U(1) fibration. In general $q = 1$, but in certain cases where $c_1(V)$ is divisible, q may be more. For instance, in the case of S^7, X is a U(1) fibration over $\mathbb{CP}^3$ and $q = 4$. In [77], the relation between ψ and A was fixed by setting the R-charge of a holomorphic four-form associated to the cone over X to two. This four-form has a dual field theory interpretation as a superpotential. The relation between A and ψ fixes the normalization of the gauge field action.

In addition to this perturbed metric, the RR four-form F_4 is also perturbed by A:

$$F_4 = \frac{3r^2}{L^3}d^3x \wedge dr - 4L^3(\star_4 dA) \wedge \omega. \qquad \text{(C3)}$$

Here $d^3x = dx^0 \wedge dx^1 \wedge dx^2$, and $\star_4$ is the Hodge dual in the AdS_4 directions only. With $A = 0$, F_4 can be thought of as the electric flux from a stack of M2-branes spanning the x^i coordinates.

With these formulae for F_4 and ds^2 in hand, we can normalize the gauge field. The 11-dimensional supergravity action is

$$\frac{1}{2\kappa^2}\int d^{11}x\sqrt{-g}R - \frac{1}{4\kappa^2}\int\left(F_4 \wedge \star F_4 + \frac{1}{3}A_3 \wedge F_4 \wedge F_4\right). \qquad \text{(C4)}$$

The first two terms both give contributions to $|F|^2$, where $F = dA$. In particular, in making A nonzero, the Ricci scalar becomes

$$R = \tilde{R} - \frac{L^2}{4}|F|^2 + \frac{21}{2L^2}, \qquad \text{(C5)}$$

where $|F|^2 = F^{AB}F_{AB}$ and $\tilde{R}$ is the scalar curvature in the AdS_4 directions. Meanwhile, the four-form produces a term of the form

$$F_4 \wedge \star F_4 = -\left(\frac{9}{L^2} + \frac{3}{2}L^2|F|^2\right)\sqrt{-g}d^{11}x. \qquad \text{(C6)}$$

We cannot simply reduce the 11-dimensional action to an effective four-dimensional action as can be seen from the form of R and $|F_4|^2$. Combining (C5) and (C6) in (C4) leads to a Maxwell term $|F|^2$ of the wrong sign. The reason Kaluza-Klein reduction does not commute with computing the equations of motion is related to the fact that the Bianchi identity $dF_4 = 0$ imposes the equation of motion $d \star F = 0$ on the gauge field.

Instead, we must reduce the 11-dimensional equations of motion and from the effective four-dimensional equations of motion reconstruct a four-dimensional action. Along with Maxwell's equations for F, the 11-dimensional equations of motion reduce to

$$R_{MN} = 2L^2\left(F_M{}^P F_{NP} - \frac{1}{4}g_{MN}|F|^2\right) - \frac{3}{L^2}g_{MN}, \qquad \text{(C7)}$$

which can be obtained from the four-dimensional action

$$S_{\rm eff} = \frac{1}{2\kappa_4^2}\int d^4x\sqrt{-g}\left(\tilde{R} - L^2F_{MN}F^{MN} + \frac{6}{L^2}\right). \qquad \text{(C8)}$$

[16] The relation to R in the body of the paper is $2L = R$.

Assuming that the four and 11-dimensional gravitational couplings are related by the volume of the compact manifold X,

$$\frac{1}{2\kappa_4^2} = \frac{(2L)^7 \,\mathrm{Vol}(X)}{2\kappa^2},$$

and using the standard normalization for κ (3.3), we find that the action for the gauge field becomes

$$-\frac{\sqrt{2}N^{3/2}}{2^3\pi^5}\,\mathrm{Vol}(X)\int d^4x\sqrt{-g_4}|F|^2. \tag{C9}$$

The volume of a seven-sphere is $\mathrm{Vol}(S^7) = \pi^4/3$.

While in the case of more highly symmetric spaces, the R-symmetry gauge field transforms under a larger group, based on the underlying Sasaki-Einstein structure, this U(1) subgroup is in some sense the most geometrically natural.

We conclude this section by explaining, for the case of S^7, which U(1) subgroup of SO(8) we have extracted. Earlier, we stated that the U(1) is normalized in reference to a holomorphic four-form on the cone over the Sasaki-Einstein space. For S^7, the cone is $\mathbb{C}^4$, and the four-form $\Omega = dX_1 \wedge dX_2 \wedge dX_3 \wedge dX_4$ where the X_a are complex coordinates on $\mathbb{C}^4$. Giving Ω R-charge two means each X_a will have R-charge one-half and will transform under the U(1) group action as $X_a \to e^{i\alpha/2}X_a$ for some phase angle α which runs from zero to 2π.

The Lie algebra for SO(8) has four generators λ_a in its Cartan subalgebra which we can choose to act on the X_a as $\exp(i\alpha\lambda_a)(X_b) = \delta_{ab}e^{i\alpha/2}X_a$. With this normalization, $\mathrm{tr}\lambda_a\lambda_b = \frac{1}{2}\delta_{ab}$. Comparing with the action of our special U(1) subgroup, we see that our U(1) Lie algebra element λ is a sum of the λ_a: $\lambda = \sum_a \lambda_a$. Thus, $\mathrm{tr}\lambda^2 = 2$.

APPENDIX D: ANALYTIC SOLUTION

At zero momentum, the mode equation (3.19) for $M(u) \equiv M_{yy}(u)$ takes the form

$$f(u)\partial_u[f(u)\partial_u M(u)] + w^2 M(u) = 0, \tag{D1}$$

with $f(u) = 1 - u^3$. By introducing a new coordinate $z = \int_0^u d\tilde{u}/f(\tilde{u})$, the equation simplifies,

$$\partial_z^2 M(z) + w^2 M(z) = 0, \tag{D2}$$

with the boundary condition $M(z=0) = 1$ at the boundary, and the outgoing condition at the horizon $z = \infty$. The solution is

$$M(z) = e^{iwz}. \tag{D3}$$

That it corresponds to outgoing waves can be seen from the fact that in the function $e^{-i\omega(t-z)}$ the wave front moves toward larger z, i.e. closer to the horizon as t increases. Therefore we find

$$M(u) = \exp\left[iw\int_0^u \frac{d\tilde{u}}{f(\tilde{u})}\right]. \tag{D4}$$

The leading asymptotics for u near zero is $M(u) = 1 + iwu$. From the AdS/CFT prescription (3.14) we immediately find

$$C_{yy}(w,0) = \Pi^T(w,0) = \Pi^L(w,0) = -i\chi w. \tag{D5}$$

This agrees with the result for conductivity in Sec. III C, as it should.

APPENDIX E: GAUGE FIELD WITH A DILATON

Consider a U(1) gauge field on a four-dimensional manifold M with an action of the form

$$S = -\frac{1}{2g_{4\mathrm{D}}^2}\int_M e^{-2\phi}F\wedge\star F. \tag{E1}$$

There are a number of interesting $2+1$ dimensional field theories which have a dual R-symmetry gauge field of this type—for example the M2-brane theory at finite R-charge chemical potential and the D2-branes in type IIA string theory. Here F is the two-form gauge field, ϕ a dilaton like scalar, and $g_{4\mathrm{D}}$ the coupling. The Maxwell equations can be written elegantly as $dF = 0$ and $d\star e^{-2\phi}F = 0$. There is an equivalent S-dual theory where the roles of F and $\tilde{F} \equiv \star e^{-2\phi}F$ are interchanged and we send $g_{4\mathrm{D}}e^{\phi} \to \tilde{g}_{4\mathrm{D}}e^{\tilde{\phi}} \equiv 1/(g_{4\mathrm{D}}e^{\phi})$:

$$S = -\frac{1}{2\tilde{g}_{4\mathrm{D}}^2}\int_M e^{2\phi}\tilde{F}\wedge\star\tilde{F}. \tag{E2}$$

The point we would like to emphasize is that when ϕ is a constant, the theory is almost self-dual in the sense that the equations of motion for F and $\tilde{F}$ are identical. When ϕ is not a constant, the equations of motion for F and $\tilde{F}$ are identical up to sending $\phi \to -\phi$.

We would like to investigate the consequences of this duality in the context of the AdS/CFT correspondence where this gauge field is interpreted as a bulk field corresponding to some global U(1) symmetry on a $2+1$ dimensional boundary theory. To this end, we assume the metric takes the diagonal form

$$ds^2 = -g_{tt}(u)dt^2 + du^2 + g_{xx}(u)(dx^2 + dy^2), \tag{E3}$$

where the metric components are only radially dependent on a coordinate we call u. By diffeomorphism invariance, we can always set $g_{uu} = 1$. The boundary is taken to be located at $u = 0$ and the interior for $u > 0$ with a horizon at $u = u_h > 0$. We will assume that as $u \to 0$, $-g_{tt} \sim g_{xx} \sim c^2/u^\alpha$ where $\alpha > -2$.

We will calculate two-point functions of the U(1) current J corresponding to this global symmetry. Introducing a vector potential $F = dA$, the retarded two-point function can be found using the method described in Sec. III B. Namely, one looks for the solution to the field equation for

A_ν of the form vector potential of the form

$$A_\nu(x,t,u) = e^{ip\cdot x} M_\nu{}^\mu(p,u) A^0_\mu(p), \tag{E4}$$

where $M_\nu{}^\mu(p,u)$ satisfies the radial component of the equation of motion for A_ν. Furthermore, $M_\nu{}^\mu(p,0) = \delta^\mu_\nu$ and $M_\nu{}^\mu$ satisfies the outgoing boundary condition at the horizon $u = u_h$. If the kinetic term for A_μ can be written as

$$-\frac{1}{2g_{4\mathrm{D}}^2}\int du\, d^3x G(u)(A'_\mu)^2 \tag{E5}$$

then

$$C^{\mu\nu}(k) = -\frac{1}{g_{4\mathrm{D}}^2}\lim_{u\to 0} G(u)\frac{\partial}{\partial u} M^{\mu\nu}(p,u). \tag{E6}$$

In particular, we take $A^\mu(x,t,u)$ to satisfy the equation of motion

$$\partial_A[e^{-2\phi}\sqrt{-g}g^{AB}g^{CD}(A_{B,D} - A_{D,B})] = 0, \tag{E7}$$

and fix a radial gauge $A_u = 0$. We also choose $p^\mu = (\omega, k, 0)$. In this gauge, the equation of motion for A_y becomes

$$\partial_u[e^{-2\phi}\sqrt{-g}g^{xx}A'_y] - k^2\sqrt{-g}(g^{xx})^2 e^{-2\phi}A_y - \omega^2\sqrt{-g}g^{xx}g^{tt}e^{-2\phi}A_y = 0. \tag{E8}$$

Because of the constraint on α, the near boundary behavior of A_y ($u \sim 0$) is governed by an expansion of the form

$$M^{yy}(p,u) = (1 + \mathcal{O}(u)) + u^{1+\alpha/2}\mathcal{B}(1 + \mathcal{O}(u)). \tag{E9}$$

In the case where α is an even integer, the two series will overlap, leading to logarithmic terms in the first series, which complicate the story but should not alter it in any fundamental way. The constant $\mathcal{B}$ is a complicated function of ω and k which is determined by fixing outgoing boundary conditions at the horizon $u = u_h$. For A_y, the function $G(u)$ in (E5) is $\sqrt{-g}g^{xx}$ which near the boundary scales as $cu^{-\alpha/2}$ where c depends on the precise form of our metric. By absorbing $\phi(0)$ into the value of $g_{4\mathrm{D}}$, we can choose $\phi(0) = 0$. From this expansion and the form of $G(u)$, clearly

$$C^{yy} = \frac{1}{g_{4\mathrm{D}}^2}(1 + \alpha/2)c\mathcal{B}. \tag{E10}$$

In our gauge, A_y can be reinterpreted as a radial magnetic field, $B_u = F_{xy} = -ikA_y$. By electric-magnetic duality, replacing ϕ with $-\phi$, the equation of motion for B_u (E8) must be the same as the equation of motion for $E_u \equiv -(\star e^{-2\phi}F)_{xy} = \sqrt{-g}g^{tt}e^{-2\phi}A'_t$:

$$\partial_u[e^{2\phi}\sqrt{-g}g^{xx}E'_u] - k^2\sqrt{-g}(g^{xx})^2 e^{2\phi}E_u - \omega^2\sqrt{-g}g^{xx}g^{tt}e^{2\phi}E_u = 0. \tag{E11}$$

We thus know that E_u has the near boundary expansion

$$E_u = E^0_u e^{ip\cdot x}[(1 + \mathcal{O}(u)) + u^{1+\alpha/2}\tilde{\mathcal{B}}(1 + \mathcal{O}(u))]. \tag{E12}$$

The tilde over $\mathcal{B}$ indicates it was derived from (E8) having replaced ϕ with $-\phi$. In the case $\phi = \text{const}$, $\mathcal{B} = \tilde{\mathcal{B}}$. We now use Gauss's law to constrain the boundary behavior of E^0_u. The equation of motion following from taking the index $C = t$ in (E7) is

$$(\sqrt{-g}g^{tt}e^{-2\phi}A'_t)' - k\sqrt{-g}g^{tt}g^{xx}e^{-2\phi}(\omega A_x + kA_t) = 0. \tag{E13}$$

From the near boundary behavior, we find that

$$E^0_u(1 + \alpha/2)\tilde{\mathcal{B}} = -\frac{k}{c}(\omega A^0_x + kA^0_t). \tag{E14}$$

We can run a similar analysis of the component A'_x and construct a full boundary action. We find that

$$S_b = -\frac{1}{2g_{4\mathrm{D}}^2}\int_{\mathbb{R}^3} d^3x\Big[\mathcal{B}c(1 + \alpha/2)(A^0_y)^2 - \frac{1}{\tilde{\mathcal{B}}c(1 + \alpha/2)}(k^2(A^0_t)^2 + \omega^2(A^0_x)^2 + \omega k A^0_t A^0_x)\Big]. \tag{E15}$$

From this normalization, we conclude that the remaining two-point functions are

$$C^{tt} = -\frac{1}{g_{4\mathrm{D}}^2}\frac{k^2}{c(1 + \alpha/2)\tilde{\mathcal{B}}}, \tag{E16}$$

$$C^{xt} = -\frac{1}{g_{4\mathrm{D}}^2}\frac{k\omega}{c(1 + \alpha/2)\tilde{\mathcal{B}}}, \tag{E17}$$

$$C^{xx} = -\frac{1}{g_{4\mathrm{D}}^2}\frac{\omega^2}{c(1 + \alpha/2)\tilde{\mathcal{B}}}. \tag{E18}$$

In the special case where ϕ is a constant and hence $\tilde{\mathcal{B}} = \mathcal{B}$, we find that

$$C^{tt}C^{yy} = -\frac{1}{g_{4\mathrm{D}}^4}k^2; \qquad C^{xt}C^{yy} = -\frac{1}{g_{4\mathrm{D}}^4}k\omega; \qquad C^{xx}C^{yy} = -\frac{1}{g_{4\mathrm{D}}^4}\omega^2. \tag{E19}$$

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