Conformal window of SU(N) gauge theories with fermions in higher dimensional representations

Dennis D. Dietrich
Institute for Theoretical Physics, Heidelberg University, Heidelberg, Germany and The Niels Bohr Institute, Copenhagen, Denmark

Francesco Sannino
Department of Physics and Chemistry, University of Southern Denmark, Odense, Denmark and The Niels Bohr Institute, Copenhagen, Denmark

(Received 27 November 2006; published 26 April 2007)

We study the phase diagram as a function of the number of colors and flavors of asymptotically free nonsupersymmetric theories with matter in higher-dimensional representations of arbitrary SU(N) gauge groups. Since matter in higher-dimensional representations screens more than in the fundamental a general feature is that a lower number of flavors is needed to achieve a near-conformal theory.

DOI: 10.1103/PhysRevD.75.085018 PACS numbers: 11.30.Qc, 11.10.Hi, 12.60.Fr, 12.60.Nz

I. INTRODUCTION

In this article we portray the phase diagram, as a function of the number of colors and the number of flavors, of nonsupersymmetric gauge field theories with matter in higher-dimensional representations of SU(N) gauge groups. Strongly interacting non-Abelian theories generally feature a coupling constant which varies with the energy scale; it *runs*. This is caused by the antiscreening due to the charged gauge bosons (gluons). This antiscreening is competing with the screening contribution from matter fields. There is another feature of strongly interacting theories which comes into play in this context, that is chiral symmetry breaking. This formation of a $\langle\bar{\psi}\psi\rangle$ condensate below the chiral symmetry breaking scale renders the fermions massive and results in their decoupling from the dynamics. In that case the antiscreening of the gluons becomes more important again. With sufficient matter content an infrared fixed point of the coupling constant can be reached before chiral symmetry breaking is triggered, that is there exists a conformal phase. For a number of flavors slightly below the value for which the conformal phase is present, the coupling constant is evolving slowly. It stays almost constant over a range of energy scales. One says it *walks* instead of *runs*. These features will be discussed in more detail below.

We are particularly interested in these walking theories: It is possible that, besides Quantum Chromodynamics (QCD), new strongly interacting theories will emerge when exploring the unknown territory beyond the standard model (SM) of particle interactions. For example, to avoid unnaturally large quantum corrections to the mass scale of the electroweak theory arising in the Higgs sector of the SM one can replace the elementary Higgs by a strongly coupled sector. This approach has been named technicolor [1]. The generation of the masses of the standard model fermions requires extended technicolor interactions, which are consistent with the observed amount of flavor-changing neutral currents and lepton number violation for technicolor theories possessing a sufficient amount of walking [2–6]. The simplest of such models which also passes the electroweak precision tests (like, for example, the experimental bounds on the oblique parameters) requires fermions in higher-dimensional representations of the technicolor gauge group [7–10]. Matter in a higher-dimensional representation screens more strongly than matter in the fundamental representation. Hence a smaller number of flavors is required in order to achieve a given amount of screening. In the recent past we have studied the gauge dynamics of fermions transforming according to the two-index representations of SU(N). Here we extend the analysis to a generic asymptotically free gauge theory with fermions in various higher-dimensional representations of the SU(N) group. Some general structures are unveiled.

After a general and fairly complete analysis (Sec. II) we explore the various ways these theories can be used to break the electroweak symmetry (Sec. III). Among other things we compute the left-right vector correlator in perturbation theory. Thanks to the results found in [11–13] and corroborated more recently by AdS/QCD-based computations [14] we know that near the conformal window the perturbative value of the oblique parameter S provides a conservative estimate, that is an upper bound. In Sec. IV, we summarize the results.

II. PHASE DIAGRAM

A. Review of the basic tools

We start our journey by reviewing the two-loop β function for a generic non-Abelian gauge theory with fermionic matter in a given representation R of SU(N) [15][1]:

[1]Notice the different normalization as compared to, for example [7]. Conveniently, below, the quadratic Casimir operator will only assume integer values. The numbers of flavors N_f^{I}, N_f^{II}, and N_f^{III}, however, are physical quantities and therefore independent of the normalization.

$$\beta(g) = -\beta_0 \frac{g^3}{(4\pi)^2} - \beta_1 \frac{g^5}{(4\pi)^4}, \tag{1}$$

$$2N\beta_0 = \tfrac{11}{3}C_2(\mathrm{G}) - \tfrac{4}{3}T(\mathrm{R}), \tag{2}$$

$$(2N)^2\beta_1 = \tfrac{34}{3}C_2^2(\mathrm{G}) - \tfrac{20}{3}C_2(\mathrm{G})T(\mathrm{R}) - 4C_2(\mathrm{R})T(\mathrm{R}). \tag{3}$$

$C_2(\mathrm{R})$ stands for the quadratic Casimir operator of the representation R,

$$2NX^a_{\mathrm{R}}X^a_{\mathrm{R}} = C_2(\mathrm{R})\mathbb{1}, \tag{4}$$

where X^a_{R} are the generators in the representation R. $C_2(\mathrm{G})$ is the quadratic Casimir operator of the adjoint representation. $T(\mathrm{R})$ is the trace normalization factor for the representation R. $T(\mathrm{R})$ is connected to the quadratic Casimir operator, $C_2(\mathrm{R})$, by [16]

$$N_f C_2(\mathrm{R})d(\mathrm{R}) = T(\mathrm{R})d(\mathrm{G}), \tag{5}$$

where $d(\mathrm{R})$ denotes the dimension of the representation, R, and, accordingly, $d(\mathrm{G})$ the dimension of the adjoint representation G, that is the number of generators. N_f stands for the number of flavors.

A theory's loss of asymptotic freedom shows itself in a change of sign of the first coefficient, β_0, of the β function. Hence, the number of flavors, $N_f^{\mathrm{I}}[\mathrm{R}]$, above which the loss occurs satisfies $\beta_0[N_f^{\mathrm{I}}(\mathrm{R})]\stackrel{!}{=}0$. Therefore, from Eqs. (2) and (5) we obtain

$$N_f^{\mathrm{I}}[\mathrm{R}] := \frac{11}{4}\frac{d(\mathrm{G})C_2(\mathrm{G})}{d(\mathrm{R})C_2(\mathrm{R})}, \tag{6}$$

which is thus proportional to the ratio of the second order indices, $d(\mathrm{R})C_2(\mathrm{R})$, of the adjoint representation G and the representation R, respectively. For matter in the adjoint representation, R = G, the number of flavors above which asymptotic freedom is lost is independent of the number of colors and equal to $\frac{11}{4}$.

Already for a smaller number of flavors than the one at which asymptotic freedom is lost, the theory develops a Banks-Zaks infrared fixed point [17]. It appears as soon as the second coefficient of the β function, β_1, changes sign, that is at $\beta_1[N_f^{\mathrm{III}}(\mathrm{R})]\stackrel{!}{=}0$. Equations (3) and (5) then lead to

$$N_f^{\mathrm{III}}[\mathrm{R}] = \frac{d(\mathrm{G})C_2(\mathrm{G})}{d(\mathrm{R})C_2(\mathrm{R})}\frac{17C_2(\mathrm{G})}{10C_2(\mathrm{G}) + 6C_2(\mathrm{R})}. \tag{7}$$

This fixed point will not come into play if, before it is reached, the coupling constant α becomes so large that chiral symmetry breaking is triggered.

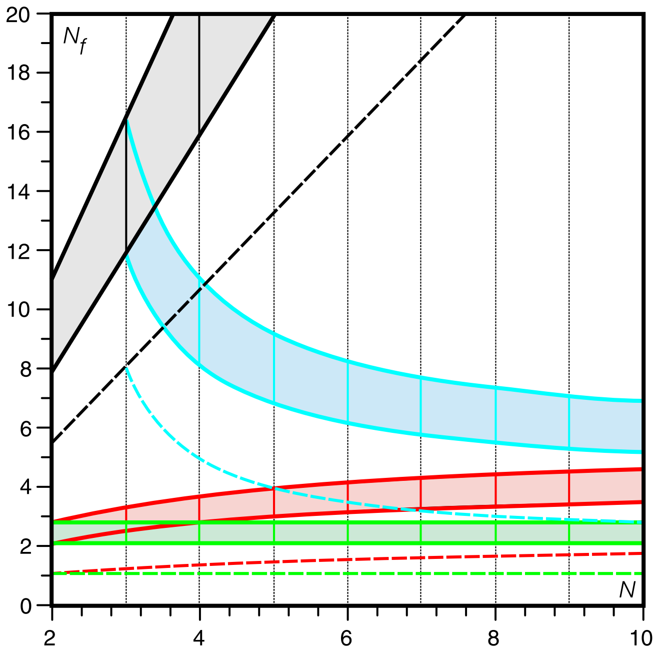
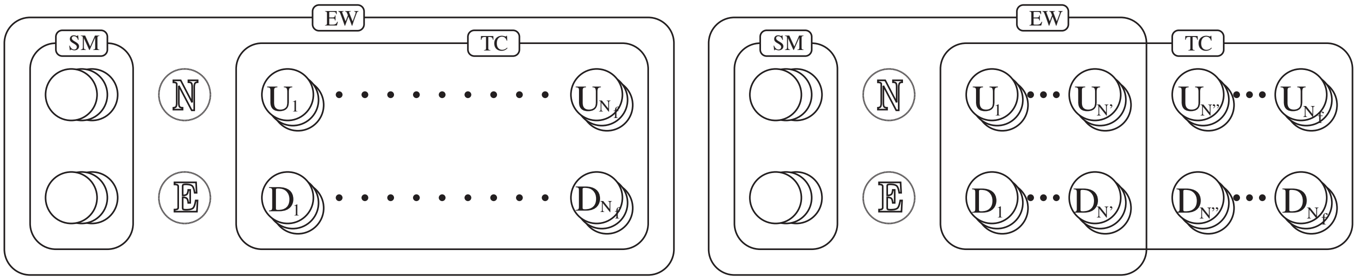


FIG. 1 (color online). Phase diagram for theories with fermions in the (from top to bottom in the plot): (i) fundamental representation (gray), (ii) two-index antisymmetric (blue), (iii) two-index symmetric (red), (iv) adjoint representation (green) as a function of the number of flavors and the number of colors. The shaded areas depict the corresponding conformal windows. The upper solid curve represents $N_f^{\mathrm{I}}[R(N)]$ (loss of asymptotic freedom), the lower $N_f^{\mathrm{II}}[R(N)]$ (loss of chiral symmetry breaking). The dashed curves show $N_f^{\mathrm{III}}[R(N)]$ (existence of a Banks-Zaks fixed point). Note how consistently the various representations merge into each other when, for a specific value of N, they are actually the same representation.

In this context, in order to determine the number of flavors $N_f^{\mathrm{II}}[\mathrm{R}]$ above which the theory becomes conformal, we employ the criterion derived in [18,19] and follow the discussion in [7]: Whether chiral symmetry can be broken depends on the relative order (with respect to the energy scale) of the values of the coupling constant at which a conformal fixed point is encountered, α_*, and where chiral symmetry breaks, α_c. When following the renormalization group flow from higher to lower energies the coupling α keeps growing as long as the β function remains finite and negative. If the value α_c is reached, the fermions decouple, their screening effect is lost, and only the antiscreening of the gauge bosons remains. In that case the conformal fixed point cannot be reached. If, on the other hand, α_* is met before chiral symmetry is broken, the coupling freezes (because the β function becomes zero). Now, in turn, the value α_c of the coupling required for chiral symmetry breaking cannot be attained.

In [18,19], the critical value α_c of the coupling constant for which chiral symmetry breaking occurs is defined as the value for which the anomalous dimension of the quark mass operator becomes unity, $\gamma\stackrel{!}{=}1$. According to [18], in



(a) Technicolor, fully gauged (b) Technicolor, partially gauged

FIG. 2. Technicolor models. The boxes depict under which gauge groups the different particles transform. The box headlined "SM" represents all standard model particles (excluding an elementary Higgs). N and E stand for a fourth family of leptons, which may or may not have to be included in order to evade a topological Witten anomaly. They have to be included, if the number of techniquark families transforming under the electroweak symmetry times the number of color degrees of freedom is an odd number. Left panel: In fully gauged technicolor models, all techniquarks transform under the electroweak symmetry. Right panel: In partially gauged technicolor a part of the techniquarks are electroweak singlets. It is conceivable that only one (minimal guaging) or several (nonminimal gauging) families of techniquarks carry electroweak charges. The latter set-up may be an alternative cure from the Witten anomaly not involving additional leptons.

ladder approximation the critical value of the coupling is given by[2]

$$\alpha_c = \frac{2\pi N}{3C_2(\mathrm{R})}. \tag{8}$$

Compared to that, the two-loop fixed point value of the coupling constant reads [7]

$$\frac{\alpha^*}{4\pi} = -\frac{\beta_0}{\beta_1}. \tag{9}$$

For a fixed number of colors the critical number of flavors for which the order of α_* and α_c changes is defined by imposing $\alpha_* \stackrel{!}{=} \alpha_c$, and is given by

$$N_f^{\mathrm{II}}[\mathrm{R}] = \frac{d(\mathrm{G})C_2(\mathrm{G})}{d(\mathrm{R})C_2(\mathrm{R})} \frac{17C_2(\mathrm{G}) + 66C_2(\mathrm{R})}{10C_2(\mathrm{G}) + 30C_2(\mathrm{R})}. \tag{10}$$

In order to evaluate the above expressions and throughout the article we use the Dynkin indices (the Dynkin labels of the highest weight of an irreducible representation) to uniquely characterize the representations and determine the relevant coefficients. We summarize, for the reader's convenience, the relevant formulae in the Appendix A.

B. Classification of asymptotically free theories

As we increase the dimension of the representation of the matter fields the screening effect of the matter compensates more and more the antiscreening effect of the gauge bosons. Eventually one loses asymptotic freedom. Here we will render quantitative this fact by constructing all theories which are asymptotically free with at least two Dirac flavors.

For two flavors the fundamental, adjoint, as well as two-index symmetric and antisymmetric representations remain asymptotically free for any number of colors. We will show below that there are no exceptions from this rule for more than nine colors. Apart from theories based on these representations the only remaining variants are the three-index antisymmetric representation which is asymptotically free with two flavors for six to nine colors[3] and the four-index antisymmetric with two flavors for eight colors. For $N \leq 9$ all such theories are listed in Table I.

That at a large number of colors only the fundamental, adjoint, two-index symmetric, and two-index antisymmetric representations survive is due to the fact that they are the only ones whose dimension grows quadratically or more slowly with the number of colors. It can thus be compensated by the quadratic growth of the dimension of the adjoint representation in the expressions for N_f^{I}. For any given representation the quadratic Casimir operators grow quadratically with (large) N and their ratio for different representations goes to a constant.

1. *Only four remaining representations for* $N > 9$

In this subsection we prove that

There exist no asymptotically free theories with at least two flavors and ten or more colors which are not contained in the fundamental, adjoint, or the two-index representations.

The utilized methods are general and can be applied to different numbers of Dirac flavors. We will consider also

[2]We investigate here whether chiral symmetry breaking occurs in a *walking* theory. Thus, corrections to the ladder approximation like vertex corrections, which take into account the change of the coupling constant, are small. Alternative methods for calculating the conformal window have been provided, e.g. in [5,17,20,21].

[3]For a number of colors less than six the three-index antisymmetric always coincides with another representation with a smaller number of indices.

TABLE I. Complete list of asymptotically free SU(N) theories with at least one family of fermions and up to nine colors. λ_* [Eq. (18)] and πS [Eq. (17)] are calculated for $N_f < N_f^{II}$ and even.

R	$\bar{\text{R}}$	N_f^{I}	N_f^{II}	N_f^{III}	πS	λ_*
(1)	≡	11	$7\frac{73}{85}$	$5\frac{27}{49}$	1	3.229
(2)	≡	$2\frac{3}{4}$	$2\frac{3}{40}$	$1\frac{1}{16}$	$\frac{1}{2}$	1124.
(10)	(01)	$16\frac{1}{2}$	$11\frac{32}{35}$	$8\frac{1}{19}$	$2\frac{1}{2}$	10.33
(20)	(02)	$3\frac{3}{10}$	$2\frac{163}{325}$	$1\frac{28}{125}$	1	11.29
(11)	≡	$2\frac{3}{4}$	$2\frac{3}{40}$	$1\frac{1}{16}$	$1\frac{1}{3}$	1124.
(100)	(001)	22	$15\frac{361}{385}$	$10\frac{126}{205}$	$4\frac{2}{3}$	20.70
(200)	(002)	$3\frac{2}{3}$	$2\frac{82}{105}$	$1\frac{71}{201}$	$1\frac{2}{3}$	5.865
(010)	≡	11	$8\frac{12}{115}$	$4\frac{52}{55}$	4	127 359.
(101)	≡	$2\frac{3}{4}$	$2\frac{3}{40}$	$1\frac{1}{16}$	$2\frac{1}{2}$	1124.
(1000)	(0001)	$27\frac{1}{2}$	$19\frac{58}{61}$	$13\frac{32}{161}$	$7\frac{1}{2}$	35.90
(2000)	(0002)	$3\frac{13}{14}$	$2\frac{747}{763}$	$1\frac{662}{1463}$	$2\frac{1}{2}$	4.284
(0100)	(0010)	$9\frac{1}{6}$	$6\frac{191}{237}$	$3\frac{514}{537}$	5	29.00
(1001)	≡	$2\frac{3}{4}$	$2\frac{3}{40}$	$1\frac{1}{16}$	4	1124.
(10000)	(00001)	33	$23\frac{283}{295}$	$15\frac{123}{155}$	11	57.57
(20000)	(00002)	$4\frac{1}{8}$	$3\frac{33}{260}$	$1\frac{53}{100}$	$3\frac{1}{2}$	2.519
(01000)	(00010)	$8\frac{1}{4}$	$6\frac{3}{20}$	$3\frac{21}{44}$	$7\frac{1}{2}$	5146.
(00100)	≡	$5\frac{1}{2}$	$4\frac{18}{145}$	$2\frac{14}{61}$	$6\frac{2}{3}$	2198.
(10001)	≡	$2\frac{3}{4}$	$2\frac{3}{40}$	$1\frac{1}{16}$	$5\frac{5}{6}$	1124.
(100000)	(000001)	$38\frac{1}{2}$	$27\frac{584}{605}$	$18\frac{125}{317}$	$30\frac{1}{3}$	87.67
(200000)	(000002)	$\frac{77}{18}$	$3\frac{2294}{9495}$	$1\frac{2168}{3663}$	$4\frac{2}{3}$	3.061
(010000)	(000010)	$7\frac{7}{10}$	$5\frac{3186}{4225}$	$3\frac{356}{1825}$	7	4.050
(001000)	(000100)	$3\frac{17}{20}$	$2\frac{8707}{9650}$	$1\frac{1941}{3890}$	$5\frac{5}{6}$	4.421
(100001)	≡	$2\frac{3}{4}$	$2\frac{3}{40}$	$1\frac{1}{16}$	8	1124.
(1000000)	(0000001)	44	$31\frac{1537}{1585}$	$20\frac{828}{829}$	40	128.6
(2000000)	(0000002)	$4\frac{2}{5}$	$3\frac{1141}{3425}$	$1\frac{851}{1325}$	6	2.752
(0100000)	(0000010)	$7\frac{1}{3}$	$5\frac{829}{1695}$	$3\frac{7}{723}$	$9\frac{1}{3}$	5.329
(0010000)	(0000100)	$2\frac{14}{15}$	$2\frac{8738}{39975}$	$1\frac{1733}{15675}$	$9\frac{1}{3}$	52.90
(1000001)	≡	$2\frac{3}{4}$	$2\frac{3}{40}$	$1\frac{1}{16}$	$10\frac{1}{2}$	1124.
(10000000)	(00000001)	$49\frac{1}{2}$	$35\frac{326}{335}$	$23\frac{106}{157}$	51	183.4
(20000000)	(00000002)	$4\frac{1}{2}$	$3\frac{516}{1265}$	$1\frac{1678}{2453}$	$7\frac{1}{2}$	2.526
(01000000)	(00000010)	$7\frac{1}{14}$	$5\frac{1016}{3395}$	$2\frac{1261}{1435}$	12	6.669
(10000001)	(10000001)	$2\frac{3}{4}$	$2\frac{3}{40}$	$1\frac{1}{16}$	$13\frac{1}{3}$	1124.

the case of one flavor. We start by analyzing the ordering of $N_f^{\text{I}}(\text{R})$ at fixed N for different representations, R.

Ordering $N_f^{\text{I}}(\text{R})$ *at fixed N.*—As all coefficients in Eqs. (A1) and (A2) are positive, we have, at a fixed number of colors, N (that is with a fixed number of Dynkin indices),

$$a_j \geq b_j \forall j \in \{1; \ldots ; N\} \Rightarrow C_2(\{a_j\}) \geq C_2(\{b_j\}) \quad \text{and} \quad d(\{a_j\}) \geq d(\{b_j\}). \tag{11}$$

Using Eq. (6) we find

$$a_j \geq b_j \forall j \in \{1; \ldots ; N\} \Rightarrow N_f^{\text{I}}(\{a_j\}) \leq N_f^{\text{I}}(\{b_j\}). \tag{12}$$

By increasing the value of any of the Dynkin indices the critical number of flavors decreases. The next step is to determine a complete set of theories with $N_f^{\text{I}}(R)$ just below two. They are to serve as universal bounds on $N_f^{\text{I}}(R)$ for all theories.

$n \geq 3$*-index antisymmetric representations.*—Let us start from the $n \geq 3$-index antisymmetric representations, $\text{A}_n(N)$. We notice from the corresponding expression for $N_f^{\text{I}}[\text{A}_n(N)]$ in Table II that for $N \geq 2n$,[4]

$$N_f^{\text{I}}[\text{A}_n(N)] > N_f^{\text{I}}[\text{A}_n(N+1)] \tag{13}$$

and

$$N_f^{\text{I}}[\text{A}_{n-1}(N)] > N_f^{\text{I}}[\text{A}_n(N)]. \tag{14}$$

We deduce that N_f^{I} shrinks if the number of colors, N, or the number of indices is increased as long as $N \geq 2n$.

Therefore, given a reference value $N_f^{\text{I}}[\text{A}_{n_0}(N_0)]$,

$$(n > n_0 \quad \text{or} \quad N > N_0) \Rightarrow N_f^{\text{I}}[\text{A}_n(N)] < N_f^{\text{I}}[\text{A}_{n_0}(N_0)]. \tag{15}$$

Picking as reference value $N_f^{\text{I}}[\text{A}_3(10)] = 1\frac{27}{28}$, we know that

$$N_f^{\text{I}}[\text{A}_n(N)] \leq 1\tfrac{27}{28} \;\forall\; n \geq 3, \qquad N \geq 10. \tag{16}$$

For a given number of colors, N, starting from all antisymmetric representations from $A_3(N)$ to $A_{N-3}(N)$, we can reach other representations by *increasing* the values of the Dynkin indices. The Dynkin representation of an n-index antisymmetric representation reads $(0\ldots010\ldots0)$, with 1 at the n-th position. Therefore, the only representations that cannot be reached in this way are $(a_1a_20\ldots0a_{N-2}a_{N-1})$ $\forall a_1, a_2, a_{N-2}, a_{N-1}$. In combination with the findings of the previous subsection this leads to the conclusion that only these latter representations may, but need not, have $N_f^{\text{I}}[\text{A}_n(N)] > 1\frac{27}{28}$.

S_3 *and (020…000).*—As can be seen from Table I we cannot obtain an equally low boundary from the fundamental, (100…000), and the two-index antisymmetric representation, (010…000). The same holds for the two-index symmetric representation, (200…000). Hence, we continue with the three-index symmetric representation, $\text{S}_3(N) = (300\ldots000)$, and the representation $\text{R}_1(N) := (020\ldots000)$. $N_f^{\text{I}}[\text{S}_3(N)]$ and $N_f^{\text{I}}[\text{R}_1(N)]$ are monotonically

[4] We are in the right-hand flank of Pascal's triangle. For $N < 2n$ the n-index antisymmetric representation is covered by an $(n' < n)$-index antisymmetric representation due to the symmetry of the result under conjugation, that is inversion of the order of the Dynkin indices. For example (0010), which is A_3, is, in this sense, equivalent to $\overline{(0010)} = (0100)$ which is A_2.

TABLE II. Characteristic quantities sorted after representations: F = fundamental, G = adjoint, S_n = n-index symmetric, A_n = n-index antisymmetric, $R_1 = (020\ldots0)$, $R_2 = (110\ldots0)$, $R_3 = (10\ldots010)$, $R_4 = (010\ldots010)$. Representations marked in boldface lead to theories that stay asymptotically free with at least two flavors for any number of colors.

	$d(\mathrm{R})$	$C_2(\mathrm{R})$	N_f^{I}	N_f^{II}	N_f^{III}
F	N	N^2-1	$\frac{11}{2}N$	$\frac{2}{5}N\frac{50N^2-33}{5N^2-3}$	$\frac{34N^3}{13N^2-3}$
G	N^2-1	$2N^2$	$2\frac{3}{4}$	$2\frac{3}{40}$	$1\frac{1}{16}$
S_n	$\frac{(N+n-1)!}{n!(N-1)!}$	$n(N-1)(N+n)$	$\frac{11N(n-1)!(N+1)!}{2(n+N)!}$	$\frac{2N(33n^2(N-1)+33nN(N-1)+17N^2)n!(N+1)!}{5n[3n^2(N-1)+3nN(N-1)+2N^2](n+N)!}$	$\frac{34N^3(n-1)!(N+1)!(n+N)!^{-1}}{3n^2(N-1)+3nN(N-1)+10N^2}$
$\mathbf{S}_2$	$\frac{N(N+1)}{2}$	$2(N-1)(N+2)$	$\frac{11}{2}\frac{N}{N+2}$	$\frac{N}{N+2}\frac{83N^2+66N-132}{20N^2+15N-30}$	$\frac{17N^3}{(N+2)(8N^2+3N-6)}$
S_3	$\frac{N(N+1)(N+2)}{6}$	$3(N-1)(N+3)$	$\frac{11N}{(N+2)(N+3)}$	$\frac{4N(-297+2N(99+58N))(N+1)!}{5(-27+N(18+11N))(N+3)!}$	$\frac{68N^3(N+1)!}{(-27+N(18+19N))(N+3)!}$
A_n	$\frac{N!}{n!(N-n)!}$	$n(N-n)(N+1)$	$\frac{11}{2}N\binom{N-2}{n-1}^{-1}$	$\binom{N-2}{n-1}^{-1}\frac{(34+66n)N^3-66n(n-1)N^2-66n^2N}{(10+15n)N^2-15n(n-1)N-15n^2}$	$\binom{N-2}{n-1}^{-1}\frac{34N^3}{10N^2-3n^2(N+1)+3nN(N+1)}$
$\mathbf{A}_2$	$\frac{N(N-1)}{2}$	$2(N-2)(N+1)$	$\frac{11}{2}\frac{N}{N-2}$	$\frac{N}{N-2}\frac{83N^2-66N-132}{20N^2-15N-30}$	$\frac{17N^3}{(N-2)(8N^2-3N-6)}$
A_3	$\frac{N(N-1)(N-2)}{6}$	$3(N-3)(N+1)$	$\frac{11N}{(N-2)(N-3)}$	$\frac{4N}{(N-2)(N-3)}\frac{116N^2-198N-297}{55N^2-90N-105}$	$n=3$ in A_n above
A_4	$\frac{N(N-1)(N-2)(N-3)}{24}$	$4(N-4)(N+1)$	$\frac{33N}{(N-2)(N-3)(N-4)}$	$\frac{6N}{(N-2)(N-3)(N-4)}\frac{149N^2-396N-528}{35N^2-90N-120}$	$n=4$ in A_n above
R_1	$\frac{(N-1)N^2(N+1)}{12}$	$4(N^2-4)$	$\frac{33}{2(N^2-4)}$	$\frac{3(149N^2-528)}{5(7N^4-52N^2+96)}$	$\frac{51N^2}{11N^4-68N^2+96}$
R_2	$\frac{(N-1)N(N+1)}{3}$	$3(N^2-3)$	$\frac{11N}{2(N^2-3)}$	$\frac{2N(-297+116N^2)}{5(81-60N^2+11N^4)}$	$\frac{34N^3}{81-84N^2+19N^4}$
R_3	$\frac{(N+1)N(N-2)}{2}$	$(3N+1)(N-1)$	$\frac{11N}{(3N+1)(N-2)}$	$\frac{4N(-33-66N+116N^2)}{5(6+27N-N^2-73N^3+33N^4)}$	$\frac{68N^3}{6+27N-17N^2-113N^3+57N^4}$
R_4	$\frac{(N+1)N^2(N-3)}{4}$	$4N(N-1)$	$\frac{11}{2N(N-3)}$	$\frac{149N-132}{5N(N-3)(7N-6)}$	$\frac{17}{18-39N+11N^2}$

decreasing functions of the number of colors. This can be seen from the explicit expressions in Table II. As reference values, we can pick $N_f^{\mathrm{I}}[\mathrm{S}_3(3)] = 1\frac{1}{10}$ and $N_f^{\mathrm{I}}[\mathrm{R}_1(4)] = 1\frac{3}{8}$, which are both smaller than the previous reference value $N_f^{\mathrm{I}}[\mathrm{A}_3(10)] = 1\frac{27}{28}$ and are situated at a smaller number of colors as well. The representations of the set $\{(a_1a_20\ldots0a_{N-2}a_{N-1})\forall a_1, a_2, a_{N-2}, a_{N-1} \in \mathbb{N}\}$, which cannot be obtained by increasing the Dynkin indices either of $\mathrm{S}_3(N)$ or of $\mathrm{R}_1(N)$ or of their conjugates are given by $(a_1a_20\ldots0a_{N-2}a_{N-1}) \quad \forall \quad (a_1, a_{N-1} \in \{0;1;2\}$ and $a_2, a_{N-2} \in \{0;1\})$. This amounts to 36 combinations before making use of symmetry properties under conjugation.

(110...000), (100...010), and (010...010).—In the next step, we exploit information on the representations $\mathrm{R}_2(N) = (110\ldots000)$, $\mathrm{R}_3(N) = (100\ldots010)$, and $\mathrm{R}_4(N) = (010\ldots010)$ in order to set limits for most of the remaining representations. To this end, we repeat the same steps as before. First we check that N_f^{I} for these representations is a decreasing function of the number of colors. This can be seen directly from the corresponding expressions in Table II. Subsequently, we pick reference values for the three cases: $N_f^{\mathrm{I}}[\mathrm{R}_2(4)] = 1\frac{9}{13}$, $N_f^{\mathrm{I}}[\mathrm{R}_3(4)] = 1\frac{9}{13}$, and $N_f^{\mathrm{I}}[\mathrm{R}_4(4)] = 1\frac{3}{8}$, respectively. They are all smaller than $N_f^{\mathrm{I}}[\mathrm{A}_3(10)] = 1\frac{27}{28}$ and lie at a smaller number of colors. The limits set by these reference values cannot be exceeded for a larger number of colors in these representations. Finally, we eliminate all representations from the remaining 36 that can be generated by adding to the Dykin indices of R_2, R_3, or R_4. This leaves but F, G, A_2, and S_2 as well as their conjugates $\bar{\mathrm{F}}$, $\bar{\mathrm{A}}_2$, and $\bar{\mathrm{S}}_2$ (G is a real representation), that is seven representations of which four are independent.

Thereby we have shown that beyond ten colors no other representations lead to asymptotically free theories with two flavors but the fundamental, the adjoint, as well as the two-index symmetric and antisymmetric representations. Below ten colors the only exceptions are given by the three-index antisymmetric representation at six to nine colors and the four-index antisymmetric at eight colors. This we have checked by explicit calculation and again by making use of the fact that N_f^{I} increases if any Dynkin index is increased.

As $N_f^{\mathrm{I}} > N_f^{\mathrm{II}} > N_f^{\mathrm{III}}$ the above findings are handed down to the lower bound for the conformal window and the existence of a Banks-Zaks fixed point.

2. One flavor

When the requirement is weakened to asymptotic freedom with one flavor, all the above steps can be repeated after choosing new reference values: $N_f^{\mathrm{I}}[\mathrm{A}_3(16)] = \frac{88}{91}$, $N_f^{\mathrm{I}}[\mathrm{S}_3(5)] = \frac{55}{56}$, $N_f^{\mathrm{I}}[\mathrm{R}_1(5)] = \frac{11}{14}$, $N_f^{\mathrm{I}}[\mathrm{R}_2(7)] = \frac{77}{92}$, $N_f^{\mathrm{I}}[\mathrm{R}_3(6)] = \frac{33}{38}$, $N_f^{\mathrm{I}}[\mathrm{R}_4(5)] = \frac{11}{20}$. This means that for the one flavor limit, no exceptions are present from 16 colors onward.

C. Conformal window

Evaluating the expression (10) for the lower bound of the conformal window for all theories, which are asymptotically free with at least two flavors, leads to the values listed in the fourth column of Table I. For the fundamental, adjoint and two-index symmetric as well as antisymmetric representations there exist theories with at least two flavors, which have not yet entered the conformal phase. This remains true beyond nine colors. From the exceptions below ten colors, the three-index antisymmetric representation at nine colors and the four-index antisymmetric representations are already conformal with less than two flavors.

The minimal number of flavors necessary for a Banks-Zaks fixed point to appear as calculated from Eq. (7) is listed in the fifth column of Table I. For values beyond nine colors, the explicit expressions in Table II can be evaluated. The present analysis exhausts the phase diagram for gauge theories with Dirac fermions in arbitrary representations as function of the number of colors and flavors (see Fig. 1).

III. WALKING TECHNICOLOR

Here we will use our findings concerning asymptotically free theories with at least one doublet of fermions to identify all physically acceptable technicolor models. In technicolor models the dynamical breaking of the electroweak symmetry from $SU_L(2) \times U_Y(1)$ to $U_{em}(1)$ is not generated by an elementary Higgs particle like in the standard model, but by chiral symmetry breaking in an additional strongly interacting sector. It is made up of techniquarks transforming under the electroweak and an additional technicolor gauge group (see Fig. 2(a)). The scale at which the chiral symmetry of the technicolor sector breaks is chosen to be the electroweak scale. Three of the emergent Goldstone bosons are absorbed as longitudinal degrees of freedom of the electroweak gauge bosons, which thus become massive. The fermion masses are generated by embedding the electroweak and technicolor gauge groups in a larger extended technicolor gauge group. The gauge bosons of the extended technicolor model couple the fermions of the standard model to the techniquarks and their condensate, which renders the standard model fermions massive.

Like all other mechanisms for electroweak symmetry breaking, technicolor has to face constraints derived from experimental data. In the case of technicolor the two main aspects are additional contributions to the vacuum polarization of the electroweak gauge bosons (oblique parameters) and flavor-changing neutral currents as well as lepton number violation due to the extended technicolor dynamics. These issues have been discussed in great detail in the literature (see, for example [22,23]). Experimental data (see, for example [24,25]) tells us that the above mentioned contributions must be small. Here, let us only recall that flavor-changing neutral currents and lepton number violation are suppressed in walking technicolor theories, that is technicolor theories with nearly conformal dynamics. Through nonperturbative effects, quasiconformality also helps reducing the techniquarks' contribution to the oblique parameters [11–14,26]. (In the absence of quasiconformal dynamics the S parameter can be larger than its perturbative value.) On top of that, potential additional Goldstone bosons, beyond the three which are absorbed as the longitudinal degrees of freedom of the electroweak gauge bosons, become very heavy, thereby alleviating bounds set by them not having been detected to date. Therefore, candidates for realistic technicolor theories should feature quasiconformal dynamics and should contribute little to the oblique parameters already at the perturbative level. In what follows, we will quantify these criteria.

Already taking into account the experimental limits on the S-parameter [27] severely constrains the set of candidates. Perturbatively, it is given by

$$S = \frac{1}{6\pi}\frac{N_f}{2}d(\mathrm{R}). \tag{17}$$

The values for S are given in Table I. Drawing the line at $S < \pi^{-1}$—somewhat arbitrarily but in accordance with the experimental limits [24,25]—leaves us with three candidates which, characterized by their Dynkin indices are: (1) with six flavors, (2) with two flavors, and (20) with two flavors. Doubling the value of the cut on the S parameter ($S < 2\pi^{-1}$) would admit two more: (11) with two flavors and (200) with two flavors.

The estimate for the lower bound (critical number of flavors) of the conformal window is based on the point where the critical coupling and the fixed point value coincide. This critical number of flavors is, in general, not an even integer. A quasiconformal physical realization of a technicolor theory is, however, constructed from complete families of techniquarks.[5]

From the difference of the two scales, the amount of walking, that is the ratio of the scale can be estimated [8,11]

$$\lambda_* \approx \exp(\pi/\sqrt{\alpha_*/\alpha_c - 1}). \tag{18}$$

λ_* is the ratio of the scale from which onwards the coupling constant stays approximately constant divided by the scale for which it starts running again. For this walking mechanism to be effective it must typically cover several decades. Setting the cut at $\lambda_* > 10^3$ leaves (2) with two flavors (see Table I). [If the weaker bound on the S parameter is chosen also (11) with two flavors survives.] Weakening the requirement on the range of the walking to $\lambda_* > 10^2$ leads to no supplementary candidates.

[5]Generalizations with an odd number of Dirac or even Weyl flavors are conceivable. A corresponding example is given in Sec. III C.

Two flavors, SU(2), adjoint representation: (2)

The technicolor theory with two techniquarks in the two-index symmetric/adjoint representation of SU(2), that is (2), has been studied in [9,10,28] and found to be in good agreement with experimental constraints. It has to contain an additional family of leptons in order to avoid the topological Witten anomaly[6] [29]. This model has a rich phenomenology owing to the SU($2N_f = 4$) flavor symmetry in the unbroken phase, which is enhanced because the matter transforms under the adjoint representation, which is real. The breaking to SO(4) leads to nine Goldstone bosons, three of which become the longitudinal degrees of freedom of the electroweak gauge bosons. The remaining six are of technibaryonic nature and may be very massive due to the intense walking of the theory. They can contribute to dark matter as might one of the additional leptons. This always depends on the hypercharge assignment for the particles beyond the standard model, which here is not fixed totally by requiring the freedom from gauge anomalies. Since, in the present theory, the fermions transform under the adjoint representation, their technicolor can be neutralized directly by technigluons, which leads to potentially phenomenologically interesting states. They are expected to have a mass of the order of the confining scale of the theory. For more details on this particular candidate see [28].

Two flavors, SU(3), adjoint representation: (11)

The theory with two techniflavors in the adjoint representation of SU(3), (11), features a similarly enhanced flavor symmetry, SU($2N_f = 4$), whose breaking to SO(4) gives rise to a total of nine Goldstone bosons with the same consequences as in the two-color case. The theory does not have to contain additional leptons. Hence the hypercharge assignment is fixed such that the techniquarks carry half-integer electrical charges with opposite signs. All objects of two and more techniquarks can be technicolor neutral and the technicolor of even a single techniquark can be neutralized by technigluons. This is in direct analogy to the two-technicolor case.

A. Partially gauged technicolor

A small modification of the traditional technicolor approach, which neither involves additional particle species nor more complicated gauge groups, allows constructing several other viable candidates. It consists in letting only one doublet of techniquarks transform nontrivially under the electroweak symmetries with the rest being electroweak singlets, as already suggested in [9] and later used in [30] (see Fig. 2(b)). Still, all techniquarks transform under the technicolor gauge group. Thereby, perturbatively, only one techniquark doublet contributes to the oblique parameter which is thus kept to a minimum for theories which need more than one family of techniquarks to be quasiconformal. It is the condensation of that first electroweakly charged family that breaks the electroweak symmetry. Additionally, the number of ungauged techniquarks, in general, need not be even. In certain cases, this allows to come closer to conformality (see Sec III B 1). There exist several more partially gauged candidates with $S < \pi^{-1}$ (see Table I): (1) with seven flavors [instead of six fully gauged flavors], (10) with 11 flavors, (100) with 15 flavors, (010) with eight flavors, (1000) with 19 flavors, and (10000) with 23 flavors. Of these last-mentioned theories with techniquarks in the fundamental representation, (10...0), only those with four or more colors walk over more than two decades. Three decades could be reached from seven technicolors and 27 techniflavors onwards, which leads to a perturbative contribution to the S parameter of $\pi S \approx 1.2$ from the two electroweakly gauged technifermions. Therefore, the parametrically admissible models in the fundamental representation are rather nonminimalist.

The theory with eight techniquarks in the two-index antisymmetric representation of SU(4), (010), on the other hand, according to Eq. (18) walks over more than five decades.

The techniquarks which are uncharged under the electroweak gauge group are natural building blocks for components of dark matter.

Eight flavors, SU(4), two-index antisymmetric representation: (010)

Among the partially gauged cases the prime candidate is the theory with eight techniflavors in the two-index antisymmetric representation of SU(4). The techniquarks of one of the four families carry electroweak charges while the others are electroweak singlets. Gauge anomalies are avoided if the two electrically charged techniquarks possess half-integer charges. The technihadron spectrum can contain technibaryons made of only two techniquarks because in the two-index antisymmetric representation of SU(4) a singlet can already be formed in that case [$(010) \otimes (010) \to (000) \oplus (101) \oplus (020)$]. Otherwise technisinglets can also be formed from four techniquarks. All technihadrons formed from techniquarks without electrical charges can contribute to dark matter. Because of the special charge assignment of the electrically charged particles (opposite half-integer charges) certain combinations of those can also be contained in electrically uncharged technibaryons. For instance we can construct the following completely neutral technibaryon:

$$\epsilon_{t_1 t_2 t_3 t_4} Q_L^{t_1 t_2, f} Q_L^{t_3 t_4, f'} \epsilon_{f f'}, \qquad (19)$$

[6] It does not allow for an odd number of families of fermions to transform under an SU(2) gauge group [here the $SU_L(2)$ of the electroweak gauge group]. The one family of (2), however, corresponds to three [$\dim(2) = 3$] technicolor copies added to the previously even number of SM fermion families.

where $\epsilon_{ff'}$ saturates the $SU(2)_L$ indices of the two gauged techniquarks and the first antisymmetric tensor ϵ is summed over the technicolor indices. We have suppressed the spin indices. If there is no technibaryon asymmetry the cross section for annihilation would be too big and the present relic density would be negligible. In the presence of a technibaryon asymmetry, however, this particle would be another candidate for dark matter (different from those formed from electroweakly neutral techniquarks), hardly detectable in any earth based experiment [28].

As (010) is a real representation the model's flavor symmetry is enhanced to $SU(2N_f = 16)$,[7] which, when it breaks to SO(16), induces 135 Goldstone bosons.[8]

It is worth recalling that the center group symmetry left invariant by the fermionic matter is a Z_2 symmetry. Hence there is a well defined order parameter for confinement [31] which can play a role in the early Universe.

B. Beyond the prime candidate

In view of the uncertainty of the experimental limits for the oblique parameters (see, for example [32]) and the approximations made here to determine the degree of walking, not all the other theories, which above have not been identified as prime candidates for a realistic walking technicolor theory, should be considered as being ruled out. While the theories which, in this analysis, stood out as most favored, would, with the highest likelihood, survive an amelioration of the experimental limits and/or a refinement of the theoretical assessment, others might emerge as being viable candidates as well. For these reasons, we address some of those in the rest of this section. At the end, we will have addressed all settings listed in Table III.

1. Fundamental

Compared to the prime examples with matter in higher-dimensional representations, models with fundamental matter must have a comparatively large number of colors and flavors in order to feature a sufficient amount of walking. In the partially gauged approach their S parameter reaches π^{-1} for six colors. Phenomenologically, the models with fundamental techniquarks can be divided into two groups with an even, respectively, odd number of colors (=dimension). Technicolor singlets are always formed by N techniquarks.

[7] It is slightly explicitly broken by the electroweak interactions. This is, of course, always the case between the techniups and technidowns, but here, additionally, there is a difference, on the electroweak level, between gauged and ungauged techniquark families.

[8] Obviously the Goldstone bosons must receive a sufficiently large mass. This is usually achieved in extended technicolor. Still, they could be copious at LHC.

TABLE III. List of possible technicolor theories. The first two columns give the representation and the number of flavors, respectively. In the next column follows the symmetry breaking pattern. $[SU_L(N_f) \times SU_R(N_f)$ always breaks to $SU(N_f)$.] Thereafter, it is marked whether a global Witten anomaly has to be taken care of by adding a family of leptons or, where this is possible, by a less minimal partial weak gauging of the theory. Finally, the amount of walking and the perturbative S parameter are listed. The prime candidates, that is the theories with the largest amount of walking at an acceptable size for the oblique corrections are marked in boldface.

R	N_f	flavor symmetry	WA	P-G	$\lg\lambda_*$	πS[a]
(1)	7	$SU(14) \to Sp(14)$	no	yes	1.3	0.3[b]
(**2**)	2	$SU(4) \to SO(4)$	yes	no	3.1	0.5
(10)	11	$SU_L(11) \times SU_R(11)$	yes	yes[c]	1.8	0.5
(20)	2	$SU_L(2) \times SU_R(2)$	no	no	1.1	1.0
(11)	2	$SU(4) \to SO(4)$	no	no	3.1	1.3
(100)	15	$SU_L(15) \times SU_R(15)$	no	yes	2.2	0.7
(200)	2	$SU_L(2) \times SU_R(2)$	no	no	0.8	1.7
(**010**)	8	$SU(16) \to SO(16)$	no	yes	5.1	1.0
(101)	2	$SU(4) \to SO(4)$	yes	no	3.1	2.5
(1000)	19	$SU_L(19) \times SU_R(19)$	yes	yes[c]	2.4	0.8
(0100)	6	$SU_L(6) \times SU_R(6)$	no	yes	1.5	1.7
(10000)	23	$SU_L(23) \times SU_R(23)$	no	yes	2.7	1.0

[a] The contribution to S from additional leptons is not included. It is of the order $\pi S \approx 1/6$. The contribution to the T parameter from non-mass-degenerate leptons helps adjusting to the correlation of the S and T parameters indicated by experimental data [9,10].

[b] This value results for a partial weak gauging involving only one doublet of the theory. If all possible, that is three doublets are gauged one has $\pi S \approx 1$.

[c] In these cases, the global Witten anomaly can be circumvented either by adding one family of leptons or by having two families of techniquarks charged under the electroweak interactions. The value for the S parameter is for the former case. In the latter, it is twice as large.

For an even number of colors the model with one electroweakly gauged techniquark doublet is free of the Witten anomaly. No additional lepton family has to be included. Opposite half-integer electrical charges for the technifermions avoid gauge anomalies. Technicolor singlets constructed from the techniquarks which are gauged under the electroweak interactions carry integer electrical charges. This also includes uncharged technibaryons which can be components of dark matter.

For an odd number of colors a single doublet of techniquarks gauged under the electroweak interactions leads to a Witten anomaly. It can be circumvented by including one additional lepton family. This allows for a more general hypercharge assignment [9] than in the case without leptons.

An alternative to an additional lepton family for circumventing a Witten anomaly is to gauge two instead of one techniquark doublet, at the cost of doubling the S parameter. For this nonminimal weak gauging the hypercharge assignment which corresponds to opposite half-integer electric charges for the techniquarks avoids gauge anomalies.

2. Two-index symmetric

Two more theories with apparently insufficient walking but which make at least the weaker bounds on the S parameter are the two flavor theories with techniquarks in the two-index symmetric representation of SU(3), (20), and SU(4), (200). None of the two needs additional leptons. Thus the hypercharge assignment is completely fixed. Three, respectively, four techniquarks are needed to form a singlet technibaryon.

3. Two flavors, SU(4), adjoint

The four technicolor model with two techniflavors in the adjoint representation has a relatively large perturbative S parameter but walks over more than three decades which helps reducing it through nonperturbative corrections. It also requires an additional lepton family in order to cure the Witten anomaly, which allows for a more general hypercharge assignment. The enhanced symmetry, SU($2N_f = 4$), breaks to SO(4) and leaves behind nine Goldstone bosons. They must be rendered massive which also necessitates a good amount of walking. As for all other adjoint models, any number of techniquarks can form a singlet and technigluons can neutralize technicolor as well.

4. Six flavors, SU(5), two-index antisymmetric

Finally, let us mention the five technicolor model with six techniflavors in the two-index antisymmetric representation. For an acceptable S parameter, only part of the techniflavors can be gauged under the electroweak interactions. No additional leptons are required. This variant does not exhibit a remarkable amount of walking.

C. Split technicolor: A review

Since one aim of this work is to provide a catalogue of various possible walking type, nonsupersymmetric gauge theories, which can be used to dynamically break the electroweak symmetry we summarize here also another possibility already appearing in [9]. There we also suggested a way of keeping the technifermions in the fundamental representation while still reducing the number of techniflavors needed to be near the conformal window. Like for the partially gauged case described above this can be achieved by adding matter uncharged under the weak interactions. The difference to section III A is that this part of matter transforms under a different representation of the technicolor gauge group than the part coupled directly to the electroweak sector. We choose it to be a massless Weyl fermion in the adjoint representation of the technicolor gauge group, that is a technigluino. The resulting theory has the same matter content as N_f-flavor super QCD but without the scalars; hence the name "split technicolor." We expect the critical number of flavors above which one enters the conformal window $N_f^{\rm II}$ to lie within the range

$$\frac{3}{2} < \frac{N_f^{\rm II}}{N} < \frac{11}{2}. \tag{20}$$

The lower bound is the exact supersymmetric value for a nonperturbative conformal fixed point [33], while the upper bound is the one expected in the theory without a technigluino. The matter content of "split technicolor" lies between that of super QCD and the standard fundamental technicolor theory.

For two colors the number of (techni)flavors needed to be near the conformal window in the split case is at least three, while for three colors more than five flavors are required. These values are still larger than the ones for theories with fermions in the two-index symmetric representation. It is useful to remind the reader that in supersymmetric theories the critical number of flavors needed to enter the conformal window does not coincide with the critical number of flavors required to restore chiral symmetry. The scalars in supersymmetric theories play an important role from this point of view. We note that a split technicolorlike theory has been used recently in [34], to investigate the strong CP problem.

Split technicolor shares some features with theories of split supersymmetry recently advocated and studied in [35,36] as possible extensions of the standard model. Clearly, we have introduced split technicolor—differently from split supersymmetry—to address the hierarchy problem. This is why we do not expect new scalars to appear at energy scales higher than the one of the electroweak theory.

IV. SUMMARY

We have presented a comprehensive analysis of the phase diagram of nonsupersymmetric vectorlike and strongly coupled SU(N) gauge theories with matter in various representations of the gauge group. We have considered models with fermions in a single representation of the gauge group, but also the case of a combination of fermions in the fundamental with fermions in the adjoint representation. As physical application we considered the dynamical breaking of the electroweak symmetry via walking technicolor. We have then taken into account constraints form electroweak precision measurements and, thereby, reduced the number of theories viable for correctly describing the dynamical breaking. Still, we find that a considerable number of strongly coupled theories are excellent candidates for breaking the electroweak symmetry dynamically.

ACKNOWLEDGMENTS

It is a pleasure to thank J. Schechter, K. Splittorff, and K. Tuominen for discussions. The work of F. S. is supported by the Marie Curie Excellence Grant under contract No. MEXT-CT-2004-013510 as well as the Danish Research Agency.

APPENDIX: BASIC GROUP THEORY RELATIONS

The Dynkin indices label the highest weight of an irreducible representation and uniquely characterize the representations. The Dynkin indices for some of the most common representations are given in Table IV. For details on the concept of Dynkin indices see, for example [37,38].

For a representation, R, with the Dynkin indices $(a_1, a_2, \ldots, a_{N-2}, a_{N-1})$ the quadratic Casimir operator reads [39]

$$C_2(\mathrm{R}) = \sum_{m=1}^{N-1}\Big[N(N-m)ma_m + m(N-m)a_m^2 + \sum_{n=0}^{m-1} 2n(N-m)a_n a_m\Big] \tag{A1}$$

TABLE IV. Examples for Dynkin indices for some common representations.

representation	Dynkin indices
singlet	(000...00)
fundamental (F)	(100...00)
antifundamental ($\bar{\mathrm{F}}$)	(000...01)
adjoint (G)	(100...01)
n-index symmetric (S_n)	(n00...00)
2-index antisymmetric (A_2)	(010...00)

and the dimension of R is given by

$$d(\mathrm{R}) = \prod_{p=1}^{N-1}\Big\{\frac{1}{p!}\prod_{q=p}^{N-1}\Big[\sum_{r=q-p+1}^{p}(1+a_r)\Big]\Big\}, \tag{A2}$$

which gives rise to the following structure

$$d(\mathrm{R}) = (1+a_1)(1+a_2)\ldots(1+a_{N-1})\Big(1+\frac{a_1+a_2}{2}\Big)\ldots\Big(1+\frac{a_{N-2}+a_{N-1}}{2}\Big)\Big(1+\frac{a_1+a_2+a_3}{3}\Big) \ldots\Big(1+\frac{a_{N-3}+a_{N-2}+a_{N-1}}{3}\Big)\ldots\Big(1+\frac{a_1+\ldots+a_{N-1}}{N-1}\Big). \tag{A3}$$

The Young tableau associated to a given Dynkin index $(a_1, a_2, \ldots, a_{N-2}, a_{N-1})$ is easily constructed. The length of row i (that is the number of boxes per row) is given in terms of the Dynkin indices by the expression $\mathrm{R}_i = \sum_i^{N-1} a_i$. The length of each column is indicated by c_k; k can assume any positive integer value. Indicating the total number of boxes associated to a given Young tableau with b one has another compact expression for $C_2(\mathrm{R})$,

$$C_2(\mathrm{R}) = N\Big[bN + \sum_i r_i^2 - \sum_i c_i^2 - \frac{b^2}{N}\Big], \tag{A4}$$

and the sums run over each column and row.

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