Gluonic phases, vector condensates, and exotic hadrons in dense QCD

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We study the dynamics in phases with vector condensates of gluons (gluonic phases) in dense two-flavor quark matter. These phases yield an example of dynamics in which the Higgs mechanism is provided by condensates of gauge (or gauge plus scalar) fields. Because vacuum expectation values of spatial components of vector fields break the rotational symmetry, it is natural to have a spontaneous breakdown both of external and internal symmetries in this case. In particular, by using the Ginzburg-Landau approach, we establish the existence of a gluonic phase with both the rotational symmetry and the electromagnetic $U(1)$ being spontaneously broken. In other words, this phase describes an anisotropic medium in which the color and electric superconductivities coexist. It is shown that this phase corresponds to a minimum of the Ginzburg-Landau potential and, unlike the two-flavor superconducting (2SC) phase, it does not suffer from the chromomagnetic instability. The dual (confinement) description of its dynamics is developed and it is shown that there are light exotic vector hadrons in the spectrum, some of which condense. Because most of the initial symmetries in this system are spontaneously broken, its dynamics is very rich.

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I. INTRODUCTION

It is expected that at sufficiently high baryon density, cold quark matter should be in a color superconducting state [1–3]. On the other hand, it was suggested long ago that quark matter might exist inside the central region of compact stars [4]. This is one of the main reasons why the dynamics of the color superconductivity has been intensively studied (for reviews, see Ref. [5]).

Bulk matter in compact stars should be in β-equilibrium, providing by weak interactions, and be electrically and color neutral. The electric and color neutrality conditions play a crucial role in the dynamics of quark pairing [6–12]. Also, in the dense quark matter, the strange quark mass cannot be neglected. These factors lead to a mismatch $\delta\mu$ between the Fermi momenta of the pairing quarks.

As was revealed in Ref. [13], the gapped (2SC) and gapless (g2SC) two-flavor color superconducting phases suffer from a chromomagnetic instability connected with the presence of imaginary Meissner masses of gluons. While the 8th gluon has an imaginary Meissner mass only in the g2SC phase, with the diquark gap $\bar{\Delta} < \delta\mu$ (an intermediate coupling regime), the chromomagnetic instability for the 4–7th gluons appears also in a strong coupling regime, with $\delta\mu < \bar{\Delta} < \sqrt{2}\delta\mu$. Later a chromo-magnetic instability was also found in the three-flavor gapless color-flavor locked (gCFL) phase [14–16].

Meissner and Debye masses are screening (and not pole) ones. It has been recently revealed in Ref. [17] that the chromomagnetic instabilities in the 4–7th and 8th gluonic channels correspond to two very different tachyonic spectra of plasmons. It is noticeable that while (unlike the Meissner mass) the (screening) Debye mass for an electric mode remains real for all values of $\delta\mu$ both in the 2SC and g2SC phases [13], the tachyonic plasmons occur both for the magnetic and electric modes [17]. The latter is important since it clearly shows that this instability is connected with vectorlike excitations: Recall that two magnetic modes correspond to two transverse components of a plasmon, and one electric mode corresponds to its longitudinal component. This form of the plasmon spectrum leads to the unequivocal conclusion about the existence of vector condensates of gluons in the ground state of two-flavor quark matter with $\bar{\Delta} < \sqrt{2}\delta\mu$, thus supporting the scenario with gluon condensates (gluonic phase) proposed in the paper [18]. While the analysis in Ref. [18] was done only in the vicinity of the critical point $\delta\mu \simeq \bar{\Delta}/\sqrt{2}$, a numerical analysis of the gluonic phase far away from the scaling region was considered in Refs. [19–21].

At intermediate energy scales of the order of the diquark condensate $\bar{\Delta} \sim \mathcal{O}(50$ MeV), the analysis of QCD dynamics is very hard. Hence the phenomenological Nambu-Jona-Lasinio (NJL) model plays a prominent role in the analysis in dense quark matter (for recent extensive studies of dense QCD in this approach, see Refs. [11,12]). The NJL model is usually regarded as a low-energy effective theory in which massive gluons are integrated out. The

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situation with dense quark matter is however quite different from that in the vacuum QCD. We will introduce gluonic degrees of freedom into the NJL model because in the 2SC/g2SC phase the gluons of the unbroken $SU(2)_c$ subgroup of the color $SU(3)_c$ are left as massless, and, under certain conditions considered below, some other gluons can be also very light. This yields the gauged NJL model.

Also, because of the presence of matter, the running of the QCD coupling constant dramatically changes. As was shown in Ref. [22], the confinement scale Λ'_{QCD} in the 2SC phase is essentially smaller than the typical scale of the diquark condensate, $\Lambda'_{QCD} \lesssim \mathcal{O}(10 \text{ MeV})$ or even much smaller. This justifies introducing the gluonic degrees of freedom into the NJL model at the energy scale around and below $\bar{\Delta}$ (and larger than Λ'_{QCD}), as was done in Ref. [18]. At such scales, the dynamics in the gluonic phase corresponds to the Higgs picture.

As we will discuss in this paper, the description of the infrared dynamics in the gluonic phase depends on the relation between two scales: Λ'_{QCD} and the value of the vector gluon condensates. If the latter is larger than Λ'_{QCD}, then the Higgs picture is appropriate even in the infrared region, similarly as it happens in the electroweak theory. Indeed, when the $SU(2)_c$ gauge symmetry becomes completely broken by the dynamics with a characteristic scale being essentially larger than Λ'_{QCD}, the strong coupling dynamics presented in the 2SC solution at the scale of order Λ'_{QCD} is washed out. However, if this characteristic scale is $\lesssim \Lambda'_{QCD}$, the confinement picture should be used. Although such a dynamics is not under control, the structure of the global symmetry should be the same both in the Higgs and confinement phases, if the Higgs fields (both vector and scalar ones) are assigned to the fundamental representation of the gauge group. In particular, the global charges of hadrons in the confinement picture should correspond to those of fundamental fields in the Higgs picture. This is one of the manifestations of the complementarity principle [23–25]. Using this approach, the existence of exotic hadrons in the gluonic phase will be shown. Moreover, we will see that dynamics with vector gluon condensates in the Higgs phase correspond to dynamics with condensates of exotic vector mesons in the confinement one.

From the viewpoint of quantum field theory, it is quite natural to expect the existence of vector condensates of gluons in order to remove such instabilities as the chromomagnetic one or those connected with tachyonic plasmons. In Ref. [18], homogeneous, i.e., independent of spatial coordinates, vector gluon condensates were considered. At the same time, because the condensates of spatial components of gluon fields break the rotational symmetry, such condensates are anisotropic. On the other hand, inhomogeneous condensates for diquark fields were studied in Refs. [26–31]. For example, the Larkin-Ovchinnikov-

Fulde-Ferrell (LOFF) phase [32] in QCD was considered in Refs. [26,28]. In this phase a diquark condensate is inhomogeneous. Some solutions with inhomogeneous diquark condensates, including the single plane-wave LOFF phase, can be considered as a special case of homogeneous vector condensates of gluons with the field strength being zero [18,33,34]. However, when the latter is not zero, as in the case of the gluonic phase [18], homogeneous vector gluon condensates cannot be traded for inhomogeneous diquark ones. Moreover, as was suggested in Ref. [17], inhomogeneous vector condensates of gluons could exist in the g2SC and gCFL phases.

The paper is organized as follows. In Sec. II, a renormalizable model with condensates of gauge fields is considered. This essentially soluble model yields a proof that such dynamics exist indeed. In Sec. III, the gauged NJL model is described. Because the diquark gap $\bar{\Delta}$ breaks the initial color $SU(3)_c$ symmetry to the $SU(2)_c$ one, it is useful to decompose the fields in the gauged NJL model with respect to the $SU(2)_c$ subgroup. This decomposition is considered in Sec. IV. Section V is the central one in this paper. In its four subsections, the dynamics in the gluonic phase is described in detail. One cannot exclude that besides this phase, other phases with vector condensates of gluons may exist in the gauged NJL model. In Sec. VI, we classify possible sets of homogeneous gluon condensates and typical symmetry breaking patterns in the corresponding phases in this model. We also describe the sets of operators relevant for constructing the Ginzburg-Landau effective theories for these dynamics. In particular, as a consequence of this analysis, it is shown that the ansatz for gluon condensates used for the gluonic phase [18] is self-consistent. In Sec. VII, the main results of the paper are summarized. In Appendices A and B, some useful formulas and relations are derived.

II. RENORMALIZABLE MODEL FOR DYNAMICS WITH VECTOR CONDENSATES

Since a dynamics with vector condensates is a rather new "territory," it would be important to have an essentially soluble model which would play the same role for such a dynamics as the linear σ models play for the conventional dynamics with spontaneous symmetry breaking with condensates of scalar fields. Fortunately, such a model exists: it is the gauged linear $SU(2)_L \times U(1)_Y$ σ-model (without fermions) with a chemical potential for hypercharge Y [35].[1]

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[1]The ungauged linear $SU(2)_L \times U(1)_Y$ σ model with a chemical potential for hypercharge [36] is a toy model for the description of the dynamics of the kaon condensate in high density QCD [37]. In particular, it realizes the phenomenon with an abnormal number of Nambu-Goldstone (NG) bosons [36], when the spontaneous breakdown of continuous symmetries leads to a lesser number of NG bosons than that required by the Goldstone theorem (for a recent discussion of this model, see Ref. [38]).

Let us briefly describe this model. It will be very useful for better understanding the dynamics in the gluonic phase. Its Lagrangian density reads (we use the metric $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$):

$$\mathcal{L} = -\tfrac{1}{4}F^{(a)}_{\mu\nu}F^{\mu\nu(a)} - \tfrac{1}{4}F^{(Y)}_{\mu\nu}F^{\mu\nu(Y)}$$
$$+ [(D_\nu - i\mu_Y\delta_{\nu0})\Phi]^\dagger(D^\nu - i\mu_Y\delta^{\nu0})\Phi - m^2\Phi^\dagger\Phi$$
$$- \lambda(\Phi^\dagger\Phi)^2, \quad (1)$$

where the covariant derivative $D_\mu = \partial_\mu - igA_\mu - (ig'/2)B_\mu$, Φ is a complex doublet field $\Phi^T = (\varphi^+, \varphi_0)$, and the chemical potential μ_Y is provided by external conditions (to be specific, we take $\mu_Y > 0$). Here $A_\mu = A^{(a)}_\mu\tau^a/2$ are $SU(2)_L$ gauge fields (τ^a are three Pauli matrices) and the field strength $F^{(a)}_{\mu\nu} = \partial_\mu A^{(a)}_\nu - \partial_\nu A^{(a)}_\mu + g\epsilon^{abc}A^{(b)}_\mu A^{(c)}_\nu$. B_μ is a $U_Y(1)$ gauge field with the field strength $F^{(Y)}_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. The hypercharge of the doublet Φ equals $+1$. This model has the same structure as the electroweak theory without fermions and with the chemical potential for hypercharge Y. Note that the terms with the chemical potential are $SU(2)_L \times U(1)_Y$ (and not $SU(2)_L \times SU(2)_R$) symmetric. This follows from the fact that the hypercharge generator Y is $Y = 2I^3_R$ where I^3_R is the third component of the right-handed isospin generator. Henceforth we will omit the subscripts L and R, allowing various interpretations of the $SU(2)$.

The model is renormalizable and for small coupling constants g, g', and λ, the tree approximation is reliable there. The chemical potential explicitly breaks the Lorentz symmetry, the symmetry of the model is $SU(2) \times U(1)_Y \times SO(3)_{\text{rot}}$. As was shown in Ref. [35], for sufficiently large values of the chemical potential μ_Y, the condensates of both the scalar doublet Φ *and* the gauge field A_μ occur. The ground state solution is given by

$$|\langle W^{(-)}_z\rangle|^2 = \frac{\mu_Y v_0}{\sqrt{2}g} - \frac{v_0^2}{4}, \qquad \langle A^{(3)}_0\rangle = \frac{v_0}{\sqrt{2}},$$
$$\langle\Phi^T\rangle = (0, v_0), \quad (2)$$

where

$$v_0 = \frac{\sqrt{(g^2 + 64\lambda)\mu_Y^2 - 8(8\lambda - g^2)m^2} - 3g\mu_Y}{\sqrt{2}(8\lambda - g^2)}, \quad (3)$$

$W^{(\mp)}_\mu = \frac{1}{\sqrt{2}}(A^{(1)}_\mu \pm iA^{(2)}_\mu)$, $\Phi^T = (\varphi^+, \varphi_0)$, and the vacuum expectation values of all other fields are equal to zero [39]. It is clear that this solution implies that the initial symmetry $SU(2) \times U(1)_Y \times SO(3)_{\text{rot}}$ is spontaneously broken down to $SO(2)_{\text{rot}}$. In particular, the electromagnetic $U(1)_{\text{em}}$, with electric charge $Q_{\text{em}} = I_3 + Y/2$, is spontaneously broken

by the condensate of W bosons, i.e., electric superconductivity takes place in this medium.[2]

Because the dynamics in this model is under control for small g, g', and λ, the model provides a proof that the dynamics with vector condensate is a real thing. Moreover, this dynamics is quite rich. In particular, as was shown in Ref. [41], there are three types of topologically stable vortices in model (1), which are connected either with photon field or hypercharge gauge field, or with both of them. As we will see below, the dynamics in the gluonic phase strikingly resembles the dynamics in this toy model being however much more complicated.

III. GAUGED NJL MODEL

We study dense two-flavor quark matter in β-equilibrium. For our purpose, it is convenient to use a phenomenological NJL model with gluons, the gauged NJL model. As was already pointed out in the introduction, this approach corresponds to the Higgs picture. The confinement picture will be considered in Subsec. V D.

For simplicity, the current quark masses and the $(\bar{\psi}\psi)^2$ interaction in chiral channels will be neglected. Then the Lagrangian density is given by

$$\mathcal{L} = \bar{\psi}(i\not{D} + \hat{\mu}_0\gamma^0)\psi + G_\Delta[(\bar{\psi}^C i\varepsilon\epsilon^\alpha\gamma_5\psi)(\bar{\psi}i\varepsilon\epsilon^\alpha\gamma_5\psi^C)]$$
$$- \tfrac{1}{4}F^{(a)}_{\mu\nu}F^{(a)\mu\nu}, \quad (4)$$

where

$$D_\mu \equiv \partial_\mu - igA^{(a)}_\mu T^a,$$
$$F^{(a)}_{\mu\nu} \equiv \partial_\mu A^{(a)}_\nu - \partial_\nu A^{(a)}_\mu + gf^{abc}A^{(b)}_\mu A^{(c)}_\nu. \quad (5)$$

Here $A^{(a)}_\mu$ are gluon fields, T^a are the generators of $SU(3)$ in the fundamental representation, and f^{abc} are the structure constants of $SU(3)$. The spinor field $\psi = \psi_{i\alpha}$ has the flavor ($i = u, d$) and color ($\alpha = r, g, b$) indices (ψ_{ur} is an up-red quark, etc.). In Eq. (4), $(\varepsilon) \equiv \epsilon^{ij}$, $(\epsilon^{ud} = +1)$, and $(\epsilon^\alpha) \equiv \epsilon^{\alpha\beta\gamma}$, $(\epsilon^{rgb} = +1)$ are the totally antisymmetric tensors in the flavor and color spaces, respectively. Note that we do

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[2]Note that because the $U(1)_Y$ symmetry is local, for a nonzero chemical potential μ_Y one should introduce a source term B_0J_0 in Lagrangian density (1) in order to make the system neutral with respect to hypercharge Y. This is necessary since otherwise in such a system thermodynamic equilibrium could not be established. The value of the background hypercharge density J_0 (representing very heavy particles) is determined from the requirement that $B_0 = 0$ is a solution of the equation of motion for B_0 (the Gauss's law) [35,40]. There exists an alternative description of this dynamics in which a background hypercharge density J_0 is considered as a free parameter and μ_Y is taken to be zero. Then Gauss's law will define the vacuum expectation value $\langle B_0\rangle$. It is not difficult to check that these two approaches are equivalent if the chemical potential μ_Y in the first approach is taken to be equal to the value $\frac{g'}{2}\langle B_0\rangle$ from the second one.

not write explicitly the free electron term, and the photon field is not introduced in the model.

In the β-equilibrium, the chemical potential matrix $\hat{\mu}_0$ for up and down quarks is:

$$\hat{\mu}_0 = \mu \mathbf{1} - \mu_e Q_{\text{em}} + \mu_8 Q_8, \qquad (6)$$

where $\mathbf{1} \equiv \mathbf{1}_c \otimes \mathbf{1}_f$, $Q_{\text{em}} \equiv \mathbf{1}_c \otimes \text{diag}(2/3, -1/3)_f$, $Q_8 \equiv \text{diag}(1/3, 1/3, -2/3)_c \otimes \mathbf{1}_f$, and μ, μ_e, and μ_8 are the quark, electron, and color chemical potentials, respectively (the baryon chemical potential μ_B is $\mu_B \equiv 3\mu$). Here the subscripts c and f mean that the corresponding matrices act in the color and flavor spaces, respectively. [Henceforth we will not show explicitly the unit matrices $\mathbf{1}$, $\mathbf{1}_c$, and $\mathbf{1}_f$.] Note that the status of the chemical potential μ_8 is somewhat different from that of μ and μ_e (in the absence of a photon field). The point is that the color neutrality condition is nothing else as the Gauss's law for the gluon field $A_0^{(8)}$ and μ_8 is expressed through $A_0^{(8)}$ as

$$\mu_8 \equiv \frac{\sqrt{3}}{2} g A_0^{(8)}. \qquad (7)$$

As was shown in Ref. [42], μ_8 is nonzero in the 2SC/g2SC phase. Although the color chemical potential is not an independent quantity in the gauged NJL model, we will keep the notation μ_8 in order to exhibit a special role of the field $A_0^{(8)}$.

Equation (6) implies that the total chemical potentials for different quarks in the 2SC/g2SC phase are

$$\mu_{ur} = \mu_{ug} = \tilde{\mu} - \delta\mu, \qquad \mu_{ub} = \tilde{\mu} - \delta\mu - \mu_8, \quad (8a)$$
$$\mu_{dr} = \mu_{dg} = \tilde{\mu} + \delta\mu, \qquad \mu_{db} = \tilde{\mu} + \delta\mu - \mu_8, \quad (8b)$$

with

$$\tilde{\mu} \equiv \mu - \frac{\mu_e}{6} + \frac{\mu_8}{3}, \qquad \delta\mu \equiv \frac{\mu_e}{2}. \qquad (9)$$

Let us now introduce the diquark field $\Phi^\alpha \sim i\bar{\psi}^C \varepsilon \epsilon^\alpha \gamma_5 \psi$. Then one can rewrite the Lagrangian density (4) as

$$\mathcal{L} = \bar{\psi}(i\slashed{D} + \hat{\mu}_0\gamma^0)\psi - \frac{|\Phi^\alpha|^2}{4G_\Delta} - \frac{1}{2}\Phi^\alpha[i\bar{\psi}\varepsilon\epsilon^\alpha\gamma_5\psi^C]$$
$$- \frac{1}{2}[i\bar{\psi}^C\varepsilon\epsilon^\alpha\gamma_5\psi]\Phi^{*\alpha} - \frac{1}{4}F_{\mu\nu}^{(a)}F^{(a)\mu\nu}. \qquad (10)$$

Without loss of generality, the diquark condensate in the 2SC/g2SC phase can be chosen along the antiblue direction: $\langle\Phi^r\rangle = 0$, $\langle\Phi^g\rangle = 0$, $\langle\Phi^b\rangle \neq 0$. In correspondence with that, in the gauged NJL model, it will be convenient to use the following (partly unitary) gauge in the 2SC/g2SC phase [43]:

$$\Phi^r = 0, \qquad \Phi^g = 0, \qquad \Phi^b \equiv \Delta \qquad (11)$$

with the field Δ being real. Then the gap $\bar{\Delta}$ in the 2SC/g2SC phase is equal to the vacuum expectation value of Δ, $\bar{\Delta} \equiv \langle\Delta\rangle$.

We now introduce the Nambu-Gor'kov spinor,

$$\Psi \equiv \begin{pmatrix} \psi \\ \psi^C \end{pmatrix}. \qquad (12)$$

The inverse propagator of Ψ with the field Δ in the 2SC/g2SC phase is given by

$$S^{-1}(P) = \begin{pmatrix} [G_0^+]^{-1} & \Delta^- \\ \Delta^+ & [G_0^-]^{-1} \end{pmatrix} \qquad (13)$$

with

$$[G_0^+]^{-1}(P) \equiv (p_0 + \tilde{\mu} - \delta\mu\tau_3 - \mu_8\mathbf{1}_b)\gamma^0 - \vec{\gamma}\cdot\vec{p}, \qquad (14)$$

$$[G_0^-]^{-1}(P) \equiv (p_0 - \tilde{\mu} + \delta\mu\tau_3 + \mu_8\mathbf{1}_b)\gamma^0 - \vec{\gamma}\cdot\vec{p}, \qquad (15)$$

and

$$\Delta^- \equiv -i\varepsilon\epsilon^b\gamma_5\Delta, \qquad \Delta^+ \equiv \gamma^0(\Delta^-)^\dagger\gamma^0 = -i\varepsilon\epsilon^b\gamma_5\Delta, \qquad (16)$$

where $P^\mu \equiv (p_0, \vec{p})$ is an energy-momentum four vector, $\tau_3 \equiv \text{diag}(1, -1)_f$ and $\mathbf{1}_b \equiv \text{diag}(0, 0, 1)_c$.

The propagator of Ψ is given by

$$S(P) = \begin{pmatrix} G^+ & \Xi^- \\ \Xi^+ & G^- \end{pmatrix} \qquad (17)$$

with

$$G^\pm = \{[G_0^\pm]^{-1} - \Delta^\mp G_0^\mp \Delta^\pm\}^{-1}, \qquad \Xi^\pm = -G_0^\mp\Delta^\pm G^\pm. \qquad (18)$$

The structure of $G^\pm$ and $\Xi^\pm$ was determined in the second paper in Ref. [13]:

$$G^\pm(P) \equiv \text{diag}(G_\Delta^\pm, G_\Delta^\pm, G_b^\pm)_c, \qquad (19)$$

where

$$G_\Delta^\pm(P) = \frac{(p_0 \mp \delta\mu\tau_3) - E^\pm}{(p_0 \mp \delta\mu\tau_3)^2 - (E_\Delta^\pm)^2}\gamma^0\Lambda_p^+$$
$$+ \frac{(p_0 \mp \delta\mu\tau_3) + E^\mp}{(p_0 \mp \delta\mu\tau_3)^2 - (E_\Delta^\mp)^2}\gamma^0\Lambda_p^-, \qquad (20)$$

$$G_b^\pm(P) = \frac{1}{(p_0 \mp \delta\mu\tau_3 \mp \mu_8) + E^\pm}\gamma^0\Lambda_p^+$$
$$+ \frac{1}{(p_0 \mp \delta\mu\tau_3 \mp \mu_8) - E^\mp}\gamma^0\Lambda_p^-, \qquad (21)$$

and

$$\Xi^\pm(P) \equiv \epsilon^b \begin{pmatrix} 0 & \Xi_{12}^\pm \\ -\Xi_{21}^\pm & 0 \end{pmatrix}_f \qquad (22)$$

with

$$\Xi_{12}^{\pm}(P) = -i\Delta\left[\frac{1}{(p_0 \pm \delta\mu)^2 - (E_\Delta^{\pm})^2}\gamma_5\Lambda_p^+\right.$$
$$\left. + \frac{1}{(p_0 \pm \delta\mu)^2 - (E_\Delta^{\mp})^2}\gamma_5\Lambda_p^-\right], \qquad (23)$$

$$\Xi_{21}^{\pm}(P) = -i\Delta\left[\frac{1}{(p_0 \mp \delta\mu)^2 - (E_\Delta^{\pm})^2}\gamma_5\Lambda_p^+\right.$$
$$\left. + \frac{1}{(p_0 \mp \delta\mu)^2 - (E_\Delta^{\mp})^2}\gamma_5\Lambda_p^-\right]. \qquad (24)$$

Here

$$E^{\pm} \equiv |\vec{p}| \pm \tilde{\mu}, \qquad E_\Delta^{\pm} \equiv \sqrt{(E^{\pm})^2 + \Delta^2},$$
$$\Lambda_p^{\pm} \equiv \frac{1}{2}\left(1 \pm \gamma^0\frac{\vec{\gamma}\cdot\vec{p}}{|\vec{p}|}\right), \qquad (25)$$

and while $G_\Delta^{\pm}$ and $G_b^{\pm}$ are 8×8 matrices in the flavor-spinor space, the 4×4 matrices $\Xi_{12}^{\pm}$ and $\Xi_{21}^{\pm}$ act only in the spinor space.

The generalization of expression (13) for the inverse propagator of Ψ both with the scalar diquark field Δ and the vector fields $A_\mu^{(a)}$ in a gluonic phase (with nonzero gluonic condensates $\langle A_\mu^{(a)}\rangle$) is straightforward:

$$S_g^{-1}(P) = \begin{pmatrix} [G_{0,g}^+]^{-1} & \Delta^- \\ \Delta^+ & [G_{0,g}^-]^{-1} \end{pmatrix}, \qquad (26)$$

with

$$[G_{0,g}^+]^{-1}(P) \equiv (p_0 + \hat{\mu}_0)\gamma^0 - \vec{\gamma}\cdot\vec{p} + g\slashed{A}^{(a)}T^a, \qquad (27)$$

$$[G_{0,g}^-]^{-1}(P) \equiv (p_0 - \hat{\mu}_0)\gamma^0 - \vec{\gamma}\cdot\vec{p} - g\slashed{A}^{(a)}(T^a)^T. \qquad (28)$$

Integrating out fermion fields, we obtain the potential including both gluon and diquark fields:

$$V = \frac{\Delta^2}{4G_\Delta} + \frac{g^2}{4}f^{a_1a_2a_3}f^{a_1a_4a_5}A_\mu^{(a_2)}A_\nu^{(a_3)}A^{(a_4)\mu}A^{(a_5)\nu}$$
$$- \frac{1}{2}\int\frac{d^4P}{i(2\pi)^4}\text{Tr}\ln S_g^{-1}. \qquad (29)$$

We will utilize the hard dense loop approximation, in which only the dominant one-loop quark contribution is taken into account, while the contribution of gluon loops is neglected. On the other hand, we keep the tree contribution of gluons in the effective potential (29). This is because we want to compare this contribution with that of hard dense loops in order to check the consistency of the hard dense loop approximation. The ground state of the system corresponds to the minimum of potential (29) and the question whether there exist gluon condensates $\langle A_\mu^{(a)}\rangle$ is a dynamical issue.

The following remarks are in order.

(a) To study the ground state in the Higgs phase, it is convenient to use the unitary gauge. The important point is that in this gauge, all auxiliary (gauge dependent) degrees of freedom are removed. Therefore in this gauge the vacuum expectations values (VEVs) $\langle A_\mu^{(a)}\rangle$ of vector fields are well-defined physical quantities. The unitary gauge in the gluonic phase, which uses and extends the constraints presented in Eq. (11), will be described in Sec. V.

(b) As we will see below, in the gluonic phase, the time-component VEVs of the gluon fields other than the 8th one are also nonzero. Because of that, it will be convenient to rewrite the effective potential (29) in a somewhat different form. Let us introduce the following matrix $\mathcal{M}_g$ in the Nambu-Gor'kov space,

$$\mathcal{M}_g \equiv S_g^{-1} - S^{-1} = \begin{pmatrix} \mu_{\check{a}}T^{\check{a}}\gamma^0 - g\vec{A}^a\cdot\vec{\gamma}T^a & 0 \\ 0 & -\mu_{\check{a}}(T^{\check{a}})^T\gamma^0 + g\vec{A}^a\cdot\vec{\gamma}(T^a)^T \end{pmatrix}, \qquad (30)$$

where

$$\mu_{\check{a}} \equiv gA_0^{\check{a}}, \qquad (\check{a} = 1, 2, \cdots, 7) \qquad (31)$$

($\mu_8 \equiv \frac{\sqrt{3}}{2}gA_0^{(8)}$ is included in S^{-1}). Expanding now the logarithmic term in Eq. (29), we find

$$V = V_\Delta(\Delta, \mu_e, \mu_8)$$
$$+ \frac{g^2}{4}f^{a_1a_2a_3}f^{a_1a_4a_5}A_\mu^{(a_2)}A_\nu^{(a_3)}A^{(a_4)\mu}A^{(a_5)\nu}$$
$$+ \sum_{n=1}^{\infty}\frac{(-1)^n}{2n}\int\frac{d^4P}{i(2\pi)^4}\text{Tr}(S\mathcal{M}_g)^n, \qquad (32)$$

where we defined the 2SC/g2SC part of the effective potential as

$$V_\Delta(\Delta, \mu_e, \mu_8) = \frac{\Delta^2}{4G_\Delta} - \frac{1}{2}\int\frac{d^4P}{i(2\pi)^4}\text{Tr}\ln S^{-1}. \qquad (33)$$

The form (32) of the effective potential will be used in our analysis below.

IV. $SU(2)_c$ DECOMPOSITION

Because the diquark condensate $\bar{\Delta} = \langle\Phi^b\rangle \equiv \langle\Delta\rangle$ breaks the initial color $SU(3)_c$ symmetry down to the $SU(2)_c$ one, it is useful to decompose the initial fields with respect to the $SU(2)_c$ subgroup. In particular, the decomposition will help to calculate systematically the effective potential (32).

The (anti)fundamental and adjoint representations of $SU(3)_c$ are decomposed with respect to the $SU(2)_c$ as:

$$\mathbf{3} = \mathbf{2} \oplus \mathbf{1}, \qquad \text{i.e.,} \qquad \begin{pmatrix} \psi_{ir} \\ \psi_{ig} \\ \psi_{ib} \end{pmatrix} = \begin{pmatrix} \psi_{ir} \\ \psi_{ig} \end{pmatrix} \oplus \psi_{ib}, \qquad (i = u, d), \tag{34}$$

$$\bar{\mathbf{3}} = \bar{\mathbf{2}} \oplus \mathbf{1}, \qquad \text{i.e.,} \qquad \begin{pmatrix} \Phi^r \\ \Phi^g \\ \Phi^b \end{pmatrix} = \begin{pmatrix} \Phi^r \\ \Phi^g \end{pmatrix} \oplus \Phi^b, \tag{35}$$

and

$$\mathbf{8} = \mathbf{3} \oplus \mathbf{2} \oplus \bar{\mathbf{2}} \oplus \mathbf{1}, \qquad \text{i.e.,} \qquad \{A_\mu^a\} = (A_\mu^{(1)}, A_\mu^{(2)}, A_\mu^{(3)}) \oplus \phi_\mu \oplus \phi_\mu^* \oplus A_\mu^{(8)}, \qquad (a = 1, 2, \cdots, 8). \tag{36}$$

Here we defined the complex doublets of the matter (with respect to the $SU(2)_c$) vector fields,

$$\phi_\mu \equiv \begin{pmatrix} \phi_\mu^r \\ \phi_\mu^g \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} A_\mu^{(4)} - iA_\mu^{(5)} \\ A_\mu^{(6)} - iA_\mu^{(7)} \end{pmatrix},$$

$$\phi_\mu^* \equiv \begin{pmatrix} \phi_\mu^{*r} \\ \phi_\mu^{*g} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} A_\mu^{(4)} + iA_\mu^{(5)} \\ A_\mu^{(6)} + iA_\mu^{(7)} \end{pmatrix}. \tag{37}$$

Then we define the field strength for the $SU(2)_c$ gauge bosons,

$$f_{\mu\nu}^{(\ell)} \equiv \partial_\mu A_\nu^{(\ell)} - \partial_\nu A_\mu^{(\ell)} + g\epsilon^{\ell mn} A_\mu^{(m)} A_\nu^{(n)} \quad (\ell, m, n = 1, 2, 3), \tag{38}$$

and the covariant derivative

$$\mathcal{D}_\mu \equiv \partial_\mu - igA_\mu^{(\ell)} \frac{\sigma^\ell}{2} \qquad (\ell = 1, 2, 3). \tag{39}$$

It will be also useful to define the combinations

$$\mathcal{A}_\mu^+ \equiv \frac{1}{\sqrt{2}}(A_\mu^{(1)} + iA_\mu^{(2)}), \qquad \mathcal{A}_\mu^- \equiv \frac{1}{\sqrt{2}}(A_\mu^{(1)} - iA_\mu^{(2)}). \tag{40}$$

Because of the presence of the electric chemical potential μ_e in the model, the chiral $SU(2)_{L,R}$ symmetry is explicitly broken down to its $U(1)$-part $U(1)_{\tau_{L,R}^3}$. Therefore the initial symmetry in the gauged NJL model is

$$SU(3)_c \times U(1)_{em} \times [U(1)_{\tau_L^3} \times U(1)_{\tau_R^3}]_\chi \times SO(3)_{rot}. \tag{41}$$

Since a photon filed was not included, the electromagnetic symmetry $U(1)_{em}$ is global. Note that the initial baryon charge is

$$\mathcal{B} = \tfrac{1}{3}\mathbf{1}_f \otimes \mathbf{1}_c = 2(Q_{em} - I_3), \tag{42}$$

where I_3 is a diagonal subgroup of the $U(1)_{\tau_L^3} \times U(1)_{\tau_R^3}$, i.e., $I_3 = \text{diag}(1/2, -1/2)_f$.

The diquark gap $\bar{\Delta}$ breaks the initial symmetry (41) down to

$$SU(2)_c \times \tilde{U}(1)_{em} \times [U(1)_{\tau_L^3} \times U(1)_{\tau_R^3}]_\chi \times SO(3)_{rot}. \tag{43}$$

The unbroken $\tilde{U}(1)_{em}$ is connected with the new electric charge

$$\tilde{Q}_{em} = Q_{em} - \tfrac{1}{\sqrt{3}}T^8, \tag{44}$$

where $T^8 \equiv \frac{1}{2\sqrt{3}}\text{diag}(1, 1, -2)_c$. The new unbroken baryon charge is

$$\tilde{\mathcal{B}} = 2(\tilde{Q}_{em} - I_3). \tag{45}$$

The charges for fermions, diquark fields, and gluons are summarized in Tables I and II.

The transformations of the fields under the gauge $SU(2)_c$ have the following form

$$\begin{pmatrix} \psi_{ir} \\ \psi_{ig} \end{pmatrix} \rightarrow \begin{pmatrix} \psi_{ir}' \\ \psi_{ig}' \end{pmatrix} = U\begin{pmatrix} \psi_{ir} \\ \psi_{ig} \end{pmatrix}, \qquad (i = u, d), \tag{46}$$

$$\begin{pmatrix} \Phi^r \\ \Phi^g \end{pmatrix} \rightarrow \begin{pmatrix} \Phi'^r \\ \Phi'^g \end{pmatrix} = U^*\begin{pmatrix} \Phi^r \\ \Phi^g \end{pmatrix}, \tag{47}$$

TABLE I. The quantum numbers of the up and down quarks and the diquark fields. In the unitary gauge the diquark fields $\Phi^{r,g}$ and $\text{Im}\Phi^b$ are absorbed into the longitudinal modes of the corresponding gluons.

	Q_{em}	$\mathcal{B}$	$\tilde{Q}_{em}$	$\tilde{\mathcal{B}}$	$\tilde{\tilde{Q}}_{em}$	$\tilde{\tilde{\mathcal{B}}}$	I_3
ψ_{ur}	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	0	0	-1	$\frac{1}{2}$
ψ_{ug}	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	0	1	1	$\frac{1}{2}$
ψ_{ub}	$\frac{2}{3}$	$\frac{1}{3}$	1	1	1	1	$\frac{1}{2}$
ψ_{dr}	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{2}$	0	-1	-1	$-\frac{1}{2}$
ψ_{dg}	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{2}$	0	0	1	$-\frac{1}{2}$
ψ_{db}	$-\frac{1}{3}$	$\frac{1}{3}$	0	1	0	1	$-\frac{1}{2}$
Φ^r	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{2}$	1	1	2	0
Φ^g	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{2}$	1	0	0	0
Φ^b	$\frac{1}{3}$	$\frac{2}{3}$	0	0	0	0	0

TABLE II.   The quantum numbers of gluons.

	Q_{em}	B	$\tilde{Q}_{em}$	$\tilde{B}$	$\tilde{\tilde{Q}}_{em}$	$\tilde{\tilde{B}}$	I_3
$\mathcal{A}_\mu^+ \equiv \frac{1}{\sqrt{2}}(A_\mu^{(1)} + iA_\mu^{(2)})$	0	0	0	0	1	2	0
$A_\mu^{(3)}$	0	0	0	0	0	0	0
$\mathcal{A}_\mu^- \equiv \frac{1}{\sqrt{2}}(A_\mu^{(1)} - iA_\mu^{(2)})$	0	0	0	0	-1	-2	0
$\phi_\mu^{*r} \equiv \frac{1}{\sqrt{2}}(A_\mu^{(4)} + iA_\mu^{(5)})$	0	0	$\frac{1}{2}$	1	1	2	0
$\phi_\mu^{*g} \equiv \frac{1}{\sqrt{2}}(A_\mu^{(6)} + iA_\mu^{(7)})$	0	0	$\frac{1}{2}$	1	0	0	0
$\phi_\mu^{r} \equiv \frac{1}{\sqrt{2}}(A_\mu^{(4)} - iA_\mu^{(5)})$	0	0	$-\frac{1}{2}$	-1	-1	-2	0
$\phi_\mu^{g} \equiv \frac{1}{\sqrt{2}}(A_\mu^{(6)} - iA_\mu^{(7)})$	0	0	$-\frac{1}{2}$	-1	0	0	0
$A_\mu^{(8)}$	0	0	0	0	0	0	0

$$\phi_\mu \rightarrow \phi'_\mu = U\phi_\mu, \tag{48}$$

$$\mathcal{A}_\mu \rightarrow \mathcal{A}'_\mu = U\mathcal{A}_\mu U^{-1} + \frac{i}{g}U\partial_\mu U^{-1}, \tag{49}$$

$$f_{\mu\nu} \rightarrow f'_{\mu\nu} = Uf_{\mu\nu}U^{-1}, \tag{50}$$

$$\mathcal{D}_\mu \rightarrow \mathcal{D}'_\mu = U\mathcal{D}_\mu U^{-1}, \tag{51}$$

where

$$\mathcal{A}_\mu \equiv A_\mu^{(\ell)}\frac{\sigma^\ell}{2}, \qquad f_{\mu\nu} \equiv f_{\mu\nu}^{(\ell)}\frac{\sigma^\ell}{2}, \qquad (\ell = 1, 2, 3) \tag{52}$$

and

$$U = \exp\left(i\theta^\ell(x)\frac{\sigma^\ell}{2}\right). \tag{53}$$

Under the new baryon symmetry connected with generator $\tilde{B}$ (45), blue quarks, the diquark fields $\Phi^{r,g}$, and the ϕ_μ field doublet transform as

$$\psi_{ib} \rightarrow \psi'_{ib} = e^{i\theta}\psi_{ib}, \qquad (i = u, d),$$
$$\Phi^{r,g} \rightarrow \Phi'^{r,g} = e^{i\theta}\Phi^{r,g}, \qquad \phi_\mu \rightarrow \phi'_\mu = e^{-i\theta}\phi_\mu, \tag{54}$$

while other fields have zero baryon charge $\tilde{B}$ (see Tables I and II.)

V. GLUONIC PHASE

Both the chromomagnetic [13] and plasmon [17] instabilities for the 4–7th gluons in the 2SC phase at $\bar{\Delta} < \sqrt{2}\delta\mu$ suggest a condensation of these gluons, i.e., the creation of a condensate of vector field ϕ_μ (37). Because the chromomagnetic instability develops in the magnetic channel, it is natural to expect that a spatial component of ϕ_μ has a VEV. In studying this condensate, one can use the freedom connected with symmetry (43) in the 2SC/g2SC

phase. By using the rotational symmetry $SO(3)_{rot}$, one can take $\langle\phi_z\rangle \neq 0$ while $\langle\phi_x\rangle = \langle\phi_y\rangle = 0$. And because of the $SU(2)_c$ symmetry, without loss of generality, we can choose $\langle A_z^{(6)}\rangle \neq 0$. This VEV breaks the $SU(2)_c$ down to nothing and the $SO(3)_{rot}$ down to the $SO(2)_{rot}$.

The following remarks are in order. (a) The complex doublet ϕ_z plays here the role of a Higgs field responsible for spontaneous breakdown of the $SU(2)_c$. The situation is similar to that taking place in the electroweak theory. The essential difference however is that now the Higgs field is a spatial component of the vector field leading also to spontaneous breakdown of the rotational symmetry. (b) In this paper, we will use the gauge in which

$$\phi_z^T = \frac{1}{\sqrt{2}}(0, \langle A_z^{(6)}\rangle + a_z^{(6)}), \tag{55}$$

where the real field $a_z^{(6)}$ describes quantum fluctuations. This constraint together with that in Eq. (11) constitute the unitary gauge: all auxiliary (gauge dependent) degrees of freedom are now removed.

A gluonic phase with such a condensate was described in the paper [18]. Since most of the initial symmetries are broken in this phase, its dynamics is rich and complicated. On the other hand, because of space shortage, the description of this phase in the paper [18] was rather brief. In this paper, we will present both a detailed description of its dynamics and present a general analysis of a possibility of the existence of other phases with vector gluon condensates in dense QCD.

A. Symmetry breaking structure and Ginzburg-Landau effective potential

Let us describe symmetry breaking structure in the gluonic phase. With a broken $SU(2)_c$, the $SU(2)_c$ gluons could have VEVs. A similar situation takes place in the gauged σ-model with a chemical potential for hypercharge described in Sec. II above: the gauge symmetry $SU(2)_L$ is broken there. Motivating by that model, we assume

$$\langle A_z^{(1)}\rangle, \langle A_0^{(3)}\rangle \neq 0. \tag{56}$$

As will be shown below, a solution with these vector condensates exists in the model indeed.

The symmetry in the 2SC/g2SC phase is that presented in Eq. (43). The VEV $\langle A_z^{(6)}\rangle$ breaks the $SU(2)_c$ but a linear combination of the generator T^3 from the $SU(2)_c$ and $\tilde{Q}_{em}$,

$$\tilde{\tilde{Q}}_{em} = \tilde{Q}_{em} - T^3 = Q_{em} - \frac{1}{\sqrt{3}}T^8 - T^3, \tag{57}$$

determines the unbroken $\tilde{\tilde{U}}(1)_{em}$. The new baryon charge is $\tilde{\tilde{B}} = 2(\tilde{\tilde{Q}}_{em} - I_3)$ [the charges $\tilde{\tilde{Q}}_{em}$ and $\tilde{\tilde{B}}$ for quarks, diquarks, and gluons are shown in Tables I and II]. However, because T^1 does not commute with T^3, the VEV $\langle A_z^{(1)}\rangle$ breaks $\tilde{\tilde{U}}_{em}(1)$. The $U(1)$ symmetry connected with the baryon charge $\tilde{\tilde{B}}$ is also broken.

After all, we have:

$$[SU(3)_c]_{\text{local}} \times [U(1)_{\text{em}} \times U(1)_{\tau_L^3} \times U(1)_{\tau_R^3}]_{\text{global}} \times SO(3)_{\text{rot}}$$

$$\xrightarrow{\bar{\Delta}} [SU(2)_c]_{\text{local}} \times [\tilde{U}(1)_{\text{em}} \times U(1)_{\tau_L^3} \times U(1)_{\tau_R^3}]_{\text{global}} \times SO(3)_{\text{rot}} \qquad (58)$$

$$\xrightarrow{\langle A_z^{(6)} \rangle} [\tilde{U}(1)_{\text{em}} \times U(1)_{\tau_L^3} \times U(1)_{\tau_R^3}]_{\text{global}} \times SO(2)_{\text{rot}} \qquad (59)$$

$$\xrightarrow{\langle A_z^{(1)} \rangle} [U(1)_{\tau_L^3} \times U(1)_{\tau_R^3}]_{\text{global}} \times SO(2)_{\text{rot}}. \qquad (60)$$

Thus, this system describes an anisotropic medium in which both the color and electric superconductivities coexist.

Let us apply the Ginzburg-Landau (GL) approach to this system near the critical point $\delta\mu \simeq \bar{\Delta}/\sqrt{2}$. The two-point function of gluons can be calculated from Lagrangian density (10). While in Ref. [13] the Debye and Meissner screening masses of the gluons in the 2SC phase were calculated, the pole masses of the corresponding light plasmons (with masses $|M| \ll |\mu|$) were analyzed in Ref. [17]. For the gluons of the unbroken $SU(2)_c$, i.e., $A_\mu^{(1)}, A_\mu^{(2)}$, and $A_\mu^{(3)}$, both Debye and Meissner masses vanish in the region $\delta\mu < \bar{\Delta}$ and there are no light plasmons in these channels. For the gluons $A_\mu^{(4)-(7)}$, the Meissner mass is approximately

$$m_{M,4}^2 = \frac{g^2 \tilde{\mu}^2}{6\pi^2} \left(1 - \frac{2\delta\mu^2}{\bar{\Delta}^2}\right), \qquad \delta\mu < \bar{\Delta}. \qquad (61)$$

Thus, near the critical point $\delta\mu = \bar{\Delta}/\sqrt{2}$, the Meissner mass for $A_\mu^{(4)-(7)}$ is very small. As $\delta\mu$ exceeds the value $\bar{\Delta}/\sqrt{2}$, $m_{M,4}^2$ becomes negative, thus signalizing a chromomagnetic instability in the 2SC solution. The pole masses of the light plasmons for the magnetic and electric modes have a similar behavior in these channels. On the other hand, around the critical point $\delta\mu = \bar{\Delta}/\sqrt{2}$, the $SU(2)_c$ singlet gluon $A_\mu^{(8)}$ is heavy. Actually, it is heavy in the whole region $\delta\mu < \bar{\Delta}$. This fact allows us to pick up the gluons $A_\mu^{(1)-(7)}$ as relevant light degrees of freedom in the low-energy effective theory around the critical point $\delta\mu = \bar{\Delta}/\sqrt{2}$.

Because the $SU(2)_c$ is a gauge symmetry, the building blocks of the GL effective action are

$$\phi_0, \quad \phi_j, \quad \mathcal{D}_0, \quad \mathcal{D}_j, \quad f_{0j}, \quad f_{jk}, \qquad (62)$$

where the indices j and k represent spatial components. The $SU(2)_c$ and $SO(3)_{\text{rot}}$ symmetries dictate the form of the general GL effective potential, which is made from these building blocks and includes operators up to the mass dimension four, i.e., relevant and marginal ones. It will be convenient to introduce the following notations:

$$B \equiv gA_z^{(6)}, \qquad C \equiv gA_z^{(1)}, \qquad D \equiv gA_0^{(3)}. \qquad (63)$$

Then, as shown in detail in Subsec. VI B below, the GL potential has the form

$$V_{\text{eff}} = V_\Delta + \frac{1}{2}M_B^2 B^2 + T_{DB}DB^2 + \frac{1}{2}\lambda_{BC}B^2C^2$$
$$+ \frac{1}{2}\lambda_{BD}B^2D^2 + \frac{1}{2}\lambda_{CD}C^2D^2 + \frac{1}{4}\lambda_B B^4, \qquad (64)$$

where V_Δ is the 2SC part of the effective potential (see Eq. (33)). Here, while the coefficients λ_B, λ_{BC}, λ_{BD}, and λ_{CD} are dimensionless, the dimension (in mass units) of the coefficient T_{DB} in the triple vertex is one. Expanding the potential V (32) with respect to B, C, and D, we can determine these coefficients.

Before realizing explicit calculations, we clarify the behavior of the effective potential (64) near the critical point. The stationary point of the effective potential (64) is given by the equations

$$\frac{\partial V_{\text{eff}}}{\partial B} = B[M_B^2 + \lambda_B B^2 + 2T_{DB}D + \lambda_{BC}C^2 + \lambda_{BD}D^2] = 0, \qquad (65)$$

$$\frac{\partial V_{\text{eff}}}{\partial C} = C[\lambda_{BC}B^2 + \lambda_{CD}D^2] = 0, \qquad (66)$$

$$\frac{\partial V_{\text{eff}}}{\partial D} = T_{DB}B^2 + \lambda_{BD}DB^2 + \lambda_{CD}C^2D = 0, \qquad (67)$$

and

$$\frac{\partial V_{\text{eff}}}{\partial \mu_e} = 0, \qquad \frac{\partial V_{\text{eff}}}{\partial \mu_8} = 0, \qquad \frac{\partial V_{\text{eff}}}{\partial \Delta} = 0. \qquad (68)$$

It will be convenient to present μ_e, μ_8, and Δ as

$$\mu_e = \bar{\mu}_e + \xi_e, \qquad (69)$$

$$\mu_8 = \bar{\mu}_8 + \xi_8, \qquad (70)$$

$$\Delta = \bar{\Delta} + \xi_\Delta, \qquad (71)$$

where the bar-quantities are from the 2SC solution, with $B = C = D = 0$. Let us assume that the origin (bifurcation point) of the solution with nonzero B, C, and D corresponds to a second order phase transition (as will become clear below, this assumption is self-consistent). Under this assumption, the analysis of Eqs. (65)–(68) was done in

Appendix A. Taking an infinitesimally small B near the critical point, it is shown there that

$$\xi_e, \xi_8, \xi_\Delta \sim \mathcal{O}(B^2) \qquad (72)$$

and that when the 2SC solution becomes unstable ($M_B^2 < 0$), a new solution occurs, *if* the parameters λ_{BC} and λ_{CD} satisfy

$$\lambda_{BC} > 0, \qquad \lambda_{CD} < 0 \qquad (73)$$

(in the next section, it will be shown that this constraint is satisfied indeed). The new solution is:

$$B_{\text{sol}} \equiv g\langle A_z^{(6)} \rangle \simeq \frac{-M_B^2}{3|T_{DB}|}\sqrt{\frac{-\lambda_{CD}}{\lambda_{BC}}},$$

$$C_{\text{sol}} \equiv g\langle A_z^{(1)} \rangle \simeq \sqrt{\frac{-M_B^2}{3\lambda_{BC}}}, \qquad D_{\text{sol}} \equiv g\langle A_0^{(3)} \rangle \simeq \frac{-M_B^2}{3T_{DB}},$$

$$(74)$$

where we neglected higher order terms in M_B^2. It is important that, as shown in Appendix A, the coefficients M_B^2, T_{DB}, λ_{BC}, and λ_{CD} in this near-critical solution are expressed through the 2SC values $\bar{\Delta}$, $\bar{\mu}_e$, and $\bar{\mu}_8$. Note that in Eq. (74) the convention $B > 0$ and $C > 0$ is chosen.

Near the critical point $M_B^2 = 0$, the solution behaves as

$$B_{\text{sol}} \propto -M_B^2, \qquad C_{\text{sol}} \propto \sqrt{-M_B^2}, \qquad D_{\text{sol}} \propto -M_B^2. \qquad (75)$$

These scaling relations are quite remarkable. While the scaling relation for C is of engineering type, those for B and D are not (the origin of this is of course in the presence of the dimensional coefficient T_{DB} in Eq. (74)). Such a scaling behavior implies that the B^4 and B^2D^2 terms in the effective potential are irrelevant near the critical point $M_B^2 = 0$. Omitting them, we arrive at the reduced effective potential:

$$\tilde{V}_{\text{eff}} = V_\Delta + \tfrac{1}{2}M_B^2 B^2 + T_{DB}DB^2 + \tfrac{1}{2}\lambda_{BC}B^2C^2 + \tfrac{1}{2}\lambda_{CD}C^2D^2. \qquad (76)$$

Let us now turn to the 2SC part V_Δ in Eq. (76). As shown in Appendix A, the difference of V_Δ in the new solution and that in the 2SC one is

$$V_\Delta(\Delta^{\text{sol}}, \mu_e^{\text{sol}}, \mu_8^{\text{sol}}) - V_\Delta(\bar{\Delta}, \bar{\mu}_e, \bar{\mu}_8) \sim \mathcal{O}(B^4). \qquad (77)$$

On the other hand, as follows from Eqs. (75) and (76), the difference $\tilde{V}_{\text{eff}} - V_\Delta \sim \mathcal{O}(B^3)$. This fact and Eq. (77) imply that in the leading approximation one can use the 2SC bar-quantities in calculating V_Δ in the reduced potential. In other words, the effective potential can be decomposed into the "constant" 2SC part V_Δ, with frozen fermion parameters, and the dynamical gluonic part:

$$\tilde{V}_{\text{eff}} \rightarrow \tilde{V}_{\text{eff}}(\bar{\Delta}, \bar{\mu}_e, \bar{\mu}_8; B, C, D)$$
$$= V_\Delta(\bar{\Delta}, \bar{\mu}_e, \bar{\mu}_8) + \tfrac{1}{2}M_B^2 B^2 + T_{DB}DB^2 + \tfrac{1}{2}\lambda_{BC}B^2C^2$$
$$+ \tfrac{1}{2}\lambda_{CD}C^2D^2. \qquad (78)$$

Equation (74) is the exact solution for the potential (78) and the energy density at the stationary point is

$$\tilde{V}_{\text{eff}}(\bar{\Delta}, \bar{\mu}_e, \bar{\mu}_8; B_{\text{sol}}, C_{\text{sol}}, D_{\text{sol}})$$
$$= V_\Delta + \frac{1}{6}M_B^2 B_{\text{sol}}^2 = V_\Delta - \frac{(-M_B^2)^3}{54T_{DB}^2}\left(-\frac{\lambda_{CD}}{\lambda_{BC}}\right) < V_\Delta. \quad (79)$$

Therefore the gluonic vacuum is more stable than the 2SC one.

In the description of the dynamics with vector condensates, there is a subtlety connected with the derivation of a *physical* effective potential, whose minima correspond to stable or metastable vacua. The point is that although the gauge symmetry is gone in the unitary gauge, the present theory still has constraints. In fact, it is a system with second-class constraints, similar to the theory of a free massive vector field A_μ described by the Proca Lagrangian (for a thorough discussion of systems with second-class constraints, see Sec. 2.3 in book [44]). In such theories, while the Lagrangian formalism can be used without introducing a gauge, the physical Hamiltonian is obtained by explicitly resolving the constraints.

In our case, this implies that to obtain the physical effective potential, one has to impose the Gauss's law constraint on the conventional effective potential $\tilde{V}_{\text{eff}}$ (78). This constraint amounts to integrating out the time-like components $A_0^{(a)}$. In the present approximation, the latter can be done by using their equations of motion, which are reduced to Eq. (67) for $D = gA_0^{(3)}$ in our case. Omitting the suppressed DB^2-term in this equation, we get

$$T_{DB}B^2 + \lambda_{CD}C^2D = 0. \qquad (80)$$

It leads to the physical effective potential without the nondynamical degree of freedom D:

$$\tilde{V}_{\text{eff}}^{\text{phys}} = V_\Delta + \frac{1}{2}M_B^2 B^2 + \frac{1}{2}\lambda_{BC}B^2C^2 - \frac{T_{DB}^2 B^4}{2\lambda_{CD}C^2}. \quad (81)$$

It is easy to show that solution (74) is a minimum by analyzing the curvature of $\tilde{V}_{\text{eff}}^{\text{phys}}$. Note that because of the constraint in Eq. (73), this potential is bounded from below.

In the next section, we will calculate M_B^2, T_{DB}, λ_{BC}, and λ_{CD}. In particular, it will be shown that constraint (73) is satisfied near the critical point.

B. Dynamics in one-loop approximation

In this subsection, we determine the GL effective potential (78) in one-loop approximation and derive the dispersion relations for quarks in the gluonic phase. The 2SC V_Δ part of the potential is known [10],

$$V_\Delta(\Delta, \mu_e, \mu_8) = \frac{\Delta^2}{4G_\Delta} - \frac{\mu_e^4}{12\pi^2} - \frac{\mu_{ub}^4}{12\pi^2} - \frac{\mu_{db}^4}{12\pi^2} - \frac{\tilde{\mu}^4}{3\pi^2}$$
$$- \frac{\Delta^2}{\pi^2}\left[\tilde{\mu}^2 - \frac{1}{4}\Delta^2\right]\ln\frac{4\Lambda^2}{\Delta^2}$$
$$- \frac{\Delta^2}{\pi^2}\left[\Lambda^2 - 2\tilde{\mu}^2 + \frac{1}{8}\Delta^2\right],$$
$$(\delta\mu < \Delta). \tag{82}$$

Here Λ is the ultraviolet cutoff in the NJL model and μ_{ub}, μ_{db}, and $\tilde{\mu}$ are given in Eqs. (8) and (9). For clarity of the presentation, the bars in Δ, μ_e, and μ_8 were omitted [$\mathcal{O}(\tilde{\mu}^2/\Lambda^2)$ and $\mathcal{O}(\Delta^2/\Lambda^2)$ and higher terms are neglected in this expression].[3] Note that the color and electrical charge neutrality conditions in the 2SC solution yield [10]

$$\delta\mu = \frac{3}{10}\mu - \frac{1}{5}\mu_8, \tag{83}$$

and

$$(\tilde{\mu}^2 + \delta\mu^2)\mu_8 = -\tilde{\mu}\Delta^2\left(\ln\frac{2\Lambda}{\Delta} - 1\right) + \tilde{\mu}(\delta\mu^2 + \mu_8^2)$$
$$- \frac{1}{3}\mu_8^3, \tag{84}$$

which is consistent with the result of Ref. [42], $\mu_8 \sim \mathcal{O}(\Delta^2/\mu)$, in the case of $\delta\mu = 0$. The size of the diquark gap Δ is essentially determined by tuning the NJL coupling constant G_Δ and cutoff Λ.

In Appendix B, after straightforward but tedious calculations of relevant one-loop diagrams from the fermion trace in Eq. (29), we find the following relations in the region $\delta\mu < \Delta$:

$$\lambda_{BC} = \frac{1}{80\pi^2}\frac{\tilde{\mu}^2}{\Delta^2}\left[-1 + 8\frac{\delta\mu^2}{\Delta^2}\left(1 - \frac{\delta\mu^2}{\Delta^2}\right)\right], \tag{85}$$

$$\lambda_{CD} = -\frac{1}{g^2} - \frac{1}{18\pi^2}\frac{\tilde{\mu}^2}{\Delta^2}, \tag{86}$$

$$T_{DB} = \frac{\mu_8}{2g^2} + \frac{\mu_8}{24\pi^2}\frac{\tilde{\mu}^2}{\Delta^2}\left(-1 + 8\frac{\delta\mu^4}{\Delta^4}\right)$$
$$+ \frac{\tilde{\mu}}{48\pi^2}\left(-1 + 4\frac{\delta\mu^2}{\Delta^2} + 8\frac{\delta\mu^4}{\Delta^4}\right). \tag{87}$$

Here the tree contribution of gluons

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[3]For realistic values $\Lambda = (1.5 - 2.0)\mu$ that we use, while the contribution of the $\mathcal{O}(\Delta^2/\Lambda^2)$ terms are parametrically suppressed, one can show that the contribution of the $\mathcal{O}(\tilde{\mu}^2/\Lambda^2)$ terms is numerically suppressed.

$$V_g \equiv -\mathcal{L}_g = -\frac{1}{2}F_{0j}^{(a)}F_{0j}^{(a)}$$
$$= -\frac{1}{2g^2}\mu_8^2 B^2 + \frac{1}{2g^2}\mu_8 DB^2 - \frac{1}{8g^2}B^2D^2$$
$$- \frac{1}{2g^2}C^2D^2 \tag{88}$$

was also taken into account. As to the coefficient M_B^2, its expression follows directly from Eqs. (61) and (88):

$$M_B^2 = \frac{1}{g^2}(-\mu_8^2 + m_{M,4}^2) = -\frac{\mu_8^2}{g^2} + \frac{\tilde{\mu}^2}{6\pi^2}\left(1 - \frac{2\delta\mu^2}{\Delta^2}\right). \tag{89}$$

We see that the coefficient λ_{CD} in (86) is definitely negative. The parameter M_B^2, which is expressed through the Meissner mass (61), is negative when

$$\delta\mu > \delta\mu_{\text{cr}}, \qquad \delta\mu_{\text{cr}} = \frac{\Delta}{\sqrt{2}}\sqrt{1 - \frac{3\pi}{2\alpha_s}\frac{\mu_8^2}{\tilde{\mu}^2}}, \tag{90}$$
$$\alpha_s \equiv \frac{g^2}{4\pi}.$$

Relation (83) and Eq. (9) yield

$$\tilde{\mu} = \frac{9}{10}\mu + \frac{2}{5}\mu_8, \tag{91}$$

and, at the critical point, we find from Eqs. (83), (84), (90), and (91) that μ_8 is approximately

$$\mu_8^{(\text{sol})} = \frac{3 - \ln\frac{200\Lambda^2}{9\mu^2}}{12 + \frac{4}{9}(\ln\frac{200\Lambda^2}{9\mu^2} - 2)}\mu. \tag{92}$$

For realistic values $\Lambda = (1.5–2.0)\mu$ and $\alpha_s = 0.75–1.0$, we obtain numerically

$$\frac{3\pi}{2\alpha_s}\frac{\mu_8^2}{\tilde{\mu}^2}\bigg|_{\mu_8=\mu_8^{(\text{sol})}} = 0.03–0.1. \tag{93}$$

This implies that the tree gluon contribution in Eq. (90) decreases the value of $\delta\mu_{\text{cr}}$ by 1.5%–5% in comparison to its value in the (nongauged) NJL model. The smallness of this correction is in accordance with the dominance of hard dense loop diagrams.

Let us now turn to the coefficient λ_{BC} (85). At the critical point $\delta\mu = \delta\mu_{\text{cr}}$, it is (henceforth we will not show explicitly the superscript (sol) in μ_8):

$$\lambda_{BC} = \frac{1}{80\pi^2}\frac{\tilde{\mu}^2}{\Delta^2}\left(1 - \frac{9\pi^2}{2\alpha_s^2}\frac{\mu_8^4}{\tilde{\mu}^4}\right). \tag{94}$$

Because the $\mu_8^4/\tilde{\mu}^4$-term is negligibly small, we conclude that the coefficient λ_{BC} is positive near the critical point. Thus, constraint (73) is satisfied indeed.

Utilizing Eqs. (85)–(87) and (89) in Eq. (74), one can obtain the solutions for B, C, and D in the near-critical region. Indeed, neglecting higher order terms in μ_8/μ in Eqs. (85)–(87) and (89), we get the approximate relations

$$M_B^2 \simeq \frac{\tilde{\mu}^2}{6\pi^2}\left(1 - \frac{\delta\mu^2}{\delta\mu_{\mathrm{cr}}^2}\right),$$

$$\lambda_{BC} \simeq \frac{9}{160\pi^2},$$

$$\lambda_{CD} \simeq -\frac{1}{4\pi\alpha_s} - \frac{1}{4\pi^2}, \quad (95)$$

$$T_{DB} \simeq \frac{\tilde{\mu}}{16\pi^2} + \frac{\mu_8}{16\pi^2}\left(3 + \frac{2\pi}{\alpha_s}\right),$$

which lead us to the near-critical solution:

$$B_{\mathrm{sol}} = \frac{\delta\mu^2 - \delta\mu_{\mathrm{cr}}^2}{\delta\mu_{\mathrm{cr}}^2} \frac{16\tilde{\mu}\sqrt{10(1 + \frac{\pi}{\alpha_s})}}{27[1 + \frac{\mu_8}{\tilde{\mu}}(3 + \frac{2\pi}{\alpha_s})]}, \quad (96)$$

$$C_{\mathrm{sol}} = \frac{\sqrt{\delta\mu^2 - \delta\mu_{\mathrm{cr}}^2}}{\delta\mu_{\mathrm{cr}}} \frac{4\sqrt{5}\tilde{\mu}}{9}, \quad (97)$$

$$D_{\mathrm{sol}} = \frac{\delta\mu^2 - \delta\mu_{\mathrm{cr}}^2}{\delta\mu_{\mathrm{cr}}^2} \frac{8\tilde{\mu}}{9[1 + \frac{\mu_8}{\tilde{\mu}}(3 + \frac{2\pi}{\alpha_s})]}. \quad (98)$$

It is noticeable that this solution describes nonzero field strengths $F_{\mu\nu}^{(a)}$ which correspond to the presence of *non-Abelian* constant chromoelectriclike condensates in the ground state:

$$\langle E_3^{(2)}\rangle = \langle F_{03}^{(2)}\rangle = \frac{1}{g}C_{\mathrm{sol}}D_{\mathrm{sol}}, \quad (99)$$

$$\langle E_3^{(7)}\rangle = \langle F_{03}^{(7)}\rangle = \frac{1}{2g}B_{\mathrm{sol}}(2\mu_8 - D_{\mathrm{sol}}). \quad (100)$$

We emphasize that while an Abelian constant electric field in different media always leads to an instability,[4] non-Abelian constant chromoelectric fields do not in many cases: For a thorough discussion of the stability problem for constant $SU(2)$ non-Abelian fields in theories with zero

baryon density, see Ref. [45]. On a technical side, this difference is connected with that while a vector potential corresponding to a constant Abelian electric field depends on spatial and/or time coordinates, a constant non-Abelian chromoelectric field is expressed through constant vector potentials, as takes place in our case, and therefore momentum and energy are good quantum numbers in the latter.

In order to illustrate the stability issue in the gluonic phase, let us consider the dispersion relations for quarks there. Because the vacuum expectation values (96)–(98) are small near the critical point and because red and green quarks are gapped in the 2SC phase, the dispersion relations for gapless blue up and down quarks are of the most interest. From Eq. (26) we find that up to the first order in B^2 they are

$$p_{ub}^0 = |\vec{p}| - \mu_{ub} + \frac{B_{\mathrm{sol}}^2}{4}\frac{1}{2|\vec{p}| + \mu_8 + \frac{\Delta^2}{2\tilde{\mu}-\mu_8}}$$

$$- \frac{B_{\mathrm{sol}}^2}{4}\frac{(p^3)^2}{\vec{p}^2}\left(\frac{1}{2|\vec{p}| + \mu_8 + \frac{\Delta^2}{2\tilde{\mu}-\mu_8}}\right.$$

$$\left. + \frac{2(|\vec{p}| - \tilde{\mu}) + \mu_8}{\Delta^2 - \mu_8^2 - 2\mu_8(|\vec{p}| - \tilde{\mu})}\right), \quad (101)$$

$$p_{db}^0 = |\vec{p}| - \mu_{db} + \frac{B_{\mathrm{sol}}^2}{4}\frac{1}{2|\vec{p}| + \mu_8 + \frac{\Delta^2}{2\tilde{\mu}-\mu_8}}$$

$$- \frac{B_{\mathrm{sol}}^2}{4}\frac{(p^3)^2}{\vec{p}^2}\left(\frac{1}{2|\vec{p}| + \mu_8 + \frac{\Delta^2}{2\tilde{\mu}-\mu_8}}\right.$$

$$\left. + \frac{2(|\vec{p}| - \tilde{\mu}) + \mu_8}{\Delta^2 - \mu_8^2 - 2\mu_8(|\vec{p}| - \tilde{\mu})}\right). \quad (102)$$

The B^2-terms in Eqs. (101) and (102) lead to nonspherical Fermi surfaces determined by the following equations:

$$|\vec{p}| = \mu_{ub} - \frac{B_{\mathrm{sol}}^2\sin^2\theta}{4}\frac{1}{2\mu_{ub} + \mu_8 + \frac{\Delta^2}{2\mu_{ub}+\mu_e+\mu_8}} - \frac{B_{\mathrm{sol}}^2\cos^2\theta}{4}\frac{\mu_e + \mu_8}{\Delta^2 + \mu_8\mu_e + \mu_8^2}, \quad \text{(blue up)} \quad (103)$$

$$|\vec{p}| = \mu_{db} - \frac{B_{\mathrm{sol}}^2\sin^2\theta}{4}\frac{1}{2\mu_{db} + \mu_8 + \frac{\Delta^2}{2\mu_{db}-\mu_e+\mu_8}} + \frac{B_{\mathrm{sol}}^2\cos^2\theta}{4}\frac{\mu_e - \mu_8}{\Delta^2 - \mu_8\mu_e + \mu_8^2}, \quad \text{(blue down)} \quad (104)$$

where we neglected higher order terms in B^2 and defined the angle θ,

$$p^3 \equiv |\vec{p}|\cos\theta. \quad (105)$$

The dispersion relations (101) and (102) clearly show that there is no instability in the quark sector in this problem.

As to bosonic degrees of freedom (gluons and composite bosons), because it is very involved to derive their derivative terms from the fermion loop in the gluonic phase, this issue is beyond the scope of this paper. It is however noticeable that there are no instabilities for bosons in a phase with vector condensates in the gauged σ-model with a chemical potential for hypercharge [35]. Although that

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[4]In metallic and superconducting media, such an instability is classical in its origin. In semiconductors and insulators, this instability is manifested in a creation of electron-hole pairs through a quantum tunneling process.

model is much simpler than the present one, its phase with vector condensates has many common features with the gluonic phase and this fact is encouraging. Note that among light collective excitations in the gluonic phase, there should be Nambu-Goldstone bosons connected with the spontaneous breakdown of the $SO(3)_{\text{rot}}$ and the $\tilde{U}(1)_{\text{em}}$ (in the presence of a photon field, the latter will be absorbed in the electric (longitudinal) part of the field).

We emphasize that these constant color condensates in the gluonic phase do *not* produce long range color *forces* acting on quasiparticles. This can be seen from the dispersion relations (101) and (102) for quarks in this model. They show that momentum and energy are conserved numbers. It would be of course impossible in the presence of long range forces. The role of these condensates is actually more dramatic: They change the structure of the ground state, making it anisotropic and (electrically) superconducting. Only in this sense, one can speak about a long range character of the condensates.

C. Searching for other solutions

Are there solutions of Eqs. (65)–(67) other than that found in the previous section and the trivial one with $B_{\text{sol}} = C_{\text{sol}} = D_{\text{sol}} = 0$? We will address this question in this subsection.

First of all, it is easy to see that there are two *formally* nontrivial solutions with $B_{\text{sol}} = 0$: (a) $C_{\text{sol}} = 0$, D_{sol}-arbitrary and (b) $D_{\text{sol}} = 0$, C_{sol}-arbitrary. However, as follows from Eq. (63), both these solutions lead to zero $SU(2)_c$ field strength $f_{\mu\nu}^{(l)}$ (38) and therefore they are gauge equivalent to the 2SC solution without vector condensates.

It is also easy to check that for $D_{\text{sol}} = 0$, Eqs. (65)–(67) lead to $B_{\text{sol}} = 0$, $C_{\text{sol}} = 0$. Therefore we conclude that both B and D have to be nonzero in a nontrivial solution. Physics underlying this conclusion is clear. When $B_{\text{sol}} \equiv g\langle A_z^{(6)}\rangle \neq 0$, the color charge densities of red and green quarks are generally different, so that $D_{\text{sol}} \equiv g\langle A_0^{(3)}\rangle \neq 0$ is also required for color neutrality (recall that $gA_0^{(3)}$ can be considered as a chemical potential μ_3 related to the third component of the color isospin).

There does exist one additional nontrivial solution with nonzero $\langle E_3^{(7)}\rangle = \langle F_{03}^{(7)}\rangle$ (see Eq. (100)):

$$C_{\text{sol}} = 0, \qquad D_{\text{sol}} = -\frac{T_{DB}}{\lambda_{BD}},$$

$$B_{\text{sol}}^2 = -\frac{M_B^2}{\lambda_B} + \frac{T_{DB}^2}{\lambda_B \lambda_{BD}}. \tag{106}$$

This solution corresponds to a phase in which while the gauge $SU(2)_c$ symmetry and the rotational $SO(3)_{\text{rot}}$ are broken, the electromagnetic $\tilde{U}(1)_{\text{em}}$ is exact. While physics in this phase is quite interesting, there is the following problem in justifying its existence. As follows from Eq. (95), the coefficient T_{DB} does not approach zero at

the critical point $M_B^2 = 0$. This and Eq. (106) imply that the values of B_{sol} and D_{sol} in this solution are also nonzero at the critical point. Therefore the solution corresponds to the first order phase transition and the GL approach is not appropriate in this case. Since the derivation of Eqs. (65)–(67), (78), and (81) was based on this approach, all these equations themselves will be modified in this phase. Therefore the question concerning the existence of solution (106) is open. It seems that a numerical analysis would be the only reliable way to answer it.

In conclusion, the following two remarks are in order. (a) Since both solutions (96)–(98) and (106) are cylindrically symmetric, it will be appropriate to call the phase corresponding to solutions (96)–(98) a *gluonic cylindrical phase* I and the phase corresponding to solution (106) a *gluonic cylindrical phase* II. (b) The fact that any nontrivial solution should have both B and D to be nonzero follows from the presence of the triple vertex $T_{DB}DB^2$ in the GL potential: for $B \neq 0$, this vertex inevitably leads to a nonzero D. One can call this a tadpole mechanism: For a given B, the diagram corresponding to such a vertex is a tadpole with a fermion loop (with B insertions), producing the coefficient T_{DB}, and with a tail being the field D.

D. Confinement picture and exotic hadrons in gluonic phase

In this subsection, we will describe some additional features of the gluonic phase. In particular, we will describe the confinement picture, which can be appropriate for the description of its dynamics in the infrared region, with the energy scale of order $\Lambda'_{\text{QCD}} \lesssim \mathcal{O}(10 \text{ MeV})$ (or even much smaller) [22]. As we will see, the dynamics in this dense medium includes light *exotic* vector mesons some of which can condense.

The gluon condensates are mostly generated at energy scales between the confinement scale in the 2SC state Λ'_{QCD} and the quark chemical potential, which is about 300–500 MeV. It is the same region where the chromomagnetic instability in the 2SC phase is created and where the hard dense loop approximation is (at least qualitatively) reliable. At such scales, gluons are still appropriate dynamical degrees of freedom and utilizing the Higgs approach with color condensates in a particular gauge is appropriate and consistent: It is a region of hard physics. Because the gluonic cylindrical phase I occurs as a result of a conventional second order phase transition, the gluon condensates are very small only in the immediate surroundings of the critical point $\delta\mu = \bar{\Delta}/\sqrt{2}$. Outside that region, their values should be of the order of the typical scale $\delta\mu \sim \bar{\Delta} \sim 50$–100 MeV. As to the gluonic cylindrical phase II, because it is connected with a first order phase transition, the situation depends on whether it is a strong or a weak one. While for the former, one could expect that the gluon condensates are of order $\delta\mu \sim \bar{\Delta}$ even at the nearest surrounding of the critical point [see Eq. (106)], for the

latter, they could be of order Λ'_{QCD} there, modulo the question of the existence of this phase (see the discussion in the previous section).

The description of the dynamics in the gluonic phases in the infrared region depends on the value of the gluon condensates. If they are essentially larger than Λ'_{QCD}, then the Higgs description is appropriate even in the infrared region, similarly as it happens in the electroweak theory. Indeed, when the $SU(2)_c$ gauge symmetry becomes completely broken by the dynamics with a characteristic scale being essentially larger than Λ'_{QCD}, the strong coupling dynamics presented in the 2SC solution at the scale of order Λ'_{QCD} is washed out. In this regard, the gluonic phases with large vector condensates are similar to the color-flavor locked (CFL) phase with a large quark chemical potential μ, where the color condensates (although not vector ones) completely break the $SU(3)_c$ color gauge symmetry [46].

But what happens if the gluon condensates are small, $\lesssim \Lambda'_{\text{QCD}}$? This regime corresponds to the near-critical dynamics in the gluonic cylindrical phase I and the confinement picture should be appropriate for the description of the infrared dynamics in this case.

In order to answer this question, note the following. As one can see in Tables I and II, the electric charge $\tilde{Q}_{\text{em}}$ and the baryon number $\tilde{\mathcal{B}}$ are integers both for gluons and quarks. Do they describe hadroniclike excitations? We believe that the answer is "yes." The point is that in models with Higgs fields in the (anti)fundamental representation of the gauge group, there is no phase transition between Higgs and confinement phases [23–25] and this is the case in the present model. Indeed, in the 2SC phase, the breakdown $SU(3)_c \to SU(2)_c$ is triggered by the diquark condensate, which is assigned to the antifundamental representation of the $SU(3)_c$, and gauge $SU(2)_c$ symmetry breaking occurs when the $SU(2)_c$ doublet vector field ϕ_μ develops the VEV. Because of that, we can apply the complementarity principle [23–25] for the description of the dynamics in the gluonic phase with small condensates in the infrared region. What matters is the existence of the unitary gauge given by constraints (11) and (55). In this gauge all gauge dependent degrees of freedom are removed.

Because of the complementarity principle, the Higgs and confinement phases provide dual, and physically equivalent, descriptions of dynamics. In particular, they provide two complementary descriptions of a spontaneous breakdown of global symmetries, such as the rotational $SO(3)$ and the electromagnetic $U(1)$ in the gluonic cylindrical phase I and the rotational $SO(3)$ in the gluonic cylindrical phase II. Following Ref. [25], we will consider the dual, gauge invariant, approach in this model and show that all the gluonic and quark fields can indeed be replaced by colorless composite ones.

The flavor quantum numbers of these composite fields are described by the conventional electric and baryon charges Q_{em} and $\mathcal{B}$. They are integers and coincide with those the operators $\tilde{Q}_{\text{em}}$ and $\tilde{\mathcal{B}}$ yield for gluonic and quark fields. The composite fields in the confinement picture should coincide with the corresponding fields in the Higgs picture in the unitary gauge in the classical approximation.

The nonzero VEVs in the Higgs picture (common for these two gluonic phases) are[5]

$$\langle \mathbf{\Phi} \rangle = (0, 0, \bar{\Delta})^T, \qquad \mu_3 \equiv D_{\text{sol}} = g\langle A_0^{(3)} \rangle,$$
$$\mu_8 = \tfrac{\sqrt{3}}{2} g\langle A_0^{(8)} \rangle, \qquad B_{\text{sol}} = g\langle A_z^{(6)} \rangle, \tag{107}$$

and thereby[6]

$$\langle iD_z \mathbf{\Phi}^* \rangle = (0, B_{\text{sol}} \bar{\Delta}/2, 0)^T,$$
$$g\langle F_{0z}^7 \rangle = B_{\text{sol}}\left(\mu_8 - \frac{\mu_3}{2} \right), \tag{108}$$
$$g\langle F_{0z} \mathbf{\Phi}^* \rangle = -i \frac{B_{\text{sol}} \bar{\Delta}}{4} (2\mu_8 - \mu_3)(0, 1, 0)^T,$$

where

$$\mathbf{\Phi} \equiv \begin{pmatrix} \Phi^r \\ \Phi^g \\ \Phi^b \end{pmatrix}, \qquad F_{\mu\nu} \equiv F_{\mu\nu}^{(a)} T^a, \qquad T^a \equiv \frac{\lambda^a}{2} \tag{109}$$

with λ^a's being the $SU(3)_c$ Gell-Mann matrices. In the classical approximation, we replace the above fields by their VEVs. Then the following composite fields can be written in terms of the elementary fields:

$$D_\mu \mathbf{\Phi}^* \to -\frac{ig}{2} \bar{\Delta} \begin{pmatrix} A_\mu^{(4)} - iA_\mu^{(5)} \\ A_\mu^{(6)} - iA_\mu^{(7)} \\ -\frac{2}{\sqrt{3}} A_\mu^{(8)} \end{pmatrix}, \tag{110}$$

$$F_{\mu z} \mathbf{\Phi}^* \to -\frac{i}{4} B_{\text{sol}} \bar{\Delta} \begin{pmatrix} A_\mu^{(1)} - iA_\mu^{(2)} \\ -A_\mu^{(3)} + \sqrt{3} A_\mu^{(8)} \\ 2iA_\mu^{(7)} \end{pmatrix}, \tag{111}$$

and

$$F_{0j} \mathbf{\Phi}^* \to -\frac{i}{4} \bar{\Delta} \begin{pmatrix} (2\mu_8 + \mu_3)(A_j^{(4)} - iA_j^{(5)}) \\ (2\mu_8 - \mu_3)(A_j^{(6)} - iA_j^{(7)}) \\ 0 \end{pmatrix}. \tag{112}$$

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[5]Recall that we do not show explicitly the superscript (sol) in the chemical potentials μ_3 and μ_8.

[6]Recall that the diquark field $\mathbf{\Phi}$ is an antitriplet under the $SU(3)_c$ symmetry.

By using the above relations, we can construct composite fermions and bosons in the confining picture. For example, the (up and down) blue quark fields can be rewritten as $\mathbf{\Phi}^T\boldsymbol{\psi}_i$ with $\boldsymbol{\psi}_i \equiv (\psi_{ir}, \psi_{ig}, \psi_{ib})^T$, $(i = u, d)$. Note that in the classical approximation the composite fields $\mathbf{\Phi}^T\boldsymbol{\psi}_i$ yield $\bar{\Delta}\psi_{ib}$. The green quarks in the confinement picture can be described by $(D_z\mathbf{\Phi}^*)^\dagger\boldsymbol{\psi}_i \to \frac{1}{2}B_{\text{sol}}\bar{\Delta}\psi_{ig}$. By using the epsilon tensor, we can rewrite the red quarks as $\epsilon^{\alpha\beta\gamma}\psi_{i\alpha}(D_z\mathbf{\Phi}^*)_\beta(\mathbf{\Phi}^*)_\gamma \to \frac{1}{2}B_{\text{sol}}\bar{\Delta}^2\psi_{ir}$. For the diquark field, only the real part of the antiblue one is a physical degree of freedom. The composite field is $\mathbf{\Phi}^\dagger\mathbf{\Phi} \to 2\bar{\Delta}\,\text{Re}\Phi^b$. For other fields, see Table III.

Similarly, we can construct the vector composite fields. For example, we find $(D_z\mathbf{\Phi}^*)^\dagger(D_\mu\mathbf{\Phi}^*) \to \frac{1}{4}B_{\text{sol}}\bar{\Delta}^2(A_\mu^{(6)} - iA_\mu^{(7)}) \sim \phi_\mu^g$, $\epsilon^{\alpha\beta\gamma}(D_\mu\mathbf{\Phi}^*)_\alpha(D_z\mathbf{\Phi}^*)_\beta \times (\mathbf{\Phi}^*)_\gamma \to \frac{1}{4}B_{\text{sol}}\bar{\Delta}^3(A_\mu^{(4)} - iA_\mu^{(5)}) \sim \phi_\mu^r$, $\epsilon^{\alpha\beta\gamma}(F_{z\mu}\mathbf{\Phi}^*)_\alpha \times (D_z\mathbf{\Phi}^*)_\beta(\mathbf{\Phi}^*)_\gamma \to \frac{1}{8}B_{\text{sol}}^2\bar{\Delta}^3(A_\mu^{(1)} - iA_\mu^{(2)}) \sim \mathcal{A}_\mu^-$, etc. We summarize them in Table IV.

Some of the vector mesons in Table IV are exotic because they carry baryon charge. (In vacuum QCD, mesons carry of course no baryon charge.) For example, the electric and baryon charges Q_{em} and $\mathcal{B}$ of vector mesons corresponding to $A_\mu^{(\pm)} = A_\mu^{(1)} \pm iA_\mu^{(2)}$ gluons are equal to ± 1 and ± 2, respectively. The origin of these exotic quantum numbers is connected with (anti)diquarks, which are constituents of these mesons (see Table IV). Indeed, (anti)-diquarks are bosons carrying the baryon charge $\pm 2/3$ and therefore are exotic themselves.

This feature has a dramatic consequence for the gluonic cylindrical phase I. Since in the Higgs description of this phase $A^{(\pm)}$ gluons are condensed (leading to the spontaneous $\tilde{U}(1)_{\text{em}}$ breakdown), we conclude that in the con-

TABLE III. Composite fermions and scalar fields in the confinement picture.

	Q_{em}	$\mathcal{B}$	I_3
$\epsilon^{\alpha\beta\gamma}\psi_{u\alpha}(D_z\mathbf{\Phi}^*)_\beta(\mathbf{\Phi}^*)_\gamma \sim \psi_{ur}$	0	-1	$\frac{1}{2}$
$(D_z\mathbf{\Phi}^*)^\dagger\boldsymbol{\psi}_u \sim \psi_{ug}$	1	1	$\frac{1}{2}$
$\epsilon^{\alpha\beta\gamma}\psi_{u\alpha}(F_{0z}\mathbf{\Phi}^*)_\beta(\mathbf{\Phi}^*)_\gamma \sim \psi_{ur}$	0	-1	$\frac{1}{2}$
$(F_{0z}\mathbf{\Phi}^*)^\dagger\boldsymbol{\psi}_u \sim \psi_{ug}$	1	1	$\frac{1}{2}$
$\mathbf{\Phi}^T\boldsymbol{\psi}_u \sim \psi_{ub}$	1	1	$\frac{1}{2}$
$\epsilon^{\alpha\beta\gamma}\psi_{d\alpha}(D_z\mathbf{\Phi}^*)_\beta(\mathbf{\Phi}^*)_\gamma \sim \psi_{dr}$	-1	-1	$-\frac{1}{2}$
$(D_z\mathbf{\Phi}^*)^\dagger\boldsymbol{\psi}_d \sim \psi_{dg}$	0	1	$-\frac{1}{2}$
$\epsilon^{\alpha\beta\gamma}\psi_{d\alpha}(F_{0z}\mathbf{\Phi}^*)_\beta(\mathbf{\Phi}^*)_\gamma \sim \psi_{dr}$	-1	-1	$-\frac{1}{2}$
$(F_{0z}\mathbf{\Phi}^*)^\dagger\boldsymbol{\psi}_d \sim \psi_{dg}$	0	1	$-\frac{1}{2}$
$\mathbf{\Phi}^T\boldsymbol{\psi}_d \sim \psi_{db}$	0	1	$-\frac{1}{2}$
$\mathbf{\Phi}^\dagger\mathbf{\Phi} \sim \text{Re}\Phi^b$	0	0	0
$(F_{0z}\mathbf{\Phi}^*)^\dagger(F_{0z}\mathbf{\Phi}^*) \sim \text{Re}\Phi^b$	0	0	0

TABLE IV. Composite vectors in the confinement picture.

	Q_{em}	$\mathcal{B}$	I_3
$\epsilon^{\alpha\beta\gamma}(F_{z\mu}^*\mathbf{\Phi})_\alpha(D_z^*\mathbf{\Phi})_\beta(\mathbf{\Phi})_\gamma \sim \mathcal{A}_\mu^+$	1	2	0
$(D_z\mathbf{\Phi}^*)^\dagger(F_{z\mu}\mathbf{\Phi}^*) + (\text{H.c.}) \sim A_\mu^{(3)}, A_\mu^{(8)}$	0	0	0
$\epsilon^{\alpha\beta\gamma}(F_{z\mu}\mathbf{\Phi}^*)_\alpha(D_z\mathbf{\Phi}^*)_\beta(\mathbf{\Phi}^*)_\gamma \sim \mathcal{A}_\mu^-$	-1	-2	0
$\epsilon^{\alpha\beta\gamma}(F_{0j}^*D_z^*\mathbf{\Phi})_\alpha(D_z^*\mathbf{\Phi})_\beta(\mathbf{\Phi})_\gamma \sim \mathcal{A}_j^+$	1	2	0
$i(D_z\mathbf{\Phi}^*)^\dagger(D_\mu D_z\mathbf{\Phi}^*) + (\text{H.c.}) \sim A_\mu^{(3)}, A_\mu^{(8)}$	0	0	0
$\epsilon^{\alpha\beta\gamma}(F_{0j}D_z\mathbf{\Phi}^*)_\alpha(D_z\mathbf{\Phi}^*)_\beta(\mathbf{\Phi}^*)_\gamma \sim \mathcal{A}_j^-$	-1	-2	0
$\epsilon^{\alpha\beta\gamma}(D_\mu^*\mathbf{\Phi})_\alpha(D_z^*\mathbf{\Phi})_\beta\mathbf{\Phi}_\gamma \sim \phi_\mu^{*r}$	1	2	0
$(D_z\mathbf{\Phi}^*)^\dagger(D_\mu^*\mathbf{\Phi}) \sim \phi_\mu^{*g}$	0	0	0
$\epsilon^{\alpha\beta\gamma}(F_{0j}^*\mathbf{\Phi})_\alpha(D_z^*\mathbf{\Phi})_\beta(\mathbf{\Phi})_\gamma \sim \phi_j^{*r}$	1	2	0
$(D_z^*\mathbf{\Phi})^\dagger(F_{0j}^*\mathbf{\Phi}) \sim \phi_j^{*g}$	0	0	0
$\epsilon^{\alpha\beta\gamma}(D_\mu\mathbf{\Phi}^*)_\alpha(D_z\mathbf{\Phi}^*)_\beta(\mathbf{\Phi}^*)_\gamma \sim \phi_\mu^r$	-1	-2	0
$(D_z\mathbf{\Phi}^*)^\dagger(D_\mu\mathbf{\Phi}^*) \sim \phi_\mu^g$	0	0	0
$\epsilon^{\alpha\beta\gamma}(F_{0j}\mathbf{\Phi}^*)_\alpha(D_z\mathbf{\Phi}^*)_\beta(\mathbf{\Phi}^*)_\gamma \sim \phi_j^r$	-1	-2	0
$(D_z\mathbf{\Phi}^*)^\dagger(F_{0j}\mathbf{\Phi}^*) \sim \phi_j^g$	0	0	0
$\mathbf{\Phi}^T(iD_\mu\mathbf{\Phi}^*) + (\text{H.c.}) \sim A_\mu^{(8)}$	0	0	0

finement picture this corresponds to a condensation of *exotic* vector mesons. In this regard, it is appropriate to mention that some authors speculated about a possibility of a condensation of vector ρ mesons in dense baryon matter [47]. The dynamics in the gluonic phase yields a scenario even with a more unexpected condensation.

VI. DYNAMICS WITH GLUON CONDENSATES: GENERAL ANALYSIS AND CLASSIFICATION

In the previous sections, the dynamics with gluon condensates connected with the instability for the 4–7th gluons in the 2SC phase was considered. The question is whether there are gluonic phases other than those two described in Sec. V. In particular, an interesting issue is the dynamics of gluon condensates connected with the instability for the 8th gluon in the g2SC phase. Moreover, our consideration of the gluonic phase in Sec. V was somewhat heuristic. We used ansatz (63) without addressing the question whether it is self-consistent, i.e., whether for VEVs B, C, $D \neq 0$ the equations of motion will or will not lead to nonzero VEVs of other gluon fields. These questions will be addressed in the present section. The general analysis we will use is based on the symmetry consideration and the GL effective theory approach.

A. Symmetry breaking structure

In this subsection, we consider possible symmetry breaking samples in the gauged NJL model and the structure of the corresponding homogeneous gluon condensates. Our strategy is the following. For each symmetry breaking sample, we will pick up the maximal set of gluon condensates consistent with it. The dynamics of course

could allow a subset of this set to be a solution. This much harder issue is intimately connected with the structure of the GL effective theory corresponding to the symmetry sample and will be discussed in Subsec. VI B.

In the present analysis, it will be convenient to use the (partly unitary) gauge (11) for the diquark field Φ^α. Then the symmetry in the 2SC phase is that presented in Eq. (43). In the general case, the homogeneous gluon condensates consist of 32 VEVs, $\langle A_\mu^{(a)} \rangle$, ($a = 1, 2, \cdots, 8$,

$\mu = 0, x, y, z$). The symmetry (43) contains 9 parameters. However, since all gluon fields are singlet with respect to the chiral group $[U(1)_{\tau_L^3} \times U(1)_{\tau_R^3}]_\chi$, only 7 parameters connected with the group $SU(2)_c \times \tilde{U}(1)_{\rm em} \times SO(3)_{\rm rot}$ matter. By using the corresponding transformations, the 32 VEVs can be reduced to 25 ones.

Let us show that these 25 VEVs can be chosen in the form:

$$\langle A_z^{(8)} \rangle, \tag{113a}$$

$$\langle \phi_x \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \langle A_x^{(4)} \rangle - i\langle A_x^{(5)} \rangle \\ \langle A_x^{(6)} \rangle - i\langle A_x^{(7)} \rangle \end{pmatrix}, \qquad \langle \phi_y \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \langle A_y^{(4)} \rangle \\ \langle A_y^{(6)} \rangle \end{pmatrix}, \qquad \langle \phi_z \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \langle A_z^6 \rangle \end{pmatrix}, \tag{113b}$$

$$\langle A_j^{(\ell)} \rangle, \qquad (\ell = 1, 2, 3, \; j = x, y, z), \tag{113c}$$

and

$$\langle A_0^{(a)} \rangle \qquad (a = 1, 2, \cdots, 8), \tag{113d}$$

while $\langle A_x^{(8)} \rangle = \langle A_y^{(8)} \rangle = \langle A_y^{(5)} \rangle = \langle A_y^{(7)} \rangle = \langle A_z^{(4)} \rangle = \langle A_z^{(5)} \rangle = \langle A_z^{(7)} \rangle = 0$.

The proof is going as follows. Because of the rotational symmetry, without loss of generality, we can choose one direction for the VEV of the spatial component of the 8th gluon, say, $\langle A_z^{(8)} \rangle \neq 0$. We may apply the $SU(2)_c$ symmetry to the same spatial component of ϕ_j and thereby obtain $\langle \phi_z^T \rangle \sim (0, \langle A_z^{(6)} \rangle)$. Note that we can still rotate ϕ_x and ϕ_y by a $SO(2)$ spatial rotation around the z-axis and vary the upper component of ϕ_j by a $U(1)$ transformation with the generator which is an appropriate linear combination of σ^3 in the color $SU(2)_c$ and the charge $\tilde{Q}_{\rm em}$. By using these transformations, we can choose $\langle \phi_y \rangle$ to be real. As a result, we can reduce general homogeneous gluon condensates to the 25 VEVs indicated above.

This set of 25 VEVs breaks symmetry (43) down to the chiral $[U(1)_{\tau_L^3} \times U(1)_{\tau_R^3}]_\chi$ [henceforth this irrelevant for our discussion chiral group will be omitted]. When we take subsets of the set in Eq. (113), typical symmetry breaking patterns are[7]:

$SU(2)_c \times \tilde{U}(1)_{\rm em} \times SO(3)_{\rm rot}$

$\quad \to SU(2)_c \times \tilde{U}(1)_{\rm em} \times SO(2)_{\rm rot}, \quad$ (phase A),

$\quad \to \tilde{U}(1)_{\rm em} \times SO(2)_{\rm rot}, \quad$ (phase B),

$\quad \to SO(2)_{\rm rot}, \quad$ (phase C),

$\quad \to SO(2)_{\rm diag}, \quad$ (phase D),

$\quad \to \tilde{U}(1)_{\rm em}, \quad$ (phase E),

$\quad \to {\rm nothing}, \quad$ (phase F).

The maximal subsets of the gluon condensates consistent with these symmetry breaking patterns are

(phase A) $\langle A_0^{(8)} \rangle \neq 0, \qquad \langle A_z^{(8)} \rangle \neq 0,$

(phase B) $\langle A_0^{(3)} \rangle \neq 0, \qquad \langle A_0^{(6)} \rangle \neq 0, \qquad \langle A_0^{(7)} \rangle \neq 0, \qquad \langle A_0^{(8)} \rangle \neq 0, \qquad \langle A_z^{(3)} \rangle \neq 0, \qquad \langle A_z^{(6)} \rangle \neq 0, \qquad \langle A_z^{(8)} \rangle \neq 0,$

(phase C) $\langle A_0^{(a)} \rangle \neq 0, \qquad (a = 1, 2, \cdots, 8), \qquad \langle A_z^{(1)} \rangle \neq 0, \qquad \langle A_z^{(3)} \rangle \neq 0, \qquad \langle A_z^{(6)} \rangle \neq 0, \qquad \langle A_z^{(8)} \rangle \neq 0,$

(phase D) $\langle A_0^{(2)} \rangle \neq 0, \qquad \langle A_0^{(8)} \rangle \neq 0, \qquad \langle A_y^{(4)} \rangle = \langle A_z^{(6)} \rangle \neq 0,$

(phase E) $\langle A_0^{(3)} \rangle \neq 0, \qquad \langle A_0^{(6)} \rangle \neq 0, \qquad \langle A_0^{(7)} \rangle \neq 0, \qquad \langle A_0^{(8)} \rangle \neq 0, \qquad \langle A_j^{(3)} \rangle \neq 0, \qquad \langle A_j^{(6)} \rangle \neq 0, \qquad \langle A_x^{(7)} \rangle \neq 0,$

          $\langle A_z^{(8)} \rangle \neq 0,$

(phase F)   All 25 VEVs in Eqs. (11a)–(11d) $\neq 0$.

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[7]It is mathematically possible to consider also the phase with $\langle A_0^{(3)} \rangle \neq 0, \langle A_0^{(8)} \rangle \neq 0$ and $\langle A_z^{(3)} \rangle \neq 0, \langle A_z^{(8)} \rangle \neq 0$, in which the unbroken subgroup is $U(1)_c \times \tilde{U}(1)_{\rm em} \times SO(2)_{\rm rot}$ with the generator σ^3 for the color $U(1)_c$. However, the Debye and Meissner screening masses for the $SU(2)_c$ gluons $A_\mu^{1,2,3}$ do not take imaginary values both for $\delta\mu < \bar{\Delta}$ and $\delta\mu > \bar{\Delta}$. Therefore there is no reason to consider such a phase.

As was pointed out in [18,33], the phase A corresponds to the single plane-wave LOFF phase [26]. In this case, both color electric and magnetic field strengths equal zero. This simplest case of the LOFF phase has been analyzed by several authors [26,28,33]. With the neutrality conditions taken into account, it was shown that the single plane-wave LOFF phase cannot resolve the chromomagnetic instability in the 2SC and g2SC regions [33].

While in the phase B the symmetry breakdown sample is the same as in the gluonic cylindrical phase II considered in Subsec. V C, in the phase C, it coincides with that in the gluonic cylindrical phase I discussed in Sec. V. The VEVs of gluon fields in the latter constitute a subset of the VEVs presented in the maximal set in the phase C. Based on the GL approach, it will be shown in Subsec. VI B that this subset is self-consistent. On the other hand, as was already pointed out in Subsec. V C, the question concerning the existence of the gluonic cylindrical phase II is open. Recall that there are nonzero chromoelectric field strengths in these phases.

In the phase D, the VEV set does not change under the $SO(2)_{diag}$ transformations with the generator which is an appropriate linear combination of σ^2 in the color $SU(2)_c$ and the generator of spatial y-z rotations. We may call this phase the *gluonic color-spin locked* (CSL) phase. In this phase, while nonzero $\langle A_0^{(2)} \rangle$ and $\langle A_0^{(8)} \rangle$ are allowed, nonzero $\langle A_0^{(1)} \rangle$ and $\langle A_0^{(3),\cdots,(7)} \rangle$ are not, because they break the $SO(2)_{diag}$ symmetry. It is noticeable that there exist both chromoelectric and chromomagnetic field strengths in the gluonic CSL phase.

In the phase E, the $\tilde{U}(1)_{em}$ is unbroken, while the rotational $SO(3)_{rot}$ symmetry is completely broken down. At last, the phase F contains the maximum number, 25, of gluon condensates. This classification provides a useful framework for studying dynamics of the gluon condensation.

In conclusion, we comment on the relation between the homogeneous gluon condensates and inhomogeneous diquark condensates. It is clear that when there are no field strengths, constant gluon condensates can be removed by using an appropriate gauge transformation. However, the point is that such gauge transformations can break constraint (11) for the diquark field. It is exactly what happens in the case of the single plane-wave LOFF state (the phase A): One can remove $\langle A_z^{(8)} \rangle$ at the cost of introducing an exponential factor with a linearly depending on z phase in the diquark field Δ. For other examples, see Ref. [34].

B. Ginzburg-Landau approach

In this section, we study the GL approach for dynamics with vector gluon condensates in the gauged NJL model. As was shown in Sec. III, in the hard dense loop approxi-mation, the potential V in the model is given in Eq. (32). However, it is very hard to perform explicitly the calculations of the term with the fermion trace in V in the case of nonzero gluon condensates (113).

The GL approach can help to overcome this difficulty in studies of near-critical dynamics in systems with a second order phase transition. In this approach, we need to pick up those local operators $\mathcal{O}_n$ with the mass dimension four and less which are invariant under the symmetry of a system. In our case, it is

$$ SU(2)_c \times \tilde{U}(1)_{em} \times [U(1)_{\tau_L^3} \times U(1)_{\tau_R^3}]_\chi \times SO(3)_{rot} \tag{114} $$

(see Eq. (43)). The GL effective action and Lagrangian, S_{eff} and $\mathcal{L}_{eff}$, are written in terms of the sum of these operators:

$$ S_{eff}[A_\mu^a; \mu, \mu_e, \Delta] = \int dx^4 \mathcal{L}_{eff}[A_\mu^a; \mu, \mu_e, \Delta], $$
$$ \mathcal{L}_{eff} = \sum_n K_n \mathcal{O}_n. \tag{115} $$

The part of the Lagrangian without derivative terms yields the effective potential V_{eff}. By analyzing the structure of the $\mathcal{L}_{eff}$, we can determine the relevant terms before providing the calculations of the full potential V (32). It essentially reduces the amount of the work.

Let us consider the GL effective Lagrangian in the gauged NJL model. Because the $SU(2)_c$ in Eq. (114) is a gauge symmetry, the building blocks of $\mathcal{L}_{eff}$ are

$$ \phi_0, \quad \phi_j, \quad \mathcal{D}_0, \quad \mathcal{D}_j, \quad f_{0j}, \quad f_{jk}, \quad A_j^{(8)}, \tag{116} $$

where the indices j and k represent spatial components. The coefficients K_n of the operators are functions of Δ, μ, μ_e, and μ_8. But when we expand the effective action with respect to μ_8, one should add the field $A_0^{(8)}$:

$$ \phi_0, \quad \phi_j, \quad \mathcal{D}_0, \quad \mathcal{D}_j, \quad f_{0j}, \quad f_{jk}, \quad A_0^{(8)}, \quad A_j^{(8)}, \tag{117} $$

and now the coefficients of the operators are functions of Δ, μ, and μ_e. For completeness, we will describe the larger set of the operators expressed through building blocks (117).

Up to the mass dimension four, we find the operators without the singlet field $A_\mu^{(8)}$,

$$\phi_0^\dagger \phi_0, \qquad \phi_j^\dagger \phi^j, \tag{118a}$$

$$\phi_0^\dagger (i\mathcal{D}_0 \phi_0), \qquad \phi_j^\dagger (i\mathcal{D}_0 \phi^j), \qquad \phi_0^\dagger (i\mathcal{D}_j \phi^j), \tag{118b}$$

$$(\phi_0^\dagger \phi_0)^2, \qquad (\phi_0^\dagger \phi_0)(\phi_j^\dagger \phi^j), \qquad (\phi_0^\dagger \phi^j)(\phi_j^\dagger \phi_0), \qquad (\phi_j^\dagger \phi^j)^2, \qquad (\phi_j^\dagger \phi^k)(\phi_k^\dagger \phi^j), \qquad (\phi_0^\dagger \phi^j)(\phi_0^\dagger \phi_j),$$

$$(\phi^{j\dagger} \phi_k)(\phi_j^\dagger \phi^k), \tag{118c}$$

$$|\mathcal{D}_0 \phi_0|^2, \qquad |\mathcal{D}_j \phi_0|^2, \qquad |\mathcal{D}_0 \phi_j|^2, \qquad |\mathcal{D}_j \phi_k|^2, \qquad \phi_j^\dagger ((\mathcal{D}_i \mathcal{D}^i \delta^{jk} + \tfrac{1}{2}\{\mathcal{D}^j, \mathcal{D}^k\})\phi_k), \qquad \phi_0^\dagger (\{\mathcal{D}^j, \mathcal{D}^0\}\phi_j),$$

$$\tag{118d}$$

and

$$\mathrm{tr}\,(f_{0j}f^{0j}), \qquad \mathrm{tr}(f_{jk}f^{jk}), \qquad i\phi_j^\dagger f^{jk}\phi_k, \qquad \phi_0^\dagger f^{0j}\phi_j. \tag{118e}$$

Including $A_\mu^{(8)}$, we also get the following operators,

$$\phi_0^\dagger \phi_0 A_0^{(8)}, \qquad \phi_j^\dagger \phi^j A_0^{(8)}, \qquad \phi_0^\dagger \phi_0 (A_0^{(8)})^2, \qquad \phi_j^\dagger \phi^j (A_0^{(8)})^2, \qquad \phi_0^\dagger \phi_0 \partial_0 A_0^{(8)}, \qquad \phi_j^\dagger \phi^j \partial_0 A_0^{(8)}, \qquad \phi_0^\dagger \phi^j \partial_j A_0^{(8)}, \tag{119a}$$

$$\phi_0^\dagger (i\mathcal{D}_0 \phi_0) A_0^{(8)}, \qquad \phi_j^\dagger (i\mathcal{D}_0 \phi^j) A_0^{(8)}, \qquad \phi_0^\dagger (i\mathcal{D}_j \phi^j) A_0^{(8)}, \tag{119b}$$

and

$$\phi_0^\dagger \phi^j A_j^{(8)}, \qquad \phi_0^\dagger \phi^j A_j^{(8)} A_0^{(8)}, \qquad \phi_0^\dagger \phi_0 (A_j^{(8)})^2, \qquad \phi_j^\dagger \phi^j (A_k^{(8)})^2, \qquad \phi^{j\dagger} \phi^k A_j^{(8)} A_k^{(8)}, \qquad \phi_0^\dagger \phi^j \partial_0 A_j^{(8)}, \qquad \phi_j^\dagger \phi^k \partial^j A_k^{(8)}, \tag{120a}$$

$$\phi_0^\dagger \phi_0 \partial^j A_j^{(8)}, \qquad \phi_j^\dagger \phi^j \partial^k A_k^{(8)}, \qquad A_j^{(8)} \phi_0^\dagger (i\mathcal{D}_0 \phi^j), \qquad A_j^{(8)} \phi_0^\dagger (i\mathcal{D}^j \phi_0), \qquad A_k^{(8)} \phi_j^\dagger (i\mathcal{D}^k \phi^j), \qquad A_k^{(8)} \phi_j^\dagger (i\mathcal{D}^j \phi^k). \tag{120b}$$

The operators including only the singlet field $A_\mu^{(8)}$ are

$$A_0^{(8)}, \qquad (A_0^{(8)})^2, \qquad (A_j^{(8)})^2, \qquad A_0^{(8)}(A_j^{(8)})^2, \qquad \text{etc.} \tag{121}$$

The effective Lagrangian is given by the sum of the above operators. It is understood that the Hermitian conjugate operators are also included in $\mathcal{L}_{\mathrm{eff}}$, and some coefficients can be complex. The following remarks are in order. (a) As has to be, the sets (118)–(120) do not contain operators corresponding to the Debye and Meissner screening mass terms for the $SU(2)_c$ gauge gluons. (b) We do not consider the parity violating term such as $\epsilon^{ijk}\phi_i^\dagger f_{0j}\phi_k$, etc. (c) It is easy to check in the hard dense loop approximation (with the one-loop fermion contributions) that due to the structure of the fermion-antifermion-gluon vertex, a n-point vertex of gluons has an even number of the ϵ^b-type vertices. Therefore we do not consider terms as $\epsilon^{\alpha\beta}(\phi_j)_\alpha(i\mathcal{D}_0\phi^j)_\beta$, etc., where α, β denote the $SU(2)_c$ indices.

We are ready to show that the GL effective potential has the form presented in Eq. (64). Indeed, now we know all relevant and marginal operators which can be constructed for the fields B, C, and D defined in Eq. (63). They are: $\phi_j^\dagger \phi^j$ corresponding to B^2, $\phi_j^\dagger (i\mathcal{D}_0 \phi^j)$ to DB^2, $|\mathcal{D}_j \phi_k|^2$ to $B^2 C^2$, $|\mathcal{D}_0 \phi_j|^2$ to $B^2 D^2$, $\mathrm{tr}(f_{0j}f^{0j})$ to $C^2 D^2$, and $(\phi_j^\dagger \phi^j)^2$, $|\phi_j^\dagger \phi^k|^2$ and $(\phi_j^\dagger \phi^k)^2$ corresponding to B^4. Since there are no other relevant and marginal operators which can be constructed from these three fields, we are led to the form in Eq. (64).

To obtain all of the coefficients K_n of the operators is a very hard task. Only some of them are known. The coef-

ficients of $\phi_0^\dagger \phi_0$, $(A_0^{(8)})^2$, and $\phi_j^\dagger \phi^j$, $(A_j^{(8)})^2$ correspond to the Debye and Meissner screening masses, respectively [13]. The coefficients of the operators $\phi_j^\dagger (i\mathcal{D}_0 \phi^j)$, $|\mathcal{D}_j \phi_k|^2$, and $\mathrm{tr}(f_{0j}f^{0j})$ correspond to the parameters T_{DB}, λ_{BC}, and λ_{CD}, respectively, in the GL potential in the gluonic phase used in Sec. V (see Eq. (76)). These parameters are calculated in Appendix B. Without $\delta\mu$, the three and four-point gluon vertices have been obtained in Ref. [48]. The important consequences of those calculations are that the coefficients of the three-point vertex operators $\phi_0^\dagger (i\mathcal{D}_0 \phi^0)$, $\phi_j^\dagger (i\mathcal{D}_0 \phi^j)$, and $\phi_0^\dagger (i\mathcal{D}_j \phi^j)$ are vanishing at the order of μ^2, (i.e., they $\sim \mathcal{O}(\mu)$), and that four-point vertex operators such as $|\mathcal{D}_j \phi_k|^2$, $\mathrm{tr}(f_{0j}f^{0j})$, and $A_k^{(8)} \phi_j^\dagger (i\mathcal{D}^k \phi^j)$ are of order μ^2. For their concrete expressions, see Ref. [48]. The results of our calculations in Appendix B are of course consistent with them.

The GL approach can be applied only to those symmetry breaking samples, discussed in Subsec. VI A, which are connected with a second order phase transition. Therefore, it cannot be applied both to the sample A (the single plane-wave LOFF state) and the sample B (the gluonic cylindrical phase II), which are related to a first order phase transition (see Refs. [26,28,33] and the discussion in Subsec. VI A).[8] On the other hand, as was shown in Sec. V, the GL approach consistently describes the gluonic

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[8]Note that as follows from Refs. [13,17], the instabilities at $\bar\Delta = \delta\mu$ in the g2SC phase (not considered in this paper), also cannot be connected with a conventional second order phase transition.

cylindrical phase I, assigned to the symmetry breaking sample C (a detailed analysis leading to this conclusion is considered in this subsection below). The discussion of the dynamics in the symmetry breaking sample D (the gluonic CSL phase) is beyond the scope of this paper.

To demonstrate the power of the GL approach, we will apply it to prove a self-consistency of the ansatz for the gluonic phase we used in Sec. V. In Subsec. V C, a tadpole mechanism for producing VEVs of gluon fields from triple vertices was discussed. Let us describe it in more detail. One can always divide a field into the VEV part and the fluctuation one,

$$A_\mu^{(\ell)} = \langle A_\mu^{(\ell)} \rangle + a_\mu^{(\ell)}, \qquad (\ell = 1, 2, 3), \quad (122\text{a})$$

$$\phi_\mu = \langle \phi_\mu \rangle + \zeta_\mu, \qquad\qquad\qquad (122\text{b})$$

$$A_\mu^{(8)} = \langle A_\mu^{(8)} \rangle + a_\mu^{(8)}. \qquad\qquad\quad (122\text{c})$$

Substituting expressions (122) for *all* fields into the GL potential expressed through operator sets (118)–(121), we obtain the VEV part, the tadpole part, which is linear with respect to fluctuations, plus higher order terms. The gap equation is the stationary condition for the effective potential and it is equivalent to vanishing the tadpole part. We emphasize that this condition is stronger than solving the gap equation for a given ansatz by taking into consideration only the fluctuations of the fields in the ansatz and ignoring the fluctuations of other ones.

The condition of vanishing the tadpole contribution has to be correct for any consistent ansatz used for VEVs: its violation indicates that the ansatz is inconsistent. While for the *maximal* ansatz, corresponding to a chosen symmetry breaking pattern, the violation would imply that the pattern cannot be realized in the model at all, for a nonmaximal one, this could just mean that the latter is not closed and one should add new VEVs for restoring its consistency.

As a concrete example, let us consider the ansatz with $\langle A_0^{(8)} \rangle \neq 0$, $\langle A_x^{(4)} \rangle \neq 0$, and $\langle A_y^{(8)} \rangle \neq 0$, which was discussed in Ref. [19], and show that it is *not* self-consistent. The point is that the operators $A_k^{(8)} \phi_j^\dagger (i\mathcal{D}^k \phi^j)$ and $\phi_j^\dagger (i\mathcal{D}_0 \phi^j)$ yield the tadpole contributions $a_y^{(3)} \langle A_y^{(8)} \rangle \langle A_x^{(4)} \rangle^2$ and $a_0^{(3)} \langle A_x^{(4)} \rangle^2$, respectively. Because there are no other tadpole terms for $a_y^{(3)}$ and $a_0^{(3)}$, the condition of vanishing the tadpole contribution leads to $\langle A_x^{(4)} \rangle = 0$, which is inconsistent with the original ansatz. If we started from the ansatz with $\langle A_0^{(3)} \rangle \neq 0$, $\langle A_0^{(8)} \rangle \neq 0$, $\langle A_y^{(3)} \rangle \neq 0$, $\langle A_x^{(4)} \rangle \neq 0$, and $\langle A_y^{(8)} \rangle \neq 0$, new tadpole contributions for $a_y^{(3)}$ and $a_0^{(3)}$ would appear. They will either allow to cancel the previous tadpole contributions without generating new ones or this modified ansatz is also inconsistent and should be in its turn extended. This process will eventually lead either to a con-

sistent ansatz or to the conclusion that this symmetry breaking pattern is not realized in the model.[9]

Without calculating the coefficients of the operators and solving the gap equations, this check helps to pick up a self-consistent ansatz. By estimating the size of the coefficients, one can also specify suppressed VEVs, as we will see below. The latter can be useful for simplifying the ansatz we work with.

Let us now turn to the gluonic phase in Sec. V. As was shown in Subsec. VI A, the phase C, having the same symmetry structure as the gluonic cylindrical phase I, can contain other VEVs than $B = g\langle A_z^{(6)} \rangle$, $C = g\langle A_z^{(1)} \rangle$, $D = g\langle A_0^{(3)} \rangle$, and $\mu_8 = \sqrt{3}/2g\langle A_0^{(8)} \rangle$ used in Sec. V. Is the ansatz including only B, C, D, and μ_8 self-consistent? Based on the GL approach, we will show that the answer to this question is affirmative. Actually, we will show that, strictly speaking, the self-consistent minimal ansatz for the gluonic cylindrical phase I is

$$\langle A_z^{(6)} \rangle, \langle A_z^{(1)} \rangle, \langle A_0^{(3)} \rangle, \langle A_0^{(8)} \rangle, \quad \text{and} \quad \langle A_0^{(4)} \rangle, \langle A_0^{(5)} \rangle. \quad (123)$$

However, it will be shown that the additional VEVs $\langle A_0^{(4)} \rangle$ and $\langle A_0^{(5)} \rangle$ are suppressed in the vicinity of the critical point.

For $B, C, D \neq 0$, the tadpole contributions for $a_0^{(4,5)}$ come from the operators $\phi_0^\dagger (i\mathcal{D}_j \phi^j)$ and $\phi_0^\dagger f^{0j} \phi_j$ in the GL effective Lagrangian with, say, coefficients K_1 and K_2, respectively. The condition of vanishing the tadpole part yields $D \sim -K_1/K_2$. On the other hand, the solution of the gap equations is given in Eq. (74). Since M_B^2 and T_{DB} are the coefficients of different operators, $\phi_j^\dagger \phi^j$ and $\phi_j^\dagger (i\mathcal{D}_0 \phi^j)$, respectively, it is hard to expect a "magic" relation (fine-tuning) among K_1, K_2, M_B^2, and T_{DB} coefficients, which would lead to the same solution for D from the gap equation. Therefore, one should introduce VEVs for $A_0^{(4,5)}$ in this case.

Noting that $A_0^{(4,5)}$ have nonzero Debye mass $m_{D,4} \sim \mathcal{O}(\mu)$, one can then estimate the solution of the gap equations for $\langle A_0^{(4,5)} \rangle$ as $\langle A_0^{(4)} \rangle$, $\langle A_0^{(5)} \rangle \sim K_1 BC/m_{D,4}^2 + K_2 BCD/m_{D,4}^2$. It is obvious that near the critical point, where $B, C, D \to 0$, the contribution of the first operator dominates. Without $\delta\mu$, the coefficients K_1 and K_2 were calculated in hard dense loop approximation in Ref. [48]. It was shown there that while K_2 is of order μ^2/Δ^2, K_1 is vanishing in this leading approximation. It implies that $K_1 \sim \mathcal{O}(\mu)$ or less, and thereby we conclude that the contribution of this operator to the effective potential is the term $B^2 C^2$ with the coefficient $\sim \mathcal{O}(1)$ at most. On the other hand, the coefficient of the $B^2 C^2$ term discussed in

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[9]This example corresponds to one of the nonmaximal ansätze for the symmetry breaking sample E considered in the previous subsection. For the choice of the VEVs in the phase E used in the present paper, $\langle A_0^{(3)} \rangle \neq 0$, $\langle A_x^{(4)} \rangle \neq 0$, and $\langle A_y^{(8)} \rangle \neq 0$ in the ansatz should be replaced by $\langle A_z^{(3)} \rangle \neq 0$, $\langle A_y^{(6)} \rangle \neq 0$, and $\langle A_z^{(8)} \rangle \neq 0$.

Sec. V is λ_{BC} in Eq. (85): $\lambda_{BC} \sim \mathcal{O}(\mu^2/\Delta^2)$. Therefore, the contributions of $\langle A_0^{(4)} \rangle$ and $\langle A_0^{(5)} \rangle$ can be ignored in the vicinity of the critical point. In addition to this, one can show that with the ansatz (123) for VEVs, there are no potentially dangerous operators in Eqs. (118)–(121) which would lead to tadpole contributions for $A_z^{(8)}$ and other additional fields from the maximal ansatz absent in Eq. (123). This allows us to conclude that the ansatz used in Sec. V is self-consistent indeed. The above example clearly shows the power of the GL approach and its relevance for the dynamics with vector condensates of gluons.[10]

VII. CONCLUSION

The gluonic phases yield an example of dynamics in gauge models with matter in which the Higgs mechanism is provided by condensates of gauge (or gauge plus scalar) fields. Because VEVs of spatial components of vector fields break the rotational symmetry, it is natural to have a spontaneous breakdown both of external and internal symmetries in this case. Dynamics in such systems are quite sophisticated. The existence of exotic hadrons in the gluonic phases is especially intriguing.

What could be directions for future studies in these phases? It is evident that it would be interesting to consider the spectrum of light collective excitations there. The results in Refs. [35,41], obtained in the gauged σ-model with a chemical potential for hypercharge (briefly discussed in Sec. II), suggest that the spectrum should be rich, containing, in particular, gapless NG modes, rotonlike and vortexlike excitations.

Another interesting direction would be to clarify whether the symmetry breaking patterns considered in Subsec. VI A, can be realized as stable, metastable, or unstable ground states in dense QCD. Recent results [49], showing that the landscape of such ground states in the gauged σ-model with a chemical potential for hypercharge is rich, are encouraging.

It is clear that it would be worth figuring out whether phases with vector condensates of gluons could exist in dense matter with three quark flavors. Recently, this possibility has been mentioned in Refs. [17,19,50,51].

It has been recently revealed that a strong enough magnetic field can influence the phase structure in dense quark matter [52].[11] It would be interesting to study this phe-

nomenon in gluonic phases, especially because one can expect the existence of vortices there [41,55].

Last but not least, it would be worth searching for a realization of a gluoniclike phase in condensed matter. Recently, there has been considerable interest in systems with coexisting order parameters (such as high T_c superconductors) in condensed matter [56]. Generating vector condensates is a natural way of creating such systems (for example, in the gluonic cylindrical phase I, electric superconductivity coexists with spontaneous rotational symmetry breaking).

APPENDIX A: STRUCTURE OF NEAR-CRITICAL SOLUTION IN GLUONIC PHASE

In this appendix, we analyze the structure of the near-critical solution with $B \neq 0$, $C \neq 0$, and $D \neq 0$ discussed in Subsec. V A. Equations (65)–(67) yield

$$D = -\frac{M_B^2}{3T_{DB}} + \left(\frac{\lambda_B \lambda_{CD}}{\lambda_{BC}} - 2\lambda_{BD} \right) \frac{D^2}{3T_{DB}},$$
$$B^2 = -\frac{\lambda_{CD}}{\lambda_{BC}} D^2, \qquad C^2 = \frac{T_{DB}}{\lambda_{BC}} D + \frac{\lambda_{BD}}{\lambda_{BC}} D^2. \tag{A1}$$

On the other hand, Eq. (68) yields:

$$\frac{\partial V_\Delta}{\partial \Delta} = -\frac{1}{2} B^2 \frac{\partial M_B^2}{\partial \Delta} - DB^2 \frac{\partial T_{DB}}{\partial \Delta} - \frac{1}{2} B^2 C^2 \frac{\partial \lambda_{BC}}{\partial \Delta}$$
$$- \frac{1}{2} C^2 D^2 \frac{\partial \lambda_{CD}}{\partial \Delta} - \frac{1}{2} B^2 D^2 \frac{\partial \lambda_{BD}}{\partial \Delta} - \frac{1}{4} B^4 \frac{\partial \lambda_B}{\partial \Delta}, \tag{A2a}$$

$$\frac{\partial V_\Delta}{\partial \mu_e} = -\frac{1}{2} B^2 \frac{\partial M_B^2}{\partial \mu_e} - DB^2 \frac{\partial T_{DB}}{\partial \mu_e} - \frac{1}{2} B^2 C^2 \frac{\partial \lambda_{BC}}{\partial \mu_e}$$
$$- \frac{1}{2} C^2 D^2 \frac{\partial \lambda_{CD}}{\partial \mu_e} - \frac{1}{2} B^2 D^2 \frac{\partial \lambda_{BD}}{\partial \mu_e} - \frac{1}{4} B^4 \frac{\partial \lambda_B}{\partial \mu_e}, \tag{A2b}$$

$$\frac{\partial V_\Delta}{\partial \mu_8} = -\frac{1}{2} B^2 \frac{\partial M_B^2}{\partial \mu_8} - DB^2 \frac{\partial T_{DB}}{\partial \mu_8} - \frac{1}{2} B^2 C^2 \frac{\partial \lambda_{BC}}{\partial \mu_8}$$
$$- \frac{1}{2} C^2 D^2 \frac{\partial \lambda_{CD}}{\partial \mu_8} - \frac{1}{2} B^2 D^2 \frac{\partial \lambda_{BD}}{\partial \mu_8} - \frac{1}{4} B^4 \frac{\partial \lambda_B}{\partial \mu_8}. \tag{A2c}$$

Because the coefficients M_B^2, T_{DB}, etc. in Eq. (A1) are the

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[10]Another example of this analysis is for the gluonic CSL phase D. One can show that with the GL operators in Eqs. (118)–(121), there are two self-consistent ansätze in that case: the maximal one and the maximal one minus the VEV of the field $A_0^{(2)}$.

[11]This phenomenon has some similarities with the phenomenon of the magnetic catalysis in vacuum field theories [53], in particular, in the vacuum QCD [54].

functions of Δ, μ_e, and μ_8, Eqs. (68) and (A1) constitute a coupled system of six equations. However, as will be shown below, near the critical point, the Eqs. (68) and (A1) decouple.

Near the critical point, $M_B^2 \approx 0$, one can expand Δ, μ_e, and μ_8 around $B = C = D = 0$:

$$\Delta = \bar{\Delta} + \xi_\Delta, \tag{A3}$$

$$\mu_e = \bar{\mu}_e + \xi_e, \tag{A4}$$

$$\mu_8 = \bar{\mu}_8 + \xi_8, \tag{A5}$$

where the bar-quantities correspond to the 2SC solution with $B = C = D = 0$. By definition of the bar-quantities, the following stationary conditions are satisfied:

$$\left. \frac{\partial V_\Delta}{\partial \Delta} \right|_{\Delta=\bar{\Delta},\mu_e=\bar{\mu}_e,\mu_8=\bar{\mu}_8} = 0,$$

$$\left. \frac{\partial V_\Delta}{\partial \mu_e} \right|_{\Delta=\bar{\Delta},\mu_e=\bar{\mu}_e,\mu_8=\bar{\mu}_8} = 0, \tag{A6}$$

$$\left. \frac{\partial V_\Delta}{\partial \mu_8} \right|_{\Delta=\bar{\Delta},\mu_e=\bar{\mu}_e,\mu_8=\bar{\mu}_8} = 0.$$

They yield

$$\bar{\mu}_e = \tfrac{3}{5}\mu - \tfrac{2}{5}\bar{\mu}_8, \tag{A7}$$

$$\left(\frac{3}{5}\mu - \frac{2}{5}\bar{\mu}_8\right)^3 - \frac{2}{9}\left(\frac{9}{10}\mu + \frac{2}{5}\bar{\mu}_8\right)^3$$
$$= \frac{1}{3}\bar{\Delta}^2\left(\frac{9}{10}\mu + \frac{2}{5}\bar{\mu}_8\right)\left(\ln\frac{4\Lambda^2}{\bar{\Delta}^2} - 2\right), \tag{A8}$$

$$\left[\left(\frac{9}{10}\mu + \frac{2}{5}\bar{\mu}_8\right)^2 - \frac{1}{2}\bar{\Delta}^2\right]\ln\frac{4\Lambda^2}{\bar{\Delta}^2}$$
$$= \frac{\pi^2}{4G_\Delta} - \Lambda^2 + 3\left(\frac{9}{10}\mu + \frac{2}{5}\bar{\mu}_8\right)^2 - \frac{1}{2}\bar{\Delta}^2. \tag{A9}$$

Note that the bar-quantities $\bar{\Delta}$, $\bar{\mu}_e$, and $\bar{\mu}_8$ are uniquely determined by the theoretical parameters G_Δ, Λ, μ in the

gauged NJL model. Expanding the left-hand side in Eq. (A2) around the bar-quantities, we obtain the gap equation for ξ_Δ, ξ_e, and ξ_8 expressed in a matrix form as

$$\begin{pmatrix} \frac{\partial^2 V_\Delta}{\partial \Delta^2} & \frac{\partial^2 V_\Delta}{\partial \mu_e \partial \Delta} & \frac{\partial^2 V_\Delta}{\partial \mu_8 \partial \Delta} \\ \frac{\partial^2 V_\Delta}{\partial \Delta \partial \mu_e} & \frac{\partial^2 V_\Delta}{\partial \mu_e^2} & \frac{\partial^2 V_\Delta}{\partial \mu_8 \partial \mu_e} \\ \frac{\partial^2 V_\Delta}{\partial \Delta \partial \mu_8} & \frac{\partial^2 V_\Delta}{\partial \mu_e \partial \mu_8} & \frac{\partial^2 V_\Delta}{\partial \mu_8^2} \end{pmatrix} \begin{pmatrix} \xi_\Delta \\ \xi_e \\ \xi_8 \end{pmatrix}$$

$$= -\frac{1}{2}B^2 \begin{pmatrix} \frac{\partial M_B^2}{\partial \Delta} \\ \frac{\partial M_B^2}{\partial \mu_e} \\ \frac{\partial M_B^2}{\partial \mu_8} \end{pmatrix} + \mathcal{O}(DB^2) + \mathcal{O}(B^2C^2) + \mathcal{O}(C^2D^2), \tag{A10}$$

where all derivatives are calculated at $\Delta = \bar{\Delta}$, $\mu_e = \bar{\mu}_e$, and $\mu_8 = \bar{\mu}_8$.

Combining Eq. (A1) with Eq. (A10), we find the following approximate solution:

$$B_{\text{sol}} \simeq \frac{-\bar{M}_B^2}{3|\bar{T}_{DB}|}\sqrt{\frac{-\bar{\lambda}_{CD}}{\bar{\lambda}_{BC}}}, \qquad C_{\text{sol}} \simeq \sqrt{\frac{-\bar{M}_B^2}{3\bar{\lambda}_{BC}}},$$

$$D_{\text{sol}} \simeq \frac{-\bar{M}_B^2}{3\bar{T}_{DB}}, \qquad \xi_\Delta^{\text{sol}}, \xi_e^{\text{sol}}, \xi_8^{\text{sol}} \sim \mathcal{O}((\bar{M}_B^2)^2), \tag{A11}$$

where higher order terms in $\bar{M}_B^2$ were neglected. Here all the coefficients $\bar{M}_B^2$, $\bar{T}_{DB}$, $\bar{\lambda}_{BC}$, and $\bar{\lambda}_{CD}$ are expressed through the 2SC values $\bar{\Delta}$, $\bar{\mu}_e$, and $\bar{\mu}_8$. This solution exists when

$$\bar{M}_B^2 < 0, \qquad \bar{\lambda}_{BC} > 0, \qquad \bar{\lambda}_{CD} < 0. \tag{A12}$$

Note that in Eq. (A11) the convention $B_{\text{sol}} > 0$ and $C_{\text{sol}} > 0$ is chosen.

Substituting solution (A11) into GL potential (64), we find that its gluonic part $V_{\text{eff}} - V_\Delta$ consists of leading terms $\sim \mathcal{O}((M_B^2)^3)$ and subleading terms $\sim \mathcal{O}((M_B^2)^4)$. In particular, the deviation of V_Δ from that in the 2SC solution is of subleading order $\sim \mathcal{O}((M_B^2)^4)$. Indeed, the second order of the Taylor expansion for V_Δ yields

$$V_\Delta(\Delta^{\text{sol}}, \mu_e^{\text{sol}}, \mu_8^{\text{sol}}) - V_\Delta(\bar{\Delta}, \bar{\mu}_e, \bar{\mu}_8) = \frac{1}{2}\begin{pmatrix} \xi_\Delta^{\text{sol}} & \xi_e^{\text{sol}} & \xi_8^{\text{sol}} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 V_\Delta}{\partial \Delta^2} & \frac{\partial^2 V_\Delta}{\partial \mu_e \partial \Delta} & \frac{\partial^2 V_\Delta}{\partial \mu_8 \partial \Delta} \\ \frac{\partial^2 V_\Delta}{\partial \Delta \partial \mu_e} & \frac{\partial^2 V_\Delta}{\partial \mu_e^2} & \frac{\partial^2 V_\Delta}{\partial \mu_8 \partial \mu_e} \\ \frac{\partial^2 V_\Delta}{\partial \Delta \partial \mu_8} & \frac{\partial^2 V_\Delta}{\partial \mu_e \partial \mu_8} & \frac{\partial^2 V_\Delta}{\partial \mu_8^2} \end{pmatrix} \begin{pmatrix} \xi_\Delta^{\text{sol}} \\ \xi_e^{\text{sol}} \\ \xi_8^{\text{sol}} \end{pmatrix} \tag{A13}$$

$$= \frac{1}{8}B_{\text{sol}}^4\begin{pmatrix} \frac{\partial M_B^2}{\partial \Delta} & \frac{\partial M_B^2}{\partial \mu_e} & \frac{\partial M_B^2}{\partial \mu_8} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 V_\Delta}{\partial \Delta^2} & \frac{\partial^2 V_\Delta}{\partial \mu_e \partial \Delta} & \frac{\partial^2 V_\Delta}{\partial \mu_8 \partial \Delta} \\ \frac{\partial^2 V_\Delta}{\partial \Delta \partial \mu_e} & \frac{\partial^2 V_\Delta}{\partial \mu_e^2} & \frac{\partial^2 V_\Delta}{\partial \mu_8 \partial \mu_e} \\ \frac{\partial^2 V_\Delta}{\partial \Delta \partial \mu_8} & \frac{\partial^2 V_\Delta}{\partial \mu_e \partial \mu_8} & \frac{\partial^2 V_\Delta}{\partial \mu_8^2} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial M_B^2}{\partial \Delta} \\ \frac{\partial M_B^2}{\partial \mu_e} \\ \frac{\partial M_B^2}{\partial \mu_8} \end{pmatrix}, \tag{A14}$$

where Eq. (A10) was used (here all derivatives are calculated at $\Delta = \bar{\Delta}$, $\mu_e = \bar{\mu}_e$, and $\mu_8 = \bar{\mu}_8$).

Omitting the subleading terms, we arrive at the reduced effective potential,

$$\tilde{V}_{\text{eff}} = V_{\Delta}(\bar{\Delta}, \bar{\mu}_e, \bar{\mu}_8) + \tfrac{1}{2}\bar{M}_B^2 B^2 + \bar{T}_{DB}DB^2 + \tfrac{1}{2}\bar{\lambda}_{BC}B^2C^2$$
$$+ \tfrac{1}{2}\bar{\lambda}_{CD}C^2D^2. \tag{A15}$$

This potential is composed of two parts: the constant 2SC part V_{Δ}, with frozen fermion parameters, and the dynamical gluonic part.

APPENDIX B: COEFFICIENTS IN GL EFFECTIVE POTENTIAL

In this appendix, we calculate the coefficients λ_{BC}, λ_{CD}, and T_{DB} of the marginal and relevant operators in the reduced GL effective potential (78). The coefficients λ_{BC} and λ_{CD} are connected with the operators $|\mathcal{D}_j\phi_k|^2$ and $\text{tr}(f_{0j}f^{0j})$, respectively. Therefore, they can be obtained from the kinetic terms of the 4–7th and 1–3rd gluons. Although the coefficient T_{DB} of the triple vertex is con-

nected with the operator $\phi_j^{\dagger}(i\mathcal{D}_0\phi^j)$, it comes from a non-hard-dense-loop part, as we will see below. Because of that, T_{DB} will be directly calculated from the corresponding three-point vertex. In passing, the coefficient λ_B in GL potential (64) is connected with the operator $(\phi_j^{\dagger}\phi^j)^2$ and cannot be reduced to a calculation of a two-point function. On the other hand, because the coefficient λ_{BD} in (64) is connected with the operator $|\mathcal{D}_0\phi_j|^2$, it can be obtained by taking time-derivatives of the vacuum polarization tensor for the 6th gluon.

One of the main difficulties in calculating these coefficients is related to the presence of the gauge fields $C \equiv gA_z^{(1)}$ and $D \equiv gA_0^{(3)}$ connected with noncommuting color matrices T^1 and T^3. In order to overcome it, the following trick will be used. We decompose the inverse Nambu-Gor'kov propagator (26) with gluon fields as

$$S_g^{-1}(X) = S_C^{-1}(X) + \mathcal{M}_{BD}, \tag{B1}$$

where

$$S_C^{-1}(X) = \begin{pmatrix} i\slashed{\partial} + (\tilde{\mu} - \delta\mu\tau_3)\gamma^0 + CT^1\gamma^3 & -i\varepsilon\epsilon^b\gamma_5\Delta \\ -i\varepsilon\epsilon^b\gamma_5\Delta & i\slashed{\partial} - (\tilde{\mu} - \delta\mu\tau_3)\gamma^0 - CT^1\gamma^3 \end{pmatrix} \tag{B2}$$

and

$$\mathcal{M}_{BD} \equiv \begin{pmatrix} DT^3\gamma^0 - BT^6\gamma^3 & 0 \\ 0 & -DT^3\gamma^0 + BT^6\gamma^3 \end{pmatrix}, \tag{B3}$$

with $X^{\mu} = (x^0, x^1, x^2, x^3) = (t, x, y, z)$ denoting space-time coordinates. The point is that since the diquark gap Δ is $SU(2)_c$-invariant, we can remove the field $C = gA_z^{(1)}$ from $S_C(X)$ by using a $SU(2)_c$ gauge transformation. [Note that since the field strength is nonzero in the gluonic phase, it is impossible to remove all gauge fields from the propagator $S_g(X)$]. In fact, one can easily find this $SU(2)_c$ gauge transformation:

$$U(X) \equiv \begin{pmatrix} e^{-iCT^1x^3} & 0 \\ 0 & e^{iCT^1x^3} \end{pmatrix} \tag{B4}$$

transforms the inverse $S_C^{-1}(X)$ in Eq. (B2) into $S^{-1}(X)$ without C, i.e.,

$$S_C^{-1}(X) = U^{\dagger}(X)S^{-1}(X)U(X). \tag{B5}$$

Let us now expand the potential V (29) in the power series with respect to $S_C(P)$ in momentum space:

$$V = V_{\Delta} + V_g + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} \int \frac{d^4P}{i(2\pi)^4} \text{Tr}(S_C\mathcal{M}_{BD})^n, \tag{B6}$$

where the identity

$$\text{Tr} \ln S_C^{-1} = \text{Tr} \ln(U S_C^{-1} U^{\dagger}) = \text{Tr} \ln S^{-1} \tag{B7}$$

was used. Note that, by construction, the V_g term is C independent.

In order to utilize expression (B6), we need to calculate the propagator $S_C(P)$ in momentum space. From Eq. (B5) we get

$$S_C(X) = U^{\dagger}(X)S(X)U(X). \tag{B8}$$

Using then the relations

$$e^{iCT^1x^3} = \begin{pmatrix} e^{iC'\sigma^1x^3} & 0 \\ 0 & 1 \end{pmatrix}_c,$$

$$e^{iC'\sigma^1x^3} = \tfrac{1}{2}(e^{iC'x^3} + e^{-iC'x^3}) + \tfrac{1}{2}\sigma^1(e^{iC'x^3} - e^{-iC'x^3}), \tag{B9}$$

with

$$C' \equiv \tfrac{1}{2}C, \tag{B10}$$

we find that the Fourier transform of $e^{iC'\sigma^1x^3}$ is given by the sum of two δ-functions:

$$\text{F.T.} e^{iC'\sigma^1x^3} = \tfrac{1}{2}(\delta(p^3 + C') + \delta(p^3 - C'))$$
$$+ \tfrac{1}{2}\sigma^1(\delta(p^3 + C') - \delta(p^3 - C')). \tag{B11}$$

This implies that the Fourier transform of $U(X)$ (B4) is

expressed through these δ-functions. And since the Fourier transform of $S(X)$ is known, the Fourier transform of $S_C(X)$ can be easily found from Eqs. (B8) and (B11):

$$S_C(P) \equiv S_{rg}(P) \oplus S_b(P), \qquad (B12)$$

with the red-green part

and the blue one

$$S_{rg}(P) = \begin{pmatrix} G_{rg}^+ & \Xi_{rg}^- \\ \Xi_{rg}^+ & G_{rg}^- \end{pmatrix} \qquad (B13)$$

$$S_b(P) = \begin{pmatrix} (\not{P} + (\tilde{\mu} - \delta\mu\tau_3 - \mu_8)\gamma^0)^{-1} & 0 \\ 0 & (\not{P} - (\tilde{\mu} - \delta\mu\tau_3 - \mu_8)\gamma^0)^{-1} \end{pmatrix} = \begin{pmatrix} G_b^+(P) & 0 \\ 0 & G_b^-(P) \end{pmatrix}. \qquad (B14)$$

While $G_b^\pm$ is given in Eq. (21), the matrix elements $G_{rg}^\pm$ and $\Xi_{rg}^\pm$ in $S_{rg}(P)$ are connected with the gapped part $S_\Delta(P)$,

$$S_\Delta(P) = \begin{pmatrix} G_\Delta^+ & \Xi_\Delta^- \\ \Xi_\Delta^+ & G_\Delta^- \end{pmatrix} \qquad (B15)$$

with

$$\Xi_\Delta^\pm(P) \equiv i\sigma^2 \begin{pmatrix} 0 & \Xi_{12}^\pm \\ -\Xi_{21}^\pm & 0 \end{pmatrix}_f, \qquad (B16)$$

where the quantities $G_\Delta^\pm$, $G_b^\pm$, $\Xi_{12,21}$ are given in Eqs. (20), (23), and (24). In fact, $G_{rg}^\pm$ and $\Xi_{rg}^\pm$ are:

$$G_{rg}^\pm(P) \equiv \tfrac{1}{2}(G_\Delta^\pm(P + P_C) + G_\Delta^\pm(P - P_C)) + \tfrac{1}{2}(\pm G_\Delta^\pm(P + P_C) \mp G_\Delta^\pm(P - P_C))\sigma^1, \qquad (B17)$$

$$\Xi_{rg}^\pm(P) \equiv \tfrac{1}{2}(\Xi_\Delta^\pm(P + P_C) + \Xi_\Delta^\pm(P - P_C)) + \tfrac{1}{2}(\pm\Xi_\Delta^\pm(P + P_C) \mp \Xi_\Delta^\pm(P - P_C))\sigma^1, \qquad (B18)$$

where P_C is the four-vector $(0, 0, 0, C/2)$. Note that $\Xi_\Delta^\pm \sigma^1 = -\sigma^1 \Xi_\Delta^\pm$ and, using Eq. (18), one can check that $S_{rg}^{-1}(P)S_{rg}(P) = 1$.

After taking the trace over the color indices, we obtain the square term for λ_{BC} and λ_{CD},

$$\begin{aligned}
\mathrm{Tr}_{c,f,s}(S_C \mathcal{M}_{BD})^2 &= \tfrac{1}{4}B^2 \mathrm{Tr}_{f,s}[(G_\Delta^+(P + P_C) + G_\Delta^+(P - P_C))\gamma^3 G_b^+(P)\gamma^3 + (G_\Delta^-(P + P_C) + G_\Delta^-(P - P_C))\gamma^3 G_b^-(P)\gamma^3] \\
&\quad + \tfrac{1}{2}D^2 \mathrm{Tr}_{f,s}[G_\Delta^+(P + P_C)\gamma^0 G_\Delta^+(P - P_C)\gamma^0 + G_\Delta^-(P + P_C)\gamma^0 G_\Delta^-(P - P_C)\gamma^0] \\
&\quad + \tfrac{1}{2}D^2 \mathrm{tr}_s[\Xi_{12}^-(P + P_C)\gamma^0 \Xi_{21}^+(P - P_C)\gamma^0 + \Xi_{12}^-(P - P_C)\gamma^0 \Xi_{21}^+(P + P_C)\gamma^0 \\
&\quad + \Xi_{21}^-(P + P_C)\gamma^0 \Xi_{12}^+(P - P_C)\gamma^0 + \Xi_{21}^-(P - P_C)\gamma^0 \Xi_{12}^+(P + P_C)\gamma^0]. \qquad (B19)
\end{aligned}$$

In fact, the B^2- and D^2-terms in Eq. (B19) correspond to the vacuum polarization tensors $\tilde{\Pi}_{44,55}^{jj}$ and Π_{11}^{00}, respectively, (see Eqs. (76) and (40) in the second paper in Ref. [13].) Therefore, the calculations of λ_{BC} and λ_{CD} are reduced to those of $\tilde{\Pi}_{44,55}^{jj}$ and Π_{11}^{00}. We find

$$\lambda_{BC} = \frac{1}{80\pi^2} \frac{\tilde{\mu}^2}{\Delta^2}\left[-1 + 8\frac{\delta\mu^2}{\Delta^2}\left(1 - \frac{\delta\mu^2}{\Delta^2}\right) + \theta(\delta\mu - \Delta)\frac{4\delta\mu\sqrt{\delta\mu^2 - \Delta^2}}{\Delta^2}\left(-1 + 2\frac{\delta\mu^2}{\Delta^2}\right)\right], \qquad (B20)$$

$$\lambda_{CD} = -\frac{1}{18\pi^2} \frac{\tilde{\mu}^2}{\Delta^2}\left[1 + \theta(\delta\mu - \Delta)\frac{\delta\mu(3\Delta^2 - 2\delta\mu^2)}{2(\delta\mu^2 - \Delta^2)^{3/2}}\right]. \qquad (B21)$$

In order to calculate the coefficient T_{DB}, the cubic term in expansion (B6) is required:

$$\begin{aligned}
\mathrm{Tr}_{c,f,s}(S_C \mathcal{M}_{BD})^3|_{B^2D} &= -\frac{3}{8}B^2 D[\mathrm{Tr}_{f,s}(G_\Delta^+(P)\gamma^3 G_b^+(P)\gamma^3 G_\Delta^+(P)\gamma^0 - G_\Delta^-(P)\gamma^3 G_b^-(P)\gamma^3 G_\Delta^-(P)\gamma^0) \\
&\quad + \mathrm{tr}_s(\gamma^3 \Xi_{12}^-(P)\gamma^0 \Xi_{21}^+(P)\gamma^3 G_b^+(P)|_{\tau_3=+} + \gamma^3 \Xi_{21}^-(P)\gamma^0 \Xi_{12}^+(P)\gamma^3 G_b^+(P)|_{\tau_3=-} \\
&\quad - \gamma^3 \Xi_{12}^+(P)\gamma^0 \Xi_{21}^-(P)\gamma^3 G_b^-(P)|_{\tau_3=+} - \gamma^3 \Xi_{21}^+(P)\gamma^0 \Xi_{12}^-(P)\gamma^3 G_b^-(P)|_{\tau_3=-})]. \qquad (B22)
\end{aligned}$$

Then T_{DB} is given by

$$T_{DB} = \frac{1}{16} \int \frac{d^4 P}{i(2\pi)^4} \sum_{e_1=\pm, e_2=\pm, e_3=\pm} [\mathcal{K}_{e_1 e_2 e_3} \mathcal{T}_{e_1 e_2 e_3}$$
$$+ \mathcal{H}_{e_1 e_2 e_3} \mathcal{U}_{e_1 e_2 e_3}],$$
(B23)

where

$$\mathcal{T}_{e_1 e_2 e_3} \equiv \mathrm{tr}_s[\gamma^0(\gamma^0 \Lambda_p^{e_1})\gamma^3(\gamma^0 \Lambda_p^{e_2})\gamma^3(\gamma^0 \Lambda_p^{e_3})] \quad (B24)$$

and

$$\mathcal{U}_{e_1 e_2 e_3} \equiv \mathrm{tr}_s[\gamma^3(\gamma^0 \Lambda_p^{e_1})\gamma^3(\gamma^5 \Lambda_p^{e_2})\gamma^0(\gamma^5 \Lambda_p^{e_3})]. \quad (B25)$$

The functions $\mathcal{K}_{e_1 e_2 e_3}$ and $\mathcal{H}_{e_1 e_2 e_3}$ are expressed through the following components of the propagators (compare with Eqs. (20)–(24)):

$$G_\Delta^\pm(P) = G_{\Delta,+,\tau}^\pm \gamma^0 \Lambda_p^+ + G_{\Delta,-,\tau}^\pm \gamma^0 \Lambda_p^-, \quad (B26)$$

$$G_b^\pm(P) = G_{b,+,\tau}^\pm \gamma^0 \Lambda_p^+ + G_{b,-,\tau}^\pm \gamma^0 \Lambda_p^-, \quad (B27)$$

with

$$G_{\Delta,+,\tau}^\pm(P) \equiv \frac{(p_0 \mp \delta\mu\tau) - E^\pm}{(p_0 \mp \delta\mu\tau)^2 - (E_\Delta^\pm)^2}, \quad (B28)$$

$$G_{\Delta,-,\tau}^\pm(P) \equiv \frac{(p_0 \mp \delta\mu\tau) + E^\mp}{(p_0 \mp \delta\mu\tau)^2 - (E_\Delta^\mp)^2}, \quad (B29)$$

$$G_{b,+,\tau}^\pm(P) \equiv \frac{1}{(p_0 \mp \delta\mu\tau \mp \mu_8) + E^\pm}, \quad (B30)$$

$$G_{b,-,\tau}^\pm(P) \equiv \frac{1}{(p_0 \mp \delta\mu\tau \mp \mu_8) - E^\mp}, \quad (B31)$$

where $\tau = \pm 1$ are eigenvalues of τ_3, and

$$\Xi_{12}^\pm(P) = -i\Delta[\Xi_{+,\tau=-}^\pm \gamma_5 \Lambda_p^+ + \Xi_{-,\tau=-}^\pm \gamma_5 \Lambda_p^-], \quad (B32)$$

$$\Xi_{21}^\pm(P) = -i\Delta[\Xi_{+,\tau=+}^\pm \gamma_5 \Lambda_p^+ + \Xi_{-,\tau=+}^\pm \gamma_5 \Lambda_p^-] \quad (B33)$$

with

$$\Xi_{+,\tau}^\pm(P) \equiv \frac{1}{(p_0 \mp \delta\mu\tau)^2 - (E_\Delta^\pm)^2}, \quad (B34)$$

$$\Xi_{-,\tau}^\pm(P) \equiv \frac{1}{(p_0 \mp \delta\mu\tau)^2 - (E_\Delta^\mp)^2}. \quad (B35)$$

The explicit expressions of $\mathcal{K}_{e_1 e_2 e_3}$ and $\mathcal{H}_{e_1 e_2 e_3}$ are:

$$\mathcal{K}_{e_1 e_2 e_3} = \sum_{\tau=\pm} [G_{\Delta,e_1,\tau}^+ G_{b,e_2,\tau}^+ G_{\Delta,e_3,\tau}^+$$
$$- G_{\Delta,e_1,\tau}^- G_{b,e_2,\tau}^- G_{\Delta,e_3,\tau}^-] \quad (B36)$$

and

$$\mathcal{H}_{e_1 e_2 e_3} = -\Delta^2[G_{b,e_1,\tau=+}^+ \Xi_{e_2,\tau=-}^- \Xi_{e_3,\tau=+}^+$$
$$+ G_{b,e_1,\tau=-}^+ \Xi_{e_2,\tau=+}^- \Xi_{e_3,\tau=-}^+$$
$$- G_{b,e_1,\tau=+}^- \Xi_{e_2,\tau=-}^+ \Xi_{e_3,\tau=+}^-$$
$$- G_{b,e_1,\tau=-}^- \Xi_{e_2,\tau=+}^+ \Xi_{e_3,\tau=-}^-], \quad (B37)$$

$$= -\Delta^2 \sum_{\tau=\pm} [G_{b,e_1,\tau}^+ \Xi_{-e_2,\tau}^+ \Xi_{e_3,\tau}^+ - G_{b,e_1,\tau}^- \Xi_{-e_2,\tau}^- \Xi_{e_3,\tau}^-], \quad (B38)$$

where we used the relations

$$\Xi_{+,\tau=-}^\pm(P) = \Xi_{-,\tau=+}^\mp(P), \qquad \Xi_{-,\tau=-}^\pm(P) = \Xi_{+,\tau=+}^\mp(P). \quad (B39)$$

We also find

$$\mathcal{T}_{+\pm-} = \mathcal{T}_{-\pm+} = 0,$$
$$\mathcal{T}_{+++} = \mathcal{T}_{---} = 2\frac{(p^3)^2}{p^2}, \quad (B40)$$
$$\mathcal{T}_{+-+} = \mathcal{T}_{-+-} = 2\left(1 - \frac{(p^3)^2}{p^2}\right)$$

and

$$\mathcal{U}_{\pm++} = \mathcal{U}_{\pm--} = 0,$$
$$\mathcal{U}_{+-+} = \mathcal{U}_{-+-} = -2\frac{(p^3)^2}{p^2}, \quad (B41)$$
$$\mathcal{U}_{++-} = \mathcal{U}_{--+} = -2\left(1 - \frac{(p^3)^2}{p^2}\right).$$

Thus,

$$T_{DB} = \frac{1}{8} \int \frac{d^4 P}{i(2\pi)^4} \left[\frac{(p^3)^2}{p^2}(\mathcal{K}_{+++} + \mathcal{K}_{---} - \mathcal{H}_{+-+} - \mathcal{H}_{-+-}) \right.$$
$$\left. + \left(1 - \frac{(p^3)^2}{p^2}\right)(\mathcal{K}_{+-+} + \mathcal{K}_{-+-} - \mathcal{H}_{++-} - \mathcal{H}_{--+}) \right], \quad (B42)$$

$$= \frac{1}{8} \sum_{\tau=\pm} \int \frac{d^4P}{i(2\pi)^4} \left\{ \frac{(p^3)^2}{p^2} [G^+_{\Delta,-,\tau} G^+_{b,-,\tau} G^+_{\Delta,-,\tau} - G^-_{\Delta,+,\tau} G^-_{b,+,\tau} G^-_{\Delta,+,\tau} + \Delta^2(G^+_{b,-,\tau} \Xi^+_{-,\tau} \Xi^+_{-,\tau} - G^-_{b,+,\tau} \Xi^-_{+,\tau} \Xi^-_{+,\tau}) \right.$$

$$+ G^+_{\Delta,+,\tau} G^+_{b,+,\tau} G^+_{\Delta,+,\tau} - G^-_{\Delta,-,\tau} G^-_{b,-,\tau} G^-_{\Delta,-,\tau} + \Delta^2(G^+_{b,+,\tau} \Xi^+_{+,\tau} \Xi^+_{+,\tau} - G^-_{b,-,\tau} \Xi^-_{-,\tau} \Xi^-_{-,\tau})]$$

$$+ \left(1 - \frac{(p^3)^2}{p^2}\right) [G^+_{\Delta,-,\tau} G^+_{b,+,\tau} G^+_{\Delta,-,\tau} - G^-_{\Delta,+,\tau} G^-_{b,-,\tau} G^-_{\Delta,+,\tau} + \Delta^2(G^+_{b,+,\tau} \Xi^+_{-,\tau} \Xi^+_{-,\tau} - G^-_{b,-,\tau} \Xi^-_{+,\tau} \Xi^-_{+,\tau})$$

$$\left. + G^+_{\Delta,+,\tau} G^+_{b,-,\tau} G^+_{\Delta,+,\tau} - G^-_{\Delta,-,\tau} G^-_{b,+,\tau} G^-_{\Delta,-,\tau} + \Delta^2(G^+_{b,-,\tau} \Xi^+_{+,\tau} \Xi^+_{+,\tau} - G^-_{b,+,\tau} \Xi^-_{-,\tau} \Xi^-_{-,\tau})] \right\}. \tag{B43}$$

After integrating over p^0, we find

$$T_{DB} = \frac{1}{24\pi^2} \int_0^\Lambda dp\, p^2 [I_1 + 2I_2] \tag{B44}$$

with

$$I_1 \equiv -\frac{E^-_\Delta - E^-}{E^-_\Delta(E^-_\Delta + E^- + \mu_8)^2} + \frac{E^+_\Delta - E^+}{E^+_\Delta(E^+_\Delta + E^+ - \mu_8)^2} + \frac{1}{2}(\theta(-E^- + \delta\mu - \mu_8) + \theta(-E^- - \delta\mu - \mu_8))$$

$$\times \left(\frac{E^-_\Delta - E^-}{E^-_\Delta(E^-_\Delta + E^- + \mu_8)^2} + \frac{E^-_\Delta + E^-}{E^-_\Delta(E^-_\Delta - E^- - \mu_8)^2}\right)$$

$$+ \frac{1}{2}\theta(-E^-_\Delta + \delta\mu)\left(\frac{E^-_\Delta - E^-}{E^-_\Delta(E^-_\Delta + E^- + \mu_8)^2} - \frac{E^-_\Delta + E^-}{E^-_\Delta(E^-_\Delta - E^- - \mu_8)^2}\right) \tag{B45}$$

and

$$I_2 \equiv \frac{E^-_\Delta + E^-}{E^-_\Delta(E^-_\Delta + E^+ - \mu_8)^2} - \frac{E^+_\Delta + E^+}{E^+_\Delta(E^+_\Delta + E^- + \mu_8)^2} + \frac{1}{2}(\theta(-E^- + \delta\mu - \mu_8) + \theta(-E^- - \delta\mu - \mu_8))$$

$$\times \left(\frac{E^+_\Delta + E^+}{E^+_\Delta(E^+_\Delta + E^- + \mu_8)^2} + \frac{E^+_\Delta - E^+}{E^+_\Delta(E^+_\Delta - E^- - \mu_8)^2}\right)$$

$$+ \frac{1}{2}\theta(-E^-_\Delta + \delta\mu)\left(\frac{E^-_\Delta - E^-}{E^-_\Delta(E^-_\Delta - E^+ + \mu_8)^2} - \frac{E^-_\Delta + E^-}{E^-_\Delta(E^-_\Delta + E^+ - \mu_8)^2}\right). \tag{B46}$$

The integrand I_2 contains only the particle-antiparticle contribution and hence should be negligible. We expand $I_{1,2}$ in the series with respect to μ_8 and obtain

$$T_{DB} = \frac{\tilde{\mu}}{48\pi^2}\left[-1 + 4\frac{\delta\mu^2}{\Delta^2} + 8\frac{\delta\mu^4}{\Delta^4} - \theta(\delta\mu - \Delta)\frac{8\delta\mu(\delta\mu^2 - \Delta^2)^{3/2}}{\Delta^4}\right]$$

$$+ \frac{\mu_8}{24\pi^2}\frac{\tilde{\mu}^2}{\Delta^2}\left[-1 + 8\frac{\delta\mu^4}{\Delta^4} - \theta(\delta\mu - \Delta)\frac{4\delta\mu(2\delta\mu^2 - \Delta^2)\sqrt{\delta\mu^2 - \Delta^2}}{\Delta^4}\right]. \tag{B47}$$

As was expected, the contribution of I_2 is suppressed indeed.

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