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Spin Hall effect of photons in a static gravitational field

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Starting from a Hamiltonian description of the photon within the set of Bargmann-Wigner equations we derive new semiclassical equations of motion for the photon propagating in a static gravitational field. These equations which are obtained in the representation diagonalizing the Hamiltonian at the order $\hbar$, present the first order corrections to the geometrical optics. The photon Hamiltonian shows a new kind of helicity-torsion coupling. However, even for a torsionless space-time, photons do not follow the usual null geodesic as a consequence of an anomalous velocity term. This term is responsible for the gravitational birefringence phenomenon: photons with distinct helicity follow different geodesics in a static gravitational field.

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I. INTRODUCTION

In the last few years many studies focused on the transport of quantum particles with spin. Indeed, manipulating spin polarization of electrons is a challenging goal in semiconductor spintronics. Achieving this goal requires the understanding of the spin transport mechanism in systems with spin-orbit (SO) interaction. It was found that in such a system a Berry phase in momentum space plays an important role by affecting both particle phase and its transport properties [1]. It is well known since the seminal work of Berry [2], that when a quantum mechanical system has an adiabatic evolution, a wave function acquires a geometric phase. It is only recently that the possible influence of the Berry phase on transport properties (in particular on the semiclassical dynamics) of several physical systems has been investigated. Semiconductors, having SO couplings greatly enhanced with respect to the vacuum case, require a theory of spin transport. However, even in the vacuum, new fundamental results concerning the semiclassical equations of motion of electrons were recently derived. For instance in [3,4], considering the Dirac equation in an external potential, it was shown that the position operator acquires a spin-orbit contribution which turns out to be a Berry connection rendering the coordinate algebraic structure noncommutative. This drastically modifies the semiclassical equations of motion and implies a topological spin transport similar to the intrinsic spin Hall effect in semiconductors [1]. A similar noncommutative algebra has been also found in the context of electrons in magnetic Bloch bands [5], leading to an anomalous velocity term.

Despite its very different nature, the photon displays many similar behaviors with electronic phenomena such as energy bands in photonic crystals and localization. These similarities stem from the wavelike nature of quantum particles. Because a photon is also a spinning particle it is important to understand if SO interaction may influence the transport of light in a similar way as electrons in vacuum or in semiconductors. It has been known for long that there is no position operator with commuting components and as a consequence photons are not localizable. Therefore, one of the main differences between photons and electrons also disappears as the coordinates of both particles have in fact noncommuting components. It is precisely this property which is at the origin of the SO interaction leaving no doubt about its contribution to the propagation of photons. Therefore these recent studies of the SO coupling in different systems taught us that one can treat both kind of particles on an equal footing. It is thus even more legitimate to wonder whether electronic phenomena have photonic counterparts.

In addition, the localization of light rays is the essential ingredient of the construction of Minkowski space-time. Also in the context of general relativity, it has been argued that the Riemannian metric is determined by the properties of light propagation [6,7]. A deeper understanding of the properties of light and, in particular, its SO interaction which is at the origin of the nonlocalizability, is thus necessary to build a quantum version of space-time.

It has already been observed that the SO coupling induces a rotation of the polarization plane of light propagating in an optical fiber with torsion [8]. This effect had been predicted long ago by Rytov and Vladimirskii [9,10] and can be interpreted in terms of Berry phase [2] in momentum space. Recently it was shown that the noncommutativity of the coordinates affects the ray of light itself in an isotropic inhomogeneous medium [3,11]. Other approaches to the spin-orbit contributions for the propagation of light in isotropic inhomogeneous media have been the focus of several other works [12] and have led to a generalization of geometric optics called geometric spinoptics [13].

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In this letter we investigate how the photon propagation in a static gravitational field, which can be seen as an anisotropic inhomogeneous medium, is affected by its spin-orbit interaction. Our photon description is based on a Dirac-like Hamiltonian in an arbitrary static gravitational field with a possible (nonspecified) torsion of space. The Bargmann-Wigner equations allow us to build the wave function of a spin one particle and to deduce the dynamical operators which satisfy unusual commutation relations in the representation where the Hamiltonian has been diagonalized at the order $\hbar$. It is in the diagonal representation (similarly to the Foldy Wouthuysen (FW) representation) that the physical content of the theory is best revealed. In particular we find that the photon semiclassical Hamiltonian shows a kind of helicity-torsion coupling resulting from the presence of Berry curvatures. From the semiclassical Hamiltonian we can deduce new equations of motion taking Berry curvatures into account. These equations correspond to the first corrections to the geometrical optics. It is found that the helicity is always conserved and that even in a torsionless space-time, photons do not follow the null geodesic due to an anomalous velocity term. The presence of this term is a general feature of the modern point of view of the topological spin transport of quantum particles [1,3,4,14]. The anomalous velocity is directly responsible for the gravitational birefringence phenomenon, i.e., photons with distinct helicity follow different geodesics in a static gravitational field. This confirms earlier claims suggesting that the absence of birefringence in Einstein's gravity might be only a consequence of the limitation of the geometrical optics limit [15]. Our computation also shows that the velocity of light is still equal to c at this order of approximation (a polarization dependent velocity would be a $\hbar^2$ phenomenon). Finally, it is worth noticing that there exists another spinorial representation of the photon Hamiltonian equivalent to the Maxwell equation [16] which contrary to the Bargmann-Wigner equations does not suffer inconsistencies due to the redundancy of the photon wave functions. However it can be seen that both approaches lead to the same semiclassical energy and dynamical operators. The advantage of the Bargmann-Wigner approach presented here is that it is more usual, more straightforward, and that it removes also the null energy eigenstates.

II. FREE PHOTON

We start with the description of the photon dynamics in the vacuum by considering the two Bargmann-Wigner equations ($r = 1, 2$) of a massless spin one particle [17]:

$$(\gamma_0^{(r)}\partial_0 + \gamma_i^{(r)}\partial^i)\Psi_{(a_1a_2)}(x) = 0 \qquad (i = 1, 2, 3), \qquad (1)$$

where $\Psi_{(a_1a_2)} = \Psi_{(a_2a_1)}$ is the symmetrized Bargmann-Wigner amplitude with a_i running from 1 to 4 the spinorial indices of the wave function (as usual the indices 0 and i refer, respectively, to the local time and space coordinates, with the flat metric $(+, -, -, -)$). The $\gamma^{(r)}$ matrix is a γ matrix acting on a_i. This symmetrized direct product of two Dirac spinors assures that the positive energy subspace forms a 3D space corresponding to the irreducible representation of angular momenta 1 deduced from the composition of two states with angular momentum $1/2$. It can be proved that Eqs. (1) are equivalent to the Maxwell equations [18].

Next, we write a Hamiltonian associated to each Bargmann-Wigner equations $\hat{H}^{(r)} = \alpha^{(r)}.\mathbf{P}$ which can be diagonalized by the product of usual FW unitary transformations [19], $U^{(r)}(\mathbf{P}) = (E + c\beta\alpha^{(r)}.\mathbf{P})/E\sqrt{2}$ such that

$$U^{(1)}(\mathbf{P})U^{(2)}(\mathbf{P})\hat{H}^{(r)}U^{(1)}(\mathbf{P})^+U^{(2)}(\mathbf{P})^+ = E\beta^{(r)}, \qquad (2)$$

with $E = Pc$ the energy of the photon in the vacuum. The wave function is also transformed and becomes $\phi_{(a_1a_2)}(\mathbf{P}) = E^{-1}(U^{(1)}(\mathbf{P})^+U^{(2)}(\mathbf{P})^+)\Psi_{(a_1a_2)}(\mathbf{P})$ in the FW representation. Whereas the quasimomentum is invariant through the action of $U^{(r)}$, i.e., $\tilde{\mathbf{p}} = U^{(1)}U^{(2)}\mathbf{P}U^{(1)+}U^{(2)+} = \mathbf{P}$, the position operator becomes $\tilde{\mathbf{r}} = i\hbar\partial_{\mathbf{p}} + \mathcal{A}^{(1)} + \mathcal{A}^{(2)}$ with a pure gauge potential $\mathcal{A}^{(r)} = i\hbar U^{(r)}(\mathbf{p})\partial_{\mathbf{p}}U^{(r)}(\mathbf{p})^+$ induced by the FW transformation. After projection of $\tilde{\mathbf{r}}$ on the positive energy subspace we get the nontrivial $SU(2)$ connection $\mathbf{A} = (\mathbf{p} \wedge \Sigma)/p^2$, and a new position operator $\mathbf{r}$ for the free photon $\mathbf{r} = \mathbf{R} + (\mathbf{p} \wedge \Sigma)/p^2$ where $\mathbf{R} = i\hbar\partial_{\mathbf{p}}$ is the canonical coordinate operator and $\Sigma = \hbar(\sigma^{(1)} + \sigma^{(2)})/2$ the spin one matrix. This definition of the position operator gives rise, through the commutation relations, to a monopole in momentum space $[x^i, x^j] = i\hbar\theta^{ij}(\mathbf{p}) = -i\hbar\varepsilon^{ijk}\lambda p_k/p^3$ where $\lambda = \pm\hbar$ is the helicity and $\theta^{ij}(\mathbf{p}) = \nabla_{p^i}A_j - \nabla_{p^j}A_i + [A_i, A_j]$ the so-called Berry curvature (the origin of this name is explained in [3], see also [20]).

We then build the Hamiltonian of the free photon of positive energy as $H = (E^{(1)} + E^{(2)})/2 = pc$ with $E^{(r)} = pc$ the positive eigenvalue of the operator $\hat{H}^{(r)}$. Therefore, for free particles, the dynamical equations of motion in the FW representation are trivially given by $d\mathbf{r}/dt = \frac{i}{\hbar} \times [\mathbf{r}, H] = \mathbf{p}c/p$ and $d\mathbf{p}/dt = \frac{i}{\hbar}[\mathbf{p}, H] = 0$ which, in particular, implies for the light velocity $v = c$. Note that the same results can be deduced from the Maxwell equations [21].

III. PHOTON IN A STATIC GRAVITATIONAL FIELD

We now extend our previous approach to the case of a photon propagating in an arbitrary static gravitational field, where $g_{0i} = 0$ for $i = 1, 2, 3$, so that $ds^2 = g_{00}(dx^0)^2 - g_{ij}dx^idx^j = 0$. Consider again the Bargmann-Wigner equations of motion with the following associated Hamiltonian of the Dirac form

$$\hat{H}^{(r)} = \sqrt{g_{00}}\alpha^{(r)}.\tilde{\mathbf{P}} + \frac{\hbar}{4}\varepsilon_{\varrho\beta\gamma}\Gamma_0^{\varrho\beta}\sigma^{\gamma} + i\frac{\hbar}{4}\Gamma_0^{0\beta}\alpha_{\beta} \quad (3)$$

and $\tilde{\mathbf{P}}$ given by $\tilde{P}_{\alpha} = h_{\alpha}^{i}(\mathbf{R})(P_i + \frac{\hbar}{4}\varepsilon_{\varrho\beta\gamma}\Gamma_i^{\varrho\beta}\sigma^{\gamma})$ with h_{α}^{i} the static orthonormal dreibein ($\alpha = 1, 2, 3$), $\Gamma_i^{\alpha\beta}$ the spin connection components and $\varepsilon_{\alpha\beta\gamma}\sigma^{\gamma} = \frac{i}{8}(\gamma^{\alpha}\gamma^{\beta} - \gamma^{\beta}\gamma^{\alpha})$. The coordinate operator is again given by $\mathbf{R} = i\hbar\partial_{\mathbf{p}}$. Note that here we consider the general case where an arbitrary static torsion of space is allowed. It is known [22] that for a static gravitational field (which is the case considered here), the Hamiltonian $\hat{H}^{(r)}$ is Hermitian. We now want to diagonalize $\hat{H}^{(r)}$ through a unitary transformation $U^{(r)}(\tilde{\mathbf{P}})$. Because the components of $\tilde{\mathbf{P}}$ depend both on operators $\mathbf{P}$ and $\mathbf{R}$ the diagonalization at order $\hbar$ is performed by adapting the method detailed in [23] to block-diagonal Hamiltonians. To do so, we first write $\hat{H}^{(r)}$ in a symmetrical way in $\mathbf{P}$ and $\mathbf{R}$ at first order in $\hbar$. This is easily achieved using the Hermiticity of the Hamiltonian which yields

$$\hat{H}^{(r)} = \frac{1}{2}(\sqrt{g_{00}}\alpha^{(r)}.\tilde{\mathbf{P}} + \tilde{\mathbf{P}}^{+}.\alpha^{(r)}\sqrt{g_{00}}) + \frac{\hbar}{4}\varepsilon_{\varrho\beta\gamma}\Gamma_0^{\varrho\beta}\sigma^{\gamma}.$$

Now for an ease of exposition we temporarily include the g_{00} is the definition of the vierbein $h_{\alpha}^{i} \rightarrow g_{00}h_{\alpha}^{i}$. The contributions of the g_{00} will be again explicitly written later on in the final result for the energy.

Semiclassical Hamiltonian diagonalization

The semiclassical diagonalization is achieved in two steps. In the first one we diagonalize at first order in $\hbar$ by considering the formal situation where $\mathbf{R}$ is considered as a parameter commuting with $\mathbf{P}$. In the second one we add the contributions due to the noncommuting character of the dynamical variables.

1. *Diagonalization when* P *and* R *commute*

The Hamiltonian $\hat{H}_0^{(r)}$ (we add the index 0 when $\mathbf{R}$ is a parameter) can then be diagonalized at first order in $\hbar$ by the following unitary FW matrix

$$U_0^{(r)}(\tilde{\mathbf{P}}) = \frac{D}{\sqrt{2(E_0^{(r)})^2}}\left(E_0^{(r)} + c\frac{1}{2}\beta(\alpha^{(r)}.\tilde{\mathbf{P}} + \tilde{\mathbf{P}}^{+}.\alpha^{(r)}) + N\right) \quad (4)$$

with $E_0^{(r)} = \sqrt{(\frac{\alpha^{(r)}.\tilde{\mathbf{P}} + \tilde{\mathbf{P}}^{+}.\alpha^{(r)}g}{2})^2}$. We introduced also the notations $N = \frac{\hbar}{4}\frac{i\alpha^{(r)}.(\mathbf{P}\times\Gamma_0)}{P}$ and $D = 1 + \frac{\hbar}{4}\beta\frac{(\mathbf{P}\times\Gamma_0)\times\mathbf{P}}{2P^3}$ with $\Gamma_{0\gamma} = \varepsilon_{\varrho\beta\gamma}\Gamma_0^{\varrho\beta}$ (here, and in the sequel, P means $(P^2)^{1/2}$, with the metric g_{ij} and P^3 is obviously defined as $(P^2)^{3/2}$).

The proof of this diagonalization relies on the following properties: For each parameter $\mathbf{R}$ the matrices h_{α}^{i} and $\Gamma_i^{\alpha\beta}$ are independent of both the momentum and position operators. The matrices β and $\alpha.\tilde{\mathbf{P}}$ anticommute and in the Taylor expansion of $E_0^{(r)}$ all terms commute with β and $\alpha^{(r)}.\tilde{\mathbf{P}} + \tilde{\mathbf{P}}^{+}.\alpha^{(\mathbf{r})}$. In this context the diagonalized Hamiltonian is equal to $U_0^{(r)}\hat{H}_0^{(r)}U_0^{(r)+}$ which reads

$$U_0^{(r)}\hat{H}_0^{(r)}U_0^{(r)+} = c\beta^{(r)}\sqrt{\left(\frac{\alpha^{(r)}.\tilde{\mathbf{P}} + \tilde{\mathbf{P}}^{+}.\alpha^{(r)}}{2}\right)^2 + \frac{\hbar(\mathbf{P}.\Gamma_0)\mathbf{P}.\sigma^{(r)}}{4P^2}} \quad (5)$$

which can also be written

$$U_0^{(r)}\hat{H}_0^{(r)}U_0^{(r)+} = c\beta^{(r)}\sqrt{\mathbf{P}^2 + \frac{\hbar}{2}\mathbf{P}.\sigma^{(r)}\varepsilon_{\varrho\beta\gamma}\Gamma_i^{\varrho\beta}h^{i\gamma} + \frac{\hbar(\mathbf{P}.\Gamma_0)\mathbf{P}.\sigma^{(r)}}{4P^2}}. \quad (6)$$

In this last expression we have neglected contributions of curvature type of order $\hbar^2$.

2. *Corrections when* P *and* R *do not commute*

Now, to diagonalize $H^{(r)}$ at the semiclassical approximation, it is shown in [23] that it is enough to apply the following FW transformation ($\mathbf{R}$ being now an operator)

$$U^{(r)}(\tilde{\mathbf{P}}) = U_0^{(r)}(\tilde{\mathbf{P}}) + X^{(r)} \quad (7)$$

with

$$X^{(r)} = \frac{i}{4\hbar}[\mathcal{A}_{P^l}^{(r)}, \mathcal{A}_{R_l}^{(r)}]U^{(r)}(\tilde{\mathbf{P}}), \quad (8)$$

where we defined the position and momentum pure gauge Berry potential $\mathcal{A}_{\mathbf{R}}^{(r)} = i\hbar U^{(r)}\nabla_{\mathbf{P}}U^{(r)+}$ and $\mathcal{A}_{\mathbf{P}}^{(r)} = -i\hbar U^{(r)}\nabla_{\mathbf{R}}U^{(r)+}$, and then to project the transformed Hamiltonian $U^{(r)}\hat{H}^{(r)}U^{(r)+}$ on the positive energy states. All expressions in $U^{(r)}(\tilde{\mathbf{P}})$ are implicitly assumed to be symmetrized in $\mathbf{P}$ and $\mathbf{R}$ and the corrective term $X^{(r)}$ must be added to restore the unitarity of $U^{(r)}(\tilde{\mathbf{P}})$ which is destroyed by the symmetrization. After projection on the positive energy subspace needed to perform the diagonalization [23] the resulting position and momentum operators can thus be written $\mathbf{r}^{(r)} = \mathbf{R} + \mathbf{A}_R^{(r)}$ and $\mathbf{p}^{(r)} = \mathbf{P} + \mathbf{A}_P^{(r)}$ where the explicit computation gives for the components

$$A_{P,k}^{(r)} = -\hbar c^2\frac{\varepsilon^{\alpha\beta\gamma}\tilde{P}_{\alpha}\sigma_{\beta}^{(r)}(\nabla_{R_k}\tilde{P}_{\gamma})}{2E^{(r)2}} + O(\hbar^2), \quad (9)$$

$$A_{R_k}^{(r)} = \hbar c^2\frac{\varepsilon^{\alpha\beta\gamma}h_{\gamma}^{k}\tilde{P}_{\alpha}\sigma_{\beta}^{(r)}}{2E^{(r)2}} + O(\hbar^2), \quad (10)$$

with $E^{(r)}$ the same as $E_0^{(r)}$ above but now $\mathbf{R}$ is an operator. Performing our diagonalization process leads us to the following expression for the positive energy operator

$$\tilde{\varepsilon}^{(r)} \simeq c\sqrt{p^2 + \frac{\hbar}{2}\mathbf{p}.\sigma^{(r)}\varepsilon_{\varrho\beta\gamma}\Gamma_i^{\varrho\beta}h^{i\gamma}} + \frac{\hbar}{4}\frac{(\mathbf{p}.\Gamma_0)\mathbf{p}.\sigma^{(r)}}{p^2} - \frac{i\hbar}{4\varepsilon^{(r)}}[\nabla_{R_l}(\alpha^{(r)}.\tilde{\mathbf{P}}), \nabla_{P^l}(\alpha^{(r)}.\tilde{\mathbf{P}})] + (\nabla_{R_l}\varepsilon^{(r)})A_{R_l}^{(r)} + (\nabla_{P^l}\varepsilon^{(r)})A_{P^l}^{(r)} + O(\hbar^2). \quad (11)$$

Some computations allow us to rewrite the right-hand side (r.h.s.) of Eq. (11) in a more familiar form. Define first the γ-component of the vector Γ_i as $\Gamma_{i,\gamma} = \varepsilon_{\varrho\beta\gamma}\Gamma_i^{\varrho\beta}(\mathbf{r})$ and the helicity $\lambda^{(r)} = \frac{\hbar\mathbf{p}.\sigma^{(r)}}{p}$. Introducing also

$$\varepsilon^{(r)} = c\sqrt{\left(p_i + \frac{\lambda^{(r)}}{4}\frac{\Gamma_i(\mathbf{r}).\mathbf{p}}{p}\right)g^{ij}\left(p_j + \frac{\lambda^{(r)}}{4}\frac{\Gamma_j(\mathbf{r}).\mathbf{p}}{p}\right)},$$

an explicit computation shows that the semiclassical energy reads

$$\tilde{\varepsilon}^{(r)} = \varepsilon^{(r)} + \frac{\lambda^{(r)}}{4}\frac{\mathbf{p}.\Gamma_0}{p} + \frac{\hbar\mathbf{B}.\sigma^{(r)}}{2\varepsilon^{(r)}} - \frac{(\mathbf{A}_R^{(r)}\times\mathbf{p}).\mathbf{B}}{\varepsilon^{(r)}},$$

where we have introduced a field $B_\gamma = -\frac{1}{2}P_\delta T^{\alpha\beta\delta}\varepsilon_{\alpha\beta\gamma}$ with $T^{\alpha\beta\delta} = h_k^\delta(h^{l\alpha}\partial_l h^{k\beta} - h^{l\beta}\partial_l h^{k\alpha}) + h^{l\alpha}\Gamma_l^{\beta\delta} - h^{l\beta}\Gamma_l^{\alpha\delta}$ the usual torsion for a static metric (where only space indices in the summations give nonzero contributions).

Interestingly, this semiclassical Hamiltonian presents formally the same form as the one of a Dirac particle in a true external magnetic field [4,23]. The term $\mathbf{B}.\sigma$ is responsible for the Stern-Gerlach effect, and the operator $\mathbf{L} = (\mathbf{A}_R\times\mathbf{p})$ is the intrinsic angular momentum of semiclassical particles. The same contribution appears also in the context of the semiclassical behavior of Bloch electrons (spinless) in an external magnetic field [5,24] where it corresponds to a magnetization term. Because of this analogy and since $T^{\alpha\beta\delta}$ is directly related to the torsion of space through $T^{\alpha\beta\delta} = h_k^\delta h^{i\alpha}h^{j\beta}T_{ij}^k$ we call $\mathbf{B}$ a magnetotorsion field.

However, this form for the energy presents the default to involve the spin rather than the helicity, this last quantity being more fundamental for a photon. Actually one can use the property $\lambda\mathbf{p}/2p = \hbar\sigma/2 - (\mathbf{A}_R\times\mathbf{p})$ to rewrite the energy as

$$\tilde{\varepsilon}^{(r)} = \varepsilon^{(r)} + \frac{\lambda^{(r)}}{4}\frac{\mathbf{p}.\Gamma_0}{p} + \frac{\lambda^{(r)}}{2\varepsilon^{(r)}}\frac{\mathbf{B}.\mathbf{p}}{p}. \quad (12)$$

We can now build the Hamiltonian as the sum of the two Hamiltonians for one-half massless spinning particle $\tilde{\varepsilon} = (\tilde{\varepsilon}^{(1)} + \tilde{\varepsilon}^{(2)})/2$. By Taylor expanding the expression of $\varepsilon^{(r)}$ we see that at the leading order in $\hbar$ the sum does recombine to give after reintroducing the g_{00} dependence

$$\tilde{\varepsilon} \simeq \varepsilon + \frac{\lambda}{4}\frac{\mathbf{p}.\Gamma_0}{p} + \frac{\lambda g_{00}}{2\varepsilon}\frac{\mathbf{B}.\mathbf{p}}{p}, \quad (13)$$

where $\varepsilon = c\sqrt{(p_i + \frac{\lambda}{4}\frac{\Gamma_i(\mathbf{r}).\mathbf{p}}{p})g^{ij}g_{00}(p_j + \frac{\lambda}{4}\frac{\Gamma_j(\mathbf{r}).\mathbf{p}}{p})}$ with $\lambda = (\lambda^{(1)} + \lambda^{(2)})/2$ the photon helicity. In the same manner, the Berry connections for the photon become $\mathbf{A}_R = \mathbf{A}_R^{(1)} + \mathbf{A}_R^{(2)}$ and $\mathbf{A}_P = \mathbf{A}_P^{(1)} + \mathbf{A}_P^{(2)}$. This allows us to write the dynamical operators at leading order in $\hbar$ as

$$\mathbf{r} = i\hbar\partial_{\mathbf{p}} + \hbar c^2\frac{\mathbf{P}\times\Sigma}{2\varepsilon^2}, \quad (14)$$

$$\mathbf{p} = \mathbf{P} - \hbar c^2\left(\frac{\mathbf{P}\times\Sigma}{2\varepsilon^2}\right)\nabla_{\mathbf{R}}\tilde{\mathbf{P}}, \quad (15)$$

where in Eqs. (14) and (15) ε^2 can be approximated by $c^2p_ig^{ij}g_{00}p_j$ at the order considered.

The semiclassical Hamiltonian equation (13) is one of the main results of this paper. It contains, in addition to the energy term ε, new contributions due to the Berry connections. Indeed, Eq. (13) shows that the helicity couples to the gravitational field through the magnetotorsion field $\mathbf{B}$ which is nonzero for a space with torsion. As a consequence, a hypothetical torsion of space may be revealed through the presence of this coupling. Note that, in agreement with [25], this Hamiltonian does not contain the spin-gravity coupling term $\Sigma.\nabla g_{00}$ predicted in [26].

From Eqs. (14) and (15) we deduce the new (noncanonical) commutations rules

$$[r^i, r^j] = i\hbar\Theta_{rr}^{ij}, \quad (16)$$

$$[p^i, p^j] = i\hbar\Theta_{pp}^{ij}, \quad (17)$$

$$[p^i, r^j] = -i\hbar g^{ij} + i\hbar\Theta_{pr}^{ij}, \quad (18)$$

where $\Theta_{\zeta\eta}^{ij} = \partial_{\zeta^i}A_{\eta^j} - \partial_{\eta^i}A_{\zeta^j} + [A_{\zeta^i}, A_{\eta^j}]$ where ζ, η mean either r or p. An explicit computation shows that at leading order

$$\begin{aligned}\Theta_{rr}^{ij} &= -\hbar c^4\frac{(\Sigma.\mathbf{p})p_\gamma}{2\varepsilon^4}\varepsilon^{\alpha\beta\gamma}h_\alpha^ih_\beta^j,\\ \Theta_{pp}^{ij} &= -\hbar c^4\frac{(\Sigma.\mathbf{p})p_\gamma}{2\varepsilon^4}\nabla_{r_i}p_\alpha\nabla_{r_j}p_\beta\varepsilon^{\alpha\beta\gamma},\\ \Theta_{pr}^{ij} &= \hbar c^4\frac{(\Sigma.\mathbf{p})p_\gamma}{2\varepsilon^4}\nabla_{r_i}p_\alpha h_\beta^j\varepsilon^{\alpha\beta\gamma}.\end{aligned} \quad (19)$$

From the additional commutation relations between the helicity and the dynamical operators $[r_i, \lambda] = [p_i, \lambda] = 0$ we deduce the semiclassical equations of motion

$$\begin{aligned}\dot{\mathbf{r}} &= (1 - \Theta_{pr})\nabla_{\mathbf{p}}\tilde{\varepsilon} + \dot{\mathbf{p}}\times\Theta_{rr},\\ \dot{\mathbf{p}} &= -(1 - \Theta_{pr})\nabla_{\mathbf{r}}\tilde{\varepsilon} + \dot{\mathbf{r}}\times\Theta_{pp}.\end{aligned} \quad (20)$$

To complete the dynamical description of the photon notice that at the leading order the helicity λ is not changed by the unitary transformation which diagonalizes the Hamiltonian so that it can be written $\lambda = \hbar\mathbf{p}.\Sigma/p$. After a short computation one can check that the helicity is always conserved

$$\frac{d}{dt}\left(\frac{\hbar \mathbf{p}.\Sigma}{p}\right) = 0 \quad (21)$$

for an arbitrary static gravitational field independently of the existence of a torsion of space.

Equations (20) are the new semiclassical equations of motion for a photon in a static gravitational field. They describe the ray trajectory of light in the first approximation of geometrical optics (GO). (In GO it is common to work with dimensionless momentum operator $\mathbf{p} = k_0^{-1}\mathbf{k}$ with $k_0 = \omega/c$ instead of the momentum [11].) For zero Berry curvatures we obtain the well-known zero order approximation of GO and photons follow the null geodesic. The velocity equation contains the by now well-known anomalous contribution $\dot{\mathbf{p}} \times \Theta^{rr}$ which is at the origin of the intrinsic spin Hall effect (or Magnus effect) of the photon in an isotropic inhomogeneous medium of refractive index $n(r)$ [3,11–13]. Indeed, this term causes an additional displacement of photons of distinct helicity in opposite directions orthogonally to the ray. Consequently, we predict gravitational birefringence since photons with distinct helicities follow different geodesics. In comparison to the usual velocity $\dot{\mathbf{r}} = \nabla_{\mathbf{p}}\tilde{\varepsilon} \sim c$, the anomalous velocity term $\mathbf{v}_\perp$ is obviously small, its order $v_\perp^i \sim c\tilde{\lambda}\nabla_{r^j} g^{ij}$ being proportional to the wave length $\tilde{\lambda}$.

The momentum equation presents the dual expression $\dot{\mathbf{r}} \times \Theta_{pp}$ of the anomalous velocity which is a kind of Lorentz force which being of order $\hbar$ does not influence the velocity equation at order $\hbar$. Note that similar equations of motion with dual contributions $\dot{\mathbf{p}} \times \Theta_{rr}$ and $\dot{\mathbf{r}} \times \Theta_{pp}$ were predicted for the wave packets dynamics of spinless electrons in crystals subject to small perturbations [24]. The complicated Eq. (20) simplifies greatly for a symmetric gravitational field as shown below.

IV. SYMMETRIC GRAVITATIONAL FIELD

As a simple application, consider the symmetric case $g_{00}g^{ij} = \delta^{ij}F^2(\mathbf{R})$. A typical example of such a metric is the Schwarzschild space-time in isotropic coordinates. For a symmetric metric one has $\mathbf{B}.\mathbf{p} = \Gamma_0 = 0$ and the semiclassical energy Eq. (13) reduces to

$$\tilde{\varepsilon} = c(pF(\mathbf{r}) + F(\mathbf{r})p)/2 \quad (22)$$

with the dynamical variables $\mathbf{r} = \mathbf{R} + \hbar\frac{\mathbf{P}\times\Sigma}{P^2}$, $\mathbf{p} = \mathbf{P}$, and the following commutation relations $[r^i, r^j] = i\hbar\Theta_{rr}^{ij} = -i\hbar\lambda\varepsilon^{ijk}p_k/p^3$, $[p^i, p^j] = 0$, $[r^i, p^j] = i\hbar g^{ij}$. As a consequence, we derive the following equations of motion

$$\dot{\mathbf{r}} = \nabla_{\mathbf{p}}\tilde{\varepsilon} + \dot{\mathbf{p}} \times \Theta_{rr}, \qquad \dot{\mathbf{p}} = -\nabla_{\mathbf{r}}\tilde{\varepsilon}. \quad (23)$$

In the symmetric case the equations of motion become simpler than in the general case, but the gravitational birefringence is still present. These equations were already postulated (but not derived) in [3] to explain the Magnus effect (the different deviation of light of distinct polarization in an inhomogeneous medium of refractive index $n(r)$) observed despite its smallness in inhomogeneous isotropic optical fibers [27] and also discussed theoretically in less general contexts and with different approaches in several other papers [11–13]. This case fits within our formalism since a gravitational field can be seen as an isotropic medium related to the metric through the relation $g^{ij} = \delta^{ij}n^{-1}(r)$. Therefore the gravitational birefringence predicted here is simply due to the Magnus effect as a consequence of the photon spin-orbit interaction. In particular this effect does not need a coupling between the electromagnetic field and a torsion term as proposed in [28].

We now apply the equations of motion to compute the deflection of polarized light by a star's gravitational field. A polarization independent result is expected by the Einstein's theory of gravitation which does not consider the anomalous velocity. With the Schwarzschild metric one has $F(R) = 1 - \frac{2GM}{R}$ [26] and for the equations of motion we get

$$\dot{\mathbf{p}} = -2GM\frac{\mathbf{r}}{r^3}p, \qquad \dot{\mathbf{r}} = \frac{\mathbf{p}}{p}F + \lambda\frac{2GM}{r^3}\frac{\mathbf{r}\times\mathbf{p}}{p^2}. \quad (24)$$

We can therefore easily compute the angle of deflection $\Delta\phi$ between ingoing and outgoing polarized rays

$$\Delta\phi = \frac{4GM}{c^2 r_0}\left(1 - \frac{\lambda}{\hbar}\frac{\tilde{\lambda}}{2\pi r_0}\right),$$

where r_0 is the smallest distance of the light ray to the central source of gravitation, M the mass of the star, $\tilde{\lambda}$ the wave length of the photon, and λ the helicity.

We observe that the deviation is helicity dependent as a consequence of the anomalous velocity, but that the effect is very small being of order $\tilde{\lambda}/r_0$ and certainly unobservable for a star like the Sun.

V. CONCLUSION

In this paper, we have diagonalized at the first order in $\hbar$ the photon Hamiltonian in a static gravitational field. This diagonal Hamiltonian displays a new interaction between the helicity and a torsion field. As a consequence the torsion of space, if any, could in principle be determined through this coupling. However, even in the absence of torsion, we found two new semiclassical equations of motion including Berry phase contributions for both dynamical variables, predicting that the photon does not follow the null geodesic due to its spinning nature. The reason is an anomalous velocity, responsible for the gravitational birefringence. This last result is in agreement with the modern point of view about the spinning particles evolution. Our results are not restricted to the gravitational field but also apply to systems with anisotropic refractive indices.

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