Extending the WMAP bound on the size of the Universe

Joey Shapiro Key and Neil J. Cornish

Department of Physics, Montana State University, Bozeman, Montana 59717, USA

David N. Spergel

Department of Astrophysical Sciences, Princeton University, Princeton, New Jersey 08544, USA

Glenn D. Starkman

Center for Education and Research in Cosmology and Astrophysics, Department of Physics, Case Western Reserve University, Cleveland, Ohio 44106–7079, USA
and Astrophysics Department, University of Oxford, Oxford OX1 3RH, United Kingdom

The topology of the Universe can leave an imprint on the cosmic microwave background (CMB) radiation. Clues to the shape of our Universe can be found by searching the CMB for matching circles of temperature patterns. A full sky search of the CMB, mapped extremely accurately by NASA's WMAP satellite, returned no detection of such matching circles and placed a lower bound on the size of the Universe at 24 Gpc. This lower bound can be extended by optimally filtering the WMAP power spectrum. More stringent bounds can be placed on specific candidate topologies by using a combination statistic. We use optimal filtering and the combination statistic to rule out the suggestion that we live in a Poincaré dodecahedral space.

I. INTRODUCTION

What is the shape of space? While this question may have once seemed more philosophical than scientific, modern cosmology has the chance to answer it using the oldest observable light in the Universe, the cosmic microwave background radiation (CMB). NASA's Wilkinson Microwave Anisotropy Probe (WMAP) has made a detailed map of the CMB sky which has been used to provide answers to many age-old questions about the nature of the Universe [1].

While it is certainly possible that the Universe extends infinitely in each spatial direction, many physicists and philosophers are uncomfortable with the notion of a universe that is infinite in extent. It is possible instead that our three dimensional Universe has a finite volume without having an edge, just as the two dimensional surface of the Earth is finite but has no edge. In such a universe, it is possible that a straight path in one spatial direction could eventually lead back to where it started. For a short enough closed path, we expect to be able to detect an observational signature revealing the specific topology of our Universe [2]. For a more thorough introduction to small universes, their philosophical underpinnings, and observational prospects, see the review by Levin [3].

II. GEOMETRY AND TOPOLOGY

An important question answered by the WMAP mission is that of the curvature of space. The matter and energy density of the Universe indicate that space is very nearly flat. The WMAP data point to a universe with a total energy density within 2% of critical [4] (at the 1-σ level). This means that, even if space in not quite flat, the radius of curvature of the Universe is at least of order the size of the observable Universe, and space can be considered to be nearly flat.

The WMAP sky also provides clues about the topology of the Universe. Observable topologies would have a characteristic fingerprint in the CMB sky. The set of possible topologies for our Universe is determined by the curvature of space. In a flat universe, the allowable topologies are restricted to a set of 18 possibilities. It has been shown that a nearly flat universe would have an observational fingerprint very similar to that of an exactly flat universe [5,6]. Cornish, Spergel, and Starkman [2] have described the CMB signature revealing the shape of space as "circles in the sky."

III. CIRCLES IN THE SKY

Circles in the CMB revealing the topology of the Universe could only be observed for a universe that admits sufficiently short closed paths. If our Universe is multiply connected and the fundamental domain fits inside the observable Universe, we expect to see the characteristic signature of the topology in the CMB. (The fundamental domain is a simply connected cell that can be used to tile space). In a universe with a multiconnected trivial topology, there exist multiple copies of objects, one in each copy of the fundamental domain. A copy of our own galaxy, solar system, and planet could possibly be observed many light-years away. Surrounding each copy of Earth would be

a copy of the last scattering surface—a 2-sphere at redshift $z \simeq 1100$. Since in a homogeneous and isotropic geometry the intersection of two 2-spheres is a circle, and these circles in the CMB are physically the same place in space, to the extent that the CMB measures the local gravitational potential, density, and temperature on the last scattering surface, we expect to find matching circles of temperature patterns in the CMB when looking in two different directions in the sky. (See our Fig. 1, and Figs. 5–8 in Ref. [7].)

The search for these matching circles of temperature patterns in the WMAP data is a huge task. The resolution of the WMAP satellite provides over 3×10^6 independent data points on the sky. Each of these points can be considered a circle center, with the circle radius ranging between $0°$ and $90°$. Once the temperature pattern around a given circle is found, it must be compared to every other circle in the sky with the same angular radius, and the matching statistic calculated for the pair. The phasing of each circle pair must also be considered, ranging from $0°$ to $360°$, since the start of the comparison can begin anywhere along the second circle. This full search is time consuming, and is still in progress.

A more efficient search can be employed when considering the WMAP result of nearly zero spatial curvature for the Universe. In a flat universe, all 17 multiconnected topologies produce some circle pairs that are back-to-back (that is, their circle centers are $180°$ apart), though the smallest circles may not be back-to-back (for example, in a half turn space the first pair of back-to-back circles in the twisted direction comes from the next-to-nearest copy of the last scattering surface). It can also be shown that, for a large radius of curvature compared to the radius of the last scattering surface, multiconnected topologies with a positive curvature produce nearly back-to-back circle pairs [5,6]. A shorter search can thus be used on the WMAP

data, where each circle is compared only to its back-to-back pair. Similarly, an almost back-to-back pair search can be implemented, looking only for matching circles between $170°$ and $180°$ from the first circle center. (At $1°$ degree resolution the $170° \rightarrow 180°$ search takes ~10 times longer than the back-to-back search.) This search has been completed for both orientable and nonorientable topologies, and no statistically significant matches were found [8].

IV. THE S STATISTIC

A matching statistic is needed to define the correlation between the temperature patterns around two circles with the same angular radius. In Ref. [8], the S statistic for circle comparison is defined,

$$S_{ij}(\alpha, \beta) = \frac{2 \sum_m m T_{im}(\alpha) T_{jm}^*(\alpha) e^{-im\beta}}{\sum_n n[|T_{in}(\alpha)|^2 + |T_{jn}(\alpha)|^2]}. \quad (1)$$

Here T_{im} denotes the mth harmonic of the temperature pattern around the ith circle. The parameters α and β denote the angular radius and the phase offset between the circles, respectively. The S statistic is normalized with a perfect match yielding an S value of 1. The comparison is done in Fourier space, in order to include appropriate weighting of different angular scales. The m weighting factor on circles plays the same role as the usual $\ell(2\ell + 1)$ weighting that appears in the full angular power spectrum: these factors take into account the numbers of degrees of freedom per mode [9]. If the m weighting is neglected, large angular scales dominate and completely drown out small-scale fluctuations. A limited search for matching circles in the CMB sky has also been performed in Ref. [10]. There, the matching statistic of [2] was used; one that does not include the appropriate m weighting
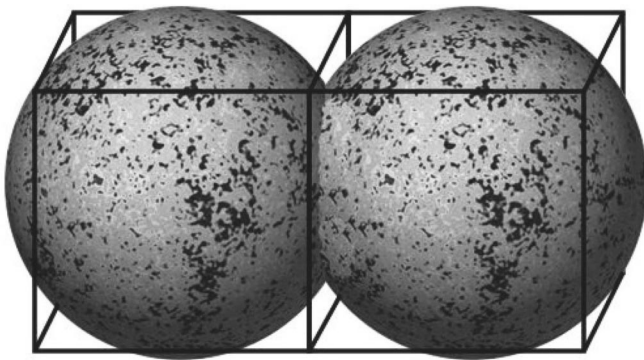

FIG. 1. In a multiply connected universe, all of space must be tiled with copies of the fundamental domain. Here the topology is toroidal, the fundamental domain is a cube, and a copy of the last scattering surface is associated with each domain. The intersections are physically the same place in space, producing matching circles of temperature patterns in two different directions as observed from the Earth.
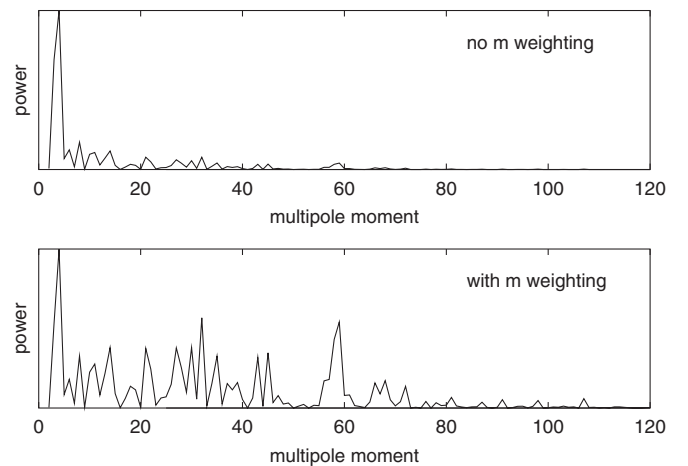

FIG. 2. The Fourier coefficients of δT, as a function of multipole moment m, around a circle with no m weighting (top panel) and including m weighting (bottom panel).

factor:

$$S \equiv \frac{\langle 2(\frac{\delta T}{T}(\alpha, \phi))_i (\frac{\delta T}{T}(\alpha, \phi + \beta))_j \rangle}{\langle (\frac{\delta T}{T})_i^2 + (\frac{\delta T}{T})_j^2 \rangle}$$

$$= \frac{2 \sum_m T_{im}(\alpha) T_{jm}^*(\alpha) e^{-im\beta}}{\sum_n [|T_{in}(\alpha)|^2 + |T_{jn}(\alpha)|^2]}. \qquad (2)$$

Here the brackets denote integration along the circle. This alternative definition of the S statistic can be calculated in position space, but the consequence of disregarding the weighting factor is that large scale temperature variations swamp any small-scale features. The importance of the m weighting is apparent in Fig. 2.

V. FALSE POSITIVE CALCULATION

An essential aspect of our search is the calculation of the threshold for a significant match. We expect this false positive line to be higher for circles with smaller angular radius, since these circles have fewer pixels used for comparison and thus have a greater chance for random matching. It has been shown in Ref. [8] that the level for a positive detection can be calculated using

$$S_{\max}^{\mathrm{fp}}(\alpha) \simeq \langle S^2 \rangle^{1/2} \sqrt{2 \ln\left(\frac{N_{\mathrm{search}}(\alpha)}{2\sqrt{\pi \ln(N_{\mathrm{search}}(\alpha))}}\right)}. \qquad (3)$$

Here N_{search} is the size of the search space, counting the number of circle comparisons made in the search at each angular radius, α. The false positive threshold is sensitive to N_{search}, increasing for larger searches. The expected value of S for random skies can be estimated from first principles, but it is better to numerically measure the expectation value from the data. The expectation value of S, $\langle S^2 \rangle^{1/2}$, and the size of the search space, N_{search}, follow from the implementation of the specific search being performed. When looking at the results from our searches, we plot a false positive threshold for which we expect fewer than 1 in 100 random skies to produce an S value above the line.

Calculating the correct value of N_{search} involves determining the number of completely independent possible searches. For example, a sky with 2° smoothing has less independent pixels than a sky with 1° smoothing and thus has fewer possible independent circle pairs to be searched. The full N_{search} value for a search should take into account the number of independent circle radii (α) searched at a given sky resolution. This was not included in Ref. [8], but we include it here by multiplying the N_{search} value at each α calculated by the search code by a factor equal to the number of independent radii searched. The false positive level must be considered for any claim of detection, and is

used to place the lower bound on the size of the Universe for no circle pair detections.

VI. FALSE NEGATIVES

Producing fully simulated CMB data sets for multiconnected spacetimes is a computationally demanding exercise, and most of our testing has been done using a single 3-torus simulation with the best-fit WMAP cosmological parameters. Since the degradation of the circle matches depends only on the cosmological parameters and not on the specific topology, we do not consider the limited number of test skies to be a significant concern. On the other hand, it does prevent us from setting rigorous bounds on the expected number of false negatives.

In a blind test of our ability to use the circle search method to detect and characterize topologies, A. Riazuelo supplied us with a simulated sky of unknown topology. As for our toroidal simulation, all appropriate CMB effects were included in the simulation. As he later informed us, the cosmological parameters were deliberately tuned to maximize those effects (e.g. ISW and Doppler) likely to decorrelate matched circles. We searched for circles pairs on which the m-weighted S statistic exceeded the circle radius (α)-dependent threshold (3). We found to our surprise that nearly all antipodal circle pairs above a certain radius were matched. This implied that the simulated universe was a projective 3-sphere (S_3/Z_2), of radius equal to that of the last scattering surface. We were informed by Riazuelo that this was the correct conclusion. Knowing the topology, we were then able to calculate the false negative rate as a function of α. Even on this deliberately difficult sky, for $\alpha > 20°$, the false negative rate fell below 5%, and for $\alpha > 25°$, it fell below 1%.

VII. THE WMAP BOUND

The full sky search for back-to-back and almost back-to-back matching circles was performed in Ref. [8], and it is apparent in Figs. 3 and 4 of that paper that no statistically significant matches were found.

A simulation of the CMB has been produced [8] that includes all relevant physics, using model parameters that give a good match to the real CMB power spectrum, but with a multiconnected topology built in. This 3-torus universe with a fundamental domain that fits inside the sphere of the CMB is used to test the search codes. The results of the back-to-back search on the simulated 3-torus universe indeed show peaks in the matching statistic, indicating matched circle pairs (Ref. [8], Fig. 1). The fact that the results of the same back-to-back and nearly back-to-back searches of the WMAP data show no such peaks can be used to place a lower limit on the size of our Universe. The intersection of the false positive threshold and the matching statistic peaks found for a simulated 3-torus universe defines this lower limit, since it indicates that the smallest

circle pairs we could expect to detect would be approximately 20°.

For decreasing α, a downward trend can be seen in the 3-torus simulation results for the height of the peak indicating a match. Photons from the last scattering surface travel toward Earth, carrying with them an imprint of the conditions at their point of origin. It is this effective temperature of the CMB photons that should match along a circle. However, two matching photons traveling different paths toward Earth will encounter different line-of-sight effects, degrading the temperature match. The combined line-of-sight effects on CMB photons can be understood by considering the physical contributions to the total power spectrum of the CMB [11] (see Fig. 3). The Integrated Sachs-Wolfe effect (ISW) degrades any possible match, while the contribution from the acoustic velocity term depends on the size of the circle pair. The velocity term is given by $\hat{n} \cdot \vec{v}$, where $\hat{n}$ is a unit vector along the line of sight and $\vec{v}$ is the velocity vector of the plasma at the surface of last scatter. For large circles this term is correlated, while for small circles this term becomes increasingly anticorrelated as the circles get smaller (see Fig. 4). This velocity term thus degrades the matching statistic value for small circles and explains the downward trend of peaks for small α.

The null result of the search for matching circles in the CMB indicates that the fundamental domain of the Universe must be at least of order the size of the surface of last scatter. The fact that the false positive line intersects the maximum peaks expected for the matching statistic at $\alpha = 20°$ means that the fundamental domain must be big

FIG. 4. Here a circle represents a cross section of the 2-sphere of the CMB. The point of intersection thus represents a point along a circle in the sky, viewed from two different copies of the Earth. It can be seen that small circles have an almost antiparallel velocity contribution, $\hat{n} \cdot \vec{v}$ (bottom panel), while large circles have an almost parallel velocity term (top panel).

enough such that only circles smaller than 20° could be produced by intersections of copies of the CMB. This, along with the best-fit values for the other cosmological parameters, places a lower bound on the size of the Universe at 24 Gpc.

VIII. EXTENDING THE BOUND

There are a couple of techniques that can be applied in an attempt to extend this bound beyond 24 Gpc (or possibly detect the topology of the Universe, lurking just below the current false positive threshold). Since we are interested in raising S values for small circles where it dips below the detection line, we can filter out from the CMB power spectrum the terms that degrade matches for small circles. This requires filtering out both the ISW effect on large angular scales and the acoustic velocity term on smaller angular scales. We thus multiply the WMAP power spectrum by a filter that includes only the regions of the power spectrum where the effective temperature dominates all other terms.

The basic shape of our filter comes from the physical theory described in Fig. 3. We then search for the best filter on the 3-torus simulation, varying the location, width, and steepness of the filter windows. Our best simulated filtered power spectrum is seen in Fig. 5.
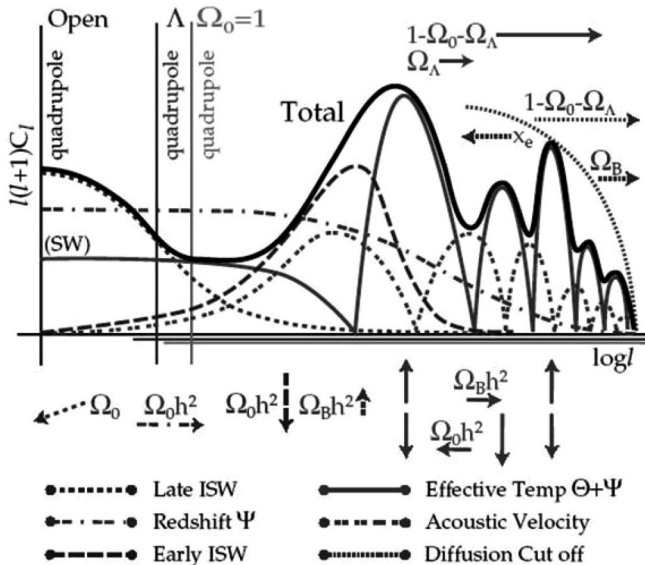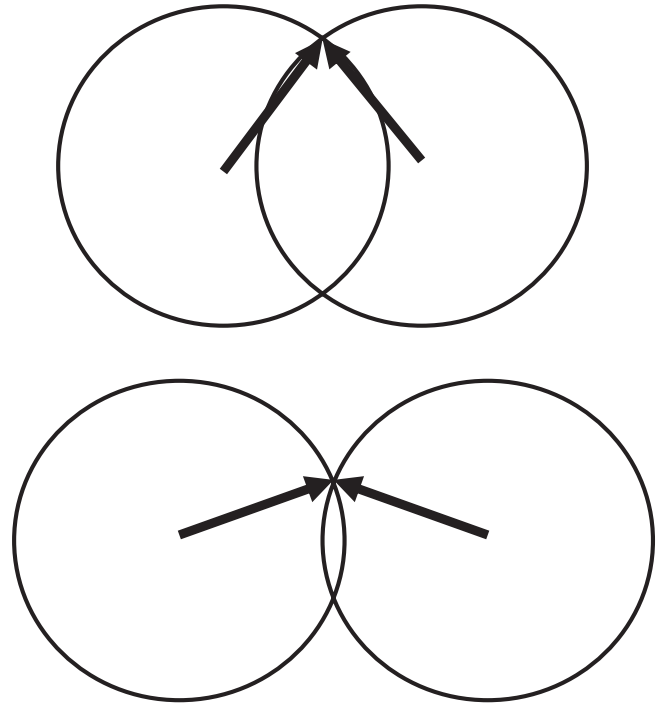

FIG. 3. Contributions to the power spectrum of the CMB. The effective temperature around circles can be correlated due to topology, but is degraded by ISW effects. The acoustic velocity also degrades correlation for small circles. (Reproduced with permission from *Nature* [11]).
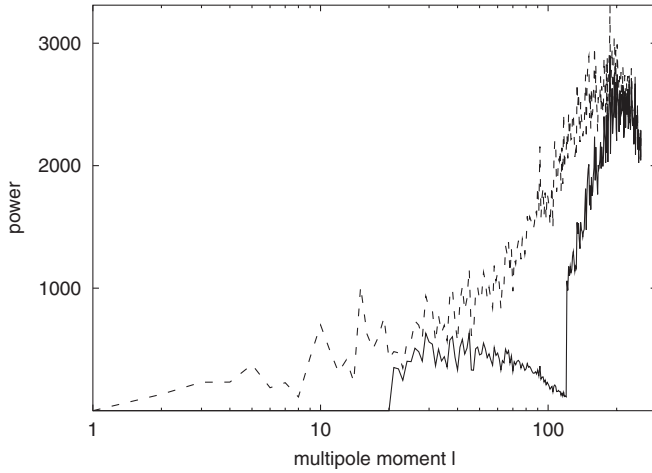
FIG. 5. The unfiltered simulated power spectrum (dotted line) and the power spectrum with regions dominated by either ISW or acoustic velocity effects filtered out (solid line).
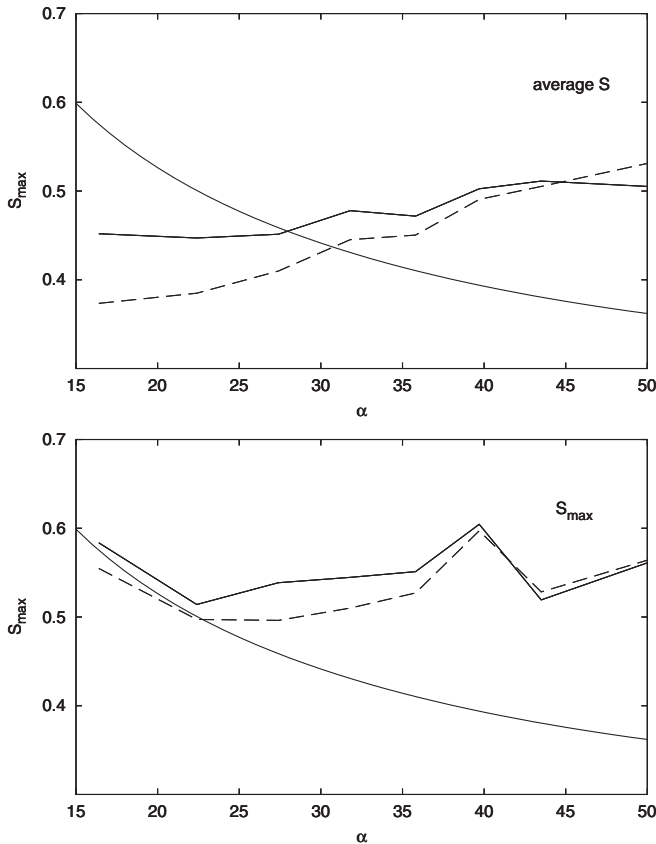

FIG. 6. The dashed lines give the S values for the unfiltered simulation, while the solid lines give the S values when the power spectrum filter is included. Top panel: The average values of the S statistic as a function of angular radius. Bottom panel: The maximum values of S as a function of angular radius. S_{max} increases by $\sim 10\%$ for small circles with filtering, but decreases for large circles, as expected. Plotted with the calculated false positive line.

The filtering indeed raises the S values for small circles in our simulated 3-torus universe and lowers the S values for large circles, as expected when filtering out the velocity term that contributes to raising S for large angular radius pairs (Fig. 6).

In general, the false positive line depends on the filter being used. A filter that keeps only the low l power, for instance, would result in a higher false positive line since there are essentially less significant pixels without the high l values, and thus a greater chance of a false match. Our power spectrum filter keeps most of the high l values, especially near the first acoustic peak, and thus does not significantly affect the false positive line. (Note: the simulated sky has $\ell_{max} = 256$.)

Another choice of filter is the use of m weighting, as discussed in Sec. IV. The S statistic calculation neglecting m weighting results in higher S values for circle pair comparisons at all angular radii, raises the false positive level, and degrades significant peaks. These effects are illustrated in Fig. 7, showing the results of a back-to-back circle search of a 3-torus CMB simulation when m weighting is neglected (top panel) and included (bottom panel). It can be seen that the search without m weighting emphasizes large angular scales, and thus only finds matching circles with $\alpha > 45°$.

Several recent papers [12–14] have called into question the efficacy of the matched circle test. In each instance the negative conclusions can be traced to the use of the un-
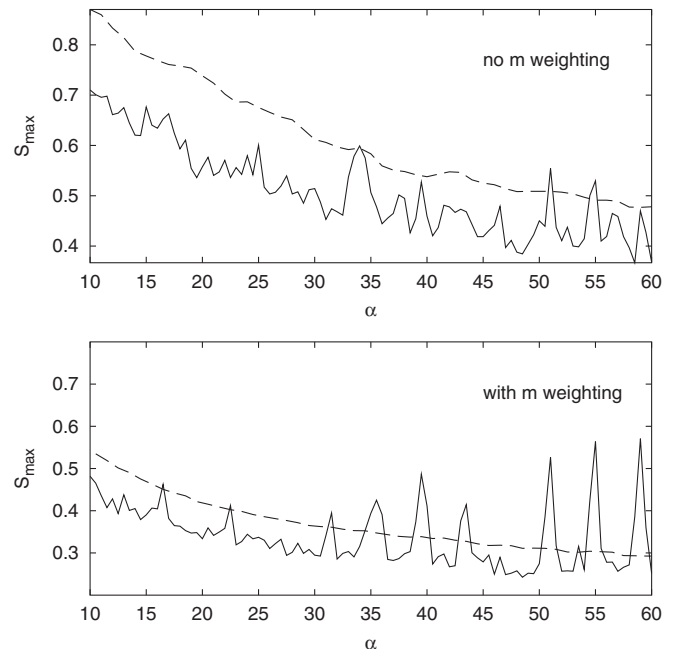

FIG. 7. A back-to-back circle search is performed on our 3-torus simulation with 1° smoothing. The top panel shows the results for a search with the matching statistic calculated without the necessary m weighting, while the bottom panel indicates a search including m weighting. The threshold line for detection for each search is also plotted.

weighted S statistic. As our searches of realistic simulated skies have shown, the properly weighted matching statistic provides a powerful tool for probing the topology of the Universe.

IX. COMBINATION STATISTIC

Another technique for probing below the limit of 20° circle pairs is employing a directed search for a specific topology. Such a search can increase the S value for small circles by stringing together several sets of circles in an expected configuration to create an effectively larger radius circle with the increased number of pixels. For a cubic 3-torus universe, for example, we expect to find at least three sets of circle pairs corresponding to the matching faces of the fundamental domain and we can predict the relative orientations of each matching set. For a cubic torus the three back-to-back circle directions are mutually orthogonal.

The combination search for this example would consist of choosing a circle center and circle radius and comparing the pixels around this circle to the circle exactly opposite it in the sky. The search then chooses an axis for a second set of back-to-back circles with the same angular radius. The location of the final circle pair is fixed once the axis has been chosen. The pixels along all three pairs are strung together to calculate the total matching statistic for the whole set. Since there are more pixels for the whole set than for one pair of circles, there is a smaller chance for a random match.

This combination statistic can be tested using our simulated 3-torus universe. There are matching circles with angular radius of about 16° in our simulation. These pairs cannot be detected in the single pair back-to-back circle search because the false positive threshold is above the peak of their matching statistic.

Our 3-torus simulation has a fundamental domain such that several copies fit within the surface of last scatter. There are thus more than three sets of circles at 16°, and our combination statistic is calculated by stringing together the pixels along four sets of circles.

It can be seen in Fig. 8 that the search for the combination statistic in our simulated 3-torus universe raises the S value above the false positive line, and we can thus make a statistically significant detection. The 3-torus combination statistic can be applied to the real WMAP data to rule out (or detect) sets of matching circles smaller than the 20° cutoff of the single circle pair search.

It is crucial to calculate the false positive line correctly for the combination search. While the combination statistic lowers the false positive level for small circles by creating effectively larger circles, it must be remembered that the combination statistic requires a larger search than looking for back-to-back circles. There is an extra degree of freedom for the orientation of the fundamental domain once the first set of circles is identified. The false positive level
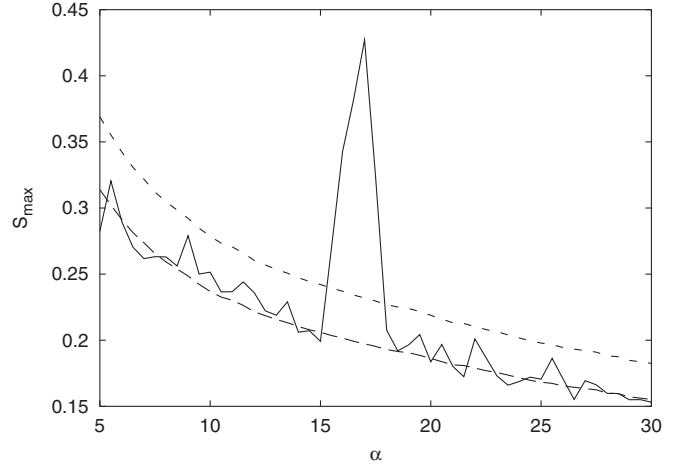

FIG. 8. The combination statistic calculated for our 3-torus simulation using the optimal filter (solid line). The upper dashed line gives the threshold for less than 1 in 100 random skies producing an S value above the line.

for the combination search can be estimated from the standard pairwise search as follows. The root-mean-square value for S_{combo} is given by $\langle S^2_{combo}\rangle^{1/2} = \langle S^2_{pair}\rangle^{1/2}/n^{1/2}$, where n is the number of circle pairs used in the combination statistic. The number of circle comparisons in the combination search, N^{combo}_{search}, scales as $N^{(d+1)/d}_{search}$, where d is the number of degrees of freedom in the pairwise search. Combining these observations in (3) yields

$$S^{fp}_{max\text{-}combo}(\alpha) \simeq \left(\frac{d+1}{nd}\right)^{1/2} S^{fp}_{max\text{-}pair}(\alpha). \qquad (4)$$

We can also estimate the combination statistic level for matching circles in terms of the pairwise values. The combination statistic for matching circles, S^{match}_{combo}, is the weighted average of the pairwise statistic for matching circles, S^{match}_{pair}, and since the total power around each circle is roughly equal, we have $S^{match}_{combo} \simeq \bar{S}^{match}_{pair}$, where the bar denotes the average. These estimates for the false positive and match level for the combination searches were found to be accurate predictors of the numerical results. Extrapolating from the values of $\bar{S}^{match}_{pair}$ seen in the upper panel of Fig. 6 and the values of S^{match}_{pair} seen in the lower panel of Fig. 7 suggests that a combination search using four or more circle pairs should be able to detect circles with radii as small as 5°.

X. POINCARÉ DODECAHEDRAL SPACE

In a universe with positive curvature, the faces of a dodecahedron can be identified to form what is known as Poincaré dodecahedral space (the ''soccer ball universe''). The face identifications for a dodecahedron involve a twist of ±36° to insure the matching up of the vertices of the face pair (Fig. 9). It has been claimed that the power
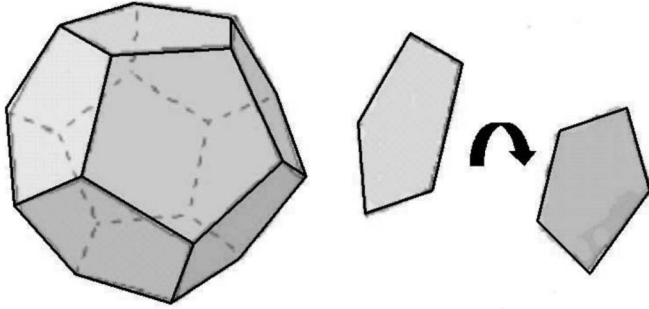
FIG. 9. The Poincaré dodecahedron is a 12-sided polygon with identical faces. This shape can tile space with no gaps only in a positively curved universe. The face identifications include a twist of ±36°.

spectrum expected for a universe with such a topology closely matches the power spectrum found for our Universe by WMAP [15].

XI. CLAIM OF A MULTICONNECTED TOPOLOGY

A claim has been made that six sets of matching circles in the CMB have been found, indicating that the Universe has the topology of the Poincaré dodecahedron [10]. The claim, however, is not accompanied by a statistical analysis of the results, and cannot be assessed until this is completed.

In Ref. [10], the combination statistic for the expected six sets of matching circles of a Poincaré dodecahedron has been computed for a range of circle radii α with a peak appearing for circles around 11°. The peak only appears when the −36° twist is used when identifying faces of the dodecahedron (cf. Ref. [10], Fig. 4). This peak is then compared to the same search using the +36° twist for face identifications (cf. Ref. [10], Fig. 5), where there appears to be no peak. It is expected that the peak would appear for only one twist value if this is an indication of the physical topology of the Universe.

Figure 10 shows that we were able to find the same peak in the matching statistic around 11° using a combination statistic. In order to reproduce the peak, we used the S statistic given in Eq. (1), a galaxy cut, and the 2° smoothing used in Ref. [10]. The locations of the six pairs in the CMB sky, shown in Fig. 11, also match those found in Ref. [10].

For proper analysis of the results, the false positive line must be calculated correctly. The results from the +36° twist search do not give the actual false positive line for detection. To get a better idea of the expected peak levels for a universe with a trivial topology, the WMAP data can be randomly scrambled. This preserves the power spectrum and distribution of temperature values while destroying any matching circles that would give an indication of topology. The same search can then be performed on many of these scrambled skies and the results plotted together to
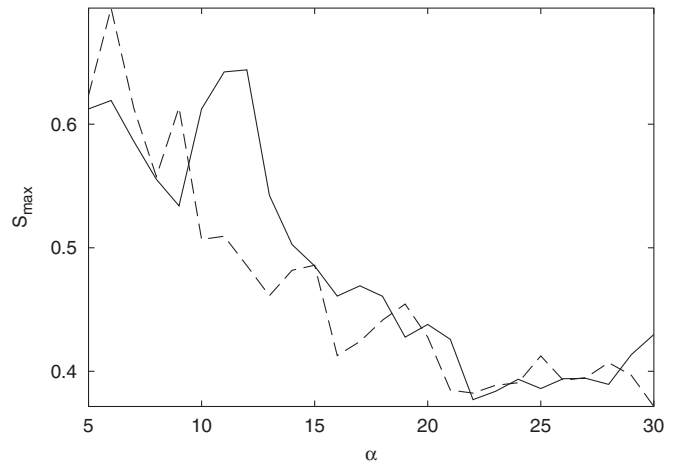

FIG. 10. Our combination statistic plotted for circle comparisons with a −36° twist (solid line) and a +36° twist (dashed line). Including 2° smoothing, cutting out the galaxy contamination, and neglecting the m weighting factor reproduces the peak found in Ref. [10].

show the expected random peak levels for a trivial topology. In the case of the position space search used in Ref. [10], the WMAP sky cannot be randomly scrambled while preserving both the power spectrum and the galaxy cut. To get a reliable measure for the expected S values, for the combination search we instead use various twists other than ±36° when identifying circle pairs. The results from the search on these random twists are plotted along with the possible detection in Fig. 12.

It can be seen in Ref. [10], Figs. 13–18, that two sets of circles include points that run through the plane of our galaxy. The CMB data for this portion of the sky are contaminated by the radiation from our galaxy. While it is possible to filter out much of this contamination, a more conservative approach is to perform a galaxy cut. The galaxy cut is easily done in position space, but is not quite as easily included in a Fourier space search. Including the galaxy cut raises the peak for matching circles, since it cuts out uncorrelated points, but it also raises the false positive
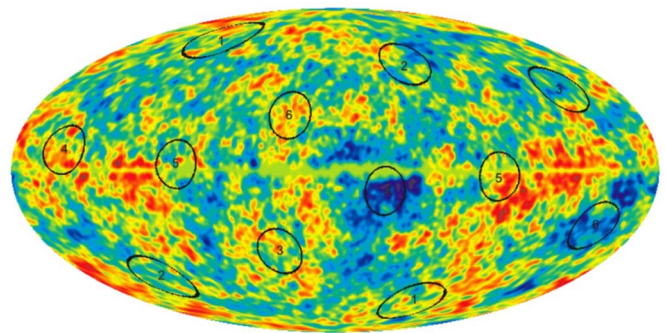

FIG. 11 (color online). The locations of the six sets of circle pairs in the CMB sky identified in Ref. [10] and found by our combination statistic search.
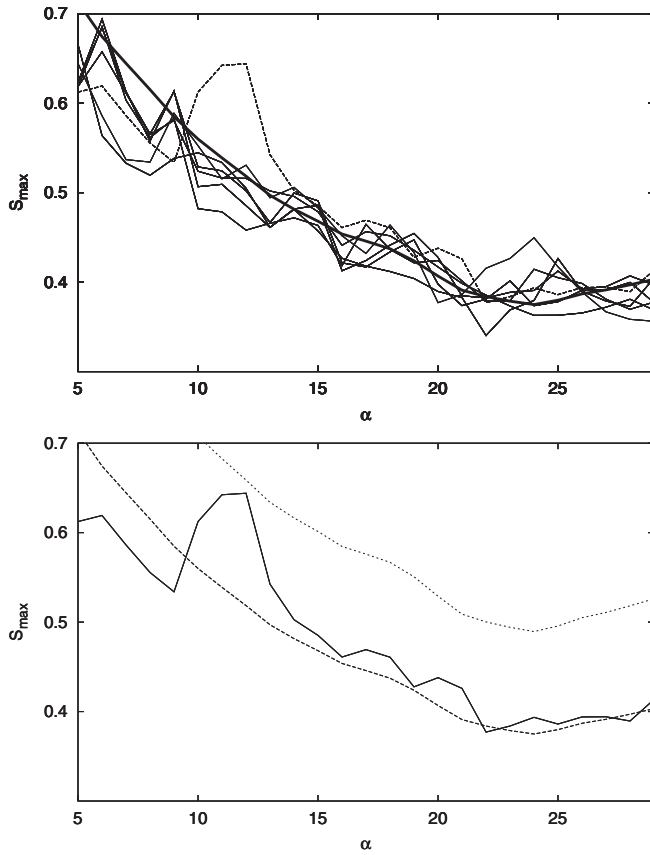
FIG. 13.   The search in Fourier space with no m weighting (solid line) matches the search in position space with no galaxy cut and no m weighting (dashed line). Including the galaxy cut raises the peak (dotted line).

FIG. 12.   Top panel: The combination search for the dodecahedral topology performed with several different twist values. The expected level for S is plotted in bold. Bottom panel: The possible detection with the $-36°$ twist (solid line) is plotted with the appropriate expected level for S (lower dashed line) and the threshold for fewer than 1 in 100 random skies producing an S value above the line (upper dotted line).

in the S values of the dodecahedron combination search (Fig. 15, top panel). The distribution of S values is shifted toward zero mean by explicitly removing the average

level, since there are less data points to compare. It can be seen in Fig. 13 that the unweighted Fourier and position space searches produce similar peaks and that the galaxy cut in position space raises the peak as expected. While Parceval's theorem states that the position space search with no galaxy cut and the Fourier search with no m weighting should produce exactly the same result, numerical error leads to a slight difference in the results.

We expect that the S values calculated for all of the sets of circle pairs compared at a given α should form a Gaussian distribution. Even if there is an outlier value of $S_{\max}$ indicating the topology of the Universe, this one value should not significantly skew the distribution of all of the nondetections at the same angular radius. Two distributions of calculated S values for the $-36°$ twist search can be seen in Fig. 14. The nonzero mean of the distributions indicates an average value for the distribution above zero, and thus a DC contribution to the temperature values around each circle pair. This contribution can be subtracted to show the true comparison of temperature variation. Simply removing the DC contribution reduces the peak
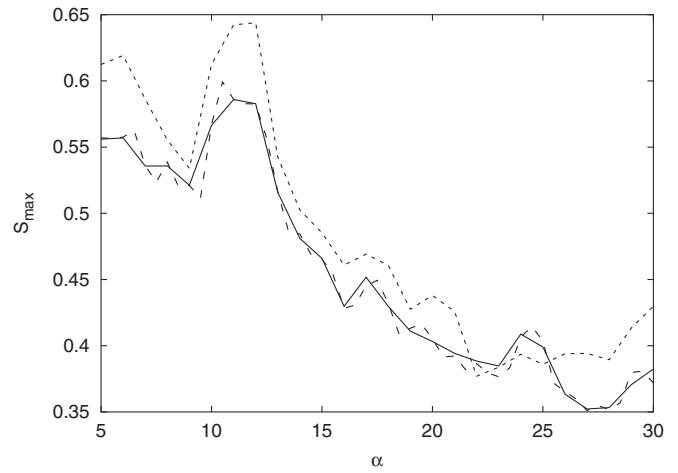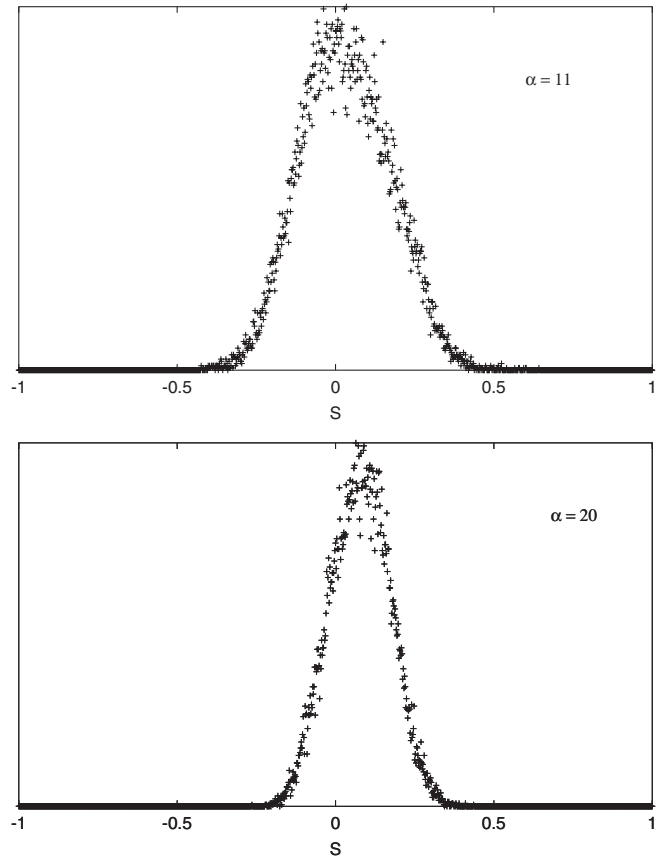

FIG. 14.   The distribution of S values calculated at $\alpha = 11°$ (top panel) and $\alpha = 20°$ (bottom panel). Both have a nonzero mean due to a DC component of the temperature values around circle pairs.
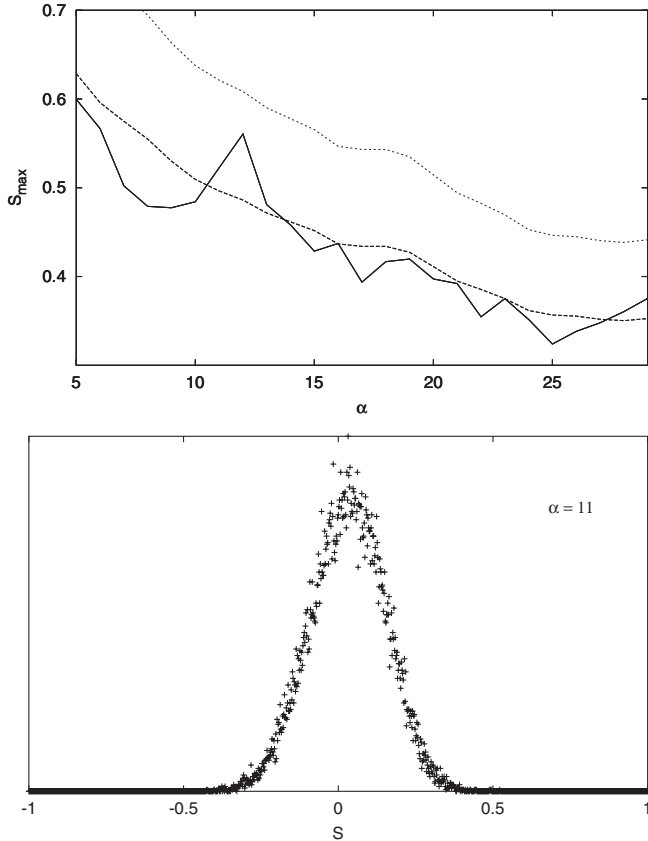
FIG. 15. Removing the DC contribution (average value) when comparing circle pairs shifts the distribution of S values toward zero mean and indicates no peak at 11°.

temperature values, but the mean is not exactly zero. This is due in part to numerical error. The S statistic calculated using m weighting [Eq. (1)] automatically takes care of this average offset.

Figure 16 shows our final analysis, using the Poincaré combination search with the appropriate m weighting and plotted with the threshold line for detection. The appropriate analysis yields no significant peak at 11°. It is thus clear that when the proper matching statistic is used, and the expected level of false positives is calculated, the detection claimed in Ref. [10] does not hold up. It seems that the observed peak is most likely due to overemphasis of large angular scales by neglecting m weighting, resulting in searching for matching circles in a sky with large patches of constant temperature. Such a sky would indeed produce randomly matched small circles.

It should also be noted that a search for matching circles has been performed in Refs. [12,13]. The authors found that their circle matching statistic was degraded considerably by the ISW and velocity contributions to the temperature fluctuations in the CMB. In Ref. [12], it was found that the matching circles of a Poincaré dodecahedron universe cannot be found by current CMB searches because of this degradation. In Ref. [13], the search includes filtering and
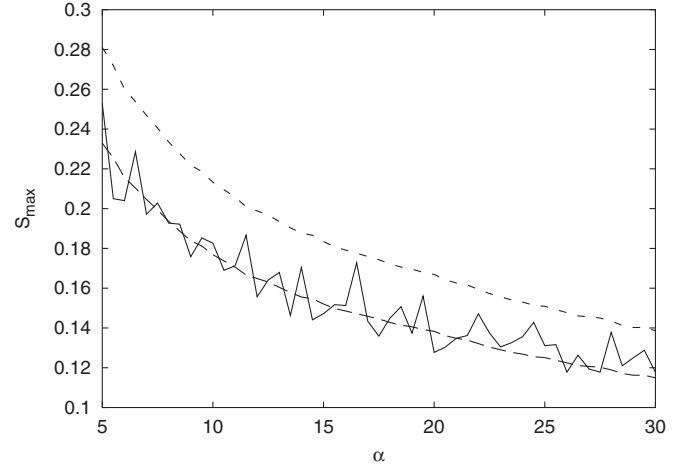

FIG. 16. Our results for the Poincaré dodecahedron combination search with optimal filtering. The dashed line indicates the detection threshold.

combines sets of expected circles, but disregards the m weighting in the S statistic, and provides no false positive thresholds. As we have shown, the use of power spectrum filtering, the combination search, and the proper S statistic that includes m weighting can indeed probe beyond contaminating factors to determine if there are matching circles in the CMB. Applying these techniques to the putative Poincaré dodecahedral model produced no statistically significant sets of matching circles. Since the smallest matching circles in our simulated sky have radii of 16.4°, we have not been able to directly establish a lower limit on the angular radii that can be probed by our combination search. However, we can extrapolate from the results shown in Figs. 6 and 8 to put the lower limit at $\alpha \simeq 5°$. This effectively rules out the Poincaré dodecahedral model as an interesting shape for the Universe.

XII. CONCLUSION

The claim that the topology of our Universe has been found to be that of the Poincaré dodecahedron does not stand up under scrutiny. The signature found in Ref. [10] disappears when one uses the proper S statistic and considers the false positive threshold. While the shape of our Universe remains a mystery, the matching circles test can be used to place a lower bound on the size of the Universe. The previous limit of 24 Gpc [8] can be extended by about 10% using filtering of the WMAP power spectrum. A full search with optimal filtering is now underway.

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