Multiple-event probability in general-relativistic quantum mechanics

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We discuss the definition of quantum probability in the context of "timeless" general-relativistic quantum mechanics. In particular, we study the probability of *sequences* of events, or multievent probability. In conventional quantum mechanics this can be obtained by means of the "wave function collapse" algorithm. We first point out certain difficulties of some natural definitions of multievent probability, including the conditional probability widely considered in the literature. We then observe that multievent probability can be reduced to single-event probability, by taking into account the quantum nature of the measuring apparatus. In fact, by exploiting the von-Neumann freedom of moving the quantum/classical boundary, one can always trade a sequence of *noncommuting* quantum measurements at different times, with an ensemble of simultaneous *commuting* measurements on the joint system+-apparatus system. This observation permits a formulation of quantum theory based only on single-event probability, where the results of the wave function collapse algorithm can nevertheless be recovered. The discussion also bears on the nature of the quantum collapse.

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I. INTRODUCTION

The general-relativistic revolution of our understanding of space and time has proven extremely effective empirically. Conventional textbook quantum mechanics (QM) and conventional quantum field theory (QFT), however, are formulated in a language which is incompatible with the general-relativistic notions of space and, especially, time. Is there a formulation of QM compatible with these notions? Such a formulation should be required, in particular, in order to provide a clear interpretative framework to any attempt to formulate a quantum theory of gravity in a form consistent with the general-relativistic understanding of space and time [1].

The main difficulty in extending QM to the general-relativistic context comes from the notion of time. (Historical, if a bit out-of-date reviews are Refs. [2,3]. See also the recent Refs. [4,5].) In conventional QM (including QFT), the independent time variable t is interpreted as an observable parameter independent from the physical system considered. States and probability amplitudes evolve deterministically in t.[1] This notion of time is not general relativistic and conflicts with general covariance, and so, if we take general relativity seriously, it is likely to be unsuitable to describe the world at the Planck scale.

In a sum-over-histories context, this problem is addressed by Hartle's generalized QM [6]. Here we work in the Hamiltonian context. We summarize below, in Sec. II, a Hamiltonian formulation of QM that does not make use of this nonrelativistic notion of time, following Ref. [7], and especially Chapter 5 of Ref. [8], to which we refer for full details and a complete discussion. Following Ref. [8], we call this formulation "general relativistic." In this formulation, the probability for observing a certain event s' if an event s was observed, is given by the modulus square of a suitably defined transition amplitude $\mathcal{P}_{s\Rightarrow s'} = |\mathcal{A}_{s\Rightarrow s'}|^2 = |\langle s'|s\rangle_{\mathcal{H}}|^2$, as explained below. This probability postulate coincides with standard QM transition probabilities in the non-general-relativistic case, but is well defined in a wider context, sufficiently general to form the basis for a timeless interpretation of a general-relativistic quantum theory.

However, a key problem was left open in Ref. [8]. QM does not provide only probabilities for single observations. It also provides probabilities for sequences of observations. For instance, it provides the probability $\mathcal{P}_{\psi\Rightarrow\psi'\psi''}$ for a system in a state $|\psi\rangle$ to be observed in a state $|\psi'\rangle$ and then in a state $|\psi''\rangle$. One way to compute this is to assume that at the time of the first measurement the state "collapses" to the state $|\psi'\rangle$. It is not clear how these probabilities for ensembles of events can be computed in general-relativistic QM. The difficulty comes from the following fact. As is well known, in QM, given a unordered ensemble of observations, its probability depends on the *time order* in which they are performed. Now, in a quantum general-relativistic context, as we recall below, the notion of time evolution, and, in particular, the notion of time order, is subtle due to the absence of an external notion of time. How do we compute, then, the probability of sequences of measurements? The problem was discussed for instance by Hartle in Ref. [9]. A strictly related problem is that the probability defined in Ref. [8] concerns only the *nondegenerate* eigenvalues of the (partial) observables. The probability of observing a *degenerate* eigenvalue was not

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[1]In QFT this is realized via the representation of the Poincaré group in the state space.

defined. In this paper we study a possible solution to this problem.

In Sec. III, we state the problem and we point out a certain number of difficulties that emerge in trying to assign a probability to sets of events in a general-relativistic context. In particular, we discuss the difficulties of two apparently ''natural'' solutions. The first is a direct generalization of the single-event probability postulate: the probability of an ensemble of events is determined by the projection on the physical Hilbert space of the subspace of the kinematical Hilbert space associated to this ensemble of events. We show that this postulate is not viable because it does not reduce to the standard QM probabilities in the nonrelativistic case. The second is the use of conditional probability, widely discussed in the literature. We point out certain difficulties with the operational definition of this probability.

In Sec. IV, we indicate a possible general way for solving the problem. This is based on the observation that a multiple-event probability, such as $\mathcal{P}_{\psi\Rightarrow\psi'\psi''}$ can always be reinterpreted as a single-event probability, once the dynamics and the quantum nature of the apparatus making the measurements are taken into account. If we do so, the time order gets naturally coded into the dynamics of the system. This strategy provides a general way for dealing with multiple-event probabilities in general-relativistic quantum mechanics.

Finally, in Sec. V we comment on the meaning of the notion of probability in the timeless case. In particular, we clarify the apparent difficulty presented by the fact that probabilities assigned to the possible values of a variable may not sum up to one. In Sec. VI we summarize our results and discuss the issues that remain open.

We illustrate the working of this technique in a companion paper [10], where we introduce a discrete model that allows us to exemplifies all the general structures introduced above. The example in Ref. [10] illustrates a notable convergence between the strategy introduced here and the generalized QM approach of Hartle, Halliwell, and others (see, for example, Refs. [6,11] and references therein).

II. GENERAL-RELATIVISTIC QUANTUM MECHANICS

We give a rapid overview of general-relativistic QM, referring to Chapter 5 of Ref. [8] for a complete discussion and full details.

A. General-relativistic classical mechanics

A classical mechanical system is described by a number of observables quantities, that we call *partial observables*. We include under this name the dependent as well as the independent variables in the equations of motion of nonrelativistic mechanics. Examples of partial observables are the time t, the position of a particle $\vec{X}$, the momentum, the energy, the spacetime coordinates x^μ in special-relativistic field theory, the four coordinates x^μ of a relativistic particle, and so on. We assume that each partial observable can be measured by a suitable physical apparatus.

In a system with a finite number n of degrees of freedom, we choose $n+1$ partial observables (typically the n Lagrangian variables plus the time variable), and form the $n+1$ dimensional *extended configuration space*, or event space, $\mathcal{C}$. The extended configuration space of a relativistic particle is the Minkowski space. The extended configuration space of a homogenous and isotropic cosmological model where a is the volume of the universe and ϕ is the matter density, is coordinatized by a and ϕ. Points in $\mathcal{C}$ are called *events* and denoted $s, s', s'', \ldots$: for instance, a point in Minkowski space $s = x^\mu = (\vec{x}, t)$, or a given value of radius of the universe and matter density $s = (a, \phi)$, define an event. Measuring a complete set of partial observables—that is, determining a point in $\mathcal{C}$—is to detect the happening of a certain event. For instance: a particle is detected in a point of Minkowski space, a certain value of radius of the universe and average energy density are measured, and so on. Each such detection describes an interaction of the system with another system, playing the role of observer.

Dynamics is then uniquely determined by a single function H, the relativistic Hamiltonian,[2] on the cotangent space $T_*\mathcal{C}$.[3] This is a symplectic space, with a natural symplectic two-form ω. The submanifold Σ defined by $H = 0$ is called the constraint surface. The integral lines of the restrictions of ω to Σ are the physical motions of the system. The space of these motions is the phase space Γ. Thus, in this context the phase space Γ is interpreted, à la Lagrange, as the space of the solutions of the equations of motion,[4] or the space of the possible ''motions'' of the system, instead than a space of initial data. Notice that a physical motion determines a continuous sequence of events in $\mathcal{C}$.

The interest of this formulation of dynamics is that the time variable t is just one of the coordinates of $\mathcal{C}$, on equal footing with the other partial observables, as required by general covariance. In a general covariant system, in fact, dynamics is not the description of how physical variables evolve in a preferred independent variable t, but rather the description of the physical relations between partial observables (see the discussion in Ref. [8], Secs. 3.1 and 3.2.4).

A special case is provided by the nonrelativistic systems. In these systems, one of the coordinates of $\mathcal{C}$ is singled out to play a special role: the time variable t, thus $\mathcal{C} = \mathcal{C}_0 \times \mathbb{R}$,

[2] It is also called ''super-Hamiltonian,'' ''Hamiltonian constraint,'' or ''scalar constraint.''

[3] In this paper, we do not consider the case of other gauge invariances, besides the reparametrization invariance generated by H.

[4] More precisely, as the space of the equivalence classes of solutions, up to gauge.

where $\mathcal{C}_0$ is the conventional configuration space; and H has the form

$$H = p_t + H_0, \tag{1}$$

where p_t is the momentum conjugate to t and H_0 is independent from p_t. H_0 is the standard nonrelativistic Hamiltonian. It is easy to see that in this case Σ is isomorphic to $\mathbb{R} \times T_*\mathcal{C}_0$ and the motions are the integral lines of the Hamiltonian flow of H_0 on $T_*\mathcal{C}_0$: that is, we recover conventional Hamiltonian mechanics. Our main interest, of course, is for systems where H does *not* have the form (1). These are the systems for which conventional QM is insufficient, and general relativity is among (the ∞ number of degrees of freedom version of) these systems.

B. General-relativistic quantum mechanics: basic structure

General-relativistic quantum mechanics is defined by a kinematical Hilbert space $\mathcal{K}$ carrying an algebra of operators corresponding to the partial observables. The dynamics is given by a generalized projection operator $\mathbb{P}$ in $\mathcal{K}$.

The relation with a classical system is as follows. A linear representation of the Poisson algebra of coordinates and momenta on $T_*\mathcal{C}$ defines the kinematical Hilbert space $\mathcal{K}$. For instance, we may take a Schrödinger representation $\mathcal{K} = L_2[\mathcal{C}]$. The relativistic Hamiltonian H defines a self-adjoint quantum operator (that we denote with the same symbol) H on $\mathcal{K}$. The kernel of H, formed by the (possibly generalized) states in $\mathcal{K}$ satisfying the "Wheeler-DeWitt" equation

$$H\psi = 0 \tag{2}$$

is called the physical state space $\mathcal{H}$. $\mathbb{P}$ is the linear (self-adjoint) operator $\mathbb{P}$: $\mathcal{K} \to \mathcal{H}$, given by

$$\mathbb{P}\,\psi = \int dn e^{-inH}\psi \tag{3}$$

(we put $\hbar = 1$) and loosely called "the projector," since $\psi \perp \mathcal{H} \Leftrightarrow \mathbb{P}\psi = 0$.[5] For periodic systems the range of integration is the period, for the others it is the real line (recall that we are only working with reparametrization invariant systems). If zero is in the discrete (respectively continuous) spectrum of H, then $\mathcal{H}$ is a proper (respectively generalized) eigenspace of $\mathcal{K}$. On the linear space $\mathcal{H}$ we consider the Hilbert structure

$$\langle \mathbb{P}s'|\mathbb{P}s\rangle_{\mathcal{H}} := \langle s'|\mathbb{P}|s\rangle, \tag{4}$$

which is well defined (on a dense subspace of $\mathcal{K}$) even when $\mathcal{H}$ is a generalized eigenspace; see for instance Ref. [12] and the discussion in Section 5.5.2 of Ref. [8]. We also write $\langle s'|s\rangle_{\mathcal{H}} \equiv \langle \mathbb{P}s'|\mathbb{P}s\rangle_{\mathcal{H}}$. Remark that since in general $\mathbb{P}$ is not a true projector, $\langle s'|\mathbb{P}|s\rangle$ may very well be different from $\langle \mathbb{P}s'|\mathbb{P}s\rangle$. In particular, this last quantity is in general divergent in the case in which $\mathcal{H}$ is a generalized eigenspace.

States in $\mathcal{K}$ have a physical interpretation,[6] as follows. If $|s\rangle \in \mathcal{K}$ is an eigenstate of a complete set of commuting partial observable (self-adjoint) operators, with eigenvalues $(a, b, c\ldots)$, then $|s\rangle$ is interpreted as describing the event in $\mathcal{C}$ with coordinates $(a, b, c\ldots)$. That is, it describes an interaction of the system with another physical system, in which the values $(a, b, c\ldots)$ are realized.

In a nonrelativistic system, where Eq. (1) holds, the Wheeler-DeWitt Eq. (2) becomes the Schrödinger equation and $\mathbb{P}$ is strictly related to the unitary evolution operator $U(t) = e^{-iH_0 t}$. As a simple example, consider a 2-state spin system with time-independent Hamiltonian H_0. Here $\mathcal{K} = \mathbb{C}^2 \otimes L[\mathbb{R}]$ is formed by states of the form $\psi_S(t)$, $S = \uparrow, \downarrow$ and spanned by the (generalized) basis $|S, t\rangle$, where $\psi_S(t) = \langle S, t|\psi\rangle$, and

$$\begin{aligned}\mathbb{P}\,|S, t\rangle &= \int_{-\infty}^{\infty} d\tau e^{-i\tau(p_t + H_0)}|S, t\rangle \\ &= \int_{-\infty}^{\infty} d\tau e^{-iH_0\tau}|S, t+\tau\rangle.\end{aligned} \tag{5}$$

$\mathbb{P}$ sends $|S, t\rangle$ into a solutions of the Schrödinger equation. The physical space $\mathcal{H}$ is given by these solutions and clearly $\mathcal{H} \sim \mathbb{C}^2$ (any initial state generates one and only one solution of the Schrödinger equation). A basis in $\mathcal{H}$ can be obtained by choosing a reference time t_0 and defining $|S\rangle := \mathbb{P}|S, t_0\rangle$. It follows easily $\mathbb{P}|S, t\rangle = U^\dagger(t - t_0)|S\rangle$. Then notice that

$$\langle S', t'|\mathbb{P}|S, t\rangle = \langle S'|e^{-iH_0(t'-t)}|S\rangle, \tag{6}$$

that is, the matrix elements of $\mathbb{P}$ are essentially the matrix elements of the unitary evolution operator.

C. Single-event probability

The main interpretation postulate of general-relativistic QM is the following (see Ref. [8]). The probability $\mathcal{P}_{s\Rightarrow s'}$ of observing the event s' if the event s was observed (we shall write $\mathcal{P}_{s\Rightarrow s'}$ simply as $\mathcal{P}_{s'}$ when there is no need of indicating the initial state) is given by the modulus square of the amplitude

$$A_{s\Rightarrow s'} = \langle s'|\mathbb{P}|s\rangle, \tag{7}$$

where the states are normalized in $\mathcal{H}$, not in $\mathcal{K}$, that is

$$\langle s|\mathbb{P}|s\rangle = \langle s'|\mathbb{P}|s'\rangle = 1. \tag{8}$$

We add here some comments to this probability postulate, referring the reader to Ref. [8] for a full discussion. There are several equivalent ways of writing this probability. We can define the projection operator

[5] If $\mathcal{K}/\mathcal{H}$ has finite dimension N then $\mathbb{P} = N\tilde{\mathbb{P}}$, with $\tilde{\mathbb{P}}$ the true projector from $\mathcal{K}$ to $\mathcal{H}$.

[6] This is the distinguishing feature of the interpretation we are considering.

$$\Pi_{s'} = \mathbb{P}|s'\rangle\langle s'|\mathbb{P} = |\mathbb{P}s'\rangle\langle\mathbb{P}s'| \tag{9}$$

and write the probability as the expectation value

$$\mathcal{P}_{s\Rightarrow s'} = |A_{s\Rightarrow s'}|^2 = \langle s|\Pi_{s'}|s\rangle, \tag{10}$$

where again $|s\rangle$ and $|s'\rangle$ are normalized by (8). Alternatively, recalling the notation $\langle s'|s\rangle_{\mathcal{H}} \equiv \langle\mathbb{P}s'|\mathbb{P}s\rangle_{\mathcal{H}} := \langle s'|\mathbb{P}|s\rangle$, and explicitly including the normalization in the expression for the probability, we can write

$$\mathcal{P}_{s\Rightarrow s'} = \frac{|\langle s'|s\rangle_{\mathcal{H}}|^2}{\langle s'|s'\rangle_{\mathcal{H}}\langle s|s\rangle_{\mathcal{H}}} = \frac{|\langle\mathbb{P}s'|\mathbb{P}s\rangle_{\mathcal{H}}|^2}{\langle\mathbb{P}s'|\mathbb{P}s'\rangle_{\mathcal{H}}\langle\mathbb{P}s|\mathbb{P}s\rangle_{\mathcal{H}}}. \tag{11}$$

Notice that this probability is a standard quantum mechanical probability computed in the *physical* Hilbert space $\mathcal{H}$, in the following sense. The states $|s\rangle$ and $|s'\rangle$ in $\mathcal{K}$ "project" down to the normalized physical states $|\psi_s\rangle = \frac{|\mathbb{P}s\rangle}{\sqrt{\langle s|\mathbb{P}|s\rangle}}$ and $|\psi_{s'}\rangle = \frac{|\mathbb{P}s'\rangle}{\sqrt{\langle s'|\mathbb{P}|s'\rangle}}$ in $\mathcal{H}$. The probability (11) is then simply the standard probability amplitude of measuring the physical state $|\psi_{s'}\rangle$ if the physical state $|\psi_s\rangle$ was measured. In particular, it can be expressed in terms of the usual Gleason formula

$$\mathcal{P}_{s\Rightarrow s'} = \mathrm{tr}_{\mathcal{H}}(\Pi_{s'}\rho_s), \tag{12}$$

where $\Pi_{s'} = |\psi_{s'}\rangle\langle\psi_{s'}|$ is the projection operator in $\mathcal{H}$, $\rho_s = |\psi_s\rangle\langle\psi_s|$ is the density matrix of the pure state $|\psi\rangle$ and $\mathrm{tr}_{\mathcal{H}}$ is the trace in the space $\mathcal{H}$. In other words, $\Pi_{s'}$ is the projector onto $|\mathbb{P}s'\rangle$, in $\mathcal{H}$, and it is a *genuine Dirac operator*. Therefore general-relativistic QM simplifies, but does not contradict, the standard Dirac treatment of a constrained system, and it is fully equivalent to the standard probabilistic interpretation in the physical state space $\mathcal{H}$. This fact, by the way, assures us that the amplitude (7) yields a probability less or equal to one.

This formulation is very close in spirit to the conventional scattering formulation of QFT, where probabilities are defined by transition amplitudes between initial and final asymptotic states, and these are defined as timeless Heisenberg states, or full "histories" of the system: see for instance Weinberg's clear discussion in Chapter 3 of Ref. [13].

This definition of probability reduces to the conventional one in the nonrelativistic case. In the spin system considered above, for instance, the states $\mathbb{P}|S, t\rangle$ are normalized and the amplitude for measuring the spin S' at time t' if S at time t was measured is

$$A_{St\Rightarrow S't'} = \langle S', t'|\mathbb{P}|S, t\rangle = \langle S'|e^{-iH_0(t'-t)}|S\rangle, \tag{13}$$

in agreement with conventional QM. More in general, in the case in which $H = p_t + H_0$, the definitions above reduce to the standard QM postulates regarding states, observables and probability.

What distinguishes this formulation from the usual one is the fact that the physical interpretation of the states in $\mathcal{H}$ is not directly determined by operators in $\mathcal{H}$, but rather by the kinematical operators in the kinematical state space $\mathcal{K}$. The relation between the kinematical operators in $\mathcal{K}$ and the physical operators in $\mathcal{H}$ is determined by the dynamics. In the case of unitary evolution, states $|s\rangle = |S, t\rangle$ and $|s'\rangle = |S, t'\rangle$ that differ only by the eigenvalue of the time operator in $\mathcal{K}$, are related by the unitary time evolution in $\mathcal{H}$. The corresponding states in $\mathcal{H}$ are eigenstates (up to an isomorphism) of the Heisenberg operators $S(t)$ and $S(t')$ related by time evolution. In the general case, this interpretation of the formalism remains viable even if we are not able to explicitly compute the Heisenberg operators, namely: *this interpretation remains viable even without need of solving the dynamics* (as is the case, for instance, with the evolving constant of motion approach). Therefore this formulation of general-relativistic QM allows us to have the conventional advantages of the conventional interpretation in $\mathcal{H}$, but at the same time exploiting the physical transparency of the operators defined in $\mathcal{K}$, as is the case for the interpretation of the classical theory.

Notice, finally, that the physical interpretation cannot be directly given in $\mathcal{K}$ without reference to $\mathcal{H}$. One way of seeing this is the failure of the hypothesis of the Gleason theorem that could justify replacing (12) with an analogous form in $\mathcal{K}$. Indeed, a key hypothesis of Gleason theorem is that the probabilities add up for the linear sum of orthogonal subspaces (see for instance Sec. 9.2.4 of Ref. [14]). This means, in particular, that the probability for the system of being detected in either of two points s' and s'' of $\mathcal{C}$ should be the sum of the probabilities of being detected in s' and s''. But this is not the case, because the system can be in s' *and* in s'', in particular, if s' and s'' are on the same motion. Thus, the hypotheses of Gleason theorem fail in $\mathcal{K}$. The interpretation postulate considered here consistently combines the simplicity of the physical interpretation of the kinematical operators in $\mathcal{K}$, with the fact that the probabilities are well defined in the physical state space $\mathcal{H}$.

However, there is one additional postulate required to define quantum theory: the collapse postulate, stating that *after* a measurement, the state changes and becomes an eigenstate of the operator associated to the measurement. The translation of this postulate to the relativistic formalism is our concern here.

III. MULTIPLE-EVENT PROBABILITY

Consider a partial observable A in $\mathcal{K}$ and let a be one of its eigenvalues. If a is nondegenerate, and $|s'\rangle$ is the corresponding eigenstate, then (7) provides the probability amplitude of measuring a. What happens if a is degenerate?

Let us say for simplicity that a is doubly degenerate, and that $|s'\rangle$ and $|s''\rangle$ are two orthogonal eigenstates having eigenvalue a, that is, they span the a eigenspace $\mathcal{K}_a$. Then, to measure the eigenvalue a, or, equivalently, to measure its associated projection operator $\pi_a = |s'\rangle\langle s'| + |s''\rangle\langle s''|$, means that we have a measuring apparatus that gives us a Yes answer if either the event s' or the event s'' happen (Yes answer corresponds to the eigenvalue 1 of π_a). In order to compute the probability of having a Yes answer, we need therefore the probability $\mathcal{P}_{s' \text{ OR } s''}$ that the event s' OR the event s'' happens.

Alternatively, suppose that we have a measuring apparatus that gives us a Yes answer if both the event s' and the event s'' happen. In order to compute the probability of having a Yes answer, we need therefore the probability $\mathcal{P}_{s' \text{ AND } s''}$ that the event s' AND the event s'' happen. The solution of one case gives immediately the solution of the other since, clearly

$$\mathcal{P}_{s' \text{ OR } s''} = \mathcal{P}_{s'} + \mathcal{P}_{s''} - \mathcal{P}_{s' \text{ AND } s''}. \tag{14}$$

There are two possibilities: either $\mathcal{P}_{s' \text{ AND } s''}$ is always zero, or not. Consider the two cases separately.

(i) *Mutually exclusive events*—If $\mathcal{P}_{s \Rightarrow (s' \text{ AND } s'')} = 0$ for any s, then s' and s'' are alternative events that cannot both happen. That is, if one happens, the probability that the other happens is zero. By Eq. (7) and *the given interpretation*, this is equivalent to

$$\langle s'|\mathbb{P}|s''\rangle = 0. \tag{15}$$

In this case, Eq. (14) gives

$$\mathcal{P}_{s' \text{ OR } s''} = \mathcal{P}_{s'} + \mathcal{P}_{s''}. \tag{16}$$

That is, the probability of s' OR s'' is simply the sum of the probabilities of s' and s''. Observe that this can be written generalizing (10) to

$$\mathcal{P}_{s \Rightarrow a} = \langle s|\Pi_a|s\rangle, \tag{17}$$

where Π_a is the orthogonal projector on the subspace $\mathcal{H}_a = \mathbb{P}\mathcal{K}_a$ in $\mathcal{H}$, which, if Eq. (15) holds, is simply given by [remember we are requiring Eq. (8)]

$$\Pi_a = |\mathbb{P}s'\rangle\langle\mathbb{P}s'| + |\mathbb{P}s''\rangle\langle\mathbb{P}s''|. \tag{18}$$

A typical example is the following. In the two-state spin system considered in the previous section, let $|s'\rangle = |\uparrow, t\rangle$ and $|s''\rangle = |\downarrow, t\rangle$. In this case, $\langle s'|\mathbb{P}|s''\rangle = \langle\uparrow|U(0)|\downarrow\rangle = 0$. The two events are mutually exclusive. Therefore $\mathcal{P}_{s' \text{ AND } s''} = 0$. The projector on the a eigenspace $\mathcal{K}_a$ is

$$\pi_a = |\uparrow, t\rangle\langle\uparrow, t| + |\downarrow, t\rangle\langle\downarrow, t|. \tag{19}$$

The projection $\mathcal{H}_a$ of $\mathcal{K}_a$ in $\mathcal{H}$ is spanned by the two orthogonal states $\mathbb{P}|\uparrow, t\rangle = U^\dagger(t - t_0)|\uparrow\rangle$ and $\mathbb{P}|\downarrow, t\rangle = U^\dagger(t - t_0)|\downarrow\rangle$, therefore

$$\Pi_a = U^\dagger(t - t_0)(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|)U(t - t_0). \tag{20}$$

In this two-state system, $\Pi_a = \mathbb{1}$ and the corresponding probability is $\mathcal{P}_a = 1$. Not so, of course, in general.

(ii) *Non exclusive events*—The interesting case is when

$$\langle s'|\mathbb{P}|s''\rangle \neq 0. \tag{21}$$

A typical example is the following. Let $|s'\rangle = |\uparrow, t'\rangle$ and $|s''\rangle = |\leftarrow, t''\rangle = \frac{|\uparrow, t''\rangle + |\downarrow, t''\rangle}{\sqrt{2}}$. In this case, $\langle s''|\mathbb{P}|s'\rangle = \langle\leftarrow|U(t'' - t')|\uparrow\rangle \neq 0$, in general. The question we are asking is: what is the probability of detecting the spin $\uparrow$ at time t' AND the spin $\leftarrow$ at t''? The problem is of course well posed: if a particle is in a certain initial state at t, what is the probability of finding it with a certain spin at time t' AND with another spin at a later time t''? This can be measured by measuring the fraction of a beam that passes through a sequences of two Stern-Gerlach apparatuses.

Now, in ordinary quantum mechanics, these probabilities depends on the *time ordering* between the events. For instance, let the initial state $|s\rangle$ be the state $|\rightarrow\rangle = \frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}}$ at time t_0, and let $U(t) = 11$ for all t. Then

$$\mathcal{P}_{s \Rightarrow (s' \text{ AND } s'')} = \begin{cases} \frac{1}{4} & \text{if } t' < t'', \\ 0 & \text{if } t'' < t'. \end{cases} \tag{22}$$

Because the sequence

$$|\rightarrow\rangle \Rightarrow |\uparrow\rangle \Rightarrow |\leftarrow\rangle \tag{23}$$

has probability 1/4; while the sequence

$$|\rightarrow\rangle \Rightarrow |\leftarrow\rangle \Rightarrow |\uparrow\rangle \tag{24}$$

cannot happen. The standard way of obtaining these probabilities in conventional quantum mechanics is via the projection postulate. For instance, say $t' < t''$, that is, case (23). We have: (i) at time t' the spin $\uparrow$ is measured with probability $|\langle\uparrow|\leftarrow\rangle|^2 = 1/2$; (ii) the state is hence projected to $|\uparrow\rangle$; (iii) at time t'' the spin $\leftarrow$ is measured with probability $|\langle\leftarrow|\uparrow\rangle|^2 = 1/2$, giving total probability $1/2 \times 1/2 = 1/4$.

Standard QM gives also, easily

$$\mathcal{P}_{s \Rightarrow (s' \text{ OR } s'')} = \begin{cases} \frac{3}{4} & \text{if } t' < t'', \\ \frac{1}{2} & \text{if } t'' < t'. \end{cases} \tag{25}$$

Comparing with Eq. (14), notice that the probabilities $\mathcal{P}_{s \Rightarrow s'}$ and $\mathcal{P}_{s \Rightarrow s''}$ relevant here (with two detectors) are different from the probabilities $\mathcal{P}_{s \Rightarrow s'}$ and $\mathcal{P}_{s \Rightarrow s''}$ relevant when only one detector is present. For instance, in the first case, we have $\mathcal{P}_{s \Rightarrow s''} = |\langle\rightarrow|\uparrow\rangle\langle\uparrow|\leftarrow\rangle|^2 + |\langle\rightarrow|\downarrow\rangle\langle\downarrow|\leftarrow\rangle|^2 = \frac{1}{2}$, because of the presence of a detector in s'; while in the absence of this, we would clearly have $\mathcal{P}_{s \Rightarrow s''} = |\langle\rightarrow|\leftarrow\rangle|^2 = 0$. This well known fact will play an important role below.

How do we recover these probabilities in relativistic QM, where we do not have a notion of time ordering in t?

A. Two false tracks

1. Taking Eq. (17) as the general probability postulate

Let us suppose we ignore for the moment the time-ordering problem, and we try to directly associate a probability to the a eigenspace $\mathcal{K}_a$, as we did in the case (i) (mutually exclusive events). Notice that the definition (17) of probability remains meaningful also in the case (ii) (non mutually exclusive events). It is therefore very tempting to suppose that the probability is still given by (17) also in this case.

In the example above, for instance, the eigenspace $\mathcal{K}_a$ is spanned by the two events considered (which are orthogonal in $\mathcal{K}$ because they are at different time), and the projection operator on $\mathcal{K}_a$ is

$$\pi_a = |\uparrow, t'\rangle\langle\uparrow, t'| + |\leftarrow, t''\rangle\langle\leftarrow, t''|. \qquad (26)$$

The projection $\mathcal{H}_a$ of $\mathcal{K}_a$ to $\mathcal{H}$ is spanned by the two states $\mathbb{P}|\uparrow, t'\rangle$ and $\mathbb{P}|\leftarrow, t''\rangle$. This time these two states are not orthogonal in $\mathcal{H}$. We can nevertheless still consider the possibility that the probability $\mathcal{P}_a$ is given by Eq. (17), where Π_a is the projection operator on the space $\mathcal{H}_a$ they span.

The probability postulate (17) is not of completely straightforward utilization in the case in which the orthogonal eigenstates of π_a are projected to nonorthogonal states in $\mathcal{H}$, namely, in the case (21), because in this case Eq. (18) is not true. This is a technical difficulty that can be addressed by standard linear algebra methods, for instance via an orthogonalization procedure. A more powerful technique is to observe that the projector on the linear space spanned by a set of possibly linearly dependent states $S = \{|u^1\rangle \dots |u^N\rangle\}$ can be written in the form

$$\Pi_a = \sum_{ij} |u^i\rangle G_{ij} \langle u^j|, \qquad (27)$$

where the sum is over any maximal subset of linear independent sates in S and the matrix G_{ij} is the inverse of their Gramm matrix $G^{ij} = \langle u^i|u^j\rangle$.

Unfortunately, however, there is a serious problem: this probability postulate is wrong because it does not reduce to the conventional and well tested probabilities of nonrelativistic QM. This can be seen from the following example. Consider the spin case mentioned above. The two kinematical states $|s'\rangle$ and $|s''\rangle$ are orthogonal in $\mathcal{K}$ (because they are at different times). If we project them down to $\mathcal{H}$, the two resulting states $|\mathbb{P}s'\rangle$ and $|\mathbb{P}s''\rangle$ are not anymore orthogonal, but they are still linearly independent. Therefore they still span a two-dimensional space. Since the space they span is $\mathcal{H}_a$, it follows that $\mathcal{H}_a$ is two-dimensional. But so is $\mathcal{H}$ in this example. Hence $\mathcal{H}_a = \mathcal{H}$. It follows immediately that Π_a is the identity operator in $\mathcal{H}$, and therefore that (17) states that the probability of measuring either s' or s'' is always equal to unity. This is in contradiction with the correct result (25) given by nonrelativistic QM.

Therefore the probability formula (17) is not correct in the case (ii) of compatible events. The difficulty appears to be in the fact that the formalism ignores that s' happens *before* s''.

2. Conditional probabilities

The idea that in the timeless case the interpretation of QM can be entirely based on conditional probabilities has been suggested in Ref. [15] and is very attractive. The idea has been widely discussed and also criticized, see for instance Ref. [2], but has recently received new attention. For instance, in Ref. [5], C. Dolby has provided an interesting and convincing reply to the criticisms in Ref. [2]. We refer therefore here to Dolby's version of the conditional probability interpretation. For full references, see the two papers quoted.

Dolby gives a postulate for the probability $\mathcal{P}(a \text{ when } b)$ that an event a happens together with (as the same time as) an event b.[7] The events a and b are described by commuting projector operators π_a and π_b in $\mathcal{K}$. Dolby's probability postulate is, in our notation (and not writing explicitly the dependence of the probability on the state),

$$\mathcal{P}(a \text{ when } b) = \frac{\langle s|\mathbb{P}\pi_a\pi_b\mathbb{P}|s\rangle}{\langle s|\mathbb{P}\pi_b\mathbb{P}|s\rangle}. \qquad (28)$$

Suppose that we have a set $\{a_1, \dots, a_n\}$ of events such that

$$\sum_n \pi_{a_n} = \mathbb{1}. \qquad (29)$$

It follows immediately from Dolby's definition that

$$\sum_n \mathcal{P}(a_n \text{ when } b) = 1. \qquad (30)$$

In order to illustrate the difficulty with this definition of probability, consider the two-state system introduced in Sec. II B, but let us imagine, for simplicity, that time is discrete. That is, the states are $\psi_S(t_n)$, $S = \uparrow, \downarrow$, where $\psi_S(t) = \langle S, t_n|\psi\rangle$, with integer n (as in the example we will discuss in the companion paper [10]). In the conventional formalism one focus on probabilities of the form $\mathcal{P}(\uparrow \text{ when } t_n)$, where the event $(\uparrow, t_n)$ is considered as one element of the set of equal-time alternatives $\mathcal{S}_{t_n} = \{(\uparrow, t_n)(\downarrow, t_n)\}$. But the general formalism does not privilege the time variable and therefore allows us to consider also probabilities of the form $\mathcal{P}(t_n \text{ when } \uparrow)$, where the event $(\uparrow, t_n)$ is considered as one element of the set of alternatives $\mathcal{S}_\uparrow = \{\dots, (\uparrow, t_{n-1}), (\uparrow, t_n), (\uparrow, t_{n+1}), \dots\}$. Let us therefore study the interpretation of these. If we take $\pi_b = \pi_\uparrow$, the projector on the $\uparrow$ eigenspace of the spin operator, and $\pi_{a_n} = \pi_{t_n}$, the projector on the t_n eigenspace of the t operator in Eq. (31), we have the probability $\mathcal{P}(t_n \text{ when } \uparrow)$ to find the particle at time t_n "when" the spin is $\uparrow$. Let us calculate this probability for a state such that, in particular, (assume

[7]Carefully distinguishing $\mathcal{P}(a \text{ when } b)$ from the conditional probability $\mathcal{P}(a \text{ if } b)$.

the dynamics is such that this is a physical state)

$$\psi_\uparrow(t_n) = \begin{cases} 1, & \text{if } n = 1, 2. \\ 0, & \text{otherwise.} \end{cases} \quad (31)$$

There are two different times at which the spin is $\uparrow$: t_1 and t_2. By symmetry and Eq. (30), we have immediately

$$\mathcal{P}(t_1 \text{ when } \uparrow) = \mathcal{P}(t_2 \text{ when } \uparrow) = \tfrac{1}{2}. \quad (32)$$

At first sight, this looks reasonable: since the particle has spin up at two different times, if the spin is up, then there might be a 50/50 probability that it is one or the other of these 2 times. But can we give a more precise definition of this probability? That is, can we give a precise operational procedure for measuring this probability? Here is where we see a difficulty. Indeed, we see two possibilities for an operational interpretation of this probability, but neither appears to work in general. The following are the two possibilities.

(i) There is a single detector at $t = t_n$, that detects the spin $\uparrow$, and we interpret $\mathcal{P}(t_1 \text{ when } \uparrow)$ as (the limit of) the ratio of the number of detections over the number of trials. This is clearly not viable, because under the conditions given, this ratio is equal to one, not to $1/2$: a detector which is on at time t_1 will *always* detect the particle, not just half of the times. How is this probability equal to one accounted for in Dolby's scheme?

We have one detector at each t_n, each sensitive only to the spin $\uparrow$. We call $N_{t_n\uparrow}$ the number of times the detector at t_n clicks, and call $N_\uparrow$ the total number of detections, and we interpret $\mathcal{P}(t_1 \text{ when } \uparrow)$ as the limit of $N_{t_1\uparrow}/N_\uparrow$. This works, because at each trial both detectors at t_1 and t_2 click, so that we get the correct $1/2$.

However, (ii) does not work in general, because of the fact that, as noticed after (25), the very presence of detectors alters quantum mechanical probabilities. For instance, suppose we have a system with four states $|i\rangle$, $i = 1, 2, 3, 4$, with a time dependent dynamics given by $\psi(t_{n+1}) = \mathcal{U}(t_n)\psi(t_n)$ where

$$\mathcal{U}(t_0) = \mathcal{U}(t_1) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$
$$\mathcal{U}(t_2) = \mathcal{U}(t_3) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}, \quad (33)$$

and $\mathcal{U}(t_n) = \mathbb{1}$ for any other n. Say the initial state is $\psi(t_0) = (|3\rangle + |4\rangle)/\sqrt{2}$. Then easily

$$\psi(t_{n\le 0}) = (|3\rangle + |4\rangle)/\sqrt{2}, \qquad \psi(t_1) = (|3\rangle + |1\rangle)/\sqrt{2},$$
$$\psi(t_2) = (|3\rangle + |4\rangle)/\sqrt{2}, \qquad \psi(t_3) = |2\rangle, \quad (34)$$
$$\psi(t_{n\ge 4}) = (|3\rangle + |4\rangle)/\sqrt{2}.$$

Hence

$$\psi_1(t_n) = \delta_n^1 \frac{1}{\sqrt{2}}, \quad (35)$$

and Dolby's probability gives

$$\mathcal{P}(t_n \text{ when } i = 1) = \delta_n^1. \quad (36)$$

But, accordingly with quantum mechanics, if we have detectors in $i = 1$ for all t_n, then at t_1 the state is projected with probability $1/2$ on $|1\rangle$, and with probability $1/2$ on $|3\rangle$. In both cases, the detector at t_3 will click with probability $1/2$, as a straightforward calculation shows. Hence the probabilities are: $1/4$ that t_1 alone clicks, $1/4$ that t_3 alone clicks, $1/4$ that t_1 and t_3 click, $1/4$ that no detector click. The ratio $N_{t_1\uparrow}/N_\uparrow$ is therefore $1/2$, different from Dolby's probability. The point is of course that the very presence of the detectors changes the wave function at later times, and this is not taken into account by Dolby's expression.

Thus, neither the operational interpretation (i), nor the operational interpretation (ii) work. What is then the meaning of this probability? We do not exclude that third consistent operational definitions of Dolby's probability could be given, but we do not yet see it. (On this problem, see Ref. [16].) Dolby appears to be aware of this difficulty; in Ref. [5] he writes "$\mathcal{P}(a \text{ when } b)$ is best thought as representing the proportion of the physical path on which a and b are simultaneously true, divided by the proportion on which b is true." This is clearly intuitively okay, but in strident contradiction with the fact that there is no way of measuring "physical paths" in quantum mechanics. The difficulty is that Dolby's probability is defined as if the detectors did not affect the quantum states; but this contradicts the very physical contents of quantum theory.

For clarity, the difficulty we are raising is not that Dolby's scheme cannot take into account situations with several measurements. Dolby defines also probabilities, such as $\mathcal{P}(a_1 \text{ when } b_1 | a_2 \text{ when } b_2)$ that do so. The point we are raising is that we do not understand how to interpret, that is, how to concretely measure, Dolby's probabilities when the b variable in Eq. (28) is not conventional time.

Thus, in spite of the attractiveness of the conditional probability interpretation, we find that its foundation is not solid. Therefore we do not see, at present, how to take it as a general basis for relativistic quantum mechanics. The present work can be seen, to a large extent, as an attempt to ameliorate the conditional probability interpretation, but putting it on firmer grounds.

IV. MULTIEVENT PROBABILITY FROM THE COUPLING OF AN APPARATUS

A. The basic idea

It is perhaps the very central physical content of quantum theory that certain questions cannot be combined by

AND and OR, even if we can do so in classical physics. For instance, we can meaningfully ask if an electron is in a certain region of space, and we can meaningfully ask if an electron has a certain energy. But asking if an electron "is in a region AND has a certain energy," makes no sense in quantum theory, beyond a certain precision. Position and energy are represented by noncommuting operators in the quantum theory, and we can assign an electron a definite position and a definite energy, but not both. Bohr has emphasized that this is related to the fact that electron manifests its position in a certain context of interaction and it manifests its energy in a different context, but there is no interaction context in which the two can be manifested together. Position and energy are incompatible observables that cannot be measured together, and therefore it is perfectly okay that we cannot assign them joint probabilities.

At the light of this well known basic consideration, let us return to the case of the measurement of the s' and s'' events, in the case (ii) of Sec. III, considered above. For concreteness, let us consider again the example in which s' represent the spin being $\uparrow$ at a time t' and s'' represents the spin being $\leftarrow$ at a later time t''. Since in the physical state space the two states $|\mathbb{P}s'\rangle$ and $|\mathbb{P}s''\rangle$ are not orthogonal, they cannot be interpreted as (nondegenerate) eigenstates of commuting operators. We are led to the tentative idea that they represent incompatible observables, to which we cannot assign joint probabilities. After all, it is hard to say that two events at different times can happen *together*.

At first sight, this sounds as an elegant solution of the problem. But at a moment of reflection, something is clearly missing: Indeed, it *does* make sense to ask the experimental question of what is the probability for a spin system to have spin $\uparrow$ at time t' AND spin $\leftarrow$ at a later time t''. This is a statement that gets a precise meaning in an appropriate measurement context. In conventional QM, it can be dealt with by separating the two measurements in time, and using the collapse algorithm to compute joint probabilities. Can we obtain the same probabilities without invoking the collapse postulate? As is well known, the answer is yes, and it follows from an analysis of the experimental situation involved in the experiment in which we measure the two spins at different times. The key is to bring the apparatus that makes the measurement into the picture.

B. Multievent probability from the single-event one in nonrelativistic QM

Let us see how this works in conventional QM. Consider an initial state $|\psi\rangle$ at time t. What is the probability of detecting the state $|\psi'\rangle$ at time t' AND the state $|\psi''\rangle$ at time $t'' > t'$? This can be computed by projecting on $|\psi'\rangle$ at time t', which gives

$$\mathcal{P}_{\psi\Rightarrow\psi'\psi''} = |\langle\psi''|U(t''-t')\Pi_{\psi'}U(t'-t)|\psi\rangle|^2. \tag{37}$$

where $\Pi_{\psi'} = |\psi'\rangle\langle\psi'|$ and all states are normalized $\langle\psi|\psi\rangle = 1$. Now, describe the apparatus measuring ψ' as a two-state system which is initially in a state $|\mathrm{No}\rangle$, which interacts with the system at the time t', jumping to the state $|\mathrm{Yes}\rangle$ if and only if the state of the system is in the state ψ'. The interacting dynamics is then given by the unitary evolution operator $U_{\psi',t'}$ defined by

$$\langle\psi'', A''|U_{\psi',t'}(t''-t)|\psi, A\rangle = \langle\psi''|U(t''-t)|\psi\rangle\delta_{A''A} \tag{38}$$

for t and t'' both larger or smaller than the interaction time t', and

$$\begin{aligned}\langle\psi'', A''|U_{\psi',t'}(t''-t)|\psi, A\rangle = \langle\psi''|U(t''-t')(&\Pi_{\psi'}\mathcal{I}_{A''A} \\ &+ (1-\Pi_{\psi'})\delta_{A''A}) \\ &\times U(t'-t)|\psi\rangle\end{aligned} \tag{39}$$

if $t'' > t' > t$, where

$$\mathcal{I}_{AA'} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{40}$$

is a matrix that flips the apparatus state. Then the question we are interested in can be rephrased as follows: "Given an initial state $|\psi, \mathrm{No}\rangle$ at time t, what is the probability of measuring the state $|\psi'', \mathrm{Yes}\rangle$ at time t''?" Notice that "the system state is $|\psi''\rangle$" and "the apparatus state is $|\mathrm{Yes}\rangle$," are *compatible* statements in quantum theory: they refer to orthogonal observables, both at time t'', that are, and can be, measured together. In other words, the question can be captured by the single-event probability amplitude

$$\mathcal{P}_{\psi\Rightarrow\psi'\psi''} = |\langle\psi'', \mathrm{Yes}|U_{\psi',t'}(t''-t)|\psi, \mathrm{No}\rangle|^2. \tag{41}$$

The same observation can also be illustrated as follows. The probability of a time-ordered sequence of events, such as Eq. (37), can be written in the form

$$\mathbb{P}_{\psi\Rightarrow\psi'\psi''} = \langle\psi|\Pi_{\psi',\psi''}|\psi\rangle, \tag{42}$$

where the history operator [6]

$$\Pi_{\psi',\psi''} = U(t-t')\Pi_{\psi'}U(t'-t'')\Pi_{\psi''}U(t''-t')\Pi_{\psi'}U(t'-t) \tag{43}$$

is *not* a projector. But this same probability *can* be written as the expectation value of a projection operator if we enlarge the state space to include the apparatus. In fact, (41) gives

$$\mathbb{P}_{\psi\Rightarrow\psi'\psi''} = \langle\psi, \mathrm{No}|\tilde{\Pi}_{\psi',\psi''}|\psi, \mathrm{No}\rangle, \tag{44}$$

where the operator

$$\tilde{\Pi}_{\psi',\psi''} = U^{\dagger}_{\psi't'}\Pi_{\psi'',\mathrm{Yes}}U_{\psi't'} \tag{45}$$

is a projector. In fact, it projects on the state $U^{\dagger}_{\psi't'}|\psi'', \mathrm{Yes}\rangle$. That is, the probability of a sequence of events can *still* be obtained from the basic probability postulate (9).

In the context of the decoherent histories approach to QM, a closely related observation has been made by Gell-Mann and Hartle [17] and Halliwell [18]. These authors show that *when there is decoherence*, the probabilities of histories may be expressed in terms of records, represented by true projection operators acting at a single moment of time. There is much in common between what we do here and the history approach, but also some subtle differences. As we will show more in detail in a forthcoming paper, the idea of describing sequences of measurements in terms of single measurements of records leads to a formalism which is essentially the same as the one of coherent histories. In this sense, there is a convergence between the approach to covariant QM that we are developing here and the approach being developed, in particular, by Halliwell (see Ref. [11] and references therein), with the same motivations as our. Seems to us, however, that there remains an important difference. The history approach was largely conceived in order to free QM from the need of making reference to the actual interactions with physical measuring apparatuses, and in order to recognize (probabilistic) systems's properties that are measurement independent. Here on the other hand we are squarely within the Copenhagen view that QM is only about properties that actualize in interactions. Thus, we consider here apparatuses whose presence is actual, and probabilities that are contextual to this presence. This is why our probabilities are well defined even in the formal absence of decoherence: we have no need to sum up probabilities that refer to different contexts, such as the presence or not of some apparatus. In the spirit of decoherent histories, on the other hand, one asks if there are probabilities that are well defined wether or not apparatuses are present; this leads to the notions of decoherence (In Ref. [18], Halliwell argues that "the physical significance of decoherence is that it ensures the storage of information about the decohering properties somewhere in the universe"). When there is decoherence, the probabilities for sequences of events that we consider here and the one considered in the decoherent histories context are related. This is the sense of the relation between our observation and the one of Gell-Mann, Hartle, and Halliwell. In the absence of decoherence, the probabilities so determined have no meaning in the decoherent histories context, but it could retain their meaning for us. However, we do not think that there is contradiction, since for us probabilities are *only* defined with respect to actual physical interactions with an external system, and the probabilities so determined make sense if the apparatuses are physically there; in which case they themselves should determine the decoherence. In other words, the case in which probabilities do not decohere is, to us, a case in which the apparatuses are not physically present and therefore, in the language used here, there is no probability at all to be computed. A more detailed comparison between these two approaches would be quite interesting to develop.

C. Multievent probability from the single-event one in relativistic QM

Let us translate the above observation in the language of general-relativistic quantum mechanics. An apparatus is now a two-state system which is initially in a state $|\mathrm{No}\rangle$, and interacts with the system at the event s', jumping to the state $|\mathrm{Yes}\rangle$ if and only if the event s' happens. The question "Given an event s, what is the probability of detecting an event s' AND an event s''?" can be rephrased as follows: "Given an event $|s, \mathrm{No}\rangle$, what is the probability of detecting the event $|s'', Yes\rangle$?" In other words, the question can be captured by a *single-event* probability amplitude

$$A_{s\Rightarrow(s' \text{ AND } s'')} = A_{s,\mathrm{No}\Rightarrow s'',\mathrm{Yes}} = \langle s'', \mathrm{Yes}|\mathbb{P}_{s'}|s, \mathrm{No}\rangle, \quad (46)$$

provided that the dynamical operator $\mathbb{P}_{s'}$ takes appropriately into account the coupling between system and apparatus at the event s'.

It is clear that this strategy works in general, for arbitrary sequences of measurements. *Sequences of incompatible measurements can always be reinterpreted as compatible measurements of apparatuses that have interacted with the system.*[8] In fact, the only operational meaning that we can ascribe to the probability that something happen at time t' AND something happen at a different time t'' refers to the probability of a simultaneous check of *records* of (one at least of) these two events. Using this strategy, any probability for sequences of observations can be reduced to a probability for eigenvalues of commuting observables, and therefore be reduced to the single-event probability (7).

Where has the information about time-ordering gone? It has been coded into the specification of the interaction between the system and the apparatus, namely, in a new projection operator $\mathbb{P}$ that includes the interaction with the apparatus. The fact that the probability of a sequence of events depends on the time ordering of the measurements can be coded into the specification of the dynamics of the physical interaction between the system and the apparatus. This observation allows us to completely reduce multi-event probabilities to single-event probability, and therefore to obtain all relevant quantum mechanical probabilities from the single probability postulate (7).

The reader might object that we have only shifted the difficulty from the problem of defining the general formalism of quantum theory to the problem of constructing a suitable dynamical operator $\mathbb{P}_{s'}$ capable of capturing the abstract idea of a "detection at the event s'." We think this objection is ill-founded, for the following reason. In deal-

[8] The idea of introducing apparatuses in order to *simultaneously* measure noncommuting observables has been considered in the context of standard quantum mechanics. See Ref. [19] and chapter 12 of Ref. [20] (and references therein). Here we exploit the same idea in order to describe sequences of noncommuting measurements, performed at *different* times. We will analyze in more detail some of the models considered in Ref. [19,20] in a forthcoming paper [21].

ing with physics the problem is not to describe abstract ideas, but to describe what we do concretely, and to give predictions about concrete experimental situations. Each concrete experimental situation has to be described by a specific dynamics, and therefore by a specific $\mathbb{P}_{s'}$ operator. The question is not whether or not a concrete system-apparatus interaction describes or not an abstract idea of measurement: the question is to find a formalism capable of describing and predicting any concrete physical situation. The problem of writing the correct dynamics describing the situation at hand is a concrete problem in the application of QM, not in the definition of its general structure. The general theory does not say what happens at different times: for every physical situation it gives the probability distribution for all the events, including those that we may wish to view as records of previous events.

D. A simple model

Let us illustrate this idea in the simple spin system model we have used so far. This shows how $\mathbb{P}$ is modified by the presence of the apparatus. Recall that here $\mathcal{K} = \mathbb{C}^2 \otimes L[\mathbb{R}]$ spanned by a (generalized) basis of states $|S, t\rangle$, $S = \uparrow, \downarrow$. Then $\mathcal{H} = \mathbb{C}^2$ and, choosing the reference time $t_0 = 0$, $\mathbb{P}|S, t\rangle = U^\dagger(t)|S\rangle$. We couple an apparatus with state space $\mathbb{C}^2$, spanned by the states $|A\rangle$, $A =$ Yes, *No*. The kinematical Hilbert space becomes $\mathcal{K} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes L[\mathbb{R}]$ spanned by a (generalized) basis of states $|S, A, t\rangle$. The modified dynamics, that includes an appropriate interaction is given, say, by the operator $\mathbb{P}_{\uparrow t'}$ defined as

$$\langle S'', A'', t''|\mathbb{P}_{\uparrow t'}|S, A, t\rangle = U^S_{S''}(t'' - t)\delta_{AA''} \qquad (47)$$

if t and t'' are both smaller or larger than t' (the interaction time); and

$$\langle S'', A'', t''|\mathbb{P}_{\uparrow t'}|S, A, t\rangle = \sum_{S'=\uparrow,\downarrow} U^{S'}_{S''}(t'' - t')(\delta^{\uparrow}_{S'} I_{A''A} + \delta^{\downarrow}_{S'}\delta_{A''A})U^S_{S'}(t' - t) \qquad (48)$$

if $t < t' < t''$. Notice that if and only if the spin is up, the apparatus flips state at time t'. That is, the dynamics defined by $\mathbb{P}_{\uparrow t'}$ is the one that describes precisely an apparatus measuring $\uparrow$ at time t'.

Equivalently, we can write (48) posing (always for $t < t' < t''$)

$$\mathbb{P}_{\uparrow t'} = \mathbb{P} \otimes \mathbb{I}_{\text{app}} + \mathbb{P}\Pi_{\uparrow t'}\mathbb{P} \otimes \tilde{I}, \qquad (49)$$

where $\langle A|\mathbb{I}_{\text{app}}|A'\rangle = \delta_{AA'}$ and $\langle A|\tilde{I}|A'\rangle = \tilde{I}_{AA'}$ is

$$\tilde{I}_{AA'} \equiv \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad (50)$$

which emphasize the fact that $\mathbb{P}_{\uparrow t'}$ is equal to $\mathbb{P}$ plus an interaction term at s'.

We can now ask what is the probability of detecting, say, the state $|\tilde{s}'\rangle = |\leftarrow, Yes, t''\rangle$ in the $\mathcal{K}$ space of the coupled (system + apparatus) system, given that the state $|\tilde{s}\rangle = |\rightarrow, \text{No}, t\rangle$ was detected. Here $\tilde{s}'$ and $\tilde{s}$ are events of the coupled (system + apparatus) system. Then (7) and (48) give the correct result obtained from conventional QM. For instance, if, say, $U(t) = 11$ [cf. (23)], we obtain $\mathcal{P}_{s\Rightarrow(s' \text{ AND } s'')} = 1/4$ if $t < t' < t''$ and $\mathcal{P}_{s\Rightarrow(s' \text{ AND } s'')} = 0$ if $t < t'' < t'$, to be compared with Eq. (22).

An example of application of this idea to a system genuinely without unitary time evolution will be given in the companion paper [10].

V. THE MEANING OF PROBABILITY

Before concluding, we discuss here a common objection to the assignment of probabilities to individual events, without reference to equal-time surfaces.

A probability is meaningful only if it assigned to an event s out of a set of alternatives $\mathcal{S} = \{s_1, s_2, \ldots\}$. In nonrelativistic QM, the probability of finding a particle in a certain spatial region R is usually understood as the probability of finding the particle in this region out of the alternatives given by the possibility of finding the particle *in other spatial regions at the same time*. That is, if $s = (R, t)$, then $\mathcal{S} = \mathcal{S}_t = \{(R_n, t)\}$ where the R_n are an ensemble of regions that cover an equal-time surface. In a general-relativistic context, there is nothing that singles out the equal-time surface. Does this imply that the probability $\mathcal{P}_s$ has no meaning in a general-relativistic context?

The answer is no, for the following reason. There exists an alternative interpretation of the probability $\mathcal{P}_s$ that does not require the set of alternatives $\mathcal{S}_t = \{(R_n, t)\}$. Consider a detector which is active in the space region R at time t. This detector has a finite probability of detecting the particle *and a finite probability of not detecting the particle*. These two alternatives define a simpler set

$$\mathcal{S}_{\text{dual}} = \{\text{Detection}, \ \text{Nondetection}\}. \qquad (51)$$

We can then interpret $\mathcal{P}_s$ as the probability that detector detects the particle, out of the two alternatives (51). This does not require the equal-time surface $\mathcal{S}_t$ to play any role.

In fact, a moment of reflection will convince the reader that this is what we truly mean by $\mathcal{P}_s$ in any realistic quantum mechanical measurement. If the set of alternatives $\mathcal{S}_t$ was the relevant one, any position measurement would only be consistent if, at the same time t, there were detectors all over the universe, all measuring whether the particle is there. This is *not* what we do when we measure if the particle is in a certain region at a certain time. What we do is to have a detector only in the region of concern and *interpret* the case of non detection as implying that the particle would have been detected by one detector elsewhere. We can do so, because in ordinary nonrelativistic quantum mechanics, we have the remarkable property that

$$\sum_n \mathcal{P}_{(R_n,t)} = 1. \qquad (52)$$

But this is a specific property of the dynamics of a particle, not a condition for $\mathcal{P}_{R_n,t}$ to be defined. $\mathcal{P}_{R_n,t}$ is defined by itself, and the probability normalization condition is simply

$$\mathcal{P}_{(R,t)} + \mathcal{P}_{\text{Non detection}} = \mathcal{P}_{\text{Detection}} + \mathcal{P}_{\text{Non detection}} = 1. \tag{53}$$

Let us express the same idea in more mathematical terms. Let $|x\rangle_t$ be an eigenvectors of the Heisenberg position operator $X(t)$, that is $X(t)|x\rangle_t = x|x\rangle_t$. The state $|x\rangle_t$ is, of course, *also* an eigenvector of the proposition operator

$$P_{xt} = |x\rangle_{tt}\langle x|. \tag{54}$$

The two operators are related by the spectral decomposition

$$X(t) = \int dxxP_{xt} \tag{55}$$

in nonrelativistic QM, but they are independently defined and they have each a physical interpretation. The operator $X(t)$ describes an ensemble of detectors covering the entire space and measuring where is the particle at time t. Its outcome is a real number x. The operator P_{xt} describes a single detector that detects whether or not the particle is in x at time t. Its outcome is a single bit: either *YES* or *NO*. Now the relation (55) implies (52): if this relation is not available, the proposition operator P_{xt} *is still well defined*, and still defines consistent probabilities for this detector outcomes.

VI. PERSPECTIVES

The main result of this work is to show that the single probability postulate (7) of general-relativistic quantum mechanics is capable of giving all probabilities of conventional quantum mechanics, including the probabilities for *sequences* of events, which are usually computed by means of the projection algorithm. This is achieved by exploiting the freedom of moving the quantum/classical boundary, emphasized by von Neumann, and assuming that (in the *nonrelativistic case*) the evolution of the system + apparatus is always unitary.

A number of points deserve to be better understood. In particular: (i) The use of this formalism in less trivial systems, where the complications connected to continuous spectrum operators and infinite volume gauge groups are more severe. (ii) The possibility of associating probabilities to arbitrary continuous regions of $\mathcal{C}$ [22–24]. This is related to the well known ‘‘time of arrival’’ problem (see for instance Ref. [25] and references therein). (iii) The extension of these ideas to field theory, and, in particular, the connection between this formalism and the boundary formalism [8,26] which is presently used [27] to compute probability amplitudes in background independent quantum gravity, in the context of loop quantum gravity [8,28,29]. (iv) Eventually, we would like to apply this formalism to physical situations where the assumption of the existence of a background geometry breaks down (for instance, see Ref. [30]). We leave these issues for further developments.

A number of tentative considerations following from the present result can nevertheless be attempted.

(i) *Time ordering* does not appear to be a fundamental structure required for the definition of quantum theory and the calculation of its probability amplitudes. In our opinion, this reinforces the hypothesis that the fundamental theory of nature can be formulated in a timeless language [31], and that temporal phenomena could be emergent [32].

(ii) In the generally covariant context, dynamics can be entirely expressed in terms of *Dirac observables*. Indeed, notice that the probability of a sequence of measurements can be written as in Eq. (10), namely, as the expectation value of the projection operator Π_s defined in Eq. (9), or Eq. (45). This operator *is* a Dirac observable of the extended system that includes the measuring devices.
In the present context, this is the answer to the long-standing problem of the description of dynamics in the ‘‘frozen-time’’ formalism; namely, in the Dirac's quantization of a system whose dynamics is expressed by constraints [2,3]. Dynamics is coded into (noncommuting) Dirac observables defined in terms of sets of interactions between (what we call) the system and (what we call) the measuring devices.

(iii) The discussion above bears also, indirectly, on the discussion on the interpretation of quantum mechanics, and on the nature of the quantum collapse.
In some interpretations of quantum mechanics, the wave function is considered a real entity that evolves unitarily, except at measurement time, when it undergoes a sudden change. In particular, some interpretations make the hypothesis that this ‘‘collapse’’ is a real physical phenomenon whose peculiar nonlocal dynamics is not yet understood. If this is the case, the full freedom of moving the quantum/classical boundary is broken, because once the collapse has happened no more interference between the two ‘‘branches’’ of a measurement outcome is possible, even in principle. If this is the case, the strategy adopted here is not viable in general, because it assumes, instead, that no true physical collapse happens at anytime.
In some others interpretations, the wave function, or the ‘‘quantum state,’’ is not considered as a real entity. Rather, only quantum ‘‘events’’ are considered real, and probabilities like $|\langle s'|s\rangle|^2$ are directly interpreted as conditional probabilities for these events to happen. In particular, in [33] these quantum events are assumed to happen at interactions between systems, and to be real only with respect to the

interacting systems themselves. From this perspective, there is no specific physical phenomenon corresponding to a quantum collapse, and the strategy considered here is viable. With respect to an external system, what happens at the interaction between system and apparatus is not a sudden change in a hypothetical real "state," but simply an entanglement between the probabilities of various outcomes of observations on the system or the apparatus. We refer to Ref. [33], and Sec. 5.6 of Ref. [8] for a discussion of this point of view.

(iv) To our knowledge, the only complete general covariant formalism for quantum theory alternative to the one we have discussed here is Hartle's generalized quantum mechanics [6]. We find an interesting convergence between Hartle's covariant sum-over histories and our results. Within Hartle's generalized quantum mechanics, probabilities for sequences of events can be expressed by means of history operators. The mean value of a history operator can be reexpressed as the mean value of a conventional projection operator on the joint system + apparatus Hilbert space. This relations will be better discussed and illustrated in the companion paper [10].

(v) Finally, in our opinion the result presented here reinforce the idea that quantum mechanics admits a very simple and straightforward generalization which is fully consistent with general relativity. And therefore that the contradictions between quantum theory and general relativity might be only apparent. Some concrete models supporting this idea will be presented elsewhere[10,21].

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