Extremal black holes in $D = 4$ Gauss-Bonnet gravity

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(Received 24 January 2007; published 17 April 2007)

We show that four-dimensional Einstein-Maxwell-dilaton-Gauss-Bonnet gravity admits asymptotically flat black hole solutions with a degenerate event horizon of the Reissner-Nordström type $AdS_2 \times S^2$. Such black holes exist for the dilaton coupling constant within the interval $0 \leq a^2 < a_{cr}^2$. Black holes must be endowed with an electric charge and (possibly) with magnetic charge (dyons) but they cannot be purely magnetic. Purely electric solutions are constructed numerically and the critical dilaton coupling is determined $a_{cr} \simeq 0.488\,219\,703$. For each value of the dilaton coupling a within this interval and for a fixed value of the Gauss-Bonnet coupling α we have a family of black holes parametrized by their electric charge. The relation between the mass, the electric charge, and the dilaton charge at both ends of the allowed interval of a is reminiscent of the Bogomol'nyi-Prasad-Sommerfield condition for dilaton black holes in the Einstein-Maxwell-dilaton theory. The entropy of the dilaton-Gauss-Bonnet extremal black holes is twice the Bekenstein-Hawking entropy.

DOI: 10.1103/PhysRevD.75.084030 PACS numbers: 04.20.Jb, 04.65.+e, 98.80.−k

I. INTRODUCTION

String theory suggests higher-curvature corrections to the Einstein equations [1–3]. Black holes in higher-curvature gravity were extensively studied during the two past decades [4,5] culminating in recent spectacular progress in the microscopic string calculations of the black hole entropy (for a review, see [6,7]). In theories with higher-curvature corrections, classical entropy deviates from the Bekenstein-Hawking value and can be calculated using Wald's formalism [8–11]. Remarkably, it still exhibits exact agreement with string theory quantum predictions at the corresponding level, both in the Bogomol'nyi-Prasad-Sommerfield (BPS) [12–24] and non-BPS [25–38] cases. In some supersymmetric models with higher-curvature terms exact classical solutions for static black holes were obtained [20,21,24]. Moreover, as was argued by Sen [39,40], knowledge of exact global black hole solutions is not necessary to be able to compare classical and quantum results: the entropy can be computed locally using the "entropy function" approach based on the typical for supergravities attractor property [25–30]. In this case it is tacitly assumed that global asymptotically flat black hole solutions exist indeed. Generically, however, the existence of local solutions does not imply the possibility to extend them to infinity as asymptotically flat black holes. Here we investigate this issue within a simple model of higher-curvature gravity.

One of the simplest four-dimensional models with higher-curvature terms is the so-called dilaton-Gauss-Bonnet (DGB) gravity which is obtained by adding to the Einstein action the four-dimensional Euler density multiplied by the dilaton exponent. As other higher-curvature theories based on topological invariants, this theory does not contain higher derivatives and thus is free of ghosts. Black hole solutions in this theory cannot be found in analytical form, but they were extensively studied perturbatively [41,42] and numerically [43–46]. More recently global properties of DGB black hole solutions were studied using the dynamical system approach [47–50]. Stability issues were discussed in [51–55]. In these papers the existence of both neutral and charged asymptotically flat solutions with a nondegenerate event horizon and without naked singularities was established. These solutions have the Schwarzschild type event horizon and do not possess an extremal limit. In this respect they differ substantially from the dilatonic black holes in the Einstein-Maxwell-dilaton (EMD) theory without the Gauss-Bonnet (GB) term [56–59]: charged dilatonic black holes do have an extremal limit in which case the event horizon shrinks to a pointlike singularity. The Bekenstein-Hawking entropy of the extremal dilatonic black holes is zero, while quantum theory suggests a nonzero result. The puzzle was solved in several supersymmetric models by showing that the horizon of the extremal dilatonic black hole is stretched to a finite radius. In the case of the DGB black holes, however, no solution

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with the *degenerate* event horizon of finite radius was found so far.

The aim of the present paper is to study this possibility in more detail. We show that, apart from the known DGB black holes with a nondegenerate event horizon, there exist electrically charged solutions with the degenerate horizon of the $AdS_2 \times S^2$ type which are asymptotically flat. These new solutions exist only in a limited range of the dilaton coupling constant. For other values of this constant, local solutions with the $AdS_2 \times S^2$ horizon cannot be continued to infinity as asymptotically flat black holes: singularity is met in a finite distance outside the horizon. Since the DGB theory does not possess S duality, magnetic solutions differ substantially from the electric ones; in particular, no purely magnetic black holes with a degenerate horizon are allowed, though dyonic solutions with a nonzero electric charge are possible.

II. 4D DILATONIC GAUSS-BONNET THEORY

The action for the four-dimensional Einstein-Maxwell-dilaton theory with an arbitrary dilaton coupling constant a modified by the DGB term reads

$$S = \frac{1}{16\pi}\int\{R - 2\partial_\mu\phi\partial^\mu\phi - e^{2a\phi}(F^2 - \alpha\mathcal{L}_{GB})\} \times\sqrt{-g}d^4x, \tag{1}$$

where $\mathcal{L}_{GB}$ is the Euler density

$$\mathcal{L}_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}. \tag{2}$$

This action contains two parameters (we use units $G = c = 1$): the dilaton coupling a and the GB coupling α. We will assume $a \geq 0$, $\alpha \geq 0$. Solutions for negative a can be obtained changing the sign of the dilaton. Note that the Maxwell term is not multiplied by α to facilitate decoupling of the GB term from the EMD action.

Consider the static spherically symmetric metrics parametrized for further convenience by three functions $w(r)$, $\rho(r)$, $\sigma(r)$:

$$ds^2 = -w(r)\sigma^2(r)dt^2 + \frac{dr^2}{w(r)} + \rho^2(r)d\Omega_2^2. \tag{3}$$

The scalar curvature and the Euler density then read

$$R = \frac{1}{\sigma\rho^2}\{-(4\sigma w\rho\rho' + \sigma w'\rho^2 + 2\sigma' w\rho^2)' + 2\sigma[\rho'(w\rho)' + 1] + 4\sigma' w\rho\rho'\}, \tag{4}$$

$$\mathcal{L}_{GB} = \frac{4}{\sigma\rho^2}\left[\frac{(w\sigma^2)'(w\rho'^2 - 1)}{\sigma}\right]'. \tag{5}$$

The corresponding ansatz for the Maxwell one-form is

$$A = -f(r)dt - q_m\cos\theta d\varphi, \tag{6}$$

where $f(r)$ is the electrostatic potential and q_m is the magnetic charge. Note that the DGB term breaks the discrete S duality which in the absence of this term is described by the transformation

$$g_{\mu\nu} \to g_{\mu\nu}, \qquad F \to e^{-2a\phi *}F, \qquad \phi \to -\phi, \tag{7}$$

where $F = dA$. It is expected therefore that properties of electric or magnetic black holes in this theory will be essentially different.

A. Reduced action and field equations

The corresponding one-dimensional Lagrangian is obtained by dropping the total derivative in the dimensionally reduced action:

$$L = \frac{\sigma}{2}[\rho'(w\rho)' + 1] + \sigma' w\rho\rho' - 2\alpha a\sigma^{-1}(\sigma^2 w)'(w\rho'^2 - 1) \times\phi' e^{2a\phi} - \frac{\sigma}{2}w\rho^2\phi'^2 + \frac{1}{2}\frac{\rho^2}{\sigma}f'^2 e^{2a\phi} - \frac{1}{2}\frac{\sigma}{\rho^2}q_m^2 e^{2a\phi}. \tag{8}$$

The corresponding equations of motion read

$$8\alpha a[w(w\rho'^2 - 1)\phi' e^{2a\phi}]' - \rho'(w\rho)' + 1 - 2w\rho\rho'' - 4\alpha a w'(w\rho'^2 - 1)\phi' e^{2a\phi} - w\rho^2\phi'^2 - \frac{\rho^2}{\sigma^2}f'^2 e^{2a\phi} - \frac{q_m^2}{\rho^2}e^{2a\phi} = 0, \tag{9}$$

$$4\alpha a\left[\frac{(w\rho'^2 - 1)\phi' e^{2a\phi}}{\sigma}\right]'\sigma - 4\alpha a\frac{(\sigma^2 w)'}{\sigma^2}\rho'^2\phi' e^{2a\phi} - \rho\rho'' + \rho\rho'\frac{\sigma'}{\sigma} - \rho^2\phi'^2 = 0, \tag{10}$$

$$4\alpha a\left[\frac{w\rho'(\sigma^2 w)'\phi' e^{2a\phi}}{\sigma}\right]' - \frac{1}{2}\left[\frac{(\sigma^2 w)'}{\sigma}\right]'\rho - (\sigma w\rho')' - \sigma w\rho\phi'^2 + \frac{\rho}{\sigma}f'^2 e^{2a\phi} + \sigma\frac{q_m^2}{\rho^3}e^{2a\phi} = 0, \tag{11}$$

$$(\sigma w\rho^2\phi')' + 2\alpha a\left[\frac{(\sigma^2 w)'(w\rho'^2 - 1)}{\sigma}\right]' e^{2a\phi} + a\frac{\rho^2}{\sigma}f'^2 e^{2a\phi} - a\sigma\frac{q_m^2}{\rho^2}e^{2a\phi} = 0, \tag{12}$$

$$\left(\frac{\rho^2}{\sigma}f' e^{2a\phi}\right)' = 0. \tag{13}$$

The Maxwell equation for the form field (13) can be directly solved

$$f'(r) = q_e\sigma\rho^{-2}e^{-2a\phi}, \tag{14}$$

where q_e is the electric charge.

B. Global symmetries and conserved quantities

The action (1) is invariant under the following three-parametric group of global transformations:

$$w \to w e^{-2(\delta+\lambda)}, \qquad \rho \to \rho e^{\delta}, \qquad r \to r e^{-\lambda} + \nu,$$
$$\sigma \to \sigma e^{\lambda}, \qquad \phi \to \phi + \frac{\delta}{a}, \qquad f \to f e^{-2\delta}. \tag{15}$$

They generate three conserved Noether currents

$$J_g := \left(\frac{\partial L}{\partial \Phi'^A}\Phi'^A - L\right)\partial_g r|_{g=0} - \frac{\partial L}{\partial \Phi'^A}\partial_g \Phi^A|_{g=0},$$
$$\partial_r J_g = 0, \tag{16}$$

where Φ^A stands for σ, w, ρ, ϕ, f, and $g = \delta, \nu, \lambda$. The conserved quantity corresponding to the parameter ν is the Hamiltonian

$$H = \frac{1}{2}\sigma[\rho'(w\rho)' - 1] + \sigma' w \rho\rho' - 2\alpha a \sigma^{-1}(\sigma^2 w)' \times (3w\rho'^2 - 1)\phi' e^{2a\phi} - \frac{1}{2}\sigma w \rho^2 \phi'^2 + \frac{1}{2}\frac{\rho^2}{\sigma} f'^2 e^{2a\phi} + \frac{1}{2}\frac{\sigma}{\rho^2} q_m^2 e^{2a\phi}. \tag{17}$$

This is known to vanish on shell for diffeomorphism invariant theories, $H = 0$. The Noether current corresponding to the parameter δ leads to the conservation equation $J_\delta = 0$, where

$$J_\delta = \frac{\sigma w \rho^2 \phi'}{a} - \frac{(\sigma^2 w)'\rho^2}{2\sigma} + 2q_e f + 2\alpha(\sigma^{-1}\{(w\rho'^2 - 1) \times [(\sigma^2 w)' - 2a\sigma^2 w \phi'] + 2aw(\sigma^2 w)'\rho\rho'\phi'\}e^{2a\phi}), \tag{18}$$

which is an Abelian counterpart of the equation given in [60].[1] The value of this integral depends on solutions. The third integral corresponding to λ is trivial: $J_\lambda = -rH$, its existence implies $H = 0$.

The above integrals of motion allow one to reduce the order of the system by two. Fixing the gauge (e.g. $\sigma = 1$) one has 3 s order equations for w, ρ, ϕ with q_e, q_m entering as parameters of this six-order system. Using the integrals, the system order can be reduced to four, with one parameter more (the fixed value of J_δ). For $q_e = 0$ one can further reduce the order to three. Introducing, for instance, new variables

$$w \to \exp(w), \qquad \rho \to \exp\left(\rho - \frac{w}{2}\right),$$
$$\phi \to \phi - \frac{1}{2a}w, \tag{19}$$

we exclude from the system the variable w (while w' and w'' still persist). For numerical computations we use the initial six-dimensional system, checking the constancy of the integrals of motion to control accuracy of the calculation.

The space of solutions is invariant under a *four-parametric* group of global transformations which consists in rescaling of the electric charge

$$q_e \to q_e e^{2\delta}, \qquad q_m \to q_m \tag{20}$$

(leaving the magnetic charge invariant), rescaling and shift of an independent variable

$$r \to r e^{(\mu/2)+\delta} + \nu, \tag{21}$$

and the following transformation of the field functions:

$$w \to w e^{\mu}, \qquad \rho \to \rho e^{\delta}, \qquad \sigma \to \sigma e^{\lambda},$$
$$\phi \to \phi + \frac{\delta}{a}, \qquad f \to f e^{(\mu/2)-\delta+\lambda}. \tag{22}$$

Note that the Lagrangian is rescaled under this 4-parameter transformation as $L \to e^{\lambda} L$, so action (1) remains invariant provided

$$\mu = -2(\delta + \lambda), \tag{23}$$

in which case we go back to (15). The shift ν is trivial and the symmetry related to λ can be frozen by the gauge choice $\sigma = 1$. Therefore, physically interesting transformations are generated by μ and δ forming the subgroup which we denote as $G(\mu, \delta)$.

C. Turning points of $\rho(r)$ and the gauge choice

Reparametrization of the radial variable r allows us to eliminate one of the three metric functions. There are two convenient gauge choices: the Schwarzschild gauge $\rho = \bar{r}$, in which the radial variable is the radius of two spheres:

$$ds^2 = -\bar{\sigma}^2(\bar{r})\bar{w}(\bar{r})dt^2 + \frac{d\bar{r}^2}{\bar{w}(\bar{r})} + \bar{r}^2 d\Omega_2^2, \tag{24}$$

and the Garfinkle-Horowitz-Strominger (GHS) gauge [59] $\sigma = 1$:

$$ds^2 = -w(r)dt^2 + \frac{dr^2}{w(r)} + \rho^2(r)d\Omega_2^2. \tag{25}$$

The coordinate transformation relating these gauges reads

$$\bar{r} = \rho(r), \qquad \bar{\sigma}^2(\bar{r})\bar{w}(\bar{r}) = w(r), \qquad \frac{1}{\bar{\sigma}(\bar{r})} = \frac{d\rho(r)}{dr}. \tag{26}$$

It becomes singular at the turning points of the function $\rho(r)$ where the derivative $\rho'(r) = 0$, so solutions containing such turning points cannot be described globally in the Schwarzschild gauge. We will see later on that the presence of turning points is typical for the DGB system, so the GHS gauge is preferable.

[1] We use this occasion to correct a misprint in Ref. [60]: the factor $e^{-2\Phi}$ is missing at the right-hand side of Eq. (23).

D. Dilaton black holes and the GB term

In the theory without the GB term, $\alpha = 0$, an electrically charged asymptotically flat black hole solution for an arbitrary dilaton coupling a reads (in the gauge $\sigma = 1$) [59]:

$$
\begin{aligned}
w(r) &= \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{(1-a^2)/(1+a^2)},\\
\rho(r) &= r\left(1 - \frac{r_-}{r}\right)^{a^2/(1+a^2)},\\
e^{2a\phi} &= e^{2a\phi_\infty}\left(1 - \frac{r_-}{r}\right)^{-((2a^2)/(1+a^2))}.
\end{aligned} \tag{27}
$$

The mass and the electric charge are given by

$$
M = \frac{r_+}{2} + \frac{1-a^2}{1+a^2}\frac{r_-}{2}, \qquad q^2 = e^{2a\phi_\infty}\frac{r_+ r_-}{1+a^2}. \tag{28}
$$

For $a = 0$ this reduces to the Reissner-Nordström solution, which in the extremal limit $r_+ = r_- = r_H$,

$$
ds^2 = -\left(1 - \frac{r_H}{r}\right)^2 dt^2 + \frac{dr^2}{(1 - \frac{r_H}{r})^2} + r^2 d\Omega_2^2, \tag{29}
$$

has a degenerate event horizon $AdS_2 \times S^2$. Note that for $a = 0$ the GB term decouples from the system, so this solution remains true in the full theory with $\alpha \neq 0$.

For $a \neq 0$, the extremal limit $r_+ = r_- = r_H$ is

$$
\begin{aligned}
ds^2 = &-\left(1 - \frac{r_H}{r}\right)^{2/(1+a^2)} dt^2 + \left(1 - \frac{r_H}{r}\right)^{-[2/(1+a^2)]} dr^2\\
&+ r^2\left(1 - \frac{r_H}{r}\right)^{2a^2/(1+a^2)} d\Omega_2^2.
\end{aligned} \tag{30}
$$

At $r = r_H$ the radius of the two-spheres shrinks, so we have a pointlike singularity. The Ricci scalar in the vicinity of this point diverges near the horizon $r = r_H$:

$$
R = \frac{2a^2 r_H^2}{(1+a^2)^2 r^4}\left(1 - \frac{r_H}{r}\right)^{-[2a^2/(1+a^2)]}, \tag{31}
$$

as well as the dilaton exponential for an electric solution:

$$
e^{2a\phi} = e^{2a\phi_\infty}\left(1 - \frac{r_H}{r}\right)^{-[2a^2/(1+a^2)]}. \tag{32}
$$

Substituting the general dilatonic black hole solution (27) to the GB term we obtain the following value at the horizon $r = r_+$:

$$
e^{2a\phi}\mathcal{L}_{GB}|_{r=r_+} \sim (r_+ - r_-)^{-\{[2(4a^2+1)]/(a^2+1)\}}. \tag{33}
$$

This expression diverges in the extremal limit $r_+ \to r_-$. Thus, it is not possible to treat the GB term perturbatively expanding in α in the vicinity of the extreme dilaton black hole. In other words, one can expect that the GB term will substantially modify the dilaton black hole solution in the extremal limit.

Summarizing the above information, we see that the DGB gravity *admits* the black hole solution with the degenerate event horizon of the $AdS_2 \times S^2$ type for $a = 0$ (the Reissner-Nordström solution (29), and *does not* admit the extremal dilaton black hole (30) for $a \neq 0$ even as an approximation for small α. So, the intriguing question arises, whether the branch of *degenerate* black holes exists in the DGB gravity which starts at $a = 0$ in the parameter space and continues to nonzero a. In the next section we analyze this possibility in detail both analytically and numerically.

III. DGB BLACK HOLES WITH $AdS_2 \times S^2$ HORIZON

We are looking for asymptotically flat solutions in the DGB theory for which the metric function $w(r)$ has double zero at some point $r = r_H$ and does not have singularities for $r > r_H$. To attack this problem numerically, one has to prove first that such solutions exist locally in the vicinity of the horizon $r = r_H$. We will show that this is true, provided some restriction on the parameters is satisfied.

A. Near-horizon expansion

Assuming the GHS gauge $\sigma = 1$, consider the series expansions around some point $r = r_H$ (supposed to be a horizon) in powers of $x = r - r_H$:

$$
w(r) = \sum_{k=1}^{\infty} w_k x^k, \qquad \rho(r) = \sum_{k=0}^{\infty} \rho_k x^k, \qquad P(r) := e^{2a\phi(r)} = \sum_{k=0}^{\infty} P_k x^k. \tag{34}
$$

The function w starts with the linear term (vanishing of w_0 means that $r = r_H$ is a horizon), two other functions have general Taylor's expansions.

Substituting these expansions into the equations of motion (9)–(12) we find local solutions of two types. The first type solution has $w_1 \neq 0$, i.e. the function w has simple zero at $r = r_H$. This corresponds to a nondegenerate horizon of the Schwarzschild type. Such local solutions and their numerical continuation to infinity were considered in some particular cases in Refs. [43–46]. Here we give more general expansion valid for both electric and magnetic charges present (in the gauge $\sigma = 1$):

$$
\begin{aligned}
w(r) &= \frac{\Gamma}{\rho_0^2 P_1}x + \frac{2a^2[\alpha(q_e^4 + P_0^2 q_m^2) + (4\Gamma P_0\alpha^2 + \rho_0^4)q_e^2 + P_0^2(4\Gamma P_0\alpha^2 - \rho_0^4 + 4\alpha q_e^2)q_m^2] + \Gamma\rho_0^6}{12\rho_0^6 a^2 P_0^2\alpha}x^2 + O(x^3),\\
\rho(r) &= \rho_0 + \frac{(\rho_0^2 P_0 - q_e^2 - P_0^2 q_m^2 - 2\alpha P_0\Gamma)P_1}{\rho_0 P_0 \Gamma}x + O(x^2), \qquad P(r) = P_0 + P_1 x + O(x^2),
\end{aligned} \tag{35}
$$

where ρ_0, P_0, and P_1 are free parameters and Γ satisfies the equation

$$48\alpha^3 a^2 P_0^2\Gamma^2 + \{\rho_0^6 - 16P_0\alpha^2 a^2[3\rho_0^2 P_0 - 2(q_e^2 + P_0^2 q_m^2)]\}\Gamma + 2a^2[2\alpha(q_e^4 + P_0^4 q_m^4) + \rho_0^2(\rho_0^2 - 12P_0\alpha)q_e^2 - \rho_0^2 P_0^2(\rho_0^2 + 12P_0\alpha)q_m^2 + 4\alpha P_0^2 q_e^2 q_m^2 + 6P_0^2\alpha\rho_0^4] = 0. \tag{36}$$

This quadratic equation has two roots $\Gamma = \Gamma_\pm$ depending on parameters $q_e, q_m, \rho_0, P_0, P_1$. The above local solutions exists for such values of parameters for which $\Gamma \neq 0$.

The second class of local solutions has $w_1 = 0$. Vanishing of w_1 means that the horizon is degenerate. Such an expansion contains only one free parameter with fixed charges. This family is disconnected from the family (35) and it was not noticed so far:

$$\begin{aligned} w(r) &= \frac{x^2}{\rho_0^2} - \frac{P_1}{6\alpha a^2\rho_0^4}[3(a^2-1)q_m^4 + 6\alpha(3a^2-2)q_m^2 + 4\alpha^2(5a^2-3)]x^3 + O(x^4),\\ \rho(r) &= \rho_0 + \frac{P_1}{4\alpha a^2\rho_0}[(a^2-1)q_m^4 + 2\alpha(3a^2-2)q_m^2 + 4\alpha^2(a^2-1)]x + O(x^2),\\ P(r) &= \frac{\rho_0^2}{2(2\alpha+q_m^2)} + P_1 x + O(x^2). \end{aligned} \tag{37}$$

Here q_m is the magnetic charge, P_1 is a free parameter, and ρ_0 is the physical radius of the horizon depending on charges as follows:

$$\rho_0^2 = \frac{2q_e(2\alpha + q_m^2)}{\sqrt{4\alpha + q_m^2}}. \tag{38}$$

Note that the dilaton coupling constant enters this expansion only through a^2, so the space of solutions is symmetric under $a \to -a$, $\phi \to -\phi$. In what follows we will assume $a \geq 0$.

The values of the integrals of motion corresponding to (37) are

$$H = \frac{1}{2\rho^2}[q_m^2 P_0 + q_e^2 P_0^{-1} - \rho^2], \qquad P_0 = \frac{\rho_0^2}{2(2\alpha + q_m^2)}, \tag{39}$$

$$J_\delta = 2q_e f_0, \tag{40}$$

where f_0 is the value of the electrostatic potential on the horizon.

From our previous analysis of the EMD black holes and the above form of the local solution one can draw the following conclusions:

(1) Black holes with $AdS_2 \times S^2$ event horizon do not exist in the $a \neq 0$ EMD theory without curvature corrections ($\alpha = 0$).

(2) Such solutions do not exist in the absence of the electric charge, while the presence of the magnetic charge is optional. S duality is thus broken as expected.

(3) This local solution is not generic (the number of free parameters is less that the degree of the system of differential equations).

For simplicity, in this paper we will focus on purely electric black holes. In this case the expansions simplify and we can give some further terms:

$$\begin{aligned} w(r) &= \frac{1}{\rho_0^2}\left[x^2 - \frac{2(5a^2-3)}{3}\left(\frac{\alpha P_1}{a^2\rho_0^2}\right)x^3 + \frac{173a^6 - 269a^4 + 99a^2 - 27}{3(5a^2-3)}\left(\frac{\alpha P_1}{a^2\rho_0^2}\right)^2 x^4\right] + O(x^5),\\ \rho(r) &= \rho_0\left[1 + (a^2-1)\left(\frac{\alpha P_1}{a^2\rho_0^2}\right)x - \frac{2a^2(a^4-6)}{(5a^2-3)}\left(\frac{\alpha P_1}{a^2\rho_0^2}\right)^2 x^2\right] + O(x^3),\\ \alpha P(r) &= \rho_0^2\left[\frac{1}{4} + a^2\left(\frac{\alpha P_1}{a^2\rho_0^2}\right)x + \frac{a^2(a^4 - 5a^2 - 3)}{(5a^2-3)}\left(\frac{\alpha P_1}{a^2\rho_0^2}\right)^2 x^2\right] + O(x^3). \end{aligned} \tag{41}$$

The electric charge is related to the horizon radius as

$$q_e = \frac{\rho_0^2}{2\sqrt{\alpha}}. \tag{42}$$

We can arrange higher order terms in such a way that a^2 enters in the denominator only in powers of the combination P_1/a^2. This facilitates taking the limit $a \to 0$. We know that in this limit there exists an exact solution which has the near-horizon expansion of the type (41), namely, the extremal Reissner-Nordström solution. Thus we expect that for the asymptotically flat solutions

$$P_1 \to 0, \quad \text{as } a \to 0. \tag{43}$$

Numerical calculations show that this is indeed the case (see Sec. III C).

Another subtlety is related to the limit of GB decoupling. Obviously, our expansion fails in this limit: for a finite charge q_e one has $\rho_0^2 = 2q_e\sqrt{\alpha} \to 0$ and, consequently, the expansion coefficients will diverge. The reason is that our expansion for w is incompatible with that for the dilaton black hole of the Einstein-Maxwell-dilaton theory. This reflects again the absence of the black hole solutions with the $AdS_2 \times S^2$ horizon in the theory without curvature corrections. Substituting q_e defined by (42) into the equations of motion (9)–(12), one can see that the GB coupling parameter α enters always in the combination $\alpha e^{2a\phi(r)}$. Thus, shifting the dilaton is equivalent to rescaling the GB term [this was in fact clear already from action (1)]. Note that in the case $a^2 = 1$ the linear term in the expansion of ρ vanishes, $\rho_1 = 0$, implying that there is no regular transformation to the gauge $\rho = \bar{r}$.

Therefore, a purely electric local solution with a fixed value of charge q_e contains one free parameter: the dilaton derivative P_1. An important issue is to determine the correct sign of P_1. To be able to interpret the region $r > r_H$ as an exterior of the black hole, one has to ensure positiveness of the derivative ρ' at the horizon. From the above expansion one finds

$$\rho'|_{x=0} = \frac{(a^2 - 1)\alpha P_1}{a^2 \rho_0} > 0. \tag{44}$$

Thus, we should take positive P_1 for $a^2 > 1$ and negative P_1 for $a^2 < 1$. It is convenient to introduce the sign parameter which ensures this:

$$\varsigma = \frac{P_1}{|P_1|} = \frac{a^2 - 1}{|a^2 - 1|}. \tag{45}$$

Now define the following combination of ρ_0 and P_1:

$$b = \frac{\alpha |P_1|}{a^2 \rho_0^2}. \tag{46}$$

It is easy to see that free parameters enter the series expansions near the horizon only through this quantity. Consider now the transformations of the expansion parameters under symmetries of the solution space (21) and (22). Since P_1 is the first derivative of the dilaton exponent, one finds that under δ transformation

$$P_1 \to P_1 e^{\delta}, \qquad b \to b e^{-\delta}, \tag{47}$$

so the quantity bx remains invariant. Thus the full set of local solutions can be generated from one particular solution with $\rho_0 = 1$, $P_1 = 1$, which we will call the normalized local solution, by the symmetry transformations with $\delta = -\ln \rho_0$ and $\mu = 2\ln(b\rho_0)$ from (22), i.e. by the group element $G[2\ln(b\rho_0), -\ln\rho_0]$. The normalized local solution does not contain free parameters at all:

$$w(r) = x^2 - \varsigma \frac{2(5a^2 - 3)}{3} x^3 + \frac{(173a^6 - 269a^4 + 99a^2 - 27)}{3(5a^2 - 3)} x^4 + O(x^4), \tag{48}$$

$$\rho(r) = 1 + \varsigma(a^2 - 1)x - \frac{2a^2(a^4 - 6)}{(5a^2 - 3)} x^2 + O(x^3), \tag{49}$$

$$\alpha P(r) = \frac{1}{4} + \varsigma(r - r_H) + \frac{(a^4 - 5a^2 - 3)}{(5a^2 - 3)} x^2 + O(x^3). \tag{50}$$

Note the presence of the sign function ς in the odd power terms. The electric charge corresponding to the normalized local solution is $q_e = 1/(2\sqrt{\alpha})$.

B. Asymptotic flatness

We are looking for asymptotically flat global solutions which satisfy the conditions $w \to 1$, $\rho/r \to 1$, $\phi \to \text{const}$ as $r \to \infty$. The subleading terms should be expandable in the power series of $1/r$. The local solution with these properties turns out to be three-parametric, depending on the ADM mass M, the dilaton charge D, and the asymptotic value of the dilaton ϕ_∞:

$$\begin{aligned} w(r) &= 1 - \frac{2M}{r} + \frac{\alpha Q_e^2}{r^2} + O(r^{-3}), \\ \rho(r) &= r - \frac{D^2}{2r} - \frac{D(2MD - \alpha a Q_e^2)}{3r^2} + O(r^{-3}), \\ \phi(r) &= \phi_\infty + \frac{D}{r} + \frac{2DM - \alpha a Q_e^2}{2r^2} + O(r^{-3}), \end{aligned} \tag{51}$$

where

$$Q_e = q_e e^{-a\phi_\infty}. \tag{52}$$

The dilaton charge can be also read from the asymptotic expansion of the dilaton exponential:

$$e^{2a(\phi - \phi_\infty)} = 1 + \frac{2aD}{r} + \frac{2aD(aD + M) - \alpha a^2 Q_e^2}{r^2} + O(r^{-3}). \tag{53}$$

The asymptotic values of two integrals of motion are

$$H = \tfrac{1}{2}(w_\infty \rho_\infty'^2 - 1), \tag{54}$$

$$J_\delta = 2q_e f_\infty - M - \frac{D}{a}. \tag{55}$$

Behavior of the global solution which starts with the normalized local solution (48)–(50) at the horizon depends only on the dilaton coupling constant a. Its existence for all a is not guaranteed *a priori*. But, for some sufficiently small values of a, we find numerically that all three functions vary smoothly with increasing x, so that w and the derivative ρ' stabilize at infinity on some constant values $w_\infty \neq 1$, $\rho'_\infty \neq 1$. Then, using the symmetries (21) and (22) of the solution space, one can rescale the whole solution to achieve the desired unit values for these parameters. More precisely, the relevant subgroup of rescalings is two-parametric: $G(\mu, \delta)$. As we have argued, two parameters μ, δ effectively replace the parameters ρ_0, P_1 of the (non-normalized) local solution (41). So one could expect that rescaling of the solution so that $w_\infty = 1$, $\rho'_\infty = 1$ would fix both quantities ρ_0, P_1 on the horizon. But from the Hamiltonian equation $H = 0$ with H given by the Eq. (54) it is easy to see that one must have $w_\infty \rho_\infty'^2 = 1$ for any solution such that $w \to w_\infty$, $\rho' \to \rho'_\infty$ asymptotically. Therefore it is enough to perform *one* but not *two* independent rescalings in order to get $w_\infty = 1$, $\rho'_\infty = 1$. Indeed, under $G(\mu, \delta)$

$$w \to w e^{\mu}, \qquad \rho_0 \to \rho_0 e^{\delta}, \qquad P_1 \to P_1 e^{\delta - \mu/2}, \qquad w\rho'^2 \to w\rho'^2. \tag{56}$$

Since the choice of μ, δ is equivalent to the choice of ρ_0, P_1, an invariance of the product $w\rho'^2$ under $G(\mu, \delta)$ means that the solution starting on the horizon with *any* ρ_0, P_1 will reach at infinity the values w_∞, ρ'_∞ satisfying $w_\infty \rho_\infty'^2 = 1$. Therefore, taking $\mu = -\ln w_\infty$, we will achieve simultaneously $w_\infty = 1$ and $\rho'_\infty = 1$. This means that asymptotically flat solutions still form a one-parameter family, a parameter being the electric charge q_e.

C. Numerical analysis

Since we know that the desired global solution exists for $a = 0$, we start with the local solution at the horizon with small a and look for numerical solutions which fit the asymptotic expansions (51). For sufficiently small a global solutions exist indeed, and, as we have explained, two basic conditions at infinity $w = 1$, $\rho' = 1$ fix only one of the two parameters ρ_0 and P_1 at the horizon. It will be convenient to leave ρ_0 (defining the electric charge) arbitrary, and to fix P_1. We will also choose the value of the GB coupling $\alpha = 1$. Then the black hole mass can be found numerically from the asymptotic expansions (51) together with the dilaton charge and the asymptotic value of the dilaton.

Typical coordinate dependence of the metric functions and the dilaton exponent are shown in Fig. 1 for some values of the dilaton coupling a. Solutions exist for

$$0 \le a < a_{\rm cr}, \qquad a_{\rm cr} \simeq 0.488\,219\,703. \tag{57}$$

Let us discuss in more detail behavior of solutions at the ends of this interval. As expected, the parameter b of the local solution at the horizon tends to unity when $a \to 0$, as shown in Fig. 2(a). This means that the first Taylor coefficient in the expansion of $\rho(x)$ becomes equal to unity (note that the sign function $\varsigma = -1$ as $a \to 0$), while all higher coefficients are zero. Therefore, assuming $\rho_0 = r_H$, we find that $\rho = r$ globally. Similarly, all terms in the expansion of $P(x)$ vanish in the limit $a \to 0$ except the constant P_0, so the dilaton exponential tends to the constant value $P = \rho_0^2/4$. Correspondingly, we find

$$\lim_{a\to 0} e^{2a\phi_\infty}/q_e \to \tfrac{1}{2}. \tag{58}$$

For w all Taylor's coefficients in (41) are nonzero and the whole series exactly reproduces an expansion

$$w(r) = \left(1 - \frac{\rho_0}{r}\right)^2 = z^2 - 2z^3 + 3z^4 + O(z^5), \qquad z = (r - \rho_0)/\rho_0. \tag{59}$$

Thus, our family of solutions begins with the extremal Reissner-Nordström metric for zero dilaton coupling a.

With fixed horizon radius ρ_0, the mass and the dilaton charge of the black holes increase with the growing dilaton coupling constant tending to infinity when a approaches $a_{\rm cr}$. The dilaton exponent, on the contrary, tends to zero in this limit. Using the symmetry of the solution space under δ transformation, one can generate the sequence of solutions with different electric charges q_e and correspondingly with different masses, dilaton charges, and ϕ_∞. Since variation of the electric charge is essentially equivalent to variation of the unique parameter ρ_0 in the horizon expan-

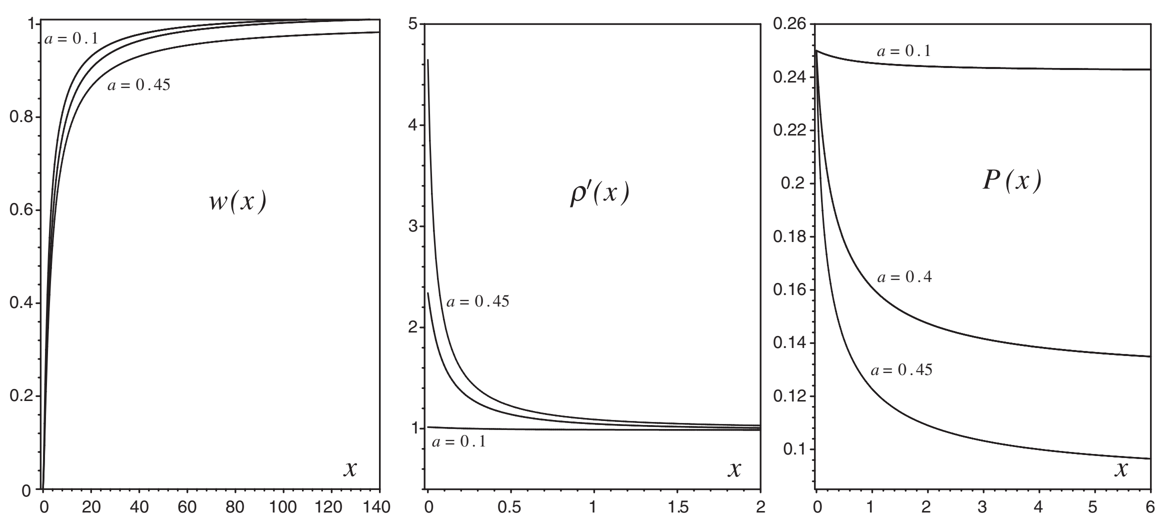

FIG. 1. The functions $w(x)$, $\rho'(x)$, $P(x)$, $x = r - r_H$, for $\rho_0 = 1$ and some values of the dilaton coupling constant: $a = 0.1$ ($P_1 = -0.01$); $a = 0.4$ ($P_1 = -0.446$); $a = 0.45$ ($P_1 = -1.18$).

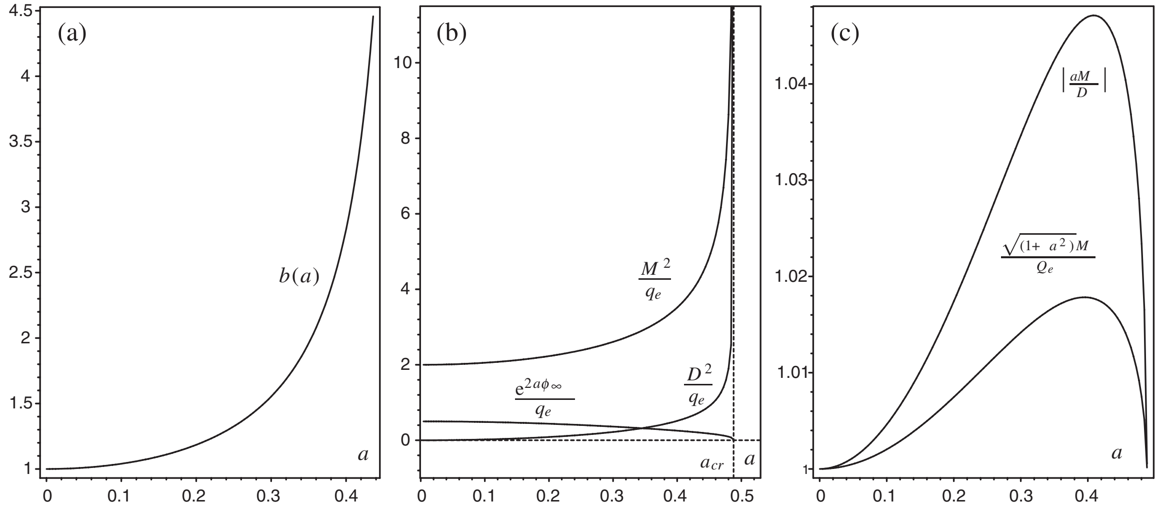

FIG. 2. (a) Dependence of the parameter b on the dilaton coupling constant a: b tends to unity for $a \to 0$ ensuring continuous transition to the extremal Reissner-Nordström solution. (b) Numerical curves $k_M(a) = M^2/q_e$, $k_D(a) = D^2/q_e$, $k_\phi(a) = e^{2a\phi_\infty}/q_e$ for $\rho_0 = 1$. The mass curve starts with the Reissner-Nordström value for $a = 0$ and diverges as $a \to a_{\rm cr}$. The dilaton charge increases from zero to infinity, while the dilaton exponential monotonically varies from the value $\frac{1}{2}$ at $a = 0$ to zero for $a \to a_{\rm cr}$. (c) The quantities $|\frac{aM}{D}|$ and $(\sqrt{1+a^2}M)/Q_e$ as functions of a: both tend to unity at the ends of the allowed interval of a.

sion, it is clear that, using δ transformation, we will generate *all* extremal solutions. Under this transformation the mass and the dilaton charge scale as e^{δ}, while the electric charge and the dilaton exponent $e^{2a\phi_\infty}$ scale as $e^{2\delta}$. Therefore the ratios

$$k_M = \frac{M^2}{q_e}, \qquad k_D = \frac{D^2}{q_e}, \qquad k_\phi = \frac{e^{2a\phi_\infty}}{q_e} \tag{60}$$

depend only on a. Their numerical graphs are presented in Fig. 2(b).

As we already discussed, the metric for $a = 0$ is known analytically. For a in the vicinity of $a_{\rm cr}$ the analytic solution is not known, but one finds that the behavior of the mass, the dilaton charge, and the rescaled electric charge Q_e exhibit some similarities at both ends of the allowed interval of a. Namely, the following two ratios stabilize at unity for both $a \to 0$ and $a \to a_{\rm cr}$ [Fig. 2(c)]:

$$\left|\frac{aM}{D}\right| \to 1, \qquad \frac{\sqrt{1+a^2}M}{Q_e} \to 1. \tag{61}$$

This corresponds to fulfillment of the following condition:

$$a^2M^2 + D^2 = \frac{2a^2}{1+a^2}Q_e^2, \tag{62}$$

which is reminiscent of the BPS condition for electrically charged black holes of the EMD theory. This feature is similar to that in another stringy generalization of EMD theory in which the Maxwell action is replaced by the Born-Infeld action, but no GB term is introduced [61].

For the values of a outside the allowed interval, solutions starting with the $AdS_2 \times S^2$ horizon are not asymptotically flat, but singular. For them the metric function $\rho(r)$ has a turning point at some finite radial coordinate $r = r_t$, such that $\rho'(r_t) = 0$, $\rho''(r_t) < 0$ (Fig. 3). This point is regular, but at a finite proper distance from it one encounters the singular turning point r_s, such that $\rho'(r_s) > 0$, $\rho''(r_s) = \infty$, where all variables have a square-root singularity, being expandable in terms of $\sqrt{r - r_s}$:

$$\begin{aligned} w &= w_s + w_1 y + w_{3/2}y^{3/2} + O(y^2),\\ y &= r - r_s,\\ \rho &= \rho_s + \rho_1 y + \rho_{3/2}y^{3/2} + O(y^2),\\ P &= P_s + P_1 y + P_{3/2}y^{3/2} + O(y^2). \end{aligned} \tag{63}$$

This local expansion contains four free parameters w_s, ρ_s, ρ_1, P_s, while other coefficients read

$$w_1 = \frac{4a^2P_s(\gamma\rho_s^2P_s + q_e^2) - \Delta^2 w_s\rho_s^4}{4P_s^2a^2\rho_s^2[\rho_1(6\Delta w_s\rho_1 - \rho_s) - 2\Delta]}, \qquad P_1 = \Delta,$$
$$P_{3/2} = \frac{(\rho_s - 4\Delta\rho_1 w_s)\rho_{3/2}}{2\gamma}, \tag{64}$$

$$w_{3/2} = \frac{\{2\Delta\rho_s^4[2\Delta\rho_1 w_s(\gamma+4) - \rho_s(5\gamma+4)] - 16a^2P_s\rho_1\gamma(\gamma\rho_1P_s\rho_s^2 + q_e^2) + \rho_s^6\rho_1\}w_s\rho_{3/2}}{8\rho_s^2a^2P_s^2\gamma^2[2\Delta(3\gamma+2) - \rho_s\rho_1]}, \tag{65}$$

where $\gamma = \rho_1^2 w_s - 1$ and Δ satisfies the equation

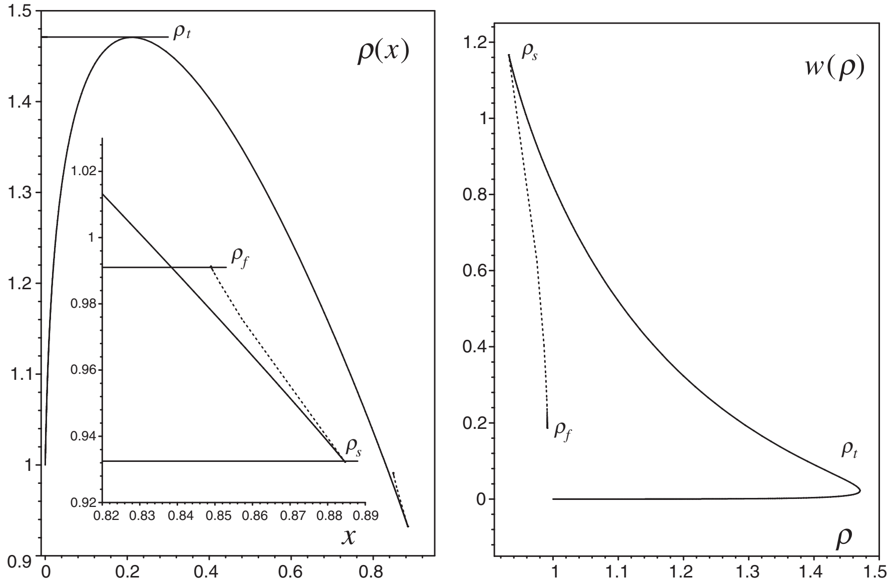

FIG. 3. The metric functions $\rho(x)$, $w(\rho)$, $(x = r - r_H)$ for the value of the dilaton coupling constant $a > a_{\rm cr}$. The function ρ has a turning point $r = r_t$ after which the naked singularity is met at a finite affine distance $(r = r_s)$. Dotted lines correspond to another solution branch which can be matched in the singularity. Numerical curves are presented for $a = 0.5$ and $\rho_0 = 1$, the corresponding value of P_1 being $P_1 = -7.746$.

$$\begin{aligned}\Delta^3\{8w_s\rho_s^4[w_s\rho_1^2(15\gamma+9)-1]\} - \Delta^2 24 w_s\rho_s^5\rho_1(3\gamma+2) \\ + \rho_s\rho_1[q_e^2 32a^2P_s\gamma + \rho_s^2(48a^2P_s^2\gamma^2-\rho_s^4)] \\ + \Delta\{2\rho_s^2[\rho_s^4(\gamma+6)+96\rho_1^2w_sa^2P_s^2\gamma^2] \\ - q_e^2 32a^2P_s(3\gamma+4)\gamma\} = 0. \qquad (66)\end{aligned}$$

An expression for $\rho_{3/2}$ is too big and is not given here. Since the second derivatives are divergent at $y = 0$, the Riemann tensor diverges as well. The divergency is localized on a sphere of finite radius, and it is rather mild: Ricci and Kretchmann scalars behave as

$$R \simeq -\frac{3(4\rho_{3/2}w_s + w_{3/2}\rho_s)}{4\rho_s}\frac{1}{\sqrt{y}}, \qquad (67)$$
$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \simeq \frac{9(8\rho_{3/2}^2w_s^2 + \rho_s^2w_{3/2}^2)}{16\rho_s^2}\frac{1}{y}.$$

The radial coordinate stops at $r = r_s$, but using an appropriate desingularization of the system (see the appendix), one can glue another patch of radial coordinate $r' \in (r_s, r_f)$ at this point, extending the manifold through the singularity. This extension is shown by dotted lines in Figs. 3. It terminates at the final singularity r_f. This situation is very similar to that described in Ref. [46] for an interior region of the nonextremal DGB black hole.

D. Thermodynamics

The temperature of the extremal DGB black hole is zero, as for the extremal solution without the GB term:

$$T = \frac{1}{2\pi}\left(\sqrt{g^{rr}}\frac{\partial\sqrt{g_{tt}}}{\partial r}\right)\Bigg|_{r=r_H} = \frac{1}{2\pi\rho_0^2}(r-r_H)|_{r=r_H} = 0. \qquad (68)$$

To calculate the entropy we apply Sen's formula [39,40] appealing to the near-horizon data. Using (41) we can write the near-horizon solution as

$$ds^2 = -\frac{(r-r_H)^2}{\rho_0^2}dt^2 + \frac{\rho_0^2}{(r-r_H)^2}dr^2 + \rho_0^2 d\Omega_2^2, \qquad (69)$$
$$\phi = \frac{1}{2a}\ln\frac{\rho_0^2}{4\alpha}, \qquad F_{[2]} = \frac{2\sqrt{\alpha}}{\rho_0^2}dt\wedge dr.$$

To apply Sen's formula [39,40], we rewrite it as follows:

$$ds^2 = v_1\left(-\hat{r}^2d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2}\right) + v_2 d\Omega_2^2, \qquad (70)$$
$$\phi = u, \qquad F_{\hat{r}\hat{t}} = e,$$

where $\hat{r} = r - r_H$, $\hat{t} = t/\rho_0^2$.

Now introduce the surface integrated Lagrangian density

$$f(u, v_1, v_2, e) = \int d\theta d\phi\sqrt{g}L, \qquad (71)$$

and evaluate it using the near-horizon data (70)

$$f(u, v_1, v_2, e) = \frac{1}{2}\left(v_1 - v_2 - 4\alpha e^{2au} + e^2\frac{v_2}{v_1}e^{2au}\right). \qquad (72)$$

The entropy function F is the Legendre transform of this function with respect to e:

$$q = \frac{\partial f}{\partial e} = e\frac{v_2}{v_1}\mathrm{e}^{2au},$$
$$F = 2\pi[qe - f(u, v_1, v_2, e)] \qquad (73)$$
$$= \pi\left(v_2 - v_1 + 4\alpha\mathrm{e}^{2au} + e^2\frac{v_2}{v_1}\mathrm{e}^{2au}\right),$$

or, in terms of q:

$$F = \pi\left(v_2 - v_1 + 4\alpha\mathrm{e}^{2au} + q^2\frac{v_1}{v_2}\mathrm{e}^{-2au}\right). \qquad (74)$$

The entropy of the extremal black hole is given by the value of the entropy function F at extremality:

$$\frac{\partial F}{\partial u} = 0, \qquad \frac{\partial F}{\partial v_1} = 0, \qquad \frac{\partial F}{\partial v_2} = 0. \qquad (75)$$

In our case, the extremality conditions (75) read

$$\mathrm{e}^{-2au}q^2 - v_2 = 0, \qquad -v_2^2 + \mathrm{e}^{-2au}q^2 v_1 = 0,$$
$$-\mathrm{e}^{-2au}q^2 v_1 + 4\alpha\mathrm{e}^{2au}v_2 = 0, \qquad (76)$$

leading to the solution

$$v_1 = 2\sqrt{\alpha}q, \qquad v_2 = 2\sqrt{\alpha}q, \qquad \mathrm{e}^{2au} = \frac{q}{2\sqrt{\alpha}}, \qquad (q > 0). \qquad (77)$$

Comparing with the local solution in our previous notation, we get $q = q_e$. Finally, substituting (77) in F one obtains the entropy

$$S = 4\pi\sqrt{\alpha}q_e = 2\pi\rho_0^2, \qquad (78)$$

which is precisely twice the Bekenstein-Hawking value. This is similar to the result of Refs. [20,21,24].

IV. DISCUSSION

In this paper, we have shown that in addition to charged black holes with nondegenerate horizons, the DBG four-dimensional gravity admits black hole solutions with the horizons of the $\mathrm{AdS}_2 \times S^2$ type. These solutions form a one-parameter family and exist in a finite range of the dilaton coupling constant a. A new family of solutions branch is disconnected from the branch of nonextremal black holes which was studied earlier. Rather, it pinches off from the extremal Reissner-Nordström black hole which *is* a solution of the full Einstein-Maxwell-dilaton-Gauss-Bonnet (EMDGB) theory for $a = 0$. Starting with zero a, we were able to find global black hole solutions interpolating between $\mathrm{AdS}_2 \times S^2$ at the horizon and Minkowski vacuum at infinity for a below some critical value which was found numerically up to several decimals as $a_{\mathrm{cr}} \simeq 0.488\,219\,703$. Near the critical value $a \to a_{\mathrm{cr}}$, the mass and the dilaton charge grow up, while their ratio saturates the BPS bound of the EMD black holes. A similar feature was observed for the charged black holes in the Einstein-Born-Infeld-dilaton (EBID) theory [61].

It is worth noting that the family of electrically charged extremal black holes in the EMDGB theory is one-parametric (q_e), while the family of the corresponding extremal solutions in the EMD theory is two-parametric (with the parameters q_e and ϕ_∞). An asymptotic value of the dilaton is no more a free parameter when the Gauss-Bonnet term is included, moreover, the dilaton exponent $\mathrm{e}^{2a\phi_\infty}$ at the threshold $a = a_{\mathrm{cr}}$ tends to zero for any finite value of the charge q_e. Therefore, modification of the extremal dilaton black hole by higher-curvature term consists not only in stretching its horizon to a finite radius, but also in fixing the value of the dilaton at infinity.

Our model can be viewed as a truncated heterotic string effective theory in four dimensions. It does not include all quadratic curvature terms, but it still shares some features relevant to more complete models, in particular, it predicts a correct entropy for extremal black holes which is twice the Bekenstein-Hawking entropy. It does not exhibit the attractor property (apart form the limit $a = 0$), and the dilaton is not frozen, but varies from some finite value at the horizon to some different finite value at infinity. It is worth noting that we add the Gauss-Bonnet term to the EMD Lagrangian in the Einstein frame. An alternative model would be adding the same Gauss-Bonnet terms to the EMD Lagrangian in the string frame. This changes the system of equations significantly, and additional work is needed to see whether this second model leads to similar results. We currently investigate this issue in more detail.

The existence of the threshold value of the dilaton coupling constant is an interesting new phenomenon which may be related to string-black hole transition as described in [62].[2] We think that the present model as well as the EBID model [61] (both bringing typical string features to gravity) can be regarded as simple toy models describing string–black hole transition. A more realistic model of heterotic string theory compactified to four dimensions by $S^1 \times T^5$ [62] does not contain a dilaton coupling constant as an order parameter, instead the role of such a parameter can be played by α'. This model contains two vector and two scalar fields, and the existing attempts to investigate it numerically are insufficient, in our opinion, to draw a conclusion as to whether it leads to a similar behavior indeed. We expect to present new results on this subject in the near future.

ACKNOWLEDGMENTS

The authors thank Soo-Jong Rey and Hideki Maeda for helpful discussion and Miguel Costa for useful correspondence. D. V. G. thanks the Department of Physics of NCU for hospitality and National Center of Theoretical Sciences

[2] We thank Miguel Costa for indicating this to us and bringing the paper [62] to our attention.

and (NCU) Center for Mathematics and Theoretic Physics for support during his visit in January 2007. The work was also supported in part by RFBR Grant No. 02-04-16949. C. M. C. and D. G. O. were supported by the National Science Council of the R.O.C. under Grant No. NSC 95-2112-M-008-003. C. M. C. was supported in part by the National Center of Theoretical Sciences and (NCU) Center for Mathematics and Theoretic Physics.

APPENDIX: DESINGULARIZATION AT THE TURNING POINT

Here we clarify the numerical procedure which allows us to continue solutions through the singular points. Rewrite the system (9)–(12) as a matrix equation of the first order

$$A\frac{d}{dr}X = B, \tag{A1}$$

where X is the six-dimensional vector consisting of the primary dynamical variables w, ρ, $e^{a\phi}$ and their first derivatives with respect to the radial coordinate. The system (A1) has a regular solution provided $\det A \neq 0$. When the solution approaches some point r_s where $\det A \rightarrow 0$, the derivative X' diverges as $O(1/\det A)$. In order to continue the solution through this point, we choose a new independent variable σ satisfying the condition

$$\dot{r}(\sigma) = \frac{dr}{d\sigma} \propto \det A. \tag{A2}$$

Then in terms of σ the matrix equation (A1) can be rewritten in the regular form

$$A\dot{X} - B\dot{r} = 0. \tag{A3}$$

This desingularization is achieved by extending the set of unknown functions to seven, considering the radial coordinate as a function $r(\sigma)$. Denoting the seven-vector $[X(\sigma), r(\sigma)]$ as $Y(\sigma)$, one can see that the tangent vector has the unit Euclidean metric norm

$$\left|\frac{dY}{d\sigma}\right| = 1, \tag{A4}$$

provided the Eq. (A3) holds:

$$|\dot{r}| = |\det A|[\det A^2 + (X' \det A)^2]^{-(1/2)}. \tag{A5}$$

Using this desingularization one can continue the solution through the singular turning point. This procedure is similar to one used in [63]. Geometrically this means gluing another coordinate patch to the solution at singularity.

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