PHYSICAL REVIEW D **75,** 084027 (2007)

Uniqueness of the Fock quantization of the Gowdy T^3 model

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(Received 23 February 2007; published 16 April 2007)

After its reduction by a gauge-fixing procedure, the family of linearly polarized Gowdy T^3 cosmologies admits a scalar field description whose evolution is governed by a Klein-Gordon type equation in a flat background in $1+1$ dimensions with the spatial topology of S^1, though in the presence of a time-dependent potential. The model is still subject to a homogeneous constraint, which generates S^1-translations. Recently, a Fock quantization of this scalar field was introduced and shown to be unique under the requirements of unitarity of the dynamics and invariance under the gauge group of S^1-translations. In this work, we extend and complete this uniqueness result by considering other possible scalar field descriptions, resulting from reasonable field reparametrizations of the induced metric of the reduced model. In the reduced phase space, these alternate descriptions can be obtained by means of a time-dependent scaling of the field, the inverse scaling of its canonical momentum, and the possible addition of a time-dependent, linear contribution of the field to this momentum. Demanding again unitarity of the field dynamics and invariance under the gauge group, we prove that the alternate canonical pairs of fieldlike variables admit a Fock representation if and only if the scaling of the field is constant in time. In this case, there exists essentially a unique Fock representation, provided by the quantization constructed by Corichi, Cortez, and Mena Marugán. In particular, our analysis shows that the scalar field description proposed by Pierri does not admit a Fock quantization with the above unitarity and invariance properties.

DOI: 10.1103/PhysRevD.75.084027 PACS numbers: 04.62.+v, 04.60.Ds, 98.80.Qc

I. INTRODUCTION

The quantization of symmetry reduced models in general relativity has been intensively studied as a tool to learn about conceptual and technical issues in quantum gravity. Specially relevant is the quantization of gravitational models with local degrees of freedom, the so-called ''midisuperspaces,'' since they retain the field character of general relativity.

In quantum cosmology, the relevance of midisuperspaces is strengthened by the fact that, in the absence of a full quantum theory of gravity, their analysis provides the most solid way to validate or derive a consistent quantum treatment of the cosmological inhomogeneities. Remarkably, the only midisuperspace model whose quantization has been studied with sufficient detail in cosmology is the family of linearly polarized Gowdy spacetimes with spatial topology of a three-torus T^3 [1–11].

Gowdy spacetimes are vacuum spacetimes that possess two spacelike and commuting Killing vectors and whose spatial sections are compact [12]. Gowdy proved that any spacetime with these properties must have spatial sections that are homeomorphic to a three-torus, a three-handle $S^1 \times S^2$, or a three-sphere S^3 (or to a manifold covered by one of the above). The case of the three-torus is particularly interesting. All classical solutions to general relativity *start* then in a spacelike singularity where the area of the two-dimensional orbits of the Killing isometries vanishes. Furthermore, this area increases monotonously in the evolution, so that one can adopt it as time coordinate. In fact (apart from convenient normalization factors) this is the standard choice of time gauge in the description of the Gowdy T^3 cosmologies [4]. The condition of linear polarization, on the other hand, implies that each Killing vector is hypersurface orthogonal, and eliminates one of the two local physical degrees of freedom of the gravitational field.

By means of a dimensional reduction employing one of the Killing vectors, the family of linearly polarized Gowdy T^3 spacetimes (that we will call Gowdy cosmologies or Gowdy model from now on) are classically equivalent to an axisymmetric massless scalar field propagating on a gravitational background in $2+1$ dimensions. Then, a quantization of this scalar field provides essentially a quantum theory for the Gowdy model. This was the procedure followed by Pierri to construct a Fock quantization of the Gowdy cosmologies [5]. However, it was soon pointed out that this quantization is not fully satisfactory:

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the classical evolution of the scalar field cannot be implemented as a quantum unitary transformation [6–8]. To recover a unitary dynamics, a different choice of ''fundamental field'' for the Gowdy model and a suitable Fock quantization of it was recently proposed [9,10]. Actually, this new choice of the scalar field is the result of a different parametrization of the metric of the Gowdy cosmologies [10]. In the following, we will refer to the field parametrization of the metric, Fock representation and quantization of the reduced Gowdy model introduced in Refs. [9,10] and later elaborated in Ref. [11] by Corichi, Cortez, Mena Marugán, and Velhinho as the CCMV ones.

In the quantization process that leads to the CCMV representation for the Gowdy model, there are three steps where one performs choices whose modification might result in an inequivalent Fock quantization [10]. First, there is the choice of gauge that allows one to eliminate most of the constraints of the model and reduce the system. Second, one makes a choice of parametrization for the reduced Gowdy metric that determines which scalar field is considered as fundamental. Finally, one has to make a choice of quantum representation for this scalar field, as systems with fieldlike degrees of freedom generally admit inequivalent quantizations. For Fock representations, this amounts to an ambiguity in the selection of the ''one-particle'' Hilbert space, which is fixed by a choice of complex structure (see e.g. Refs. [13–15]). The aim of the present work is to demonstrate that, if the choice of gauge for the Gowdy cosmologies is fixed, the CCMV quantization for the resulting reduced model is essentially unique under a set of natural requirements. This will complete previous work presented in Ref. [11] which already proves the uniqueness of the Fock representation with respect to the choice of complex structure. We will extend that analysis to take into account different parametrizations of the reduced Gowdy metric which select distinct scalar fields as the fundamental object to be quantized.

Several reasons justify the importance of this result. On the one hand, the obtained uniqueness guarantees that the physics of the quantum cosmological model does not depend on the choice of parametrization or the particular Fock representation selected, providing significance to the predictions. On the other hand, even if one adopted a distinct kind of quantization, not necessarily equivalent to a Fock one, like e.g. a polymerlike quantization [16], there ought to exist a regime in which a Fock quantum theory were recovered. This condition can hardly be used to control the acceptable quantizations of the system unless one can specify such a Fock representation. Finally, the result has a conceptual interest by itself, since it shows that it is possible to attain uniqueness even for nonstationary systems and in the framework of standard quantum field theory.

The paper is organized as follows. In Sec. II we first pose the problem, discussing the freedom available in the choice of the scalar field as the basic fieldlike variable for the reduced Gowdy model. Under reasonable demands, this freedom consists just in time-dependent canonical transformations in the reduced phase space that scale the field by a positive function of time and its momentum by the inverse factor. In addition, the momentum is allowed to get a linear contribution of the field, with a time-dependent coefficient. Section III reviews the CCMV quantization of the reduced Gowdy model. We then analyze in Secs. IV and V the alternate Fock quantizations obtained by adopting different choices of fundamental field. In addition to a unitary implementation of the dynamics, we demand that these representations satisfy a natural condition concerning the only remaining constraint of the reduced model. Namely, we require invariance under the corresponding gauge group. The proof that all such quantizations are equivalent to the CCMV one is presented in Secs. VI and VII. Finally, we conclude and summarize our results in Sec. VIII. Two appendixes are added. In Appendix A we explain some calculations employed in our uniqueness proof. Appendix B proposes a criterion to fix the linear contribution of the field to the momentum.

II. THE CONTEXT

Let us start with the metric of the Gowdy spacetimes in coordinate systems adapted to the two axial Killing vector fields, so that these are identified as ∂_σ and ∂_δ for certain coordinates σ, $\delta \in S^1$. Given the hypersurface orthogonality of these Killing vectors, the induced metric can be parametrized in terms of three fields that depend only on the (nonnegative) time coordinate $t > 0$ and one spatial coordinate $\theta \in S^1$ [8]. These fields describe the norm of one of the Killing vectors (e.g. ∂_δ), the area of the orbits of the group of isometries, and the scale factor of the metric induced on the set of group orbits. For instance, the parametrization used by CCMV in Ref. [10] is

$$ds^2 = e^{\bar{\gamma}-(\xi/\sqrt{\tau})-\xi^2/(4\tau)}(-\tau^2\underset{\sim}{N}^2dt^2 + [d\theta + N^\theta dt]^2) + \tau^2 e^{-\xi/\sqrt{\tau}}d\sigma^2 + e^{\xi/\sqrt{\tau}}d\delta^2, \quad (1)$$

where $\underset{\sim}{N}$ is the densitized lapse function, N^θ the nonvanishing component of the shift vector, and the three fields that parametrize the metric are ξ, $\tau > 0$, and $\bar{\gamma}$.

The model is subject to the θ-diffeomorphisms and Hamiltonian constraints. The standard gauge fixing for the θ-diffeomorphisms imposes the homogeneity of the phase-space variable that generates conformal transformations of the metric induced on the set of group orbits. Actually, this condition fixes only the inhomogeneous part of the θ-diffeomorphisms constraint. The homogeneous part, C_0, which generates S^1-translations, remains as a constraint on the system. After a partial reduction, the phase-space variable used in our gauge-fixing condition and its canonically conjugate variable are determined except for their zero modes. These zero modes are described

by a pair of canonically conjugate homogenous variables (Q, P) [8,10]. Finally, the system is deparametrized by choosing as time coordinate the area of the orbits of the isometry group, apart from a proportionality factor. The subsequent reduction leads to a system whose degrees of freedom correspond just to one scalar field plus the "point-particle" canonical pair (Q, P), and that is subject to the constraint C_0. With a convenient selection of the proportionality factor in our choice of time, the dynamics of the field sector can be decoupled from the homogenous pair (Q, P). These two homogenous variables are in fact constants of motion. Finally, both with the CCMV parametrization of the metric or with the one adopted in Refs. [5,8], the field dynamics is given by a Klein-Gordon type equation that is invariant under S^1-translations, though explicitly time dependent.

Let us describe the reduced model in more detail, e.g. for the CCMV parametrization (1). The gauge-fixing conditions are then $P_{\bar{\gamma}} = \oint d\theta P_{\bar{\gamma}}/(2\pi) := -e^P$ (restricted to solutions with $P \in \mathbb{R}$) and $\tau = te^P$ (time gauge). Here, $P_{\bar{\gamma}}$ is the momentum canonically conjugate to $\bar{\gamma}$. The reduced metric, expressed in the CCMV parametrization, is obtained from Eq. (1) with $\tau = te^P$ and [10]

$$2\pi\bar{\gamma}e^P = -Q - i\sum_{n=-\infty, n\neq 0}^{\infty} \oint d\bar{\theta}\frac{e^{in(\theta-\bar{\theta})}}{n} P_\xi \xi' + tH, \tag{2}$$

where P_ξ is the canonical momentum of the remaining field ξ, the prime stands for the derivative with respect to θ, and H is the (reduced) Hamiltonian that generates the evolution [17]:

$$H = \frac{1}{2}\oint d\theta\left[P_\xi^2 + (\xi')^2 + \frac{1}{4t^2}\xi^2\right]. \tag{3}$$

The associated field equation is

$$\ddot{\xi} - \xi'' + \frac{\xi}{4t^2} = 0, \tag{4}$$

with the derivative with respect to t denoted by a dot. Finally, the only constraint of the reduced system is

$$C_0 = \frac{1}{\sqrt{2\pi}}\oint d\theta P_\xi \xi' = 0. \tag{5}$$

Of course, when parametrizing the reduced metric in terms of a scalar field and taking this object as the variable to be quantized, one is introducing a choice of (part of the) set of basic variables. However, the scalar field parametrization of the reduced metric of the Gowdy model is certainly not unique. In this sense, there is no scalar field theory canonically associated with the gauge-fixed Gowdy model, but rather an infinity of them.

Nonetheless, it is most reasonable to consider only field parametrizations satisfying certain amenable properties. We discuss now the class of field parametrizations analyzed in this work, determined by a set of natural requirements. For definiteness, we take the CCMV parametrization as the reference one, and express alternate parametrizations in terms of it. First, we consider exclusively scalar fields which provide a local and (explicitly) coordinate-independent parametrization of the norm of the Killing vector ∂_δ on each section of constant time in the reduced model [possibly together with the variables that describe the point-particle degree of freedom]. In this way, the allowed field reparametrizations are local and commute both with the isometry group and with the gauge group of translations in $\theta \in S^1$. In particular, this guarantees that the corresponding field dynamics is local and θ-independent, so that the invariance under S^1-translations is preserved. Besides, the second-order field equation should be kept linear and homogenous, so that the space of solutions remains a linear space. Finally, it is convenient to preserve the decoupling between the field-like and point-particle degrees of freedom (see however our comments below). With these premises, the possible field redefinitions in the reduced Gowdy model consist in scalings of the field ξ by a function depending exclusively on time [18].

Therefore, from now on we will concentrate our discussion on time-dependent scalings of the field in the reduced model. This type of scalings can always be completed into a time-dependent canonical transformation in the reduced phase space. The canonical momentum of the field suffers the inverse scaling. We will also allow for a linear contribution of the field to the new momentum, with a time-dependent coefficient. This contribution is local, preserves the decoupling with the point-particle degrees of freedom, and is compatible with all linear structures on phase space, as well as with S^1-translation invariance.

It is not difficult to check that the process of first fixing the gauge and then performing one of the above time-dependent canonical transformations is equivalent to carrying out first a time-independent canonical transformation in the unreduced phase space (with the role of time coordinate played by the corresponding internal time variable) and afterwards the gauge fixing. One could further ask whether these canonical transformations in the unreduced phase space correspond just to field reparametrizations of the unreduced metric (without including the momenta). This will be the case only if the transformation is a contact one in the unreduced configuration space of metric fields. However, taking Eq. (1) as reference, we see that scalings by functions F of τe^{-P} (the internal time) will depend on the momentum variable P unless the scaling is trivial. Nonetheless, notice that the Klein-Gordon Eq. (4) obtained after reduction is not modified if the field is multiplied by a function of P, which is a constant of motion. If we allow for this kind of multiplication, the change of field can be regarded as a contact transformation in the unreduced configuration space if and only if there exists a function

L such that $L(P)F(\tau e^{-P})$ is independent of P. This happens only if $F(\tau) = \tau^a$ for a certain power $a \in \mathbb{R}$ [then $L(P) = e^{aP}$]. In fact, this occurs in the case of the field parametrization employed in Ref. [5], which can be obtained with $F(\tau) = 1/\sqrt{\tau}$ (see Appendix A in Ref. [10]). Multiplication by the function $L(P)$ leaves, nevertheless, a trace in the reduced Hamiltonian [see e.g. Eq. (3)], so that it actually couples the dynamics of the fieldlike and the point-particle degrees of freedom of the reduced model [19].

Summarizing, for canonical transformations in the reduced phase space that scale the field by a power of the time coordinate, and only for them, the transformation can be understood as the result of a change of field parametrization of the unreduced metric followed, after reduction, by multiplication by a constant of motion in order to decouple the fieldlike and the point-particle degrees of freedom. We will however maintain the generality of our analysis and consider all reasonable scalar field parametrizations of the reduced metric, so that we will not restrict our discussion to this specific subfamily of scalings. In fact, we will see that our results do not depend on whether one imposes or not this restriction.

Owing to the time dependence of the considered canonical transformations in the reduced phase space, the choice of fundamental scalar field can have a large impact on the quantization of the reduced Gowdy model. In fact, since two candidate fields are related by a time-dependent scaling, the evolution of both sets of variables is effectively different. It may then happen that, upon quantization, the dynamics of one of the fields admits a unitary implementation, whereas the dynamics of the other does not. Note also that, if one declares a certain field description to be fundamental, a quantization based on another field (related to the first one by a time-dependent transformation) can be seen as a quantization of the fundamental field using seemingly awkward time-dependent variables, instead of the natural field variables. However, in the context of the Gowdy model there is *a priori* an inherent freedom to choose the field parametrization of the reduced metric. Thus, any proposal to single out a field parametrization should be based on criteria such as the feasibility of the quantization and its consistency.

Given the central role that the unitarity of the evolution plays in the quantum theory (particularly within the Hilbert space approach), it is certainly desirable that the selected field parametrization allows, upon quantization, a unitary implementation of the classical evolution of the scalar field. As we have commented, the CCMV formulation admits a Fock quantization that satisfies this condition. Besides, the remaining constraint in the scalar field theory C_0 is naturally quantized. So, the outcome of Refs. [9,10] is a consistent, rigorous quantization of the gauge-fixed Gowdy cosmologies with unitary evolution.

An important issue, related to the question of unitarity of the dynamics, is the uniqueness of the quantum theory. For the reduced Gowdy model with the CCMV parametrization, it has been demonstrated that the proposed Fock quantization is indeed unique, under the following conditions on the quantum representation of the scalar field [11]. First, one demands a unitary implementation of the classical evolution. Second, one asks for a natural invariant implementation of the constraint C_0, in the sense that the Fock state—or the complex structure—that defines the field representation is required to be invariant under the gauge group of S^1-translations generated by the constraint [20]. The CCMV representation satisfies these conditions and it turns out that any representation which does so is unitarily equivalent to it. Thus, as long as the field parametrization of the reduced Gowdy model is fixed (and the invariance condition is fulfilled), the requirement of unitary dynamics selects a unique Fock quantization.

In the present work we will considerably deepen this uniqueness result by showing that it is maintained when the alternate scalar field parametrizations of the reduced Gowdy model discussed above are allowed. In principle, it might happen that a unitary dynamics could be achieved in a certain Fock quantization of some different field description, and that the new quantum theory be physically distinct from the CCMV one. For instance, this would occur if the quantum operators corresponding to the CCMV scalar field in the new description failed to define a representation equivalent to that introduced in Refs. [9,10]. We will show that this is not the case: for any scalar field parametrization, if a Fock representation exists satisfying the unitary implementability of the corresponding dynamics and the invariance under the gauge group of S^1-translations, it is guaranteed that the evolution of the CCMV field is well defined and unitary in the new description. By the results of Ref. [11], the representation is then the same as that of Refs. [9,10] (modulo unitary equivalence).

III. CCMV QUANTIZATION

We will now briefly review the scalar field formulation of the reduced Gowdy model obtained in Refs. [9,10] and its proposed quantization. We obviate the point-particle degrees of freedom because, being finite in number, they play no role in the discussion of the uniqueness of the quantization. For the same reason, we also obviate the homogeneous mode of the field (see below).

We remember that the fieldlike degrees of freedom of the reduced model are described in the CCMV parametrization by the field ξ and its momentum P_ξ. Its dynamics is governed by the time-dependent Hamiltonian (3), which is invariant under the group of S^1-translations:

$$T_\alpha\colon \theta \mapsto \theta + \alpha \quad \forall\ \alpha \in S^1. \tag{6}$$

These translations are gauge symmetries of the reduced model, generated by the only constraint that remains on the system, namely C_0.

Taking into account that the canonical fields ξ and P_ξ are periodic in θ, we can expand them in Fourier series:

$$\xi(\theta, t) = \sum_{n=-\infty}^{\infty} \xi_n(t) \frac{e^{in\theta}}{\sqrt{2\pi}},$$
$$P_\xi(\theta, t) = \sum_{n=-\infty}^{\infty} P_\xi^n(t) \frac{e^{in\theta}}{\sqrt{2\pi}}. \tag{7}$$

The Fourier coefficients $\xi_n(t)$ and $P_\xi^{-n}(t)$ are canonically conjugate pairs of variables, which alternatively describe the degrees of freedom of the system. Since the field ξ and its momentum are real, the Fourier coefficients satisfy the reality conditions $\xi_n^*(t) = \xi_{-n}(t)$ and $[P_\xi^n(t)]^* = P_\xi^{-n}(t)$. Here, the symbol $*$ denotes complex conjugation.

As we mentioned, we will obviate the zero modes of these fields for simplicity. To describe all other modes we introduce the set of variables

$$b_m(t) := \frac{m\xi_m(t) + iP_\xi^m(t)}{\sqrt{2m}},$$
$$b_{-m}^*(t) := \frac{m\xi_m(t) - iP_\xi^m(t)}{\sqrt{2m}}, \tag{8}$$

together with their respective complex conjugate $b_m^*(t)$ and $b_{-m}(t)$, where $m \in \mathbb{N}$ is any strictly positive integer. Besides, we will assemble them in the column vectors

$$B_m(t) := (b_m(t), b_{-m}^*(t), b_{-m}(t), b_m^*(t))^T. \tag{9}$$

The symbol T denotes the transpose.

The variables $\{B_m(t)\}$ simply acquire a phase under the action of translations T_α:

$$b_{\pm m}(t) \mapsto e^{\pm im\alpha} b_{\pm m}(t), \qquad b_{\mp m}^*(t) \mapsto e^{\pm im\alpha} b_{\mp m}^*(t).$$

The classical evolution of the system is expressed in terms of these variables as follows [9]. Evolution from data $\{B_m(t_0)\}$ at a certain instant of time t_0 to $\{B_m(t)\}$ at a different time t is given by a classical evolution operator $U(t, t_0)$, which, for these variables, takes the block-diagonal form

$$B_m(t) = U_m(t, t_0) B_m(t_0), \qquad U_m(t, t_0) = W(x_m) W(x_m^0)^{-1}, \tag{10}$$

with $x_m := mt$, $x_m^0 := mt_0$, and

$$W(x) = \begin{pmatrix} \mathcal{W}(x) & \mathbf{0} \\ \mathbf{0} & \mathcal{W}(x) \end{pmatrix},$$
$$\mathcal{W}(x) = \begin{pmatrix} c(x) & d(x) \\ d^*(x) & c^*(x) \end{pmatrix},$$
$$d(x) := \sqrt{\frac{\pi x}{8}} \left[\left(1 + \frac{i}{2x}\right) H_0^*(x) - iH_1^*(x) \right], \tag{11}$$
$$c(x) := \sqrt{\frac{\pi x}{2}} H_0(x) - d^*(x).$$

Here, $\mathbf{0}$ is the zero 2×2 matrix and H_j ($j = 0, 1$) is the jth order Hankel function of the second kind [21]. Since $|c(x)|^2 - |d(x)|^2 = 1$, the map defined by $U_m(t, t_0)$ is a Bogoliubov transformation.

One can check that the evolution matrices $U_m(t, t_0)$ are then block diagonal in 2×2 blocks [as $W(x)$ above], with the two diagonal blocks being equal to the same 2×2 matrix $\mathcal{U}_m(t, t_0)$:

$$\mathcal{U}_m(t, t_0) := \begin{pmatrix} \alpha_m(t, t_0) & \beta_m(t, t_0) \\ \beta_m^*(t, t_0) & \alpha_m^*(t, t_0) \end{pmatrix},$$
$$\alpha_m(t, t_0) := c(x_m) c^*(x_m^0) - d(x_m) d^*(x_m^0), \tag{12}$$
$$\beta_m(t, t_0) := d(x_m) c(x_m^0) - c(x_m) d(x_m^0).$$

For further calculations we also note that, from Eq. (11) and the asymptotic behavior of the Hankel functions [21], the functions $d(x)$ and $c(x) - e^{i\pi/4} e^{-ix}$ tend to zero in the limit $x \to \infty$. In particular, it then follows that, for every fixed t_0 and t, the sequences $\{\beta_m(t, t_0)\}$ and $\{\alpha_m(t, t_0) - e^{-im(t-t_0)}\}$ vanish in the limit $m \to \infty$.

The CCMV quantization of the reduced Gowdy model is defined by using a representation for ξ on a fiducial Fock space which allows a unitary implementation of the dynamics as well as of the group of S^1-translations [9,10]. This quantization is of the Fock type, i.e. it is defined by a Hilbert space structure in phase space (or in the space of smooth solutions), which in turn is uniquely determined by a complex structure. The resulting Hilbert space is the so-called one-particle space, from which the quantum Fock space is constructed (see e.g. [14,22]).

The procedure to introduce this quantization is the following. We first fix, once and for all, a reference time t_0 and identify the phase space as the space of Cauchy data at $t = t_0$, expressed e.g. by the linear combinations of Fourier components $\{B_m(t_0)\}$ defined by Eqs. (8) and (9) for $t = t_0$. In order to simplify the notation, we will denote $\{B_m(t_0)\}$ simply as $\{B_m\}$ from now on, understanding the evaluation at the reference time t_0. Thus, the fields at the instant t_0 play in our case the same role as the time-zero fields in standard quantum field theory in Minkowski spacetime. However, owing to the compactness of the spatial manifold S^1, the quantum counterparts of the Fourier components need not be smeared in Fourier space, i.e. one obtains well defined operators $\hat{\xi}_n(t_0)$ and $\hat{P}_\xi^n(t_0)$ (satisfying the reality conditions), as well as $\hat{b}_m$, $\hat{b}_{-m}$ and their corresponding adjoints (like for their classical counterpart $\{B_m\}$, evaluation at $t = t_0$ is implicitly understood for these operators in the following). The two sets of operators are of course related like in Eq. (8) for $t = t_0$. On the other hand, operators like $\hat{\xi}(\theta, t_0)$ remain formal, with a well defined meaning assigned only to appropriately smeared fields.

The complex structure J_0 selected in Refs. [9,10] to carry out the quantization takes the form of a block-diagonal matrix in the basis $\{B_m\}$, with 4×4 blocks

$(J_0)_m = \mathrm{diag}(i, -i, i, -i)$. With this choice of complex structure, the variables $\{B_m\}$ are quantized as the annihilation and creation operators of the Fock representation. Then the corresponding Fock vacuum is characterized by the conditions $\hat{b}_m|0\rangle = \hat{b}_{-m}|0\rangle = 0 \quad \forall\ m \in \mathbb{N}$ [23]. Besides, the invariance of the complex structure J_0 under the group of S^1-translations (6) guarantees an invariant unitary implementation for this gauge group. One thus obtains unitary operators $\hat{T}_\alpha\ \forall\ \alpha \in S^1$ which leave the vacuum invariant and satisfy $\hat{T}_\alpha^{-1}\hat{b}_m\hat{T}_\alpha = e^{im\alpha}\hat{b}_m$ (with similar actions on $\hat{b}_{-m}$ and $\hat{b}^*_{\pm m}$). Therefore the vacuum is annihilated by the generator of the unitary group $\hat{T}_\alpha$, which is the quantum constraint operator.

Most importantly, the classical dynamics turns out to admit also a unitary implementation in this Fock representation [9,10]. At this stage, it is worth recalling that, given a Fock space defined by a complex structure J, one can obtain on it a unitary implementation of a symplectic transformation A if and only if its antilinear part $A_J = (A + JAJ)/2$ is Hilbert-Schmidt on the one-particle Hilbert space (determined by J) or, equivalently, if and only if $J - AJA^{-1}$ is Hilbert-Schmidt [24]. In the case of the family of symplectic transformations defined by the classical dynamics, namely, the linear transformations $U(t, t_0)$ for each t, and considering the CCMV representation for the reduced Gowdy model, the Hilbert-Schmidt condition for the existence of a unitary implementation amounts to demanding that $\sum_{m=1}^{\infty} |\beta_m(t, t_0)|^2$ be finite $\forall\ t > 0$. This square summability condition is indeed satisfied, as shown in Refs. [9,10]. Hence, there exist unitary operators $\hat{U}(t, t_0)$ such that, $\forall\ m \in \mathbb{N}$,

$$\hat{U}^{-1}(t, t_0)\hat{b}_m\hat{U}(t, t_0) = \alpha_m(t, t_0)\hat{b}_m + \beta_m(t, t_0)\hat{b}^\dagger_{-m},$$
$$\hat{U}^{-1}(t, t_0)\hat{b}^\dagger_{-m}\hat{U}(t, t_0) = \beta^*_m(t, t_0)\hat{b}_m + \alpha^*_m(t, t_0)\hat{b}^\dagger_{-m}.$$

To conclude, let us also remind the reader that two complex structures J and J' give rise to unitarily equivalent Fock representations if and only if $J - J'$ is a Hilbert-Schmidt operator on the one-particle Hilbert space defined by J (or J') (see e.g. Refs. [13,25]). We will say that such complex structures are equivalent, $J \sim J'$. Therefore, a symplectic transformation A is unitarily implementable on a Fock space defined by J if and only if the complex structures J and AJA^{-1} are equivalent.

IV. ALTERNATE FIELD FORMULATIONS

We will now start to investigate alternate quantizations of the reduced Gowdy model derived from other reasonable field parametrizations of the reduced metric. According to our discussion in Sec. II, we consider different scalar field formulations that are obtained by a time-dependent scaling of the CCMV field ξ. In the reduced phase space, the reformulation of the model is provided by a time-dependent canonical transformation of the type

$$\varphi := F(t)\xi, \qquad P_\varphi := \frac{P_\xi}{F(t)} + G(t)\xi, \tag{13}$$

where (φ, P_φ) are the new canonical fieldlike variables (to be considered now as fundamental), and $F(t)$ and $G(t)$ are real continuous functions on $\mathbb{R}^+$ (actually these functions should be differentiable, so that the differential formulation of the field theory is not spoiled). In order to avoid introducing spurious singularities, we require that $F(t)$ vanish nowhere. Hence the sign of this function is constant. As a particular example, the scalar field formulation of the reduced Gowdy model employed by Pierri [5] (and used afterwards in Refs. [6–8,26]) is related to the CCMV one by a canonical transformation of the above form with $F(t) = 1/\sqrt{t}$ and $G(t) = -1/(2\sqrt{t})$ [10].

Since the canonical pair (φ, P_φ) is obtained from (ξ, P_ξ) by a time-dependent transformation, the classical evolution of these pairs is different. Completing the Hamiltonian (3) with the time derivative (with respect to the explicit time dependence) of the generator of the canonical transformation (13) [27], one finds that the evolution of the canonical pair (φ, P_φ) is generated by the "new Hamiltonian" $H_\varphi = \oint d\theta \mathcal{H}_\varphi$ defined by the density

$$\begin{aligned}\mathcal{H}_\varphi &:= \frac{P_\xi^2}{2} + \frac{(\xi')^2}{2} + \frac{\dot{F}}{F}\xi P_\xi + \frac{\xi^2}{2}\left(\frac{1}{4t^2} - \dot{G}F + G\dot{F}\right)\\ &= \frac{P_\varphi^2 F^2}{2} + \frac{(\varphi')^2}{2F^2} + \frac{\dot{F} - GF^2}{F}\varphi P_\varphi\\ &\quad + \varphi^2\left\{\frac{G^2}{2} + \frac{1}{2F^2}\left(\frac{1}{4t^2} - \dot{G}F - G\dot{F}\right)\right\}.\end{aligned} \tag{14}$$

Here, we have not displayed the time dependence of $F(t)$ and $G(t)$ to simplify the notation.

As in the case of (ξ, P_ξ), we fix the reference time equal to t_0 and denote the classical evolution operator corresponding to the pair (φ, P_φ) by $\tilde{U}(t, t_0)$. In order to quantize the classical fieldlike variables (φ, P_φ), attaining a unitary implementation of the corresponding dynamics, we need to select a complex structure J_{t_0} (on the space of Cauchy data) at time $t = t_0$ such that

$$\tilde{U}(t, t_0)J_{t_0}\tilde{U}^{-1}(t, t_0) \sim J_{t_0}, \quad \forall\ t > 0. \tag{15}$$

Remembering that, for every symplectic transformation A, $J \sim J'$ if and only if $AJA^{-1} \sim AJ'A^{-1}$, one can express the unitary implementability condition (15) in the equivalent form

$$\tilde{U}(t, t')J_{t_0}\tilde{U}^{-1}(t, t') \sim J_{t_0}, \quad \forall\ t, t' > 0. \tag{16}$$

Similarly, defining $J_t := \tilde{U}(t, t_0)J_{t_0}\tilde{U}^{-1}(t, t_0)$, we have

$$J_{t'} = \tilde{U}(t', t)J_t\tilde{U}^{-1}(t', t) \sim J_t, \quad \forall\ t, t' > 0. \tag{17}$$

Condition (16), which was the unitary implementability condition explicitly used in Ref. [11], guarantees that the evolution between any two arbitrarily chosen times is

unitary with respect to J_{t_0} [28], whereas condition (17) states that the evolution provides a map between a family $\{J_t\}$ of equivalent complex structures. In the present work we follow the standard approach embodied by Eq. (15), which we take as the unitary implementation condition.

On the other hand, as we said in Sec. II, we will require that the complex structure J_{t_0} be invariant under the gauge group of S^1-translations (6). This is equivalent to considering only Fock representations for which this group belongs to the unitary group of the one-particle Hilbert space, ensuring an invariant unitary implementation of the gauge group, as in the case discussed in Sec. III. We will refer to such representations as translation invariant representations, or quantizations.

For the sake of conciseness, in the following we will discuss only canonical transformations of the type (13) such that the pairs (φ, P_φ) and (ξ, P_ξ) coincide at the fixed reference time t_0. In other words, we will study the case $F(t_0) = 1$ and $G(t_0) = 0$. It is not difficult to realize that this implies no loss of generality. In fact, any transformation of the form (13) can be decomposed as a time-dependent transformation which equals the identity at $t = t_0$, and an additional time-independent transformation with no impact on our discussion. To be precise, transformation (13) can be performed in the following two steps. First, we introduce the canonical pair:

$$\tilde{\xi} := f(t)\xi, \qquad P_{\tilde{\xi}} := \frac{P_\xi}{f(t)} + g(t)\xi, \tag{18}$$

with

$$f(t_0) = 1, \qquad g(t_0) = 0. \tag{19}$$

Second, the pair (φ, P_φ) is obtained from $(\tilde{\xi}, P_{\tilde{\xi}})$ by a time-independent transformation [29]:

$$\varphi = F(t_0)\tilde{\xi}, \qquad P_\varphi = \frac{P_{\tilde{\xi}}}{F(t_0)} + G(t_0)\tilde{\xi}. \tag{20}$$

It is clear that a quantization of the field theory described by the pair (φ, P_φ) is a quantization of the system associated with $(\tilde{\xi}, P_{\tilde{\xi}})$, and vice versa, since the relation between the two pairs is a local linear transformation with constant coefficients. In particular, the coefficients of transformation (20) are time-independent and θ-independent. Thus, given a translation invariant quantization corresponding to one of the pairs, with unitary dynamics, one immediately obtains a quantization with the same properties corresponding to the other pair. The quantum field operators for the two pairs are of course related by the straightforward quantum counterpart of Eq. (20), whereas the quantum evolution operators and translation operators are actually the same in both cases.

Thus, from now on we will analyze the consequences of demanding a unitary implementation of the dynamics for the pair $(\tilde{\xi}, P_{\tilde{\xi}})$, with respect to translation invariant Fock representations. After the derivation of the unitary implementability condition in explicit form, the proof of our uniqueness result will be split into two parts. We will first show that a unitary dynamics for the pair $(\tilde{\xi}, P_{\tilde{\xi}})$ can be achieved only if the function $f(t)$ in Eq. (18) is the constant unit function [equivalently, unitary dynamics for (φ, P_φ) is reached only if the function $F(t)$ in Eq. (13) is constant]. We will then prove the uniqueness of the quantization for those cases in which unitarity is attained.

Note that the CCMV quantization already provides a representation of the time t_0-fields corresponding to the pairs $(\tilde{\xi}, P_{\tilde{\xi}})$ and (φ, P_φ). Clearly, in the case of $(\tilde{\xi}, P_{\tilde{\xi}})$ the t_0-quantum fields coincide with the CCMV ones, whereas in the (φ, P_φ) case the (Fourier components of the) fields are related by the expressions $\hat{\varphi}_n(t_0) = F(t_0)\hat{\xi}_n(t_0)$ and $\hat{P}^n_\varphi(t_0) = [1/F(t_0)]\hat{P}^n_{\tilde{\xi}}(t_0) + G(t_0)\hat{\xi}_n(t_0)$, where $\hat{\xi}_n(t_0)$ and $\hat{P}^n_{\tilde{\xi}}(t_0)$ are the CCMV operators. We will see that the dynamics of the pair $(\tilde{\xi}, P_{\tilde{\xi}})$ with $f(t) = 1$ [or (φ, P_φ) with constant $F(t)$] is unitarily implementable in the CCMV representation. Most importantly, we will show that whenever the dynamics of $(\tilde{\xi}, P_{\tilde{\xi}})$ can be implemented unitarily, the corresponding translation invariant Fock representation also provides a unitary implementation of the dynamics of the pair (ξ, P_ξ), and is therefore unitarily equivalent to the CCMV representation by the results of Ref. [11].

V. UNITARITY CONDITION

Let us consider then the field description corresponding to the canonical pair $(\tilde{\xi}, P_{\tilde{\xi}})$ (18), with the real functions $f(t)$ and $g(t)$ satisfying conditions (19). Note that, given the continuity and nonvanishing of $f(t)$, we now have $f(t) > 0$ $\forall\ t > 0$.

As in the case of (ξ, P_ξ), we perform the Fourier decomposition (7) for the new canonical pair $(\tilde{\xi}, P_{\tilde{\xi}})$, and introduce corresponding variables $\{\tilde{B}_m(t)\}$, like in Eqs. (8) and (9). In agreement with our above remarks, we note that the set $\{\tilde{B}_m\} := \{\tilde{B}_m(t_0)\}$ coincides with $\{B_m\}$ since our canonical transformation is the identity at the reference time. Thus, the same kinematical variables $\{B_m\}$ are used in the quantization of the two field descriptions of the model, (ξ, P_ξ) and $(\tilde{\xi}, P_{\tilde{\xi}})$. The classical evolution in the new description is different, as we have commented. From the definition of $\{\tilde{B}_m(t)\}$ and Eqs. (18) and (19), one can check that the evolution matrices $\mathcal{U}_m(t, t_0)$ introduced in Eq. (12) are now replaced with

$$\tilde{\mathcal{U}}_m(t, t_0) = C_m(t)\mathcal{U}_m(t, t_0), \tag{21}$$

where

$$C_m(t) := \frac{1}{2}\begin{pmatrix} f_+(t) + i\frac{g(t)}{m} & f_-(t) + i\frac{g(t)}{m} \\ f_-(t) - i\frac{g(t)}{m} & f_+(t) - i\frac{g(t)}{m} \end{pmatrix}, \tag{22}$$

$$f_{\pm}(t) := f(t) \pm \frac{1}{f(t)}. \tag{23}$$

The matrices $C_m(t)$ actually describe the canonical transformation (18) in the variables $\{B_m(t)\}$, and so $C_m(t_0) = \mathbf{1}$.

A straightforward calculation shows that

$$\tilde{\mathcal{U}}_m(t, t_0) = \begin{pmatrix} \tilde{\alpha}_m(t, t_0) & \tilde{\beta}_m(t, t_0) \\ \tilde{\beta}_m^*(t, t_0) & \tilde{\alpha}_m^*(t, t_0) \end{pmatrix}, \tag{24}$$

with

$$2\tilde{\alpha}_m(t, t_0) := f_+(t)\alpha_m(t, t_0) + f_-(t)\beta_m^*(t, t_0) + i\frac{g(t)}{m}[\alpha_m(t, t_0) + \beta_m^*(t, t_0)], \tag{25}$$

$$2\tilde{\beta}_m(t, t_0) := f_+(t)\beta_m(t, t_0) + f_-(t)\alpha_m^*(t, t_0) + i\frac{g(t)}{m}[\alpha_m^*(t, t_0) + \beta_m(t, t_0)], \tag{26}$$

where α_m and β_m are defined in Eq. (12).

According to our previous comments, in order to achieve an admissible quantization of the fieldlike variables $(\tilde{\xi}, P_{\tilde{\xi}})$, one looks for complex structures J (at time t_0) that are invariant under S^1-translations and, $\forall t > 0$, lead to a unitary implementation of the evolution given by Eq. (24). At this point, one can employ a result proven in Ref. [11], namely, that every such invariant complex structure J is related to J_0 by a symplectic transformation, where J_0 is the complex structure used in the CCMV quantization of Refs. [9,10]. Explicitly, every invariant complex structure can be expressed as $J = K_J J_0 K_J^{-1}$, where K_J is block diagonal in the basis $\{B_m\}$, with 4×4 blocks of the form

$$(K_J)_m = \begin{pmatrix} (\mathcal{K}_J)_m & \mathbf{0} \\ \mathbf{0} & (\mathcal{K}_J)_m \end{pmatrix}, \qquad (\mathcal{K}_J)_m = \begin{pmatrix} \kappa_m & \lambda_m \\ \lambda_m^* & \kappa_m^* \end{pmatrix}, \qquad |\kappa_m|^2 = 1 + |\lambda_m|^2. \tag{27}$$

On the other hand, a symplectic transformation A admits a unitary implementation with respect to a complex structure $J = K_J J_0 K_J^{-1}$ if and only if $K_J^{-1} A K_J$ is unitarily implementable with respect to J_0. Thus, the condition for unitary implementation of the classical dynamics (24) is that the antilinear part of the symplectic transformation defined by the matrices

$$(\mathcal{K}_J)_m^{-1}\tilde{\mathcal{U}}_m(t, t_0)(\mathcal{K}_J)_m = (\mathcal{K}_J)_m^{-1} C_m(t)\mathcal{U}_m(t, t_0)(\mathcal{K}_J)_m$$

must be Hilbert-Schmidt in the Hilbert space defined by J_0, $\forall\ t > 0$. This in turn translates into the following square summability condition: the dynamics of the fieldlike variables $(\tilde{\xi}, P_{\tilde{\xi}})$ has a unitary implementation with respect to an invariant complex structure $J = K_J J_0 K_J^{-1}$ if and only if the sequence $\{\tilde{\beta}_m^J(t, t_0)\}$, with

$$\tilde{\beta}_m^J(t, t_0) := (\kappa_m^*)^2\tilde{\beta}_m(t, t_0) - \lambda_m^2\tilde{\beta}_m^*(t, t_0) + 2i\kappa_m^*\lambda_m \,\mathrm{Im}[\tilde{\alpha}_m(t, t_0)], \tag{28}$$

(where Im denotes the imaginary part) is square summable for all strictly positive t, i.e. if and only if the sum $\sum_{m=1}^{\infty} |\tilde{\beta}_m^J(t, t_0)|^2$ exists $\forall\ t > 0$.

VI. "NO-GO" RESULT FOR TIME-DEPENDENT SCALINGS

We will now prove that, for the sequence $\{\tilde{\beta}_m^J(t, t_0)\}$ to be square summable, it is necessary that the scaling function $f(t)$ in the transformation (18) be constant. As we will see, the square summability condition fails strongly otherwise, in the sense that $\tilde{\beta}_m^J(t, t_0)$ does not even go to zero $\forall\ t > 0$ when $m \to \infty$. We will therefore obtain $f(t) = f(t_0) = 1$ $\forall\ t > 0$ as a necessary condition for unitarity.

Let us consider the related sequence $\{\tilde{\beta}_m^J(t, t_0)/(\kappa_m^*)^2\}$. Since $|\kappa_m|^2 \geq 1$ by Eq. (27), we have

$$\left| \frac{\tilde{\beta}_m^J(t, t_0)}{(\kappa_m^*)^2} \right|^2 \leq |\tilde{\beta}_m^J(t, t_0)|^2, \quad \forall\ m \in \mathbb{N}, \quad \forall\ t > 0. \tag{29}$$

Therefore $\{\tilde{\beta}_m^J(t, t_0)/(\kappa_m^*)^2\}$ must be square summable whenever $\{\tilde{\beta}_m^J(t, t_0)\}$ is. In particular, a necessary condition for unitarity is that $\tilde{\beta}_m^J(t, t_0)/(\kappa_m^*)^2$ tend to zero in the limit $m \to \infty$, $\forall\ t > 0$. We will now analyze the consequences of this condition.

Using again Eq. (27), we conclude that $|\lambda_m/\kappa_m| \leq 1$ $\forall\ m \in \mathbb{N}$. Then, one can check from Eq. (28) that all the time-independent factors appearing in $\tilde{\beta}_m^J(t, t_0)/(\kappa_m^*)^2$ are bounded. On the other hand, since $|\alpha_m(t, t_0)|$ and $|\beta_m(t, t_0)|$ have well defined limits when $m \to \infty$ for every fixed t [see discussion below Eq. (12)], they also form bounded sequences for each $t > 0$. As a consequence, the contribution of the terms that contain $g(t)$ in Eqs. (25) and (26) [which provide $\tilde{\alpha}_m(t, t_0)$ and $\tilde{\beta}_m(t, t_0)$] are (at most) of order $1/m$. Hence, the corresponding contribution in $g(t)$ to $\tilde{\beta}_m^J(t, t_0)/(\kappa_m^*)^2$ is also of this order and thus tends to zero when $m \to \infty$. This means that, up to corrections of order $1/m$ that are negligible for large m, we can work with the approximation

$$\begin{aligned} 2\tilde{\alpha}_m(t, t_0) &\approx f_+(t)\alpha_m(t, t_0) + f_-(t)\beta_m^*(t, t_0), \\ 2\tilde{\beta}_m(t, t_0) &\approx f_+(t)\beta_m(t, t_0) + f_-(t)\alpha_m^*(t, t_0). \end{aligned} \tag{30}$$

Let us consider the dominant terms of these expressions when $m \to \infty$, and let us call them $\tilde{\alpha}_m^0$ and $\tilde{\beta}_m^0$. They can be easily deduced using that $\alpha_m - e^{-imT}$ and β_m tend to zero in this limit, where $T := t - t_0 > -t_0$. Thus,

$$2\tilde{\alpha}_m^0(t, t_0) = f_+(t)e^{-imT}, \qquad 2\tilde{\beta}_m^0(t, t_0) = f_-(t)e^{imT}. \tag{31}$$

Employing again that the time-independent coefficients

entering $\tilde{\beta}^J_m(t,t_0)/(\kappa^*_m)^2$ are bounded, we conclude that this sequence vanishes in the limit $m \to \infty$ if and only if so does the corresponding sequence obtained by replacing $\tilde{\alpha}_m$ and $\tilde{\beta}_m$ with $\tilde{\alpha}^0_m$ and $\tilde{\beta}^0_m$, namely, the sequence with elements

$$\frac{\tilde{\beta}^{0J}_m(t,t_0)}{(\kappa^*_m)^2} := \tilde{\beta}^0_m(t,t_0) - \frac{\lambda^2_m}{(\kappa^*_m)^2}\tilde{\beta}^{0*}_m(t,t_0) + 2i\frac{\lambda_m}{\kappa^*_m}\,\mathrm{Im}[\tilde{\alpha}^0_m(t,t_0)]. \tag{32}$$

Hence, as a necessary condition for a unitary implementation of the dynamics, $\tilde{\beta}^{0J}_m(t,t_0)/(\kappa^*_m)^2$ must tend to zero when $m \to \infty$.

By substituting expressions (31) for $\tilde{\alpha}^0_m$ and $\tilde{\beta}^0_m$, we then conclude that the following two real sequences (defined with $m \in \mathbb{N}$), which give the real and imaginary parts of $\tilde{\beta}^{0J}_m(t,t_0)/(\kappa^*_m)^2$, must vanish in the limit $m \to \infty\ \forall\ t > 0$:

$$\left(\mathrm{Im}\left[\frac{\lambda_m}{\kappa^*_m}\right]f_+(t) - \mathrm{Im}\left[\frac{\lambda^2_m}{(\kappa^*_m)^2}\right]\frac{f_-(t)}{2}\right)\sin(mT) + \left(1 - \mathrm{Re}\left[\frac{\lambda^2_m}{(\kappa^*_m)^2}\right]\right)\frac{f_-(t)}{2}\cos(mT), \tag{33}$$

and

$$\left(\left\{1 + \mathrm{Re}\left[\frac{\lambda^2_m}{(\kappa^*_m)^2}\right]\right\}\frac{f_-(t)}{2} - \mathrm{Re}\left[\frac{\lambda_m}{\kappa^*_m}\right]f_+(t)\right)\sin(mT) - \mathrm{Im}\left[\frac{\lambda^2_m}{(\kappa^*_m)^2}\right]\frac{f_-(t)}{2}\cos(mT). \tag{34}$$

Here, the symbol Re denotes the real part.

Actually, as we show in Appendix A, it is impossible that the imaginary part of $\tilde{\beta}^{0J}_m(t,t_0)/(\kappa^*_m)^2$ [given by Eq. (34)] tends to zero $\forall\ t > 0$, as required, if the time-independent coefficients of the cosine terms in the above expressions,

$$1 - \mathrm{Re}\left[\frac{\lambda^2_m}{(\kappa^*_m)^2}\right] \quad \text{and} \quad \mathrm{Im}\left[\frac{\lambda^2_m}{(\kappa^*_m)^2}\right], \tag{35}$$

tend to zero simultaneously on any subsequence $S \subset \mathbb{N}$ (i.e. for $m \in S \subset \mathbb{N}$). This places us in an adequate position to prove that a necessary condition for the dynamics of the fieldlike variables $(\tilde{\xi}, P_{\tilde{\xi}})$ to admit a unitary implementation with respect to some invariant complex structure is that the scaling function f be constant.

Let us start by taking $T = 2\pi q/p$, where q and p are arbitrary integers subject only to the condition that $2\pi q/p > -t_0$. For each fixed p, we then consider the subsequence $S_p := \{m = np, n \in \mathbb{N}\}$. Since the terms (33) and (34) tend to zero when $m \to \infty\ \forall\ T > -t_0$ (i.e. $\forall t > 0$), the same happens on each S_p for every q. Thus, taking into account that $\sin(2\pi nq) = 0$ and $\cos(2\pi nq) = 1$, one obtains that both

$$\left(1 - \mathrm{Re}\left[\frac{\lambda^2_{np}}{(\kappa^*_{np})^2}\right]\right)f_-\left(t_0 + \frac{2\pi q}{p}\right) \tag{36}$$

and

$$\mathrm{Im}\left[\frac{\lambda^2_{np}}{(\kappa^*_{np})^2}\right]f_-\left(t_0 + \frac{2\pi q}{p}\right) \tag{37}$$

must tend to zero as $n \to \infty$ for all possible values of p and q. However, since we know that the time-independent coefficients in these expressions cannot have simultaneously a zero limit on any subsequence S_p [see Appendix A], our conditions can only be fulfilled if $f_-(t_0 + 2\pi q/p)$ vanishes $\forall\ p, q$ or, equivalently, if

$$f^2\left(t_0 + \frac{2\pi q}{p}\right) = 1, \quad \forall\ q, p. \tag{38}$$

But, given that the set $\{t_0 + 2\pi q/p\}$ is dense on the half line of positive numbers and $f^2(t)$ is a continuous function, this implies that $f^2(t)$ must be the unit constant function. Using again the continuity of $f(t)$ and that $f(t_0) = 1$, we then see that $f(t)$ itself must be the unit function. This ends our proof.

In conclusion, we have shown that, with a (translation) invariant complex structure, no unitary implementation of the dynamics can be achieved unless transformation (18) is actually a simple redefinition of the momentum:

$$\tilde{\xi} = \xi, \qquad P_{\tilde{\xi}} = P_\xi + g(t)\xi. \tag{39}$$

Let us end the section with the following remark. If we now turn to the general field parametrization (13), it follows from our comments in Sec. IV that a necessary condition for a unitary implementation of the corresponding dynamics is that the function $F(t)$ in Eq. (13) be constant, $F(t) = F(t_0)\ \forall\ t > 0$ [30]. Thus, one can already conclude that the CCMV choice of fundamental field ξ for the reduced Gowdy model (and ignoring for the moment the choice of momentum) is essentially unique if a unitary dynamics is to be achieved. No time-dependent scaling of this field is allowed. In particular, this shows that the field version employed by Pierri [5] admits no unitary implementation of the dynamics with respect to any of all the possible invariant complex structures.

VII. EQUIVALENCE OF REPRESENTATIONS

We will now focus our discussion on the remaining transformations (39) and show that the dynamics of the fieldlike variables $(\tilde{\xi}, P_{\tilde{\xi}})$ is unitary if and only if so is the dynamics of (ξ, P_ξ).

For this unitarity, it is still necessary that expressions (33) and (34), now particularized to $f(t) = 1$ [that is, with $f_+(t) = 2$ and $f_-(t) = 0$], tend to zero when $m \to \infty$ for all strictly positive values of t. We then arrive at the necessary conditions

$$\mathrm{Im}\left[\frac{\lambda_m}{\kappa_m^*}\right]\sin(mT) \to 0, \tag{40}$$

$$\mathrm{Re}\left[\frac{\lambda_m}{\kappa_m^*}\right]\sin(mT) \to 0 \tag{41}$$

for every $T > -t_0$. Thus, in order to avoid the false conclusion that $\sin^2(mT)$ goes to zero on a subsequence of positive integers e.g. $\forall\ T \in [0, 2\pi]$ (like in the calculations explained in Appendix A), it is necessary that both $\mathrm{Im}[\lambda_m/\kappa_m^*]$ and $\mathrm{Re}[\lambda_m/\kappa_m^*]$ tend to zero. So, $|\lambda_m|^2/|\kappa_m|^2$ must vanish in the limit $m \to \infty$. Using Eq. (27), this means that $1/|\kappa_m|^2$ must approach the unit, what implies that the sequence $\{\kappa_m\}$ has to be bounded.

Let us then start again from Eq. (28), analyzing the original condition of square summability of $\{\tilde{\beta}^J_m\}$ required for the unitary implementation of the dynamics. Recalling again Eq. (27) and the fact that the sequence $\{\kappa_m\}$ is bounded, we see that all time-independent coefficients appearing in expression (28) for $\tilde{\beta}^J_m$ are bounded. In addition, from Eqs. (25) and (26) with $f(t) = 1$, we have

$$\tilde{\alpha}_m(t, t_0) = \alpha_m(t, t_0) + i\frac{g(t)}{2m}[\alpha_m(t, t_0) + \beta^*_m(t, t_0)],$$
$$\tilde{\beta}_m(t, t_0) = \beta_m(t, t_0) + i\frac{g(t)}{2m}[\alpha^*_m(t, t_0) + \beta_m(t, t_0)].$$

When the above expressions are introduced in Eq. (28) for $\tilde{\beta}^J_m$, one immediately sees that the contribution of terms in $g(t)$ are automatically square summable, owing to the fact that all terms proportional to $g(t)$ come with a factor of $1/m$, that the time-independent coefficients in $\tilde{\beta}^J_m$ are bounded, and that $\alpha_m(t, t_0)$ and $\beta_m(t, t_0)$ are also bounded $\forall\ t > 0$. The condition for a unitary dynamics is then the square summability of the remaining contribution to $\tilde{\beta}^J_m$, namely

$$\beta^J_m(t, t_0) := (\kappa^*_m)^2\beta_m(t, t_0) - \lambda^2_m\beta^*_m(t, t_0) + 2i\kappa^*_m\lambda_m\,\mathrm{Im}[\alpha_m(t, t_0)]. \tag{42}$$

But this term $\beta^J_m(t, t_0)$ is precisely the β-coefficient corresponding to the antilinear part of the classical evolution operator (12) for the canonical pair (ξ, P_ξ) with the choice of complex structure $J = K_J J_0 K_J^{-1}$ [11]. Therefore, the dynamics of the pair $(\tilde{\xi}, P_{\tilde{\xi}})$ is unitarily implementable with respect to an invariant complex structure if and only if the dynamics of the CCMV fieldlike variables (ξ, P_ξ) admits a unitary implementation with respect to the same structure. One can now invoke the results of Ref. [11], where it was proven that any invariant complex structure J which allows a unitary implementation of the dynamics of (ξ, P_ξ) provides a quantum representation which is unitarily equivalent to that determined by J_0, i.e. the CCMV representation constructed in Refs. [9,10].

Summarizing, we have demonstrated that there is a unique (equivalence class of) translation invariant Fock representation(s) of the fields at time $t = t_0$ such that the evolution of the canonical pair of fields given by transformation (39), for any function $g(t)$, is unitary implementable. This representation is the one constructed in Refs. [9,10] and is determined by the complex structure J_0. Furthermore, as explained in Sec. IV, this conclusion applies as well to any field parametrization defined by a transformation of the form (13) with a nonnegative constant function $F(t) = F(t_0)$ and any function $G(t)$.

In particular, no new quantum representations appear when one looks for unitary implementations of the dynamics of the transformed canonical pair $(\tilde{\xi}, P_{\tilde{\xi}})$. The quantization defined by J_0 already gives a unitary implementation of such dynamics, and there are no more (inequivalent) quantizations.

It is worth noticing that, on general grounds, given any representation which allows a unitary dynamics for the two canonical pairs (ξ, P_ξ) and $(\tilde{\xi}, P_{\tilde{\xi}})$, there is a well defined quantum version of the momentum redefinition (39) provided by the time-dependent unitary operator $\hat{U}^{-1}(t, t_0)\hat{\tilde{U}}(t, t_0)$, where $\hat{U}(t, t_0)$ and $\hat{\tilde{U}}(t, t_0)$ are, respectively, the quantum evolution operators corresponding to the dynamics of the pairs (ξ, P_ξ) and $(\tilde{\xi}, P_{\tilde{\xi}})$. Our result is, however, much stronger: different field descriptions are not only unitarily related for a given representation, but there is actually a unique (equivalence class of) translation invariant representation(s) admitting a unitary dynamics.

VIII. SUMMARY AND CONCLUSIONS

We have analyzed the uniqueness of the Fock quantization of the family of linearly polarized Gowdy T^3 cosmologies after its reduction by a gauge-fixing procedure which removes all the constraints except for a homogeneous one. This constraint generates translations on the coordinate $\theta \in S^1$ that, together with the time coordinate t, parametrize the set of orbits of the isometry group. The phase space of this reduced model can be viewed as that corresponding to a point-particle degree of freedom and a scalar field. With a suitable parametrization of the induced metric, this field satisfies a Klein-Gordon equation on a fiducial flat $1 + 1$ background subject to a time-dependent potential, which is invariant under the gauge group of S^1-translations. Besides, one can choose the canonical momentum of this field in such a way that the Hamiltonian density that generates the dynamics is quadratic both in the field and in its momentum (without crossed terms): this is the CCMV field formulation introduced in Refs. [9,10] for the description of the phase space of the reduced Gowdy model.

In a previous work, it was shown that the Fock quantization of this field formulation, which depends on the choice of complex structure, is unique under some natural

requirements. More precisely, if one demands that the complex structure be invariant under S^1-translations, so that every element of the gauge group is represented by a unitary operator that leaves the Fock vacuum invariant, then any Fock quantization admitting a unitary implementation of the field dynamics is unitarily equivalent to the CCMV quantization, which was obtained with a particular choice of complex structure J_0. In the present paper we have extended this uniqueness result to cover all reasonable Fock quantizations of the reduced Gowdy model by considering also the freedom available in the choice of the field description of the system. Specifically, we have studied local field reparametrizations of the induced metric in the reduced model which are independent of the spatial coordinates (so that they commute with the isometry and gauge groups), respect the decoupling with the point-particle degrees of freedom (attained in the CCMV parametrization), and whose dynamics is governed by a homogeneous Klein-Gordon type field equation. Such reparametrizations amount to a time-dependent scaling of the scalar field. Its canonical momentum is scaled by the inverse factor and, in principle, may also get a time-dependent linear contribution in the field.

We have concentrated our discussion in the case when such a linear, time-dependent canonical transformation of the CCMV variables coincides with the identity at the fixed reference time $t = t_0$, which determines the Cauchy surface with respect to which the quantum representation of the fields is constructed. The most general situation can be obtained from this case by combining it with a time-independent canonical transformation which produces constant linear combinations of the t_0-fields and does not affect the conclusions about uniqueness. For the case of time-dependent transformations which are the identity at t_0, we have then proven that the new canonical pair of fieldlike variables admits a Fock quantization, defined by an invariant complex structure (under S^1-translations) and providing a unitary implementation of the field dynamics, if and only if the scaling function is the unit function. In particular, this demonstrates once and for all that there exists no Fock quantization with these properties for the scalar field formulation of the reduced model adopted by Pierri [5,8].

Moreover, even in the remaining case of no scaling (i.e. a unit scaling function), where only the canonical momentum differs from that of the CCMV description [see Eq. (39)], we have shown that the Fock representation of the t_0-fields corresponding to the transformed canonical pair is unique, in the sense that, if it is defined by an invariant complex structure and admits a unitary dynamics, it is unitarily equivalent to the CCMV representation determined by the complex structure J_0. No new (inequivalent) translation invariant representations with unitary dynamics appear by adopting a canonical momentum different from that of the CCMV formulation.

Furthermore, it is possible to eliminate the freedom in the choice of canonical momentum by including an additional requirement on the quantization. Namely, one further demands that there exists a choice of complex structure such that the Fock vacuum of the corresponding representation belongs to the domain of the generator of the evolution. This condition is convenient in practice, because it allows one to calculate the action of the evolution operator on the vacuum (and on the *n-particle* states) by expanding it in powers of the generator. Appendix B shows that, with this additional demand, one can actually fix the canonical pair of fieldlike variables so that it coincides with the CCMV pair.

Therefore, we conclude that all Fock quantizations obtained with a reasonable field description of the reduced Gowdy model are unitarily equivalent under natural requirements. In this sense, the CCMV quantization of the Fock type introduced in Refs. [9,10] is unique. On the other hand, if one considered instead the unreduced Gowdy model [see metric (1)], rather than its gauge-fixed and reduced version, there would still be freedom in the choice of gauge. Nonetheless, the gauge adopted is certainly well motivated both from a geometrical and a physical point of view. The θ-diffeomorphism gauge freedom has been fixed, except for the group of translations, by requiring the homogeneity of the phase-space variable that generates conformal transformations of the two-metric induced on the set of group orbits. The homogeneous part of this variable is a known Dirac observable of the Gowdy cosmologies (i.e. it commutes with all the constraints of the unreduced model) [31]. In addition, the phase-space variable chosen as time coordinate, apart from a multiplicative factor, is the area of the orbits of the group of isometries, which expands monotonously in the evolution of the cosmological solutions and whose gradient has a timelike character that is invariant under coordinate transformations.

Finally, it is worth emphasizing that our uniqueness result provides an example of a cosmological system in which, without abandoning standard quantum field theory, one can single out a preferred quantization by requiring suitable symmetry and consistency conditions. In the considered case of the reduced Gowdy model, this strongly supports the conclusion that the physical consequences that can be derived from the CCMV quantization are meaningful and not an artifact of the scalar field description and Fock representation adopted for the system.

ACKNOWLEDGMENTS

The authors are thankful to A. Ashtekar, M. Varadarajan, L. J. Garay, and M. Martín-Benito for helpful conversations and discussions. This work was supported by the Spanish MEC Projects No. FIS2005-05736-C03-02 and No. FIS2006-26387-E/, and the Portuguese FCT Project No. POCTI/FIS/57547/2004.

APPENDIX A: A PROOF FOR TIME-DEPENDENT SCALINGS

We want to prove that, if

$$1 - \mathrm{Re}\left[\frac{\lambda_m^2}{(\kappa_m^*)^2}\right] \quad \text{and} \quad \mathrm{Im}\left[\frac{\lambda_m^2}{(\kappa_m^*)^2}\right] \tag{A1}$$

tend both to zero on a subsequence $S \subset \mathbb{N}$ (namely, for $m \in S \subset \mathbb{N}$), it is impossible that the imaginary part of $\tilde{\beta}^{0J}_m(t, t_0)/(\kappa_m^*)^2$ has a vanishing limit $\forall\ t > 0$. We recall that expression (A1) provides the time-independent coefficients of the cosine terms of the real and imaginary parts of $\tilde{\beta}^{0J}_m(t, t_0)/(\kappa_m^*)^2$, given by Eqs. (33) and (34). Let us remind the reader also that $t_0 > 0$ is fixed and that we call the difference of times $T := t - t_0$.

In order to prove our statement, we first note that, if the two coefficients (A1) tend to zero on certain subsequence $S \subset \mathbb{N}$, then $(\mathrm{Re}[\lambda_m/\kappa_m^*])^2$ must tend to 1 on S. This can be seen by summing the square of the two coefficients for each $m \in S$, which gives $(1 - |\lambda_m/\kappa_m^*|^2)^2 + 4(\mathrm{Im}[\lambda_m/\kappa_m^*])^2$. Since this expression must tend to zero on S, we get that $|\lambda_m/\kappa_m^*|$ tends to 1 and $\mathrm{Im}[\lambda_m/\kappa_m^*]$ to zero. But then $(\mathrm{Re}[\lambda_m/\kappa_m^*])^2$ tends to 1 on S as we anticipated.

Let us then suppose that when $m \to \infty$, the imaginary part of $\tilde{\beta}^{0J}_m(t, t_0)/(\kappa_m^*)^2$, displayed in Eq. (34), vanishes $\forall\ t > 0$. Thus, it does so on any possible subsequence of positive integers m. Let us now suppose that there is a particular subsequence $S \subset \mathbb{N}$ such that the coefficients (A1) tend both to zero on S. Given that the term which multiplies $\mathrm{Im}[\lambda_m^2/(\kappa_m^*)^2]$ in Eq. (34), namely $f_-(t) \times \cos(mT)/2$, is bounded for every particular value of t, we conclude that

$$\left(\left\{1 + \mathrm{Re}\left[\frac{\lambda_m^2}{(\kappa_m^*)^2}\right]\right\}\frac{f_-(t)}{2} - \mathrm{Re}\left[\frac{\lambda_m}{\kappa_m^*}\right]f_+(t)\right)\sin(mT)$$

must have a zero limit on S, $\forall\ t > 0$. Moreover, since we have that $1 - \mathrm{Re}[\lambda_m^2/(\kappa_m^*)^2]$ also tends to zero on S, we get that

$$\left(-\mathrm{Re}\left[\frac{\lambda_m}{\kappa_m^*}\right]f_+(t) + f_-(t)\right)\sin(mT) \tag{A2}$$

must tend to zero on $S\ \forall\ t > 0$.

In addition, as we have seen above, $(\mathrm{Re}[\lambda_m/\kappa_m^*])^2$ necessarily tends to 1 on S. Then, there exists at least one subsequence $S' \subset S$ such that $\mathrm{Re}[\lambda_m/\kappa_m^*]$ tends to 1 or to -1 on S'. In any of these cases, given that $S' \subset S$, the sequence (A2) must tend to zero on S' and (recalling the definition of $f_\pm$) we obtain that either $\sin(mT)f(t)$ or $\sin(mT)/f(t)$ (or both) have a zero limit on some subsequence $S' \subset \mathbb{N}\ \forall\ t > 0$. Thus, since $f(t)$ is continuous and nonvanishing, $\sin(mT)$ must tend to zero on $S'\ \forall\ t > 0$, and therefore $\forall\ T > -t_0$. In particular, this implies that $\sin^2(mT)$ tends to zero on $S'\ \forall\ T \in [0, 2\pi]$. However, this last conclusion is false. For instance, Lebesgue dominated convergence [32] would then imply that $\int_0^{2\pi} dT \sin^2(mT)$, which is clearly equal to π for all nonzero integers m, has to converge to zero on S'. This indicates a contradiction. Therefore, since the imaginary part of $\tilde{\beta}^{0J}_m(t, t_0)/(\kappa_m^*)^2$ must tend to zero, one can exclude the possibility that the two sequences of time-independent coefficients appearing in Eq. (A1) can both converge to zero on any subsequence $S \subset \mathbb{N}$.

APPENDIX B: A CRITERION FOR THE CHOICE OF CANONICAL MOMENTUM

We have seen that there exists some freedom in the definition of the momentum canonically conjugate to the CCMV field ξ [see Eq. (39)], although this freedom does not result in the availability of new (inequivalent) Fock quantizations for the reduced Gowdy model. We will now introduce a possible criterion to remove this freedom and select a preferred canonical momentum.

Our starting point is a time-dependent canonical transformation of the form $\tilde{\xi} = \xi$ and $P_{\tilde{\xi}} = P_\xi + g(t)\xi$ where the function $g(t)$ is (at least) continuous and vanishes at the reference time t_0. In addition to our requirements of invariance under S^1-translations and unitarity of the dynamics, we will demand that the complex structure J that determines the Fock representation for the canonical pair $(\tilde{\xi}, P_{\tilde{\xi}})$ be such that the associated vacuum belongs to the domain of the generator of the evolution in the Schrödinger picture. This additional requirement on the vacuum is of practical interest since it is necessary to render meaningful the action of the evolution operator (in the Schrödinger picture) on the dense subspace formed by the *n-particle* states when one expands this operator as a formal series in powers of its generator.

The classical generator $H_{\tilde{\xi}}$ of the dynamics of the canonical pair $(\tilde{\xi}, P_{\tilde{\xi}})$ can be easily obtained from Eq. (14) by setting $F(t) = 1$ and $G(t) = g(t)$. In terms of the CCMV pair, this generator reads

$$H_{\tilde{\xi}} = \frac{1}{2}\oint d\theta\left[P_\xi^2 + (\xi')^2 + \xi^2\left(\frac{1}{4t^2} - \dot{g}(t)\right)\right]. \tag{B1}$$

In the basis $\{B_m(t)\}$ introduced in Eq. (9), the classical generator is thus $H_{\tilde{\xi}} = H^0_{\tilde{\xi}} + H_{\tilde{\xi}}[t|\{B_m(t)\}]$ where $H^0_{\tilde{\xi}}$ denotes the contribution of the zero modes and [11]

$$H_{\tilde{\xi}}[t|\{B_m\}] := \sum_{m=1}^{\infty}\{[m + \tilde{\rho}_m(t)][b_m^* b_m + b_{-m}^* b_{-m}] + \tilde{\rho}_m(t)[b_m^* b_{-m}^* + b_m b_{-m}]\}, \tag{B2}$$

$$\tilde{\rho}_m(t) = \frac{1}{2m}\left[\frac{1}{4t^2} - \dot{g}(t)\right]. \tag{B3}$$

On the other hand, we remember that, from our discussion in Sec. V, the variables $\{\tilde{B}_m(t)\}$ corresponding to the canonical pair $(\tilde{\xi}, P_{\tilde{\xi}})$ are related to $\{B_m(t)\}$ by the matrices

$C_m(t)$ obtained from Eq. (22) with $f(t) = 1$, namely $B_m(t) = C_m^{-1}(t)\tilde{B}_m(t)\ \forall\ m \in \mathbb{N}$. In terms of $\{\tilde{B}_m(t)\}$ we then get

$$H_{\tilde{\xi}} = H^0_{\tilde{\xi}} + H_{\tilde{\xi}}[t|\{C_m^{-1}(t)\tilde{B}_m(t)\}]. \tag{B4}$$

Therefore, in the Schrödinger picture and obviating the contribution of the zero modes (which are a finite-dimensional system), the generator of the dynamics of the fieldlike variables $(\tilde{\xi}, P_{\tilde{\xi}})$ is $:H_{\tilde{\xi}}[t|\{C_m^{-1}(t)\hat{B}_m\}]: + D$, where we have used $\tilde{B}_m(t_0) = B_m(t_0) := B_m$, $\hat{B}_m$ denotes the operator counterpart of B_m obtained with the complex structure J (for simplicity, we will obviate the use of a more accurate notation such as $\hat{B}^J_m$ that would make explicit this fact), the dots denote normal ordering with respect to J, and D is a c-number representing a possible zero-point energy.

As we have commented in Sec. V, any invariant complex structure is related with J_0 by means of a time-independent symplectic transformation, $J = K_J J_0 K_J^{-1}$, with K_J given in Eq. (27) [11]. Furthermore, we have shown in Sec. VII that, if the dynamics of the canonical pair $(\tilde{\xi}, P_{\tilde{\xi}})$ is unitarily implementable with respect to J, then K_J admits a unitary implementation in the quantum representation determined by the complex structure J_0. Therefore, there exist unitary operators $\hat{K}_J$ such that

$$\hat{K}_J \hat{B}_m \hat{K}_J^{-1} = (\mathcal{K}_J)^{-1}_m \hat{B}_m := \hat{A}_m. \tag{B5}$$

In the corresponding basis $\{A_m\}$, with

$$(\mathcal{K}_J)^{-1}_m B_m := A_m := (a_m, a^*_{-m}, a_{-m}, a^*_m)^T, \tag{B6}$$

the complex structure J has the same matrix form as J_0 in the original basis $\{B_m\}$. In other words, J is block diagonal in terms of $\{A_m\}$, with 4×4 blocks equal to $(J)_m = \mathrm{diag}(i, -i, i, -i)$. The vacuum $|0\rangle_J$ associated with the complex structure J is simply the state annihilated by the operators $\hat{a}_m$ and $\hat{a}_{-m}\ \forall\ m \in \mathbb{N}$. In total, we arrive at the following expression for the quantum generator of the dynamics of the canonical pair $(\tilde{\xi}, P_{\tilde{\xi}})$ (modulo the contribution of the zero modes):

$$\hat{H}_{\tilde{\xi}}(t) := :H_{\tilde{\xi}}[t|\{C_m^{-1}(t)(\mathcal{K}_J)_m \hat{A}_m\}]: + D, \tag{B7}$$

where the normal ordering is that corresponding to the annihilation and creation operators $\{\hat{A}_m\}$.

A straightforward calculation shows then that

$$||\hat{H}_{\tilde{\xi}}(t)|0\rangle_J||^2 = |D|^2 + \sum_{m=1}^{\infty} |\gamma_m(t)|^2, \tag{B8}$$

where

$$\begin{aligned}
\gamma_m(t) &= 2m\kappa_m(t)\lambda^*_m(t) + \tilde{\rho}_m(t)[\kappa_m(t) + \lambda^*_m(t)]^2, \\
\kappa_m(t) &= \kappa_m - i(\kappa_m + \lambda^*_m)\frac{g(t)}{2m}, \\
\lambda_m(t) &= \lambda_m - i(\kappa_m + \lambda_m)\frac{g(t)}{2m}.
\end{aligned} \tag{B9}$$

Here $\{\kappa_m = \sqrt{1 + |\lambda_m|^2}\}$ (which are real) and $\{\lambda_m\}$ are the time-independent coefficients of the symplectic transformation K_J [see Eq. (27)]. We also remember that the sequence $\{\lambda_m\}$ is square summable, because K_J is unitarily implementable with respect to J_0.

We see from Eq. (B8) that, for $|0\rangle_J$ to belong to the domain of $\hat{H}_{\tilde{\xi}}(t)$ at any positive value of t, the sequence $\{\gamma_m(t)\}$ must be square summable $\forall\ t > 0$. It then follows that $2m\lambda^*_m(t)$ must be negligible compared with $1/\sqrt{m}$ when $m \to \infty\ \forall\ t > 0$, because this factor is either of order $1/m$ (i.e., its product by m is bounded) or it gives the leading term in $\gamma_m(t)$, which has to be square summable. But this implies that $\lambda_m(t)$ must be negligible compared with $1/m^{3/2}\ \forall\ t > 0$. Since the time-dependent part of $\lambda_m(t)$ is $-i(\kappa_m + \lambda_m)g(t)/(2m)$, which is of order $g(t)/m$, it is necessary that $g(t)$ be constant, so that this contribution can be compensated by the time-independent part. Therefore, we conclude that $g(t) = g(t_0) = 0$.

This singles out the momentum $P_{\tilde{\xi}} = P_{\xi}$ of the CCMV formulation. In the case of the CCMV canonical pair, our condition on the vacuum is satisfied with the choice of complex structure J_0 [assuming $|D| < \infty$ in Eq. (B7)]. In order to see this note that, with K_J being the identity and $g(t) = 0$, one obtains $\gamma_m(t) = 1/(8mt^2)$, which is clearly square summable $\forall\ t > 0$.

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[20] The use of invariant complex structures is common to previous works on the Gowdy model, and there is no motivation to generalize this procedure when looking for a quantization of the constraint C_0, given that a large class of invariant complex structures exists. Moreover, unitary implementation of classical symmetries in a natural invariant way, whenever available, is a standard feature of quantum field theory.
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[29] Clearly, $f(t)$ and $g(t)$ have the same regularity properties as $F(t)$ and $G(t)$, and $f(t)$ vanishes nowhere. Explicitly, $f(t) = F(t)/F(t_0)$ and $g(t) = F(t_0)G(t) - G(t_0)F(t)$.
[30] The same conclusion can be obtained directly from the discussion presented in Sec. VI if one starts with transformation (13). In that case, Eq. (21) must be replaced with $\tilde{\mathcal{U}}_m(t, t_0) = C_m(t)\mathcal{U}_m(t, t_0)C_m^{-1}(t_0)$ because $C_m(t_0) \neq \mathbf{1}$ now. This affects some of the formulas given in Sec. VI but does not modify the conclusion.
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