

# “Mass without mass” from thin shells in Gauss-Bonnet gravity

Elias Gravanis<sup>1</sup> and Steven Willison<sup>2,\*</sup><sup>1</sup>*Department of Physics, Kings College London, United Kingdom*<sup>2</sup>*Centro de Estudios Científicos (CECS), Casilla 1469 Valdivia, Chile*

(Received 8 February 2007; published 13 April 2007)

Five tensor equations are obtained for a thin shell in Gauss-Bonnet gravity. There is the well-known junction condition for the singular part of the stress tensor intrinsic to the shell, which we also prove to be well defined. There are also equations relating the geometry of the shell (jump and average of the extrinsic curvature as well as the intrinsic curvature) to the nonsingular components of the bulk stress tensor on the sides of the thin shell. The equations are applied to spherically symmetric thin shells in the vacuum. The shells are part of the vacuum; they carry no energy tensor. We classify these solutions of “thin shells of nothingness” in the pure Gauss-Bonnet theory. There are three types of solutions, with one, zero, or two asymptotic regions, respectively. The third kind of solutions are wormholes. Although vacuum solutions, they have the appearance of mass in the asymptotic regions. It is striking that in this theory, exotic matter is not needed in order for wormholes to exist—they can exist even with no matter.

DOI: [10.1103/PhysRevD.75.084025](https://doi.org/10.1103/PhysRevD.75.084025)

PACS numbers: 04.50.+h, 02.40.–k, 04.20.Gz

## I. INTRODUCTION

The dynamics of thin shells in Einstein’s theory of gravity is described by a set of five tensor equations [1]. One is an algebraic relation between the jump in the extrinsic curvature and the intrinsic stress-energy tensor (the junction condition). Four more relate the geometry of the shell (extrinsic curvature on each side as well as the intrinsic curvature) to the value of the bulk stress-energy tensor on the sides of the thin shell. A fact about Einstein’s theory is that if the intrinsic stress-energy tensor of the shell vanishes then there is no jump in the extrinsic curvature. This comes from the junction condition.

In this paper, a similar analysis is performed for the Gauss-Bonnet theory of gravity in five dimensions, which is quadratic in the Riemann curvature. A qualitative difference is that the junction condition [2,3] does not imply zero jump for the extrinsic curvature when the energy tensor of the thin shell vanishes [4]. This is because the junction condition is nonlinear in the (extrinsic and intrinsic) shell curvatures. If the bulk tensor also vanishes we obtain a nonsmooth kink solution to the vacuum field equations and can be thought of as some kind of soliton. Then the other four equations describe the dynamics of that object. One of them implies the existence of a covariantly conserved, symmetric tensor  $\tilde{Q}^a_b$  on the shell. It involves the jump of the extrinsic curvature, whereas its counterpart in Einstein theory does not.

These results are applied to a simple example. We consider spherically symmetric shells in pure Gauss-Bonnet gravity. A complete classification of non-null solutions with solitonic shells, both static and time dependent, is given. A particularly striking type of solution is when two exterior regions are matched. Vacuum thin-shell wormhole solutions are found in which the stress tensor on the shell is

zero. The concept of “mass without mass” [5] is shown to be realized in this context. The exterior solution is that of the exterior of a massive object, but the massive object is excised and replaced with another exterior region connected by a wormhole throat which is a “thin shell of nothingness.”

It is argued that these conclusions should also be true in Einstein-Gauss-Bonnet and Lovelock gravity generally.

*Notation:* Capital roman letters  $A, B$ , etc. represent five-dimensional tensor indices. Lower case roman letters  $a, b$ , etc. represent four-dimensional tensor indices on the tangent space of the world sheet of the shell.

### A. Thin shells in Einstein’s theory

First, we review the formalism in general relativity (GR) due to W. Israel [1]. Let  $\Sigma$  be a hypersurface of codimension 1 (the world sheet of the shell) on which the stress tensor is a delta function:

$$T_{AB} = \begin{pmatrix} S_{ab} \delta(\Sigma) & \emptyset \\ \emptyset & 0 \end{pmatrix}, \quad (1)$$

where  $S^a_b$  is the intrinsic stress tensor on the shell. The  $\delta(\Sigma)$  is a Dirac delta function with support on the shell. To simplify the presentation, we shall assume that tensors are written in a basis  $e_A = (e_a, n)$  which is adapted to the shell so that  $e_a$  are tangent vectors to  $\Sigma$  and  $n$  is normal to  $\Sigma$ . The shell divides the space-time into two regions, which are denoted by  $M_+$  and  $M_-$ .

In Einstein’s gravity, such a concentration of matter will produce a discontinuity in the first derivative of the metric. This is given covariantly by introducing the extrinsic curvature of the shell. This is defined as  $K_{ab} = e_a \cdot \nabla_{e_b} n$ . Intuitively, it measures the tangential rate of change of the normal vector along the surface  $\Sigma$ .

The integration of the tangential-tangential components of the Einstein equation across the infinitesimal width of

\*Electronic address: [steve@cecs.cl](mailto:steve@cecs.cl)

the shell gives:

$$-\sigma(\Delta K_b^a - \delta_b^a \Delta K) = S_b^a, \quad (2)$$

where  $\Delta K_b^a \equiv (K^+)_b^a - (K^-)_b^a$  is the jump in the extrinsic curvature across the shell. The factor  $\sigma = +1$  for a space-time-like shell (with a spacelike normal vector) and  $\sigma = -1$  for a spacelike shell (timelike normal vector). The projection of the normal-tangential components of the bulk Einstein equation gives:

$$-\sigma(\Delta K_b^a - \delta_b^a \Delta K)_{;a} = 0, \quad (3)$$

and

$$-\sigma(\tilde{K}_b^a - \delta_b^a \tilde{K})_{;a} = 0, \quad (4)$$

where  $\tilde{K}_b^a \equiv (K^+)_b^a + (K^-)_b^a$  is the sum of the extrinsic curvature on each side of the shell. The semicolon denotes the intrinsic covariant derivative on the shell. Also, the projection of the normal-normal components of Einstein's equation gives:

$$\frac{\sigma}{4} \Delta K_c^a \tilde{K}_d^b \delta_{ab}^{cd} = 0, \quad (5)$$

$$-R + \frac{\sigma}{4} \{\tilde{K}_c^a \tilde{K}_d^b + \Delta K_c^a \Delta K_d^b\} \delta_{cd}^{ab} = 0, \quad (6)$$

where  $R$  is the *intrinsic* Ricci scalar of the shell. The antisymmetrized Kronecker delta is defined as:  $\delta_{cd}^{ab} = \delta_c^a \delta_d^b - \delta_c^b \delta_d^a$ .

The first Israel junction condition (2) says that the effect of the singular matter on the geometry is thus encoded in the discontinuity of the extrinsic curvature. This condition can clearly be inverted to determine the jump in the extrinsic curvature in terms of the intrinsic stress tensor. (Indeed, if one uses this to replace  $S_b^a$  in Eqs. (3)–(6), the expressions given in Ref. [1] are recovered). This one-to-one correspondence between  $S_b^a$  and  $\Delta K_b^a$  arises because the Einstein equation is linear in the curvature.

## B. Thin shells in Gauss-Bonnet gravity

A generalization of Einstein's theory which is not linear in the curvature was given by Lovelock [6]. In five or higher space-time dimensions it gives second order field equations. Indeed, it is the most general second order metric theory of gravity and can be thought of as a natural correction to Einstein gravity in more than four dimensions. In five dimensions the Lagrangian is:

$$\mathcal{L} = c_0 \sqrt{-g} d^5 x + c_1 \mathcal{R} \sqrt{-g} d^5 x + c_2 \mathcal{L}_{\text{GB}}. \quad (7)$$

The first term is the cosmological constant, the second is the Einstein-Hilbert term, the third is the so-called Gauss-Bonnet term which is quadratic in curvature:

$$\mathcal{L}_{\text{GB}} := (\mathcal{R}^2 - 4\mathcal{R}_{AB}\mathcal{R}^{AB} + \mathcal{R}_{ABCD}\mathcal{R}^{ABCD})\sqrt{-g}d^5x.$$

Above, the calligraphic script  $\mathcal{R}$  is used to denote the five-dimensional curvatures.

Solutions with a hypersurface of codimension one with a Gauss-Bonnet term have been studied extensively in the context of brane worlds [4,7,8], inspired by string theory. A covariant junction condition, the analogue of (2) was derived in Refs. [2,3] using an action principle. Also, covariant equations of motion have been derived from decomposition of the bulk field equations [9]. This approach provides an alternative derivation of the junction condition. Also other covariant equations for the brane have been derived in the literature, with the emphasis being upon finding an effective theory of gravity on the brane. A modified Einstein equation for the intrinsic metric on the hypersurface has been obtained [9], in which there are nonlinear corrections involving the extrinsic curvature and a nonlocal piece coming from the Weyl tensor in the bulk. The generalization of (3), which says that the intrinsic stress tensor on the hypersurface is covariantly conserved, is well known [2]. Other works on equations of motion for branes in Einstein-Gauss-Bonnet theory are Refs. [10].

However, it seems that the analysis in the style of Israel's five equations has not been done. Such a set of equations is of course just an alternative formalism to that of Ref. [9], but it shows some interesting information which may be hidden in other formalisms. In the next section, we present this analysis for a shell embedded in a bulk in which the field equations for the pure Gauss-Bonnet theory ( $c_0 = c_1 = 0$ ) hold. The analysis is given for arbitrary matter on the shell and no matter in the bulk (the more general case of Einstein-Gauss-Bonnet theory with matter in the bulk is given in Appendix B).

Spherically symmetric shell solutions in the vacuum in the five-dimensional Einstein-Gauss-Bonnet theory were considered in Ref. [8]. These kinds of solutions are of interest in cosmology because there is a spatially homogeneous cosmological metric induced on the shell, with an expansion factor governed by a modified Friedmann equation. The solutions were restricted to  $\mathbb{Z}_2$  symmetry, where the metric on one side of the shell is the mirror image of the other side. When this assumption is dropped, solutions are generally very complicated [11]. However, in Refs. [12,13], general thin shells in spherically symmetric space-times were examined for a certain class of Lovelock theories. In those references a Hamiltonian treatment of thin shells in GR [14] was generalized to Lovelock gravity.

There is a curious possibility, which is not possible in Einstein's theory. Because of the nonlinearity in curvature, it is possible to have a hypersurface where there is a discontinuity in the extrinsic curvature without any stress tensor as source. In other words, there is a thin shell made of nothing, where  $S_b^a = 0$  but  $\Delta K_b^a \neq 0$ . These kind of solutions were considered in Refs. [4] (also an example in 11-dimensional Chern-Simons gravity was studied in Ref. [15]) and we shall call them "solitonic shells." It happens that, for these kinds of solutions, the junction

conditions can be resolved, without assuming  $\mathbb{Z}_2$  symmetry, in a relatively simple way.

In Sec. III, explicit solitonic shell solutions are found. We shall restrict ourselves to consider only the Gauss-Bonnet term, i.e. the coefficients  $c_0$  and  $c_1$  shall be set to zero. This theory, which we shall call pure Gauss-Bonnet gravity, arises as the torsion-free sector of Chern-Simons theory of the Poincaré group in five dimensions  $ISO(4, 1)$  [16,17]. It can be thought of as generalization of the interpretation of 2 + 1-dimensional general relativity as a Chern-Simons theory for the Poincaré group [18,19].

This theory has no Newtonian limit, so, like GR in three dimensions, it should properly be regarded as a toy model for studying qualitative features of gravity. The advantage for us is that the spherically symmetric bulk solutions and the junction condition take a very simple form. We are able to classify all of the spherically symmetric solitonic thin-shell solutions, without assuming  $\mathbb{Z}_2$  symmetry.

Here, the focus shall not be on cosmology on the shell. The main interest will be in wormhole solutions which behave in a sense like material particles even though they are not massive solutions. That is, instead of being the universe, the shell should perhaps be thought of as a kind of particle.

Although only the pure Gauss-Bonnet theory is considered explicitly, we comment on the generalization to general Lovelock theory in Sec. IV.

## II. THE FIVE EQUATIONS FOR A SHELL IN GAUSS-BONNET GRAVITY

Let us for now concentrate on the Gauss-Bonnet term, setting  $c_0$  and  $c_1$  to zero and  $c_2 = 1$  in the action (7). The field equation of pure Gauss-Bonnet gravity is:

$$-\frac{1}{8}\delta_{BD_1\dots D_4}^{AC_1\dots C_4}\mathcal{R}^{D_1D_2}{}_{C_1C_2}\mathcal{R}^{D_3D_4}{}_{C_3C_4}=T^A{}_B. \quad (8)$$

Let us find the analogue of Israel's five equations (2)–(6) for Gauss-Bonnet gravity. Since the origin of these equations is clear, we shall just state here the results. The proof is given in Appendix B.

Here the results are summarized for the case where the bulk energy tensor is zero. First we define the following symmetric tensor:

$$Q^a{}_b \equiv K_f^c(2\sigma R_{gh}^{de} - \frac{4}{3}K_g^d K_h^e)\delta_{bcde}^{afgh}. \quad (9)$$

Also we define  $\Delta Q_b^a \equiv (Q^+)^a_b - (Q^-)^a_b$ , the jump across the shell and  $\tilde{Q}_b^a \equiv (Q^+)^a_b + (Q^-)^a_b$ , the sum of  $Q_b^a$  evaluated on each side.

The integration of the tangential-tangential components of the field equation (8) across the infinitesimal width of the shell gives the junction condition [2,3]:

$$\Delta Q^a{}_b = -2S_b^a. \quad (10)$$

The projection of the normal-tangent components of the bulk field equations onto the shell tell us that the intrinsic

stress tensor is covariantly conserved [2]:

$$\Delta Q^a{}_{b;a} = 0 \Rightarrow S^a{}_{b;a} = 0, \quad (11)$$

and also that the tensor  $\tilde{Q}^a{}_b$  is covariantly conserved on the shell:

$$\tilde{Q}^a{}_{b;a} = 0. \quad (12)$$

The projection of the normal-normal component of the field equation gives:

$$-\frac{3}{8}\{\tilde{K}_b^a\Delta Q^b{}_a + \Delta K_b^a\tilde{Q}^b{}_a\} + \frac{\sigma}{2}\tilde{K}_e^a\Delta K_f^b R_{gh}^{cd}\delta_{abcd}^{efgh} = 0, \quad (13)$$

$$\frac{1}{2}R^{ab}{}_{ef}R^{cd}{}_{gh}\delta_{abcd}^{efgh} - \frac{3}{8}\{\tilde{K}_b^a\tilde{Q}^b{}_a + \Delta K_b^a\Delta Q^b{}_a\} + \frac{\sigma}{4}\{\tilde{K}_e^a\tilde{K}_f^b + \Delta K_e^a\Delta K_f^b\}R_{gh}^{cd}\delta_{abcd}^{efgh} = 0. \quad (14)$$

Equations (10)–(14) are the five equations characterizing the shell. The first two are already known. The last two are rather complicated and perhaps not very useful in describing shells (although they may be useful in the Hamiltonian formalism of Gauss-Bonnet gravity—see below). On the other hand, Eq. (12) has some surprising consequences which have gone unnoticed.

Because of the nonlinearity of the Gauss-Bonnet theory, one cannot solve algebraically for the jump in extrinsic curvature in terms of the intrinsic stress tensor. There are two independent quantities  $\Delta Q^a{}_b$  and  $\tilde{Q}^a{}_b$ , which can be expressed as:

$$\Delta Q^a{}_b = \Delta K_f^c(2\sigma R_{gh}^{de} - \frac{1}{3}\Delta K_g^d\Delta K_h^e - \tilde{K}_g^d\tilde{K}_h^e)\delta_{bcde}^{afgh},$$

$$\tilde{Q}^a{}_b = \tilde{K}_f^c(2\sigma R_{gh}^{de} - \frac{1}{3}\tilde{K}_g^d\tilde{K}_h^e - \Delta K_g^d\Delta K_h^e)\delta_{bcde}^{afgh}.$$

These quantities both depend nonlinearly on  $\Delta K^a{}_b$ . Only one of these is determined by the stress tensor, *but both are covariantly conserved*.

Note that when the surface  $\Sigma$  is spacelike,  $Q_b^a$  arises naturally in the Hamiltonian formalism. It is proportional to the momentum canonically conjugate to the spatial metric. Equations (11) and (12) say that the extrinsic curvature can jump in a way that conserves the constraint  $\mathcal{H}_a = 0$ . Equations (13) and (14) say that any discontinuity must preserve the constraint  $\mathcal{H}_\perp = 0$ . Expressions for  $\mathcal{H}_a$  and  $\mathcal{H}_\perp$  in Lovelock gravity were first given in Ref. [20]. The dynamical part of the field equations in the vacuum says that a discontinuity must obey  $\Delta Q_b^a = 0$ .

The above five equations are for the pure Gauss-Bonnet theory. For the more general action (7), the generalization is straightforward. It is simply a linear combination of the terms appearing in the Israel equations with those of the Gauss-Bonnet theory. This will be given explicitly in the appendix in Eqs. (B13)–(B17).

### III. SOLITONIC SPHERICAL SHELLS IN PURE GAUSS-BONNET GRAVITY

Let us consider the pure Gauss-Bonnet theory, with just the quadratic Lovelock term in the action. This choice remains largely unstudied, no doubt because it does not include the Einstein-Hilbert term. The theory is not in any sense a small correction to general relativity. However, it offers a useful toy model in which to study thin shells, finding exact solutions. In this section, solitonic shells are found in the spherically symmetric background.

It is useful to use differential form notation. A brief explanation of this is given in Appendix A. In this notation, the field equation is:

$$c_2 \Omega^{AB} \wedge \Omega^{CD} \epsilon_{ABCD} = -2T_F, \quad (15)$$

where  $\Omega^{AB}$  is the curvature two-form and  $T_A$  is the stress-energy four-form.

The spherically symmetric vacuum solution is:

$$ds^2 = -dt^2 + \frac{dr^2}{\beta^2} + r^2 d\Omega^2. \quad (16)$$

Here  $d\Omega^2$  is the line element of the unit three-sphere. This is a special case of the solution of Boulware and Deser [21] for Einstein-Gauss-Bonnet,  $\beta$  being the constant of integration. This space-time was discussed recently in Ref. [22] but we are not aware of any previous detailed study of this metric in the literature.

A basis is chosen such that the vielbein and spin connection take the form:

$$E^0 = dt, \quad E^1 = dr/\beta, \quad E^i = r\tilde{E}^i, \quad (17)$$

$$\omega^1{}_i = -\beta\tilde{E}^i, \quad \omega^i{}_j = \tilde{\omega}^i{}_j. \quad (18)$$

*Notation:*  $\tilde{E}^i$  and  $\tilde{\omega}^i{}_j$  are the intrinsic vielbeins and spin connection on the three-sphere. The lower case Latin indices from the middle of the alphabet run from 2 to 4.

Note that, although the constant factor  $\beta$  looks innocuous, *this space-time is not locally flat*. The nonvanishing part of the curvature two-form is:

$$\Omega^{ij} = (1 - \beta^2)\tilde{E}^i \wedge \tilde{E}^j. \quad (19)$$

The five-dimensional Ricci scalar is  $\mathcal{R} = 6(1 - \beta^2)/r^2$ . For  $\beta \neq 1$  there is clearly a curvature singularity at  $r = 0$ .

Even though the curvature diverges, the singularity is well behaved in the sense that the field equations are well defined under integration. From (15) the energy tensor is zero everywhere outside the origin  $r = 0$ . The mass can be calculated unambiguously by integrating the field equations in a ball centered around  $r = 0$ , using the Gauss-Bonnet theorem. The result is:

$$m = 16\pi^2 c_2 \left[ 1 - \frac{|\beta|}{2}(3 - \beta^2) \right]. \quad (20)$$

It is natural to take  $c_2$  to be positive, so that all solutions

have positive mass (see Fig. 1). For convenience, a choice of units is made so that  $16\pi^2 c_2 = 1$ .

#### A. Nonsmooth static vacuum solutions

Let us try to match two different point particle metrics on a timelike hypersurface  $\Sigma$  at a constant radius. The surface divides space-time into two regions,  $M_+$  and  $M_-$ . The metric in each region is given by:

$$ds_+^2 = -dt^2 + \frac{dr_+^2}{\beta_+^2} + r_+^2 d\Omega^2, \quad (21)$$

$$ds_-^2 = -dt^2 + \frac{dr_-^2}{\beta_-^2} + r_-^2 d\Omega^2, \quad (22)$$

and the hypersurface is located at  $r_+ = r_- = r_0$ , a constant so that the induced metric is continuous.

For a static shell, the choice of vielbeins (17) provides a frame adapted to  $\Sigma$ , i.e.  $E^a = (E^0, E^i)$  are dual to the intrinsic frame on  $\Sigma$  and  $E^1$  is dual to the normal vector,  $n = \beta_{\pm} \partial_{r_{\pm}}$ . The normal vector is, by convention, chosen to point from  $M_+$  to  $M_-$ . Note that, in our conventions, the orientation of the embedding of  $\Sigma$  into  $M_{\pm}$  is determined by the sign of  $\beta_{\pm}$ . There are three choices (see Fig. 2):

*Type I*—If  $\beta_-$  and  $\beta_+$  are both positive, the global structure is the same as for the smooth solution. The radial coordinate in  $M_-$  is decreasing as one moves away from  $\Sigma$  and the radial coordinate in  $M_+$  increases as one moves away from  $\Sigma$ . The region  $M_-$  is the interior and contains the point singularity. The region  $M_+$  is the exterior. If  $\beta_+$  and  $\beta_-$  are both negative, the global structure of the space-time is the same but with the roles of  $M_+$  and  $M_-$  swapped.

*Type IIa*—If  $\beta_-$  is positive and  $\beta_+$  is negative, two interior regions are joined together to form a spatially closed universe. Each region contains a point source.

*Type IIb*—If  $\beta_-$  is negative and  $\beta_+$  is positive, two exterior regions are joined together. There are two asymptotic regions  $r_+ \rightarrow \infty$  and  $r_- \rightarrow \infty$  and no point sources. These are wormhole space-times.

If  $\beta_+ \neq \beta_-$  the spin connection is discontinuous, corresponding to a singular curvature. In Einstein's theory,

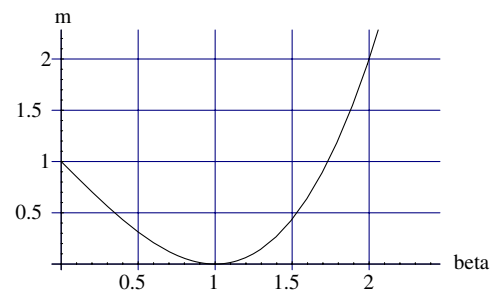
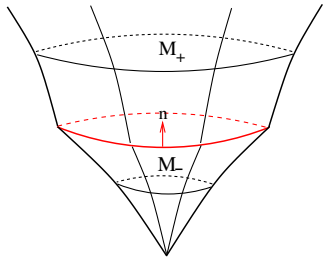
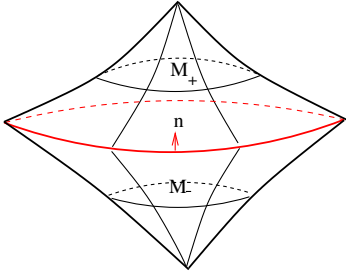


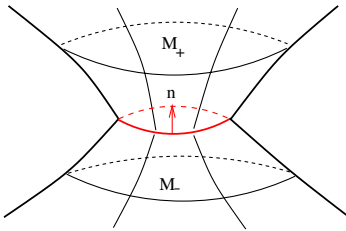
FIG. 1 (color online). The mass as a function of  $|\beta|$  in units such that  $16\pi^2 c_2 = 1$ . Note that  $m(\sqrt{3}) = m(0)$  and  $m(2) = 2$ . The mass is non-negative for all values of  $|\beta|$ . For  $0 < m \leq 1$  there are two values of  $|\beta|$  producing the same mass.



**Type I** .  $\text{sign}(\beta_+) = \text{sign}(\beta_-)$ .



**Type IIa)** .  $\beta_-$  positive and  $\beta_+$  negative.



**Type IIb)** .  $\beta_-$  negative and  $\beta_+$  positive.

FIG. 2 (color online). The three types of matching. The time direction is suppressed. Also two spatial dimensions are suppressed so that 3-spheres are represented by circles.

such a discontinuity on a timelike surface could never be a vacuum solution. However, in the Gauss-Bonnet theory, the matching conditions (10) tell us that the stress-energy tensor located on the hypersurface vanishes if

$$Q_a^+ - Q_a^- = 0,$$

where  $Q_a$  is defined in Eq. (B6).

The only nonzero components of the second fundamental form are  $\theta^{1i} = -\beta \tilde{E}^i$ . The intrinsic geometry of the hypersurface is  $\mathbb{R}^1 \times S^3$  so the nonvanishing components of intrinsic curvature is  $\Omega_{||}^{ij} = \tilde{E}^i \wedge \tilde{E}^j$ . The only component of  $Q_a$  is:

$$Q_0 = -4\beta(1 - \frac{1}{3}\beta^2)\tilde{E}^i \wedge \tilde{E}^j \wedge \tilde{E}^k \epsilon_{0ijk}.$$

[In tensor language, the only components of the extrinsic curvature are  $K_j^i = -\beta \delta_j^i$  and the only component of  $Q_b^a$  is  $Q_0^0 = -43!\beta(1 - \frac{1}{3}\beta^2)$ .]

The junction condition reduces to:

$$3\beta_+ - \beta_+^3 = 3\beta_- - \beta_-^3. \quad (23)$$

There are two useful alternative ways to use Eq. (23). We can use it either to derive a junction condition in terms of the metric parameters  $\beta_{\pm}$  or a condition in terms of the masses. Let us first find the condition in terms of  $\beta_{\pm}$ . We note that (23) factorizes to give either  $\beta_+ - \beta_- = 0$  (which is trivial—the metric is matched smoothly) or

$$\beta_+^2 + \beta_-^2 + \beta_+\beta_- - 3 = 0. \quad (24)$$

The above tells us which metrics can be matched together at a static surface of constant  $r$ . It clearly has nontrivial solutions which are described by an ellipse in the parameter space  $(\beta_+, \beta_-)$  (see Fig. 3).

Alternatively, we can use formula (20) to express (23) in terms of the masses. There are two cases:

Type I The masses must be the same:

$$m_+ = m_- \quad \text{if } \text{sign}(\beta_+) = \text{sign}(\beta_-). \quad (25)$$

Type II There is a condition on the sum of the masses:

$$m_+ + m_- = 2 \quad \text{if } \text{sign}(\beta_+) = -\text{sign}(\beta_-). \quad (26)$$

As shown in Fig. 1, the cubic form of the mass formula (20) leads to an ambiguity in the metric. If the mass is in the range  $0 < m < 1$ , there are two possible solutions—one with  $\beta$  between 0 and 1 and another solution with  $\beta$  between 1 and  $\sqrt{3}$ . Thus, there exist nonsmooth matchings of type I for masses less than 1.

The matchings of type II can occur if one of the  $\beta$ 's is between 0 and  $\sqrt{3}$  and the other is between  $\sqrt{3}$  and 2, i.e. for masses less than 2.

## B. Nonsmooth time-dependent solutions

In the previous example, we matched vacuum solutions at a static surface. It is also possible to match at a surface which is not static. We shall restrict ourselves to solutions which respect the spherical symmetry of the smooth solutions: the hypersurface is defined by  $r$  some function of  $t$ . There are three possibilities: timelike, spacelike, or null surfaces. Here only the timelike and spacelike cases will be considered.

### 1. Timelike

There are two regions with metrics given by (22) and (23). They are joined at a hypersurface which is an expanding or shrinking 3-sphere. The hypersurface  $\Sigma$  is defined parametrically by  $r_- = a(\tau)$ ,  $r_+ = a(\tau)$  and  $t_+ = T_+(\tau)$ ,  $t_- = T_-(\tau)$ . Here  $a$  is some function of  $\tau$ , a time coordinate on  $\Sigma$ . The induced metric is:

$$ds_{\Sigma_{\pm}}^2 = -d\tau^2 \left( \dot{T}_{\pm}^2 - \frac{\dot{a}^2}{\beta_{\pm}^2} \right) + a^2(\tau)d\Omega^2.$$

We require that  $ds_{\Sigma_+}^2 = ds_{\Sigma_-}^2$  for a continuous metric. Note that  $a$  must be the same function on both sides because it is the radius of curvature of the 3-sphere. It is natural to choose  $\tau$  to be the proper time coordinate on  $\Sigma$  so that

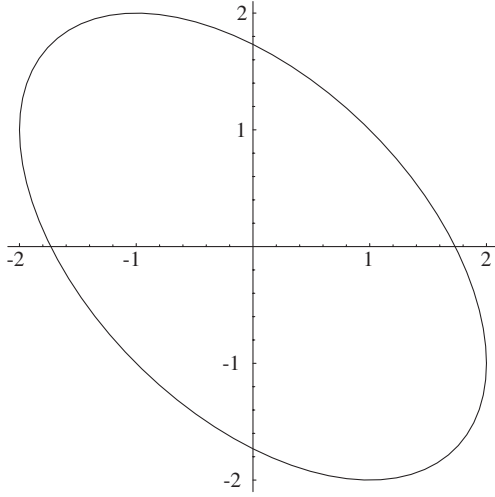


FIG. 3. The static vacuum solutions of the junction condition describe an ellipse in the space of  $\beta_+$  and  $\beta_-$ . The top right and bottom left quadrants correspond to a matching of type I with standard orientation. The top left quadrant corresponds to solutions of type IIa, the “closed universe.” The bottom right quadrant corresponds to solutions of type IIb, wormholes.

$$\dot{T}_{\pm}^2 - \frac{\dot{a}^2}{\beta_{\pm}^2} = 1 \text{ and}$$

$$ds_{\Sigma}^2 = -d\tau^2 + a^2(\tau)d\Omega^2.$$

In order to evaluate the junction conditions, we introduce a frame adapted to  $\Sigma$ . The intrinsic vielbeins are  $(E^{\hat{0}}, E^i)$ , where  $E^{\hat{0}} = d\tau$  and the normal is  $E^{\hat{1}}$ . They are related to the vielbeins (18) by the Lorentz transformation:

$$\begin{pmatrix} E^{\hat{0}} \\ E^{\hat{1}} \end{pmatrix} = \begin{pmatrix} \sqrt{1 + \dot{a}^2/\beta_{\pm}^2} & -\dot{a}/\beta_{\pm} \\ -\dot{a}/\beta_{\pm} & \sqrt{1 + \dot{a}^2/\beta_{\pm}^2} \end{pmatrix} \begin{pmatrix} E_{\pm}^0 \\ E_{\pm}^1 \end{pmatrix}.$$

The vielbeins tangent to the 3-sphere,  $\tilde{E}^i$ , are not transformed. In this basis, the second fundamental form is:

$$\theta_{\pm}^{\hat{1}i} = -\sqrt{\beta_{\pm}^2 + \dot{a}^2}\tilde{E}^i, \quad \theta_{\pm}^{\hat{1}\hat{0}} = -\frac{\ddot{a}}{\sqrt{\beta_{\pm}^2 + \dot{a}^2}}d\tau.$$

There are now two nonzero equations coming from the junction conditions, but they are not independent. The first is  $\Delta Q_{\hat{0}} = 0$  which gives:

$$\left[ \text{sign}(\beta) \sqrt{\beta^2 + \dot{a}^2} \left( 1 + \frac{2}{3}\dot{a}^2 - \frac{1}{3}\beta^2 \right) \right]_{\pm}^{\pm} = 0, \quad (27)$$

where the square bracket  $[\dots]_{\pm}^{\pm}$  denotes the difference in the argument evaluated on each side of  $\Sigma$ . The other equation,  $\Delta Q_i = 0$  gives simply the derivative with respect to  $\tau$  of the first.

Squaring (27) and solving gives:

$$\dot{a}^2 = \frac{(\beta_+^2 + \beta_-^2 + \beta_+\beta_- - 3)(\beta_+^2 + \beta_-^2 - \beta_+\beta_- - 3)}{3(\beta_+^2 + \beta_-^2 - 2)}. \quad (28)$$

[It can be checked separately using (27) that  $\beta_+^2 + \beta_-^2 = 2$  is not a solution except in the trivial case where  $\beta_+ = \beta_- = 1$ . To obtain (28) we have divided through by a common factor of  $(\beta_+^2 - \beta_-^2)$ . In principle, the case  $\beta_+^2 - \beta_-^2 = 0$  should also be verified separately. However, it turns out that this case is correctly described by Eq. (28) and the set of inequalities given below.] Time-dependent solutions only exist when  $\dot{a}^2 > 0$ . This occurs when an odd number of the three inequalities:

$$\begin{aligned} \beta_+^2 + \beta_-^2 + \beta_+\beta_- - 3 &> 0, \\ \beta_+^2 + \beta_-^2 - \beta_+\beta_- - 3 &> 0, \\ \beta_+^2 + \beta_-^2 - 2 &> 0, \end{aligned} \quad (29)$$

are satisfied. Since we have squared the junction condition, we must plug the solution back into (27) in order to determine the relative orientation consistent with the solution. A consistent solution will obey an even number of the three inequalities:

$$\begin{aligned} \beta_+\beta_- &> 0, \quad 2\beta_+^2 + \beta_-^2 - 3 > 0, \\ \beta_+^2 + 2\beta_-^2 - 3 &> 0. \end{aligned} \quad (30)$$

## 2. Spacelike

Following a similar analysis for the case of a spacelike surface gives the junction condition:

$$[\text{sign}(\beta) \sqrt{\dot{a}^2 - \beta^2} (\frac{2}{3}\dot{a}^2 - 1 + \frac{1}{3}\beta^2)]_{\pm}^{\pm} = 0. \quad (31)$$

This can be squared and solved for  $\dot{a}$  to give:

$$\dot{a}^2 = -\frac{(\beta_+^2 + \beta_-^2 + \beta_+\beta_- - 3)(\beta_+^2 + \beta_-^2 - \beta_+\beta_- - 3)}{3(\beta_+^2 + \beta_-^2 - 2)}. \quad (32)$$

Inserting this value of  $\dot{a}^2$  back into (31), the consistency of the solution tells us that an even number of the inequalities (30) must be satisfied. Also, the condition that the square root be real gives:

$$\beta_+^2 + \beta_-^2 - 2 < 0. \quad (33)$$

The set of inequalities for spacelike and timelike shells is depicted graphically in Fig. 4.

The solutions with  $\Sigma$  spacelike represent a breakdown of determinism. The extrinsic curvature can jump instantaneously from one value to another in a way which is not predicted by the initial conditions. Note that, restricting ourselves to spherically symmetric metrics, such jumps are ruled out for masses greater than 1 (i.e.  $\beta > \sqrt{2}$ ).

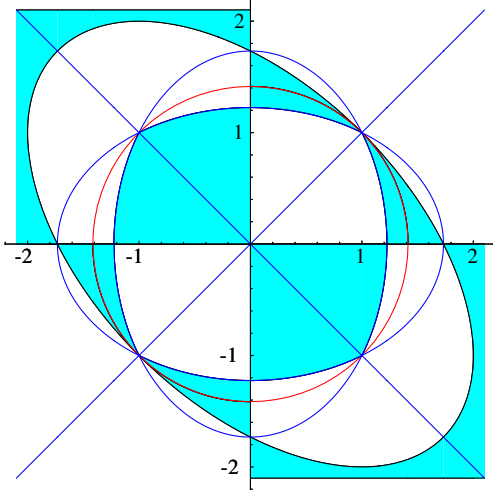


FIG. 4 (color online). The shaded regions are the allowed solutions. The circle of radius  $\sqrt{2}$  divides the solutions where  $\Sigma$  is spacelike (inside) from those where  $\Sigma$  is timelike (outside). The solutions where  $\Sigma$  is static lie on the large ellipse. The top left and bottom right quadrants are shaded everywhere outside this ellipse.

### C. “Mass without mass” and conserved quantities

Let us consider the solutions of type IIb, the wormholes. These solutions contain no point sources, i.e. the stress tensor is everywhere zero. An observer in the region  $M_+$  feels a space-time as if there were a spherically symmetric source of mass  $m_+$  on the other side of the shell. If he moves across the shell, instead of accessing a source he feels a mass  $m_-$  behind him. These wormholes illustrate the concept of “mass without mass.” The nontrivial topology of the vacuum solution creates the illusion of having a massive particle.

In Sec. II it was shown that there are two covariantly conserved symmetric tensors on the shell,  $\Delta Q_b^a$  and  $\tilde{Q}_b^a$ . Equivalently, this can be stated that  $(Q^+)_b^a$  and  $(Q^-)_b^a$  are covariantly conserved independently of each other. In certain cases then, such as if there exists a Killing vector on the hypersurface, we can define a conserved quantity associated with  $\tilde{Q}_b^a$ .

Recall the static and nonstatic solutions of Sec. III B. In the nonstatic solutions the vector  $e_{\hat{0}} = \partial_\tau$  is not a Killing vector on the hypersurface. However, it is still true that  $i^* dQ_0^\pm = 0$ , so the quantities  $\int_{S^3} Q_0^\pm$  are conserved with  $\tau$ , where  $S^3$  is any 3-sphere given by  $\tau = \text{constant}$ . The vacuum matching amounts to  $Q_a^+ - Q_a^- = 0$ . Note that  $\tilde{Q}_a \equiv Q_a^+ + Q_a^- = 2Q_a^+ = 2Q_a^-$ . We define the conserved quantity:

$$\tilde{q} \equiv \frac{c_2}{2} \int_{S^3} \tilde{Q}_0. \quad (34)$$

In the static case one finds for the wormhole solutions

$$\tilde{q} = m_+ - m_- = 2m_+ - 2. \quad (35)$$

This is what the total energy of the shell would have been, as measured from  $M_+$ , had the two regions been matched with the other orientation, i.e. with  $M_-$  replaced by an interior region of mass  $m_-$ . The sign of this charge is somewhat arbitrary. An observer in  $M_+$  would naturally define it with a plus sign but an observer in  $M_-$  with a minus sign.

Consider now the nonstatic wormhole solutions with a timelike shell  $\Sigma$  described in subsection III B. We have

$$(Q^+)_0^{\hat{0}} = -4 \cdot 3! \sqrt{\beta_+^2 + a^2} (1 + a^2 - \frac{1}{3}(\beta_+^2 + a^2)).$$

The other components vanish (by  $\dot{a} = \text{constant}$ ). A similar formula holds for  $Q_0^-$ . For the general case the result is rather nontransparent. Consider the case where the metric in  $M_-$  is flat, i.e.  $\beta_- = -1$ . The result simplifies considerably and one finds, using (28),

$$\tilde{q} = 2 \left( \frac{\beta_+^2 - 1}{3} \right)^{3/2}.$$

Expressed in terms of the speed  $v = dr/dt_-$  measured by the Minkowski observer, this reads:

$$\tilde{q} = 2(1 - v^2)^{-3/2}.$$

Note that in the static case  $v = 0$  we have  $m_+ = 2$  from the formula (26). The nonstatic result is modified by the inverse relativistic factor of the volume of the isotropically expanding 3-sphere.

As noted above  $\tilde{q}$  tells us about the asymmetry of the vacuum wormhole. It is interesting that in the static case it vanishes for unit masses  $m_+ = m_- = 1$ . One can check that  $\tilde{q}$  vanishes in the nonstatic wormhole too when  $m_+ = m_-$ . More generally, in the nonstatic case,  $\tilde{q}$  vanishes on the small ellipses shown in Fig. 4. Also  $\tilde{q}$  goes to infinity at the circle  $\beta_+^2 + \beta_-^2 = 2$  (and also at  $\beta_+ \rightarrow \infty$  and  $\beta_- \rightarrow \infty$ ). It is tempting to conjecture that  $\tilde{q}$  is a kind of gravitational energy of the solitonic shell. This is somewhat speculative but the structure of diagram 4 gives some support to the conjecture. Since the circle represents the limit of the timelike shell solutions in which the speed of the shell approaches the speed of light, it is natural that this energy should go to infinity there.

Alternatively we may say the following. In the usual sense, there is no matter in the vacuum wormhole—the stress-energy tensor is zero everywhere. There is though “mass without mass.” There are two disconnected asymptotic regions in the space-time and no universal notion of mass.  $\tilde{Q}_b^a$  measures this disagreement between asymptotic observers. For the thin-shell wormhole solutions we have found, the conserved quantity  $\tilde{q}$  is nicely expressed in terms of the speed of the shell and  $m_+, m_-$ .

More generally, consider an arbitrary space-time containing a thin shell. A geometrical construction of such a space-time is as follows. Take two space-times which contain submanifolds of codimension 1 which are diffeo-

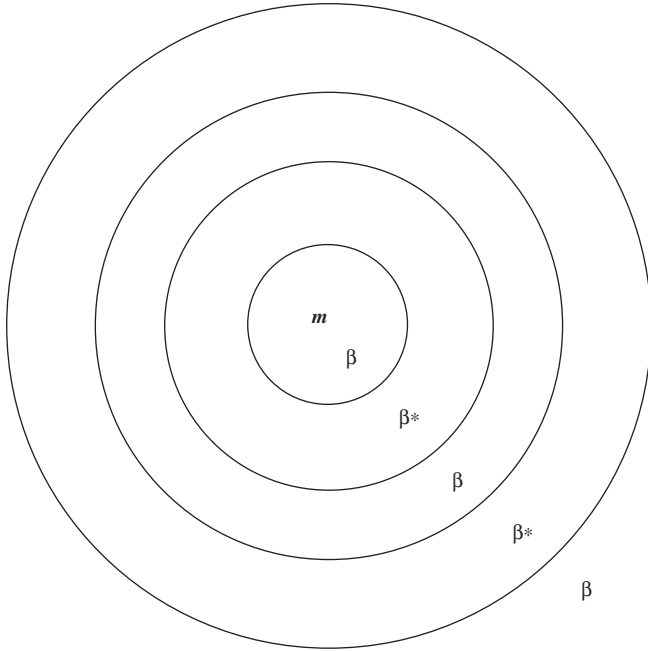


FIG. 5. The hypersurfaces separating regions with  $\beta$  and  $\beta^*$  carry zero energy tensor. For static shells these betas satisfy  $\beta^2 + \beta\beta^* + (\beta^*)^2 = 3$ . A spherically symmetric space-time around a mass  $m$  is infinitely degenerate. This is true for all masses below the critical value  $m_{\text{crit}} = 2$ . For more massive particles gravity is simpler.

morphic to  $\Sigma$ . We can cut and paste in various ways. Let us say that we cut out the region to the right of  $\Sigma$  in the first manifold and cut out the region left of  $\Sigma$  in the second and make the pasting in that way. Now alternatively we could cut out the region left of  $\Sigma$  on both manifolds, flip the orientation of one of them, and paste in that way. What effect does this have on the equations of motion of the shell? It is easy to see that the only effect on the equations of motion for the shell is to swap the orientation of one of the normal vectors which implies  $\Delta K_b^a \leftrightarrow \tilde{K}_b^a$ . Under this transformation, the five equations (10)–(14) are unchanged, except that  $\Delta Q_b^a$  and  $\tilde{Q}_b^a$  swap roles. So  $\tilde{Q}_b^a$  is what the stress tensor on the shell would have been were the orientation of the opposite type, i.e. it measures the energy difference between configurations related by  $\Delta K_b^a \leftrightarrow \tilde{K}_b^a$ . A more detailed study is needed to make this notion more precise.

#### IV. COMMENTS

Spherically symmetric solitonic thin-shell solutions have been classified in the pure Gauss-Bonnet theory in five dimensions. The results are summarized in Fig. 4.

We wish to emphasize two points. First, the pure Gauss-Bonnet theory, with only the quadratic Lovelock term, is not a physical theory. Second, the essential principle that vacuum wormholes can mimic the effect of a mass, cap-

tured here in a simple way, should indeed generalize to more realistic models.

Let us first expand upon the first point. The pure Gauss-Bonnet theory has no Newtonian limit: indeed one can see from the form of the spherically symmetric metric (16) that  $g_{tt} = -1$ . A test particle without angular momentum will feel no central potential.

The theory is also extremely degenerate. One well-known degeneracy of this theory is the absence of a perturbation theory about Minkowski space background. Any perturbation about  $\Omega^{AB} = 0$  is a solution of  $\Omega^{AB} \wedge \Omega^{CD} \epsilon_{ABCDE} = 0$  to first order. Thus, the second variation of the action about a Minkowski background is trivial, i.e. the propagator vanishes.

The point particle metric (16) has another kind of strange degeneracy: it is a solution of the field equations if  $d\Omega^2$  is the metric of a *quite arbitrary* three-manifold. The three-sphere can be replaced with a spheroid, a hypercube, or anything with the topology of a sphere and still be a solution of a point source with a given mass. This is highly counterintuitive—one would expect that a single point source would determine a spherically symmetric space-time. This very interesting arbitrariness is something which merits further investigation.

The solitonic shell solutions are a third example of degeneracy: the radius at which the static solitons are located is arbitrary. It is thus possible to have a space-time composed of different regions with different  $\beta$ 's in concentric layers. A single mass can produce any one of an infinite variety of space-times, with the layers being matched at arbitrary constant radius (see Fig. 5). The degeneracy is particularly striking for Minkowski space. There exists a static solution of type II which matches Minkowski space with a space-time with mass = 2. Now Minkowski space is spherically symmetric about every point so one can put such solitonic shells centered about any point. By combining matchings of type IIa and IIb a very exotic vacuum solution can be constructed as in Fig. 6.

Let us now turn to the second point. The existence of the solitonic shell wormholes does not depend upon the choice of pure Gauss-Bonnet theory. The essential feature is the

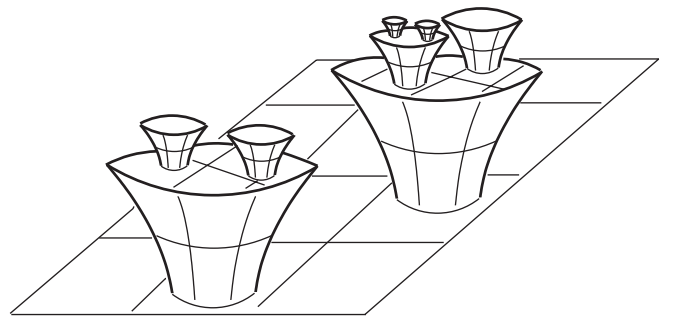


FIG. 6. An exotic vacuum(!) solution of static solitonic shells in Minkowski space, showing the extreme degeneracy of Minkowski space in the pure Gauss-Bonnet theory.



nonlinearity of the junction conditions in the jump in the curvature. Thus, such solutions should exist in the Einstein-Gauss-Bonnet theory and the feature of mass without mass should also be exhibited in that theory. There is a covariantly conserved quantity which plays the role of  $\tilde{Q}_b^a$ , as can be seen from Eq. (B15). A thin-shell vacuum wormhole whose throat is a small sphere can be interpreted as a particle. For such an interpretation to be meaningful, the location of the throat should be stable. In the pure Gauss-Bonnet theory the stability analysis leads to a strange situation: the junction condition gives  $\dot{r} = \text{constant}$ . The static solutions are absolutely fixed in place but at an arbitrary radius. This is special to pure Gauss-Bonnet in which the shell does not accelerate. In Einstein-Gauss-Bonnet the radius of the shell will be like a particle in some nontrivial potential. Indeed, such solutions in the full Einstein-Gauss-Bonnet theory have been found and will be reported in a separate paper of S.W. with C. Garraffo and G. Giribet [23].

The solutions for the toy model (pure Gauss-Bonnet) have a simple structure captured in Fig. 4. This is especially simple because of the close relation between the field equation and the Gauss-Bonnet theorem. Indeed, for the static case the mass in a given region is exactly equal to the integral of the Euler density in that region. Let us see how this comes about: For the static solutions the space-time is a trivial product of a spatial four-manifold with the time direction  $M = M^4 \times \mathbb{R}^1$ . The Gauss-Bonnet theorem for the spatial section  $M^4$  takes the form:

$$\int_{M^4} \Omega^{AB} \wedge \Omega^{CD} \epsilon_{ABCD0} - \int_{\partial M^4} Q_0 = -32\pi^2 \chi(M^4).$$

The first term on the left is proportional to the integral of  $T_0^0$ . The boundary term is the same  $Q_0$  which appears in the junction conditions. This explains the mass formula (20) and its simple relation to the junction conditions for the static shell. Also we see why the solutions of type IIa, the closed universe, have a sum of masses equal to 2, the Euler number of the spatial manifold, just as for G. R. in  $2 + 1$  dimensions.

In the full Einstein-Gauss-Bonnet theory, solitonic shell solutions should have a much more complicated and rich structure. A useful intermediate step between our toy model and the full theory is the case of pure Gauss-Bonnet with cosmological constant. There one can explore nontrivial features (horizons, etc.) in a simple setting. This is an open problem.

Finally, some comments on the meaning of the solutions are in order.

Space-like solitonic shells mean lack of determinism, which is rather a generic feature of Lovelock gravity. For our spherically symmetric ansatz, these solutions are eliminated for  $\beta > \sqrt{2}$ , i.e.  $m > 1$ . The circle in Fig. 4 provides a nice separation between the timelike solitonic shells,

whose behavior is determined by initial conditions, and the spacelike (instanton) shells.

The solutions show that when the Gauss-Bonnet term is included, wormhole solutions can exist without an exotic stress tensor as the source. Indeed here the stress tensor vanishes. This is in marked contrast to the situation in Einstein's theory. Thin-shell wormholes were first studied in Einstein gravity in [24]. Also some effects of the Gauss-Bonnet term as a correction were studied in [25]. The fact that wormholes require “exotic matter” in Einstein gravity was already discussed in [26]. Wormhole solutions with matter source in Einstein-Gauss-Bonnet have been considered in the past [27,28]. There is even another example of a vacuum wormhole which is already known [29]. This is a smooth wormhole and exists in the Lovelock theory with a special choice of coefficients such that the uniqueness theorem for the Boulware-Deser solution does not hold [8,30]. The wormholes found in this present work are nonsmooth, the curvature which defines the throat is localized in a delta function at the shell.

The wormhole solutions found here exemplify the concept of “mass without mass.” It would be interesting to see if, when one considers the Gauss-Bonnet-Maxwell theory, wormholes can be found with “charge without charge.” The field equations of the Gauss-Bonnet theory allow for nonvanishing torsion. Perhaps, by considering wormholes with torsion, one can create the illusion of a source for the torsion, “spin without spin.”

Whilst this work was in the final stages, a paper appeared treating “matter without matter” in Gauss-Bonnet theory [31], although in a somewhat different context of compactified models in six and higher dimensions.

## ACKNOWLEDGMENTS

We thank E. Ayon-Beato, G. Dotti, C. Garraffo, A. Giacomini, G. Giribet, J. Oliva, M. Ramirez, R. Troncoso, and J. Zanelli for useful discussions. S.W. was supported in part by FONDECYT Grant No. 3060016. Institutional support to the Centro de Estudios Científicos (CECS) from Empresas CMPC is gratefully acknowledged. CECS is funded in part by grants from the Millennium Science Initiative, Fundación Andes, the Tinker Foundation.

## APPENDIX A: SOME DEFINITIONS

In studying Lovelock gravity, it is useful to introduce the differential form notation [32]. We introduce the vielbein  $E^A$  and the spin connection  $\omega^{AB}$ . The curvature two-form is

$$\Omega^{AB} \equiv d\omega^{AB} + \omega^A_C \wedge \omega^{CB} = \frac{1}{2} \mathcal{R}^{AB}_{CD} E^C \wedge E^D.$$

In this notation, the Gauss-Bonnet term is:

$$\mathcal{L}_{GB} = \Omega^{AB} \wedge \Omega^{CD} \wedge E^F \epsilon_{ABCDF}.$$

In this article, it is assumed that there is no torsion. The spin connection is the Levi-Civita connection, i.e. an im-

explicit function of the vielbein. The explicit variation with respect to the spin connection is a total derivative which contributes nothing to the field equations. Euler-Lagrange variation of the action w.r.t. the vielbein gives the field equation:

$$\Omega^{AB} \wedge \Omega^{CD} \epsilon_{ABCD} = -2T_F. \quad (\text{A1})$$

On the right-hand side,  $T_F$  is the stress-energy four-form coming from the matter part of the action. This is the dual of the stress-energy tensor, which we take to be of the form:

$$T_{AB} = \begin{pmatrix} S_{ab} \delta(\Sigma) & T_{Nb} \\ T_{aN} & T_{NN} \end{pmatrix}. \quad (\text{A2})$$

We consider a single hypersurface  $\Sigma$  which divides the bulk space-time into two regions  $M_1$  and  $M_2$ . It is helpful to use the basis  $e_A = (e_a, n)$  adapted to the hypersurface such that  $e_a$  are tangential vectors and  $n$  is a normal vector. The vielbeins  $E^A = (E^a, E^N)$  are the dual basis of one-forms. We shall assume that the normal vector  $n$  can be spacelike ( $\sigma \equiv n \cdot n = -1$ ) or timelike ( $\sigma = 1$ ) but not null.

At the hypersurface there is a Levi-Civita connection associated with each region:  $\omega_+^{AB}$  and  $\omega_-^{AB}$ , respectively. Let  $i^*$  denote the pullback of differential forms onto  $\Sigma$ . Then the intrinsic connection on  $\Sigma$  is

$$\omega_{\parallel}^{ab} = i^* \omega_+^{ab} = i^* \omega_-^{ab}.$$

Let us define

$$\theta_+^{AB} \equiv \omega_+^{AB} - \omega_{\parallel}^{AB}.$$

The second fundamental form on  $\Sigma$  induced by  $M_+$  is  $i^* \theta_+^{AB}$  and has components  $i^* \theta_+^{Na} = i^* \omega_+^{Na}$  and  $i^* \theta_+^{ab} = 0$ . It is related to the extrinsic curvature tensor by:

$$i^* \theta_N^a = K^a_b E^b. \quad (\text{A3})$$

Similarly, the second fundamental form induced by  $M_-$  is denoted by  $i^* \theta_-^{AB}$ .

## APPENDIX B: PROOF OF THE FIVE EQUATIONS

In analyzing the field equations, it is useful to introduce a test field,  $\lambda^A$ , which is an arbitrary vector valued one-form. The field equations are:

$$\epsilon(\Omega \Omega \lambda) = -2T_A \lambda^A, \quad (\text{B1})$$

where, to simplify notation, it is convenient to omit indices which are all contracted with the epsilon tensor, e.g.:

$$\epsilon(\Omega \Omega \lambda) := \Omega^{AB} \wedge \Omega^{CD} \wedge \lambda^F \epsilon_{ABCD}.$$

The derivation of the five equations by decomposing the field equations and using the Bianchi identities is purely a technical one. Applying the five-dimensional Bianchi identities when the curvature has been decomposed into intrinsic and extrinsic curvature can be a mess. One elegant way around the problem is to borrow from the textbook [33]

proof of the Chern-Weil theorem. This has the advantage that the proof generalizes easily to Lovelock theory in arbitrary dimensions. Let  $\omega_t$  be a connection which interpolates between  $\omega_+$  and  $\omega_{\parallel}$ ,

$$\omega_t = (1-t)\omega_{\parallel} + t\omega_+. \quad (\text{B2})$$

Similarly, one can also interpolate between  $\omega_-$  and  $\omega_{\parallel}$ . Then, using the Bianchi identity  $D(\omega_t)\Omega_t = 0$ , the following identity can be derived:

$$\epsilon(\Omega_+ \Omega_+ \lambda) - \epsilon(\Omega_{\parallel} \Omega_{\parallel} \lambda) = 2 \int_0^1 dt \epsilon(D(\omega_t)\{\theta_+ \Omega_t\} \lambda), \quad (\text{B3})$$

where  $D(\omega_t)$  is the covariant exterior derivative and  $\Omega_t^{AB} = d\omega_t^{AB} + \omega_{tC}^A \omega_t^{CB}$  the curvature with respect to the interpolating connection. The above expression can also be rewritten in terms of the covariant derivative with respect to the intrinsic connection:

$$\begin{aligned} \epsilon(\Omega_+ \Omega_+ \lambda) - \epsilon(\Omega_{\parallel} \Omega_{\parallel} \lambda) &= 2 \int_0^1 dt \epsilon(D(\omega_{\parallel})\{\theta_+ \Omega_t\} \lambda) \\ &\quad + 2 \int_0^1 dt t \theta_+^{AB} \Omega_t^{CD} \\ &\quad \times \epsilon_{ABCDE} \theta_{+F}^E \lambda^F. \end{aligned} \quad (\text{B4})$$

Note that in the above,  $D(\omega_{\parallel})$  is a five-dimensional derivative operator. Its projection along the basis of tangential one-forms is the intrinsic covariant derivative. Its projection along  $E^N$  is just the partial derivative in the normal direction  $n^\mu \partial_\mu$ .

We can break down (B4) into various components:

- (i)  $\lambda$  is a normal one-form with a tangential vector index, i.e. the normal-tangent component of the field equations. In this case the second term on the left and the second term on the right do not contribute and we obtain

$$i^* \epsilon(\Omega_+ \Omega_+)_a = i^* D(\omega_{\parallel}) Q_a^+. \quad (\text{B5})$$

We have defined the useful quantity  $Q_e$ :

$$Q_e \equiv i^* 4 \theta_N^b (\sigma \Omega_{\parallel}^{cd} - \frac{1}{3} \theta_N^c \theta_N^d) \epsilon_{Nbcde}. \quad (\text{B6})$$

Note that this quantity is closely related to the boundary term for the Gauss-Bonnet action for a manifold with boundary [34].

- (ii)  $\lambda$  is a normal one-form with a normal vector index, i.e. the normal-normal component of the field equations. In this case we get

$$\begin{aligned} i^* \epsilon(\Omega_+ \Omega_+)_N &= \# \mathcal{H}_{\perp}, \\ \# \mathcal{H}_{\perp} &\equiv i^* (-\sigma \Omega_{\parallel}^{bc} + \theta_N^b \theta_N^c) \\ &\quad \times (-\sigma \Omega_{\parallel}^{de} + \theta_N^d \theta_N^e) \epsilon_{Nbcde}. \end{aligned} \quad (\text{B7})$$

The above formula can be obtained immediately, without reference to (B4), by using the Gauss equation.

- (iii)  $\lambda$  is a tangential one-form with a normal vector index. This gives the same as case (i) (in the absence of torsion, the stress tensor is symmetric).
- (iv)  $\lambda$  is a tangential one-form with tangential vector index. Integrating this across a region of infinitesimal thickness across  $\Sigma$  gives the known junction conditions [2,3]:

$$\Delta Q_a := -2S_a. \quad (\text{B8})$$

We can now substitute the expressions (B5) and (B7) into the field equations. It is most instructive to evaluate the field equations on the left and the right and then to consider the sum and the difference:

$$i^*D(\omega_{\parallel})S_a = i^*\Delta T_a, \quad (\text{B9})$$

$$i^*D(\omega_{\parallel})\tilde{Q}_a = -2i^*\tilde{T}_a, \quad (\text{B10})$$

$$\Delta\#\mathcal{H}_{\perp} = -2i^*\Delta T_N, \quad (\text{B11})$$

$$\widetilde{\#\mathcal{H}}_{\perp} = -2i^*\tilde{T}_N. \quad (\text{B12})$$

The five equations (B8)–(B12) are the equations of motion of  $\Sigma$  in the absence of torsion, written in terms of differential forms.

Using Eq. (A3) one obtains:

$$Q_b = K^c{}_f(2\sigma R_{gh}^{de} - \frac{4}{3}K^d{}_g K^e{}_h)\epsilon_{Nbcde}E^f \wedge E^g \wedge E^h.$$

Dualizing with respect to  $\epsilon^{Nafgh}$  one obtains  $Q^a{}_b$  given by expression (9). Dualizing  $\#\mathcal{H}_{\perp}$  one obtains the scalar:

$$\begin{aligned} \mathcal{H}_{\perp} &= \left(\frac{\sigma}{2}R^{ab}{}_{ef} - K^a{}_e K^b{}_f\right)\left(\frac{\sigma}{2}R^{cd}{}_{gh} - K^c{}_g K^d{}_h\right)\delta^{efgh} \\ &= \left(\frac{1}{4}R^{ab}R^{cd}{}_{gh} + \frac{\sigma}{2}K^a{}_e K^b{}_f R^{cd}{}_{gh}\right)\delta^{efgh} \\ &\quad - \frac{3}{4}K^a{}_b Q^b{}_a. \end{aligned}$$

Thus, dualizing Eqs. (B8)–(B12) gives the five equations in tensor form. We can combine with the Einstein term to give the conditions for the general Einstein-Gauss-Bonnet theory described by the action (7),

$$-c_1\sigma(\Delta K_b^a - \delta_b^a\Delta K) - \frac{c_2}{2}\Delta Q_b^a = S_b^a, \quad (\text{B13})$$

$$-c_1\sigma(\Delta K_b^a - \delta_b^a\Delta K)_{;a} - \frac{c_2}{2}\Delta Q^a{}_{b;a} = \Delta T^N{}_b, \quad (\text{B14})$$

$$-c_1\sigma(\tilde{K}_b^a - \delta_b^a\tilde{K})_{;a} - \frac{c_2}{2}\tilde{Q}^a{}_{b;a} = \tilde{T}^N{}_b, \quad (\text{B15})$$

$$\frac{c_1\sigma}{4}\Delta K_c^a \tilde{K}_d^b \delta_{ab}^{cd} - \frac{c_2}{2}\Delta\mathcal{H}_{\perp} = \Delta T^N, \quad (\text{B16})$$

$$-c_0 - c_1\left(R - \frac{\sigma}{4}\{\tilde{K}_c^a \tilde{K}_d^b + \Delta K_c^a \Delta K_d^b\}\delta_{cd}^{ab}\right) - \frac{c_2}{2}\tilde{\mathcal{H}}_{\perp} = \tilde{T}^N. \quad (\text{B17})$$

We show now that the junction condition (B8) is well defined. The thin-shell limit is well defined in the following sense: starting from a thick shell, in the limit that its thickness becomes zero, the results are insensitive to the way in which the limit is taken. This has been discussed at some length in the literature (e.g. [35]) but it is worth giving a precise statement of this here, since the subject still causes some confusion. To see this more explicitly, let us define a family of metrics  $g_{AB}^{(\alpha)}$ , parametrized by a positive number  $\alpha$ , which describe a thick shell of characteristic thickness  $\propto 1/\alpha$ . We can foliate the neighborhood of the shell into tangential slices and a normal vector  $N^{(\alpha)}$ . Let us define  $\theta^{(\alpha)} \equiv \omega^{(\alpha)} - \omega_{\parallel}$ , where  $\omega_{\parallel}$  is the intrinsic connection induced on the slice. In the limit  $\alpha \rightarrow \infty$ , the one-form  $\theta^{(\alpha)}$  tends towards something discontinuous: it is equal to  $\theta_+$  in one region and  $\theta_-$  in the other region. So the components of  $\theta$  become discontinuous.

Let us now look at what happens to the field equations,  $\epsilon(\Omega\Omega\lambda) = -2T_A\lambda^A$ . From the identity (B3) we obtain

$$\begin{aligned} \epsilon(\Omega\Omega\lambda) - \epsilon(\Omega_{\parallel}\Omega_{\parallel}\lambda) &= d\left\{2\int_0^1 dt \epsilon(\theta\Omega_t\lambda)\right\} \\ &\quad + 2\int_0^1 dt \theta^{AB}\Omega_t{}^{CD} \\ &\quad \times \epsilon_{ABCDE}D(\omega_t)\lambda^E. \end{aligned} \quad (\text{B18})$$

We integrate the identity (B18) over the thick shell. The potentially singular terms are those which contain the normal derivative of  $\theta^{(\alpha)}$ . Everything else is smooth or remains finite. The first term on the right-hand side (r.h.s.) of (B18) gives the junction condition (B8). The claim is that this term contains all the singular terms. From the second term on the r.h.s. of (B18) the normal derivative of  $\theta^{(\alpha)}$  appears as

$$\epsilon_{ABCDE}\theta_a{}^{AB}\partial_N\theta_b{}^{CD}E^N \wedge E^a \wedge E^b \wedge D(\omega_t)\lambda^E, \quad (\text{B19})$$

using an adapted frame  $(E^N, E^a)$  on the foliation. The index  $\alpha$  will be dropped from now on. Now  $\theta_a{}^{AB}$  are the components of the second fundamental form of a slice in the foliation: one of the indices  $A, B$  is a normal index. Thus two indices contracted to the antisymmetric symbol  $\epsilon_{ABCDE}$  are normal. The quantity (B19) vanishes identically. The integral of the second term in the r.h.s. of (B4) goes to zero for  $\alpha \rightarrow \infty$ . The discontinuity of  $\theta$  is contained in a total derivative. The singular part of the field equations is well defined as a Dirac  $\delta$  distribution.

Let us say a few more words. The equations of motion  $\epsilon(\Omega\Omega\lambda)$  have a singular term of the form  $\psi\phi\partial_N\chi$ , where  $\psi$ ,  $\phi$  and  $\chi$  are three different components of  $\theta$ . In general, these components will converge in the weak sense to a Heaviside type of distribution  $H$  in different ways, so the

product does not in general tend towards an unambiguous distribution (i.e. it is not exactly of the form  $H^2(x)\delta(x) = \delta(x)/3$ , where  $\delta$  is the Dirac  $\delta$  distribution, as would happen with a single function discontinuous at  $x = 0$ ). However, as discussed above, the simple fact is that they always appear in a combination which is a total derivative  $\partial_N(\psi\phi\chi)$ . This product of functions in the brackets is a single function which converges to  $H$ . Its derivative is unambiguously defined as a distribution in the limit as  $\delta$ .

A corollary of this and of the comments above is that the integral of the field equations of a physically thin shell is well described by these equations, i.e. if we allow for the shell to have a little thickness then in a first order approximation the results do not change if we change the configuration in the interior. The integrated stress tensor  $S_{ab}$  is unambiguously related to  $\theta_+$  and  $\theta_-$ , the values of  $\theta$  on each side of the shell.

- 
- [1] W. Israel, *Nuovo Cimento B* **44S10**, 1 (1966); **48**, 463(E) (1967); **44**, 1 (1966).
- [2] S. C. Davis, *Phys. Rev. D* **67**, 024030 (2003).
- [3] E. Gravanis and S. Willison, *Phys. Lett. B* **562**, 118 (2003).
- [4] K. A. Meissner and M. Olechowski, *Phys. Rev. Lett.* **86**, 3708 (2001); A. Iglesias and Z. Kakushadze, *Int. J. Mod. Phys. A* **16**, 3603 (2001).
- [5] C. W. Misner and J. A. Wheeler, *Ann. Phys. (N.Y.)* **2**, 525 (1957).
- [6] D. Lovelock, *J. Math. Phys. (N.Y.)* **12**, 498 (1971).
- [7] J. E. Kim, B. Kyae, and H. M. Lee, *Nucl. Phys.* **B582**, 296 (2000); **B591**, 587(E) (2000); I. Low and A. Zee, *Nucl. Phys.* **B585**, 395 (2000); N. E. Mavromatos and J. Rizos, *Phys. Rev. D* **62**, 124004 (2000); I. P. Neupane, *J. High Energy Phys.* 09 (2000) 040; J. E. Kim, B. Kyae, and H. M. Lee, *Phys. Rev. D* **64**, 065011 (2001).
- [8] C. Charmousis and J. F. Dufaux, *Classical Quantum Gravity* **19**, 4671 (2002).
- [9] K. i. Maeda and T. Torii, *Phys. Rev. D* **69**, 024002 (2004).
- [10] C. Barrabes and W. Israel, *Phys. Rev. D* **71**, 064008 (2005); A. N. Aliev, H. Cebeci, and T. Dereli, *Classical Quantum Gravity* **23**, 591 (2006); T. Kobayashi, T. Shiromizu, and N. Deruelle, *Phys. Rev. D* **74**, 104031 (2006).
- [11] K. Konya, gr-qc/0605119.
- [12] G. A. S. Dias, S. Gao, and J. P. S. Lemos, *Phys. Rev. D* **75**, 024030 (2007).
- [13] J. Crisostomo, S. del Campo, and J. Saavedra, *Phys. Rev. D* **70**, 064034 (2004).
- [14] J. Crisostomo and R. Olea, *Phys. Rev. D* **69**, 104023 (2004).
- [15] M. Hassaine, R. Troncoso, and J. Zanelli, *Phys. Lett. B* **596**, 132 (2004).
- [16] A. H. Chamseddine, *Phys. Lett. B* **233**, 291 (1989).
- [17] A. H. Chamseddine, *Nucl. Phys.* **B346**, 213 (1990).
- [18] A. Achucarro and P. K. Townsend, *Phys. Lett. B* **180**, 89 (1986).
- [19] E. Witten, *Nucl. Phys.* **B311**, 46 (1988).
- [20] C. Teitelboim and J. Zanelli, *Classical Quantum Gravity* **4**, L125 (1987).
- [21] D. G. Boulware and S. Deser, *Phys. Rev. Lett.* **55**, 2656 (1985).
- [22] R. G. Cai and N. Ohta, *Phys. Rev. D* **74**, 064001 (2006).
- [23] C. Garraffo, G. Giribet, and S. Willison (unpublished).
- [24] M. Visser, *Nucl. Phys.* **B328**, 203 (1989).
- [25] M. Thibeault, C. Simeone, and E. F. Eiroa, *Gen. Relativ. Gravit.* **38**, 1593 (2006).
- [26] M. S. Morris, K. S. Thorne, and U. Yurtsever, *Phys. Rev. Lett.* **61**, 1446 (1988); A more recent reference is: M. Visser, S. Kar, and N. Dadhich, *Phys. Rev. Lett.* **90**, 201102 (2003).
- [27] K. Ghoroku and T. Soma, *Phys. Rev. D* **46**, 1507 (1992).
- [28] B. Bhawal and S. Kar, *Phys. Rev. D* **46**, 2464 (1992).
- [29] G. Dotti, J. Oliva, and R. Troncoso, *Phys. Rev. D* **75**, 024002 (2007).
- [30] R. Zegers, *J. Math. Phys. (N.Y.)* **46**, 072502 (2005).
- [31] H. Maeda and N. Dadhich, *Phys. Rev. D* **75**, 044007 (2007).
- [32] T. Eguchi, P. B. Gilkey, and A. J. Hanson, *Phys. Rep.* **66**, 213 (1980); B. Zumino, *Phys. Rep.* **137**, 109 (1986).
- [33] Y. Choquet-Bruhat, C. DeWitt-Morette, and M. Dillard-Bleick, *Analysis, Manifolds and Physics* (North-Holland, Amsterdam, 1982).
- [34] R. C. Myers, *Phys. Rev. D* **36**, 392 (1987).
- [35] N. Deruelle and C. Germani, *Nuovo Cimento Soc. Ital. Fis.* **118B**, 977 (2003).