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Tunnelling from Gödel black holes

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We consider the spacetime structure of Kerr-Gödel black holes, analyzing their parameter space in detail. We apply the tunnelling method to compute their temperature and compare the results to previous calculations obtained via other methods. We claim that it is not possible to have the closed timelike curve (CTC) horizon in between the two black hole horizons and include a discussion of issues that occur when the radius of the CTC horizon is smaller than the radius of both black hole horizons.

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I. INTRODUCTION

There has been a fair amount of activity in recent years studying Gödel-type solutions to 5d supergravity [1–14]. Various black holes embedded in Gödel universe backgrounds have been obtained as exact solutions [2,4,10] and their string-theoretic implications make them a lively subject of interest. For example Gödel-type solutions have been shown to be T-dual to pp-waves [3–5]. Since closed timelike curves (CTCs) exist in Gödel spacetimes, these solutions can be used to investigate the implications of CTCs for string theory [6,7,9,11].

The black hole solutions are of the Schwarzschild-Kerr type embedded in a Gödel universe [4]. A study of their thermodynamic behavior [12,14] has indicated that the expected relations of black hole thermodynamics are satisfied. Making use of standard Wick-rotation methods, their temperature has been shown to equal $\kappa/2\pi$ (where κ is the surface gravity) their entropy to equal $A/4$ (where A is the surface area of the black hole) and the first law of thermodynamics to be satisfied.

We consider in this paper an analysis of the Kerr-Gödel spacetime, employing the tunnelling method [15–29] to analyze its thermodynamic properties. The tunnelling method is a semiclassical approach to black hole radiation that allows one to calculate the temperature in a manner independent of the traditional Wick-rotation methods of temperature calculation. As such it provides a useful cross-check on the thermodynamic properties of these objects and has been shown to be quite robust, having been applied to a variety of different spacetimes such as the Kerr and Kerr-Newmann cases [18,19,22], black rings [20], the 3d BTZ black holes (a class of 3d black holes originally discovered by Bañados, Teitelboim, and Zanelli) [16,21], and the Vaidya [27] and Taub-NUT spacetimes [22]. The presence of CTCs merits consideration of the applicability of the tunnelling method to Kerr-Gödel spacetimes. Because of the presence of a CTC horizon (in addition to the usual black hole horizons) some qualitatively new features appear. Our investigation of these spacetimes is in large part motivated by the fact that these new features provide additional tests as to the robustness of the tunnelling approach.

We begin by reviewing the Kerr-Gödel spacetime and some of its properties. We then describe, in Sec. III, properties of its parameter space and show that either the CTC horizon is outside both black hole horizons, inside both black hole horizons, or in coincidence with one of the horizons. We claim that it is not possible for the CTC horizon to be strictly in between the two black hole horizons, a property previously overlooked in discussions of this spacetime [14]. We then quickly review the tunnelling method and apply it to calculate the temperature of Kerr-Gödel spacetimes, showing consistency with previous results. We extend our investigation further insofar as we include a brief discussion of the issues that occur when the CTC horizon is inside the black hole horizons.

II. REVIEW OF 5D KERR-GÖDEL SPACETIMES

The 5d Kerr-Gödel spacetime has the metric [4]

$$ds^2 = -f(r)\left(dt + \frac{a(r)}{f(r)}\sigma_3\right)^2 + \frac{dr^2}{V(r)} + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2) + \frac{r^2 V(r)}{4f(r)}\sigma_3^2, \tag{1}$$

$$A = \frac{\sqrt{3}}{2} j r^2 \sigma_3, \tag{2}$$

where

$$f(r) = 1 - \frac{2m}{r^2}, \qquad a(r) = jr^2 + \frac{ml}{r^2},$$

$$V(r) = 1 - \frac{2m}{r^2} + \frac{16j^2m^2}{r^2} + \frac{8jml}{r^2} + \frac{2ml^2}{r^4},$$

and the σ's are the right-invariant one-forms on SU(2), with Euler angles (θ, ϕ, ψ):

$$\sigma_1 = \sin\phi d\theta - \cos\phi \sin\theta d\psi,$$

$$\sigma_2 = \cos\phi d\theta + \sin\phi \sin\theta d\psi, \qquad \sigma_3 = d\phi + \cos\theta d\psi.$$

This metric may be obtained by embedding the Kerr black

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hole metric (with the two possible rotation parameters set to the same value i.e. $l_1 = l_2 = l$) in a 5d Gödel universe.

This metric and gauge field satisfy the following 4 + 1-dimensional equations of motion:

$$R_{\mu\nu} = 2\left(F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{6}g_{\mu\nu}F^2\right),$$
$$D_\mu F^{\mu\nu} = \frac{1}{2\sqrt{3}}\tilde{\epsilon}^{\alpha\beta\gamma\mu\nu}F_{\alpha\beta}F_{\gamma\mu},$$

where

$$\tilde{\epsilon}_{\alpha\beta\gamma\mu\nu} = \sqrt{-g}\epsilon_{\alpha\beta\gamma\mu\nu}.$$

Some other useful ways to write the metric (1) are the expanded form

$$ds^2 = -f(r)dt^2 - 2a(r)dt\sigma_3 + g(r)\sigma_3^2 + \frac{dr^2}{V(r)} + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2),$$

where

$$g(r) = \frac{r^2V(r) - 4a^2(r)}{4f(r)} = -j^2r^4 + \frac{1 - 8mj^2}{4}r^2 + \frac{ml^2}{2r^2}, \tag{3}$$

and the lapse-shift form

$$ds^2 = -N^2dt^2 + g(r)\left(\sigma_3 - \frac{a(r)}{g(r)}dt\right)^2 + \frac{dr^2}{V(r)} + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2), \tag{4}$$

where

$$N^2 = f(r) + \frac{a^2(r)}{g(r)} = \frac{r^2V(r)}{4g(r)}.$$

When the parameters j and l are set to zero, the metric simply reduces to the 5d Schwarzschild black hole, whose mass is proportional to the parameter m. The parameter j is the Gödel parameter and is responsible for the rotation of the spacetime; when $m = l = 0$ the metric reduces to that of the 5d Gödel universe [1]. The parameter l is related to the rotation of the black hole. When $j = 0$ this reduces to the 5d Kerr black hole with the two possible rotation parameters (l_1, l_2) of the general 5d Kerr spacetime set equal to l. When $l = 0$, the solution becomes the Schwarzschild-Gödel black hole. The metric is well behaved at the horizons and the scalars only become singular at the origin. It has been noted recently that the gauge field is not well behaved at the horizons [14] although it is possible to pass to a new gauge potential that is well behaved. When $g(r) < 0$ then ∂_ϕ will be timelike, indicating the presence of closed timelike curves since ϕ is periodic. The point at which $g(r) = 0$ is where the lapse (N^2) becomes infinite, implying that nothing can cross over to the CTC region from the region without CTC's. This property is implied by the geodesic solutions for Schwarzschild-Gödel found in [4], but we will argue later in the paper that this is a general property of Kerr-Gödel. The lapse vanishes when $V(r) = 0$; these points correspond to the black hole horizons.

The function $f(r)$ is equal to zero when $r = \sqrt{2m}$, corresponding to an ergosphere. The angular velocity of locally nonrotating observers is given by $\Omega = \frac{d\phi}{dt} = \frac{a(r)}{g(r)}$ with $\Omega_H = \frac{a(r_H)}{g(r_H)}$ denoting the angular velocity of the horizon. There is a special choice of parameters that will cause the angular velocity at the horizon to vanish (besides the trivial $l = j = 0$). When $l = -4jm$ then $V(r) = 0$ has solutions at $r^2 = 2m$ and $r^2 = 16j^2m^2$. The function $a(r)$ will be equal to zero for $r^2 = 2m$. Consequently, Ω_H will vanish for the choice $l = -4jm$ at the horizon $r = \sqrt{2m}$.

For the case $l = 0$, there is only one black hole horizon located at $r_H = \sqrt{2m(1 - 8j^2m)}$. Clearly $1 > 8j^2m$ for the horizon to be well defined. A standard Wick-rotation approach yields a temperature $T_H = \frac{1}{2\pi\sqrt{2m(1-8j^2m)^3}}$ for the Schwarzschild-Gödel black hole [12], where the horizon has angular velocity $\Omega_H = \frac{4j}{(1-8j^2m)^2}$. There will be no CTC's for $r < r_{\rm CTC} = \frac{\sqrt{(1-8mj^2)}}{2j}$ [the region where $g(r) > 0$], and the condition $r_{\rm CTC} > r_H$ corresponds to $1 > 8j^2m$. Hence for $l = 0$ the CTC horizon is always outside of the black hole horizon. This property is not true for $l \neq 0$ and in the next section we will investigate the conditions under which the CTC horizon is no longer outside of the black hole horizons.

III. ANALYSIS OF 5D KERR-GÖDEL

A. Parameter space of the 5d Kerr-Gödel

We will start by examining the parameter space of the 5d Kerr-Gödel spacetimes. The functions of interest are $g(r) = 0$, which determine the location of the CTC horizon, and $V(r) = 0$, which determines the black hole horizons. We wish to find out how the horizons behave in terms of the parameters l and j. To simplify the analysis we will reparameterize as follows:

$$J = j\sqrt{8m}, \qquad L = \frac{l}{\sqrt{2m}}, \qquad x = \frac{r^2}{2m}, \tag{5}$$

so $x = 1$ at the ergosphere $(r^2 = 2m)$, $J^2 = 1$ when $8mj^2 = 1$, and the special case $l = -4jm$ corresponds to the choice $L = -J$.

The equations $V(r) = 0$ and $g(r) = 0$ now correspond, respectively, to the equations

$$\frac{1}{x^2}(x^2 - (1 - J^2 - 2LJ)x + L^2) = 0, \tag{6}$$

$$-\frac{m}{2x}(J^2x^3 - (1-J^2)x^2 - L^2) = 0. \quad (7)$$

There are two solutions to the quadratic Eq. (6), and there is only one real solution to (7) when L is nonzero [it can be shown that when $L = 0$ only the single nonzero solution of (7) is relevant]. The solutions of (6) and (7) are, respectively,

$$x_\pm = \frac{1}{2}\left((1 - J^2 - 2LJ) \pm \sqrt{(1 - J^2 - 2LJ)^2 - 4L^2}\right), \quad (8)$$

$$x_{\rm CTC} = \frac{C(J,L)}{6J^2} + \frac{2(1-J^2)^2}{3J^2C(J,L)} + \frac{1-J^2}{3J^2}, \quad (9)$$

where

$$C(J,L) = \left[108L^2J^4 + 8(1-J^2)^3 + 12J^2\sqrt{3L^2(27L^2J^4 + 4(1-J^2)^3}\right]^{1/3}.$$

The black hole is extremal when $x_+ = x_-$ and will occur when $J = \pm 1$, $J = -2L + 1$, $J = -2L - 1$. All three horizons will coincide when $J = -L = \pm 1$. Note that black hole horizons only exist when $x_\pm > 0$ since the horizon radii $r_\pm = \sqrt{2mx_\pm}$.

In Fig. 1 we show a 3d plot of $\sqrt{x_{\rm CTC}} - \sqrt{x_+}$ in terms of L and J. Note that when $J^2 > 1$ the value of $\sqrt{x_{\rm CTC}} - \sqrt{x_+}$ is negative so the CTC horizon is inside the black hole horizon. In order to get a feel for how the horizons behave it is useful to plot all three horizons (inner, outer, and CTC) together for special values of J. The choices of J that are interesting are $J = -L$ (which is when $\Omega_H = 0$ at the horizon located at $x = 1$), and the extremal cases $J = 1$ and $J = -2L - 1$. These plots are shown in Figs. 2, 3(a), and 3(b), respectively. Notice that for Fig. 2 the CTC horizon is either outside both of r_+ and r_- (i.e. the black hole horizons) or inside both r_+ and r_- (this is also trivially true for the other two plots since they are extremal black holes). In all three plots the change from CTCs outside the black hole horizons to inside the horizons occurs when you go beyond the points $J = -L = \pm 1$.

We claim that is not possible to have the CTC horizon located in between the two black hole horizons. Assuming the contrary, consider the problem of finding values of J and L when the CTC horizon is in between the two black hole horizons. We first look for solutions when the CTC horizon is in coincidence with one of the black hole horizons. We find that $x_{\rm CTC} = x_\pm$ when the equation

$$(3J^2 + 2JL - 2)^2 + 4J^2 - 5J^4 = 0 \quad (10)$$

holds. Notice that $J = -L = \pm 1$ are solutions to (10).

An analysis of the curve resulting from the left-hand side of (10) indicates that when both $J^2 < 1$ and $L^2 < 1$ then the CTC horizon is coincident with the outer horizon; on either side of this curve the CTC horizon is outside both black hole horizons. When both $J^2 > 1$ and $L^2 > 1$ then the CTC horizon is coincident with the inner horizon, and on either

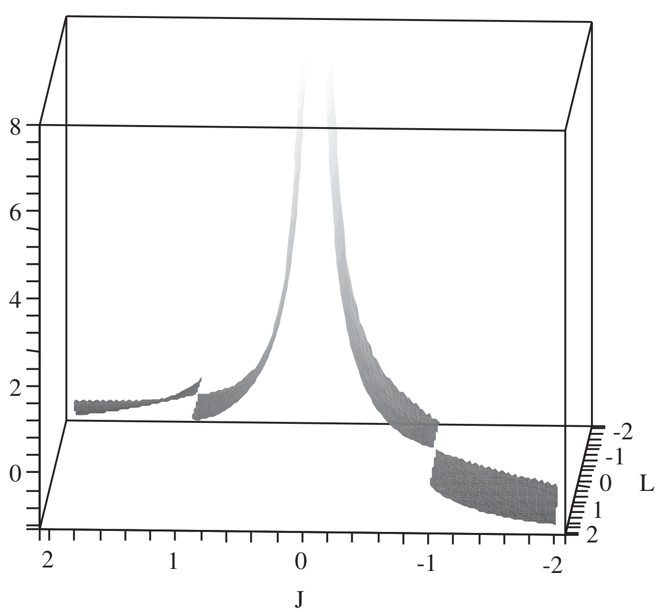

FIG. 1. 3d plot of $\sqrt{x_{\rm ctc}} - \sqrt{x_+}$ in terms of L and J (i.e. compares location of the CTC horzion to largest black hole horizon) Note: Regions when $J^2 > 1$ are negative which means the CTC horizon is inside the black hole horizon. The peak at $J = 0$ corresponds to the infinite CTC horizon and indicates regular 5d Kerr.

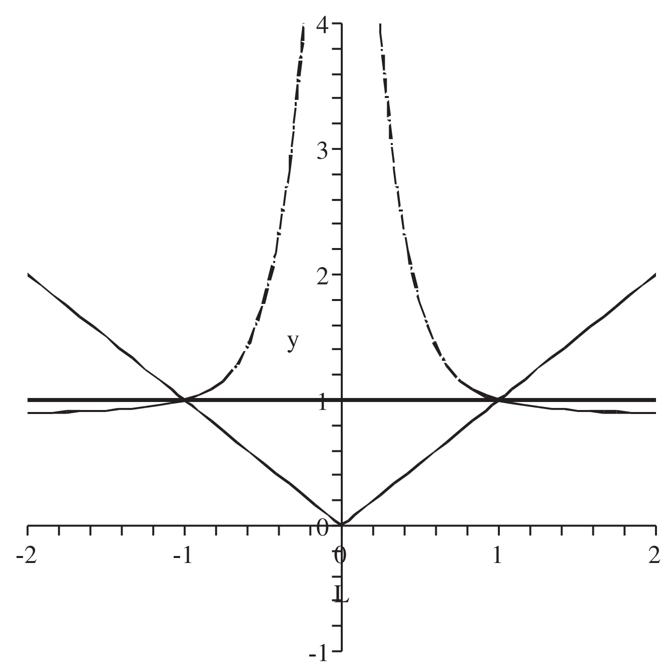

FIG. 2. Plots of $\sqrt{x_{\rm ctc}}$, $\sqrt{x_-}$, $\sqrt{x_+}$ it terms of L and when $J = -L$, Note: Dashed line corresponds to $\sqrt{x_{\rm ctc}}$ and solid lines are $\sqrt{x_-}$, $\sqrt{x_+}$. Notice the CTC's horizon is either outside both horizons or inside both horizons but never in between.

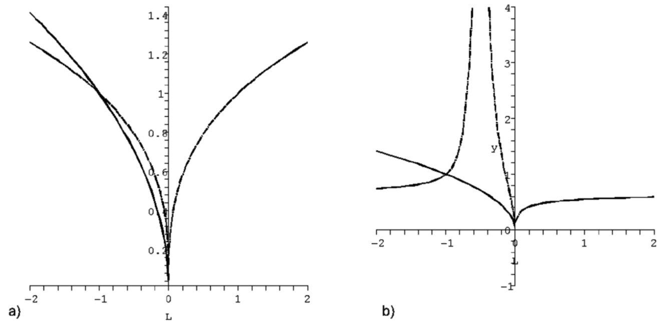

FIG. 3. Plots of $\sqrt{x_{\rm ctc}}$, $\sqrt{x_-}$, $\sqrt{x_+}$ in terms of (a) L when $J = 1$, (b) L when $J = -2L - 1$ Note: The dashed line corresponds to $\sqrt{x_{\rm CTC}}$ and solid lines are $\sqrt{x_-}$, $\sqrt{x_+}$ (these are extremal cases so $\sqrt{x_-} = \sqrt{x_+}$). Notice, black hole horizons do not exist for (a) when $L > 0$ and (b) when $L > 0$. Also in (b) $\sqrt{x_{\rm CTC}}$ is infinite at $L = -\frac{1}{2}$ because $J = 0$ which is 5d Kerr.

side of this curve the CTC horizon is inside both r_+ and r_-. In all other regions of parameter space the metric (4) has naked singularities. Figure 4 illustrates this behavior in terms of J and L. In the gray region the CTC horizon is outside both black hole horizons. In the black region the CTC horizon is inside both r_+ and r_-. The white line corresponds to the curves resulting from (10). In the white region the metric has no black hole horizons and naked singularities are present.

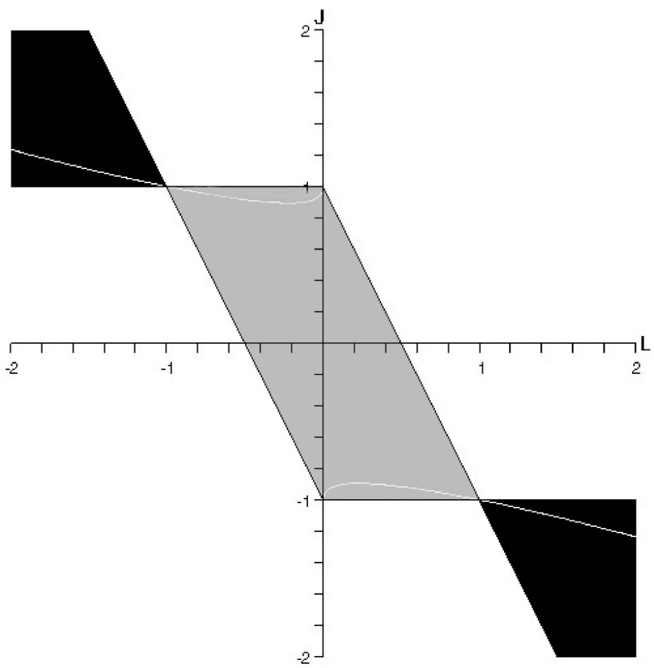

FIG. 4. Plot of the horizon behavior in terms of J and L. The white region corresponds to naked singularities (no black hole horizons). In the gray region the CTC horizon is outside both black hole horizons. In the black region the CTC horizon is inside both black hole horizons. The white line corresponds to the special case when the CTC horizon is in coincidence with a black hole horizon (outer horizon in gray region, inner horizon in black region, and both at the special points $J = -L = \pm 1$).

An alternate verification for the fact that the CTC horizon is never in between the black hole horizons may be obtained by substituting $x_\pm$ into $g(x)$, which shows that when $J^2 < 1$ then $g(x_-) > 0$, $g(x_+) \geq 0$ and for $J^2 > 1$ then $g(x_-) \leq 0$, $g(x_+) < 0$ (plots not shown). Conceptually it is easy to see why this property must be true by looking at the definition (3) of the function $g(r)$, which defines where the CTC horizon must be located. If $r = r_{\rm CTC}$ then $g(r_{\rm CTC}) = 0$ which implies $r_{\rm CTC}^2 V(r_{\rm CTC}) = 4a^2(r_{\rm CTC})$. For this equality to be true then $V(r_{\rm CTC})$ must be positive since every other term in the equation is positive. Since $V(r_{\rm CTC})$ cannot be negative then $r_{\rm CTC}$ cannot be in between r_- and r_+.

Another property worth mentioning is the location of the black hole horizons in relation to $x = 1$ (i.e. the ergosphere $r = \sqrt{2m}$). When $J^2 < 1$ then $x_\pm \leq 1$ so the horizons are inside the ergosphere. When $J^2 > 1$ then $x_\pm \geq 1$ so the "horizons" are outside the ergosphere. Indeed when $J^2 > 1$ the surfaces $x_\pm = 1$ are not actually horizons, though we have been using this term as a counterpart to the $J^2 < 1$ case. Henceforth we shall refer to this as the "other region" of parameter space.

Our finding that the CTC horizon can never be in between the two black hole horizons is contrary to assumptions made in previous work [14]. However, the resultant thermodynamics is not significantly altered, as all main results consider only the situation when the CTC horizon is outside the black hole. In the next two sections we will discuss the properties of the black hole region and the other region of parameter space.

B. Black hole region of parameter space ($J^2 < 1$)

This is the region that is well understood and can be simply regarded as a Kerr black hole embedded in a Gödel spacetime, with the CTC horizon outside of the black hole horizons. To better understand this case we will take a look at the geodesics in the (t, r, ϕ) plane (with θ and ψ fixed). The metric becomes

$$ds^2 = -\frac{r^2 V(r)}{4g(r)} dt^2 + g(r)\left(d\phi - \frac{a(r)}{g(r)} dt\right)^2 + \frac{dr^2}{V(r)}. \tag{11}$$

Note that $g(r_H) \geq 0$ for the choice of parameters $(-1 \leq J \leq 1, -\frac{1}{2} - \frac{J}{2} \leq L \leq \frac{1}{2} - \frac{J}{2})$ that we are considering. For convenience we impose the further restriction that $L \neq \frac{-3J^2+2+\sqrt{5J^4-4J^2}}{2J}$ and $L \neq \frac{-3J^2+2-\sqrt{5J^4-4J^2}}{2J}$ so that $g(r_H) > 0$ and the CTC horizon is strictly outside the outer black hole horizon.

The tangent vector to a geodesic is given by

$$u^\alpha = [\dot{t}, \dot{r}, \dot{\phi}],$$

where the dot denotes the derivative with respect to the affine parameter λ. For this metric ∂_t and ∂_ϕ are Killing vectors so in general the energy and angular momentum for these geodesics are, respectively,

$$E = \frac{r^2V(r) - 4a^2(r)}{4g(r)}\dot{t} + a(r)\dot{\phi},$$
$$\ell = -a(r)\dot{t} + g(r)\dot{\phi}.$$

We are interested in geodesics with $\ell = 0$. Note that for constant r the quantity $d\chi = d\phi - \frac{a(r)}{g(r)}dt$ is constant (i.e. $\frac{d\chi}{d\lambda} = 0$); for $r = r_H$ these correspond to geodesics for which $\chi = \phi - \Omega_H t$ is constant on the horizon [recall $\Omega_H = \frac{a(r_H)}{g(r_H)}$].

Setting $\ell = 0$ yields $\frac{a(r)}{g(r)}\dot{t} = \dot{\phi}$ and $E = \frac{r^2V(r)}{4g(r)}\dot{t}$. For null geodesics we find $\dot{r} = \pm\frac{rV(r)}{2\sqrt{g(r)}}\dot{t}$, so that

$$u^\alpha_\pm = K_\pm\left[\frac{g(r)}{V(r)r^2}, \pm\frac{\sqrt{g(r)}}{2r}, \frac{a(r)}{V(r)r^2}\right], \tag{12}$$

where the plus/minus signs refer to outgoing/ingoing geodesics. When $l = 0$ this can be solved explicitly, and we recover the results for geodesic motion examined in Ref. [4]. The $K_\pm$'s are constants related to the energy $E = \frac{K_\pm}{4}$. Choosing the normalization

$$g_{\alpha\beta}u^\alpha_+ u^\beta_- = -1$$

at some point $r = r_0$, we obtain

$$K_+K_- = \frac{2r_0^2V(r_0)}{g(r_0)},$$

and for convenience we pick $K_- = 1$, $K_+ = \frac{2r_0^2V(r_0)}{g(r_0)}$. The expansion scalar for null geodesics is

$$\Theta(u_\pm) = \pm\frac{K_\pm g'(r)}{4r\sqrt{g(r)}},$$

and we see that for outgoing null rays there is a sign difference between geodesics starting inside the horizon ($r_0 < r_H$) and geodesics starting outside ($r_0 > r_H$), with no such change for ingoing geodesics, as expected for a trapped surface at $r = r_H$.

We can also say useful things about the CTC boundary. It occurs when $g(r_{\rm CTC}) = 0$, so the expansions are infinite there. Furthermore $\frac{dr}{dt}$ is infinite and $\frac{dr}{d\lambda} = 0$ there, implying that null geodesics cannot cross the CTC boundary. These results are consistent with the observations for the Schwarzschild-Gödel ($l = 0$) case [4]: null geodesics will take infinite coordinate time t to go between the black hole horizon and CTC boundary. The CTC boundary is reached in finite affine parameter λ although once the null ray reaches the CTC horizon it spirals back toward the black hole.

C. The other region of parameter space ($J^2 > 1$)

When $x_{\rm CTC} < x_-$, i.e. the CTC boundary is the innermost surface, it is unclear what sort of object we now have. For convenience we shall continue to use the term horizon to signify $x_\pm$, and the term ergosphere to denote the surface $x = 1$ ($r = \sqrt{2m}$), mindful of potential abuses of language. Both horizons are now outside of the ergosphere, but the CTC boundary can either be inside or outside of this surface, depending on the choice of parameters. For example for $J = 1.5$, $L = -2$, $x_{\rm CTC} > 1$ (and $x_{\rm CTC} < x_- < x_+$) but $x_{\rm CTC} < 1$ for $J = 2$, $L = -2$.

To understand the causal properties of this spacetime we shall consider the metric for fixed θ and ψ for convenience. Consider the behavior of

$$\begin{aligned} ds^2 &= -\frac{r^2V(r)}{4g(r)}dt^2 + g(r)\left(d\phi - \frac{a(r)}{g(r)}dt\right)^2 + \frac{dr^2}{V(r)} \\ &= \frac{r^2V(r)}{4|g(r)|}dt^2 - |g(r)|\left(d\phi + \frac{a(r)}{|g(r)|}dt\right)^2 + \frac{dr^2}{V(r)}, \end{aligned} \tag{13}$$

where in the outer region $r > r_+$ we see that $g(r) < 0$ and $V(r) > 0$, and so we have rewritten the metric for fixed (θ, ψ) in the 2nd line above. For $r_- < r < r_+$ then $g(r) < 0$ and $V(r) < 0$.

We can define a new coordinate $\chi = \phi - \frac{a(r_0)}{g(r_0)}t$ for some $r_0 > r_+$ and the metric is now

$$\begin{aligned} ds^2 &= (-f(r) + g(r)\Omega^2 - 2a(r)\Omega)dt^2 \\ &\quad + 2(g(r)\Omega - a(r))dtd\chi + g(r)d\chi^2 + \frac{dr^2}{V(r)}, \end{aligned} \tag{14}$$

where $\Omega = \frac{a(r_0)}{g(r_0)}$. For a fixed value of $r = r_0 > r_+$ this metric simplifies to

$$ds^2 = +\frac{r_0^2V(r_0)}{4|g(r_0)|}dt^2 - |g(r_0)|d\chi^2. \tag{15}$$

Since $g_{tt} > 0$, $g_{\chi\chi} < 0$ we see that χ functions as the time coordinate, but only near $r = r_0$. For any given $r_0 > r_+$ it is possible to choose such a time coordinate χ in a neighborhood of r_0 and the metric always has signature $(- + + + +)$.

When $r_- < r < r_+$ the signature of the metric becomes $(- - - + +)$ and so this region is not a physical spacetime. There is no choice of coordinate transformation that will allow the metric to have correct signature. This is easily seen by expanding the metric near r_+

$$\begin{aligned} ds^2 &= +\frac{r_+^2V'(r_+)}{4|g(r_+)|}(r - r_+)dt^2 + 2(g'(r_+)\Omega_+ - a'(r_+)) \\ &\quad \times (r - r_+)dtd\chi - |g(r_+)|d\chi^2 + \frac{dr^2}{V'(r_+)(r - r_+)} \end{aligned}$$

indicating that the metric changes signature as r passes through r_+ from above. There is a conical singularity at $r = r_+$ that is removed by imposing a periodicity on t of $\frac{r_+ V'(r_+)}{8\pi\sqrt{|g(r_+)|}}$. Hence the region $r > r_+$ is a regular spacetime everywhere permeated by closed timelike curves due to the periodicity of ϕ and t.

Setting now $r < r_-$ we need to consider two distinct cases depending on where the "ergosphere" ($r_{\rm ergo} = \sqrt{2m}$) is located with respect to the CTC horizon. These are $r_{\rm ergo} < r_{\rm CTC} < r_-$ (represented by the black region in Fig. 5 in terms of L and J parameters) and $r_{\rm CTC} < r_{\rm ergo} < r_-$ (represented by the gray region in Fig. 5 in terms of L and J parameters).

We will start with the $r_{\rm ergo} < r_{\rm CTC} < r_-$ case and we will restrict ourselves to CTC region $r_{\rm CTC} < r < r_-$ so that $g(r) < 0$ and $V(r) > 0$. So for this case the metric is once again in the same form as (13). It is possible for an arbitrary r_0 ($r_{\rm CTC} < r_0 < r_-$) to choose a coordinate $\chi = \phi - \frac{a(r_0)}{g(r_0)} t$ and the metrics (14) and (15) will be valid in this region. Expanding the metric near r_- gives

$$\begin{aligned} ds^2 = &+ \frac{r_-^2 |V'(r_-)|}{4|g(r_-)|}(r_- - r)dt^2 - 2(g'(r_-)\Omega_- \\ &- a'(r_-))(r_- - r)dt d\chi - |g(r_-)| d\chi^2 \\ &+ \frac{dr^2}{|V'(r_-)|(r_- - r)}, \end{aligned}$$

again showing that when $r_- < r < r_+$ the signature of the metric becomes $(- - - + +)$. Removal of the conical singularity at $r = r_-$ is achieved by imposing a periodicity on t of $\frac{r_- |V'(r_-)|}{8\pi\sqrt{|g(r_-)|}}$. Notice that this differs from that imposed in the $r > r_+$ region, as expected for two regions that are disconnected spacetimes. Referring back to (14), for an arbitrary choice of χ we see that $g_{\chi\chi} \to 0$ as the CTC horizon is approached. At the CTC horizon g_{tt} can be either positive or negative depending on the Ω that defined χ. Examining (14) at $r = r_{\rm CTC}$, it is clear that g_{tt} will be positive if (a) Ω has an opposite sign to $a(r_{\rm CTC})$ [in general this will be true since $\Omega = \frac{a(r_0)}{g(r_0)}$ and $g(r_0)$ is negative] and (b) $|\Omega| > \frac{f(r_{\rm CTC})}{2|a(r_{\rm CTC})|}$ (i.e. r_0 must be chosen to be close enough to $r_{\rm CTC}$ so that this inequality will be satisfied); otherwise g_{tt} will be negative at the CTC horizon. So for an arbitrary choice of parameters L and J it will not be possible to choose a single coordinate χ for which one can write the metric in a form in which ∇t is spacelike everywhere between $r_{\rm CTC}$ and r_-. However, for an arbitrary r_0 we can choose a coordinate χ so that in a neighborhood of r_0 the metric can be written with ∇t spacelike.

The $r_{\rm ctc} < r_{\rm ergo} < r_-$ case is a little more interesting due to the presence of the ergosphere. Outside of the ergosphere the analysis remains the same as the previous $r < r_-$ case. Inside the ergosphere at any given r_0 one can still choose $\chi = \phi - \frac{a(r_0)}{g(r_0)} t$. However, when $r_{\rm ctc} < r < r_{\rm ergo}$ it is sufficient to choose $\Omega = 0$ because $f(r)$ is negative and the metric

$$\begin{aligned} ds^2 &= -f(r)dt^2 - a(r)dt d\phi + g(r)d\phi^2 + \frac{dr^2}{V(r)} \\ &= |f(r)|dt^2 - a(r)dt d\phi + g(r)d\phi^2 + \frac{dr^2}{V(r)} \end{aligned}$$

is such that ϕ is the time coordinate inside the ergosphere.

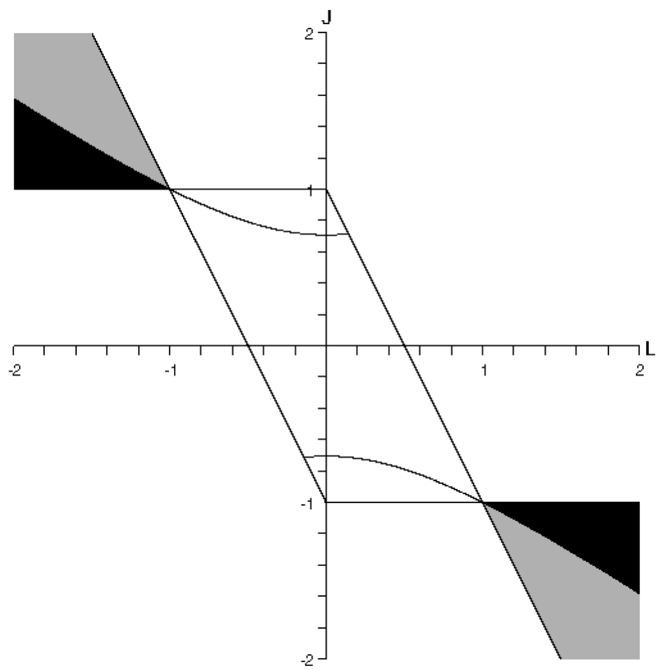

FIG. 5. Plot of horizon behavior in terms of J and L. The curve corresponds to $r_{\rm CTC} = \sqrt{2m}$. In the black region $r_{\rm CTC} > \sqrt{2m}$ and in the gray region $r_{\rm CTC} < \sqrt{2m}$.

IV. TEMPERATURE FROM TUNNELING

We now examine the application of the tunnelling method to the Kerr-Gödel spacetime. For the case when the CTC horizon is outside the black hole horizons the temperature has been computed previously by other means [12,13], allowing us to compare these with the tunnelling results. We can also see if any tunnelling occurs from the CTC horizon.

A. Review of the tunneling method

The tunnelling method is a semiclassical approach that considers a particle idealized as a spherical wave of matter emitted from inside the horizon to outside. From the WKB approximation the tunneling probability for the classically forbidden trajectory of the s-wave from inside to outside the horizon is

$$\Gamma \propto \exp(-2\,{\rm Im} I) \qquad (16)$$

(here $\hbar$ is set equal to unity). Expanding the action in terms of the particle energy, the Hawking temperature is recovered at linear order. In other words for $2I = \beta E + O(E^2)$ this gives

$$\Gamma \sim \exp(-2\,\mathrm{Im}I) \simeq \exp(-\beta E). \tag{17}$$

From this point there are two approaches that can be used to calculate the imaginary part of the action, referred to as the null geodesic method and the Hamilton-Jacobi Ansatz (refer to [22] for more details on the two approaches). We will only use the null geodesic method in this paper.

The imaginary part of the action for an outgoing s-wave (which follows a radial null geodesic) from r_{in} to r_{out} is expressed as

$$I = \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} p_r dr = \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} \int_0^{p_r} dp'_r dr, \tag{18}$$

where r_{in} and r_{out} are the respective initial and final radii of the black hole. The trajectory between these two radii is the barrier the particle must tunnel through. Note the local nature of this calculation: the tunnelling probability only depends on an integration from r_{in} to r_{out}. In fact only the near horizon form of the black hole metric is required in order to calculate the tunneling probability (and hence the black hole temperature) [16,22]. So a stationary observer anywhere outside the black hole would be able to observe the emission and measure the temperature. This is in contrast with the Wick-rotation method which requires a (scalar) field at infinity to be in equilibrium with the black hole in order to get the temperature. The presence of the CTC horizon at large distances renders the foundations of this latter approach somewhat questionable.

We assume that the emitted s-wave has energy $\omega \ll M$ and that the total energy of the spacetime was originally M. Invoking conservation of energy, to this approximation the s-wave moves in a background spacetime of energy $M \to M - \omega'$. In order to evaluate the integral, we employ Hamilton's equation $\dot{r} = \frac{dH}{dp_r}|_r$ to switch the integration variable from momentum to energy ($dp_r = \frac{dH}{\dot{r}}$), giving

$$I = \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} \int_M^{M-\omega} \frac{dr}{\dot{r}} dH = \int_0^{\omega} \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} \frac{dr}{\dot{r}} (-d\omega'), \tag{19}$$

where $dH = -d\omega'$ because total energy $H = M - \omega'$ with M constant. Note that $\dot{r}$ is implicitly a function of $M - \omega'$. Since $\omega \ll M$ it is possible to rewrite the expression in terms of an expansion of ω. To first order this gives

$$\begin{aligned} I &= \int_0^{\omega} \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} \frac{dr}{\dot{r}(r, M - \omega')} (-d\omega') \\ &= -\omega \int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} \frac{dr}{\dot{r}(r, M)} + O(\omega^2) \simeq \omega \int_{r_{\mathrm{out}}}^{r_{\mathrm{in}}} \frac{dr}{\dot{r}(r, M)}. \end{aligned} \tag{20}$$

To proceed further we will need to estimate the last integral. First we note that $r_{\mathrm{in}} > r_{\mathrm{out}}$ because black holes decrease in mass as energy is emitted; consequently the radius of the event horizon decreases. We therefore write $r_{\mathrm{in}} = r_H(M) - \epsilon$ and $r_{\mathrm{out}} = r_H(M - \omega) + \epsilon$ where $r_H(M)$ denotes the location of the event horizon of the original background spacetime before the emission of particles. Henceforth the notation r_H will be used to denote $r_H(M)$. Note that with this generalization no explicit knowledge of the total energy or mass is required since r_H is simply the radius of the event horizon before any particles are emitted.

We pause to discuss a few technical points connected with rotating spacetimes [16,22]. In general the emitted s-wave could carry angular momentum ℓ; if it has energy E then the tunnelling probability to the lowest order would be

$$\Gamma \simeq \exp(-\beta(E - \Omega_H \ell)),$$

where Ω_H is the angular velocity of the black hole horizon. For this tunnelling probability to make sense we must require $E - \Omega_H \ell > 0$. This inequality corresponds to the s-wave being able to escape from the ergosphere. For calculating the temperature β^{-1} it is sufficient to restrict to $\ell = 0$ s-waves.

B. Temperature calculation

Turning now to calculation of the black hole temperature [15–29], recall that the full metric in lapse-shift form is (4). To employ the null geodesic method it is convenient to write the metric in a Painlevé form so that the null geodesic equations convey the semipermeable nature of the black hole horizon (i.e. that it is easy to cross into the black hole but classically they cannot escape). In order to simplify the equations we rewrite the metric by defining $\chi = \phi - \Omega_H t$. We are interested in geodesics that have no angular momentum ($\ell = 0$) so we set $d\chi = 0$ (and for convenience also $d\theta = d\psi = 0$), yielding

$$ds^2 = -\frac{r^2 V(r)}{4g(r)} dt^2 + \frac{dr^2}{V(r)}.$$

We easily can rewrite this in Painlevé form via the following transformation:

$$t \to t - \frac{2\sqrt{g(r)}}{rV(r)} \sqrt{1 - V(r)} dr,$$

giving (for constant χ, θ, and ψ) the following Painleve metric:

$$ds^2 = -\frac{r^2 V(r)}{4g(r)} dt^2 + \frac{r}{\sqrt{g(r)}} \sqrt{1 - V(r)} dr dt + dr^2.$$

We need to know how the $\ell = 0$ null geodesics behave for this metric in order to solve for the imaginary part of the action using Eq. (20). The radial null geodesic equation is given by

$$\frac{dr}{dt} = \frac{r}{2\sqrt{g(r)}} (\pm 1 - \sqrt{1 - V(r)}), \tag{21}$$

where $+$ denotes outgoing and $-$ denotes ingoing geodesics (notice that $\frac{dr}{dt} = 0$ at the horizon for outgoing geodesics and $\frac{dr}{dt}$ is nonzero for ingoing geodesics). Inserting (21) into (20) we find that $1/\frac{dr}{dt}$ has a first order pole at the horizon with residue $\frac{4\sqrt{g(r_H)}}{r_H V'(r_H)}$. So solving the integral we find

$$\mathrm{Im}I = \frac{4\pi\omega\sqrt{g(r_H)}}{r_H V'(r_H)} + O(\omega^2),$$

$$\Gamma \sim \exp(-2\,\mathrm{Im}I) \simeq \exp\left(-\frac{8\pi\sqrt{g(r_H)}}{r_H V'(r_H)}\omega\right).$$

This corresponds to a temperature

$$T = \frac{r_H V'(r_H)}{8\pi\sqrt{g(r_H)}}, \tag{22}$$

$$T = \frac{m(r_H^2(1 - 8j^2m - 4ml) - 2l^2)}{\pi r_H^3\sqrt{-4j^2r_H^6 + (1 - 8j^2m)r_H^4 + 2ml^2}}. \tag{23}$$

This temperature is the same as that obtained using Wick-rotation methods [13,14]; when $l = 0$ it reduces down to the Schwarzschild-Gödel temperature found in [12]. Note that the expression for the temperature diverges when $g(r_H) = 0$, which occurs when the CTC horizon is coincident with the outer horizon. The temperature is not defined when $g(r_H) < 0$, an unsurprising result considering the analysis of the other region of parameter space and the fact the when the CTC horizon is inside the r_- and r_+ horizons the derivation used is not valid. Not only is t not the correct time coordinate, but it is unclear how to even define tunnelling from inside r_+ because the region $r_- < r < r_+$ is not a spacetime.

Consider next what happens if we try to apply the tunnelling method to the CTC horizon. From (21) we know that $\frac{dr}{dt} \rightarrow \infty$ as $r \rightarrow r_{\mathrm{ctc}}$. This means that $1/\frac{dr}{dt}$ is simply zero at the CTC horizon. Since $1/\frac{dr}{dt}$ has no poles at the CTC horizon it means there is no tunnelling at the CTC horizon.

V. CONCLUSIONS

In this paper we have reviewed some of the general properties of the Kerr-Godël spacetime, performing detailed analysis of its parameter space. There are two distinct classes. One is the class $J^2 < 1$, corresponding to black holes for which the CTC horizon r_{CTC} is exterior to the black hole horizons at r_+ and r_-. When $J^2 > 1$ we obtain the other class (the "other" region of parameter space), for which the CTC horizon is inside both of the other surfaces r_+ and r_-. We find that these are the only two possibilities (apart from naked singularities); there is no "in between" region where $r_+ > r_{\mathrm{CTC}} > r_-$, contrary to previous expectations [14].

Despite the presence of CTCs, we find that the tunnelling method applied to the black hole region of parameter space yields a temperature consistent with previous calculations made via Wick-rotation methods. We also find (when $r_{\mathrm{ctc}} > r_+$) that there is no tunnelling through the CTC horizon. We have discussed technical problems that occur in trying to apply the tunnelling method to the other region of parameter space due the fact that the region $r_- < r < r_+$ does not have the correct signature. Higher-order corrections and applications of the method to nonradial null rays remain as interesting problems to explore.

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