PHYSICAL REVIEW D **75,** 083514 (2007)

Vacuum energy and spectral function sum rules

Alexander Y. Kamenshchik

Dipartimento di Fisica and INFN, Via Irnerio 46, 40126 Bologna, Italy,
and L.D. Landau Institute for Theoretical Physics of the Russian Academy of Sciences, Kosygin Street 2, 119334 Moscow, Russia

Alessandro Tronconi, Gian Paolo Vacca, and Giovanni Venturi

Dipartimento di Fisica and INFN, Via Irnerio 46, 40126 Bologna, Italy

(Received 19 December 2006; published 13 April 2007)

We reformulate the problem of the cancellation of the ultraviolet divergencies of the vacuum energy, particularly important at the cosmological level, in terms of a saturation of spectral function sum rules which leads to a set of conditions on the spectrum of the fundamental theory. We specialize the approach to both Minkowski and de Sitter space-times and investigate some examples.

DOI: 10.1103/PhysRevD.75.083514 PACS numbers: 98.80.Cq, 04.62.+v, 11.55.Hx, 95.36.+x

I. INTRODUCTION

The energy density of the vacuum, and in general the cosmological constant problem, has always attracted much attention [1,2]. The recent discovery of cosmic acceleration [3] has stimulated the discussion about what kind of dark energy could be responsible for this phenomenon [4]. The cosmological constant is the most natural candidate. The enigma arises because the zero point energy associated with quantum fields and naively calculated by introducing a cutoff at the Planck scale appears to be some 118 orders of magnitude bigger than the observable value of the cosmological constant energy density. Many explanations have been put forward for this [5], prominent among them are symmetry mechanisms and, in particular, supersymmetry. Indeed, since the contributions to the ground state energy of fermions and bosons have opposite signs unbroken supersymmetry leads to a vanishing vacuum energy. Unfortunately supersymmetry, if it exists, certainly must be broken and at this point one appears to still be left with the cosmological constant problem. One lesson one learns however is that the number of fermionic and bosonic degrees of freedom must be equal, if one wants to have an exact cancellation at least of the quartically ultraviolet divergent part of the vacuum energy.[1]

In order to employ what one has learned from the above, one may seek inspiration from the study of other symmetry breaking, in particular, chiral symmetry. Indeed, if chiral $SU(2) \times SU(2)$ was an exact symmetry, one would have expected the ρ meson to be accompanied by an axial-vector meson of the same mass, which certainly is not the case. A way out of this was found through the introduction of an asymptotic chiral symmetry [6] leading to spectral function sum rules which are related to the short-distance behavior of products of vector and axial-vector currents [7]. The approximate saturation of the spectral functions with suitable combinations of particles leads to a physically satisfactory relation between vector and axial-vector meson masses.

We wish to follow an analogous procedure for the case of vacuum energies. Indeed, we shall first obtain expressions for the large momentum behavior of the vacuum energy in de Sitter space for arbitrary mass, spin 1/2 fermions and spin 0 and spin 1 bosons. This is done in the next section. Subsequently we shall, through the use of spectral functions, represent the contributions of an arbitrary number of fermions and bosons. On examining the contributions of all particle species to the vacuum energy and requiring that all divergent contributions cancel, we obtain constraints between diverse spectral functions and their moments. This of course leads to constraints on the mass spectra of the particles which approximately saturate the sum rules. These constraints are studied in Sec. III. Lastly our results are summarized and discussed in the Conclusions.

II. VACUUM ENERGY OF FREE FIELDS

In what follows we shall consider a flat Friedmann universe with the Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)dx^i dx^i, \tag{1}$$

where the cosmological radius $a(t)$ obeys the de Sitter expansion law: $a(t) = a_0 e^{\mathcal{H}t}$. We shall calculate the ultraviolet divergent contributions to vacuum energy for free fields on this spatially flat de Sitter background.

A. Bosonic fields

Free bosonic field dynamics, on a Friedmann-Robertson-Walker (FRW) space-time, is known to be equivalent [8] to that of an infinite system of time dependent harmonic oscillators (TDHO). Subsequently, at the quantum level, one can express the vacuum expectation value of the field Hamiltonian in terms of the so-called Ermakov-Pinney variables [9–11]. These variables are in-

[1]The finite part of the vacuum energy (which is difficult to calculate explicitly) could be responsible for the small observable value of the cosmological constant.

troduced to define the invariant operators one uses to build the Hilbert space of the solutions for the TDHO. Since we should be interested in the ultraviolet divergent contributions to the Bunch-Davies vacuum energy [12], we shall finally consider the modes up to an UV cutoff, and, in particular, restrict the analysis to trans-Planckian k-modes of the fields, namely, the limit $z \equiv \mathcal{H}^{-1}k/a \gg 1$.

1. Method of invariants

When the action of a quantum system can be written in the form of some TDHO

$$S = \int dt \frac{1}{2}\left(\frac{\dot{q}^2}{F(t)} - G(t)q^2\right) \tag{2}$$

with the Hamiltonian given by

$$\hat{H} = \frac{F}{2}\hat{p}^2 + \frac{G}{2}\hat{q}^2, \tag{3}$$

where $\hat{p} = \dot{\hat{q}}/F$, F may be interpreted as an inverse effective mass of the harmonic oscillator while G is the mass multiplied by the effective frequency squared. The solutions to the Schrödinger equation are eigenstates of the quadratic invariant operator $\hat{I}$ defined by

$$\frac{\partial \hat{I}}{\partial t} = i[\hat{I}, \hat{H}]. \tag{4}$$

The quadratic invariant operator for the case (3) is given by

$$\hat{I} = \frac{1}{2}\left[\frac{\hat{q}^2}{\rho^2} + \left(\rho\hat{p} - \frac{\dot{\rho}\,\hat{q}}{F}\right)^2\right], \tag{5}$$

where $\rho = \sqrt{F}x$, x is the Ermakov-Pinney variable satisfying the equation [10]

$$\ddot{x} + \Omega^2 x = \frac{1}{x^3}, \tag{6}$$

and $\Omega^2 = FG + \frac{\ddot{F}}{2F} - \frac{3\dot{F}^2}{4F^2}$. Both the invariant (5) and the Hamiltonian (3) can be written in terms of time dependent creation-annihilation operators but, in the time dependent case, the Hilbert space of the solutions of the Schrödinger equation is generated by the linear invariant creation-annihilation operators; energy eigenstates are related to these solutions by a Bogoliubov rotation depending on time through ρ, F, and G. The invariant vacuum $|0_I\rangle$ and the invariant operator (5) itself are not unique but depend on the initial conditions chosen for the Ermakov-Pinney equation; in any case the expectation value of the Hamiltonian (3) with respect to it can be expressed as

$$\langle 0_I|\hat{H}|0_I\rangle = \frac{1}{4}\left[\frac{\dot{\rho}^2}{F} + G\rho^2 + \frac{F}{\rho^2}\right] \tag{7}$$

and is a function of time. Setting $\rho^2(t_0) = \sqrt{F(t_0)/G(t_0)}$ and $\dot{\rho}(t_0) = 0$, the invariant vacuum $|0_I\rangle$ (which is related to the Bunch-Davies vacuum) coincides with the Hamiltonian vacuum $|0\rangle$ at time t_0.

Technically the full quantum evolution is determined once the Ermakov-Pinney equation is solved. The exact solution of the Ermakov-Pinney equation is known if the homogeneous, linear equation associated with it is solvable; otherwise the Ermakov-Pinney equation itself can be solved perturbatively and gives the correct adiabatic series.

In particular, when handling free field evolution in de Sitter space-time [13], the adiabatic parameter should be chosen to be the z defined above. In the ultraviolet regime, one must set $x = \xi/\sqrt{z}$, where ξ satisfies

$$\frac{d^2\xi}{dz^2} + \frac{1}{z^2}\left[\mathcal{H}^{-2}\Omega^2(z) + \frac{1}{4}\right]\xi = \frac{1}{\mathcal{H}^2\xi^3}, \tag{8}$$

and the modified Ermakov-Pinney equation (8) gives the correct adiabatic solution.

2. Real scalar field

The action for a minimally coupled, massive scalar field $\Phi(\mathbf{x}, t)$ is

$$S[\Phi] = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}m^2\Phi^2\right]; \tag{9}$$

if one considers the Fourier transform of the field

$$\Phi(\mathbf{x}, t) \equiv \frac{1}{\sqrt{2V}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}[\phi_1(\mathbf{k}, t) + i\phi_2(\mathbf{k}, t)], \tag{10}$$

where $\int d\mathbf{x} = V$ and $\phi(\mathbf{k}, t)$ is a real function, the action (9) can be rewritten as

$$S = \frac{1}{2}\sum_{i=1,2}\sum_{\mathbf{k}}\int \frac{a^3}{2}dt(\dot{\phi}_i(\mathbf{k})^2 - \omega_k^2\phi_i(\mathbf{k})^2), \tag{11}$$

where $\omega_k^2 \equiv \frac{\mathbf{k}^2}{a^2} + m^2$. We note that one can eliminate the overall factor 1/2 in (11) by observing that $\phi_1(\mathbf{k}) = \phi_1(-\mathbf{k})$ and $\phi_2(\mathbf{k}) = -\phi_2(-\mathbf{k})$ thus restricting the sum over the subspace $\mathbf{k}^+$ of the independent degrees of freedom. For each independent mode $\mathbf{k}$, i, one then obtains the following Hamiltonian:

$$\hat{H}_i(\mathbf{k}) = \frac{\hat{\pi}_i(\mathbf{k})^2}{2a^3} + \frac{a^3\omega_k^2\hat{\phi}_i(\mathbf{k})^2}{2}, \tag{12}$$

where $\hat{\pi}_i(\mathbf{k}) \equiv a^3\dot{\hat{\phi}}_i(\mathbf{k})$; the full Hamiltonian can then be written as $\hat{H}_S = \sum_{i=1,2}\sum_{\mathbf{k}^+}\hat{H}_i(\mathbf{k})$. Finally, to calculate the vacuum expectation value (7), in the continuum limit

$$\frac{1}{V}\sum_{\mathbf{k}} \overset{V\to\infty}{\longrightarrow} \frac{1}{(2\pi)^3}\int k^2 dk d\Omega, \tag{13}$$

it is sufficient to just keep $\hat{\phi} \equiv \hat{\phi}_1$ and integrate over the complete solid angle

$$\frac{\langle 0_I|\hat{H}_S|0_I\rangle}{V} \overset{V\to\infty}{\longrightarrow} \frac{1}{2\pi^2}\int_0^\infty k^2 E_0^S(k)dk = \frac{1}{2\pi^2}\int_0^\infty k^2\langle 0_I|\left(\frac{\hat{\pi}(\mathbf{k})^2}{2a^3} + \frac{a^3\omega_k^2\hat{\phi}(\mathbf{k})^2}{2}\right)|0_I\rangle dk. \tag{14}$$

For the case of a scalar field, one has, for a mode $\mathbf{k}$, $F_S = a^{-3}$, $G_S = a^3\omega_k^2$, and the Ermakov-Pinney equation (6) can be solved exactly with

$$x_S = \sqrt{\frac{\pi H_\nu^{(1)}(z)H_{\nu^*}^{(2)}(z)}{2\mathcal{H}}}, \tag{15}$$

where the $H_\nu^{(i)}(z)$ are the Hankel functions and $\nu = \sqrt{\frac{9}{4} - \frac{m^2}{\mathcal{H}^2}}$. We note that the solution (15) has been chosen to fulfil the requirement that $|0_I\rangle$ coincides with Bunch-Davies vacuum. One can finally calculate $E_0^S(k)$ exactly and evaluate it in the $z \gg 1$ limit:

$$E_0^S(k) \overset{z\gg1}{=} \frac{k}{2a} + \frac{(\mathcal{H}^2 + m^2)a}{4k} + \frac{m^2(2\mathcal{H}^2 - m^2)a^3}{16k^3} + o\left(\frac{1}{z^3}\right). \tag{16}$$

3. *Real vector field*

The Proca action for a minimally coupled, massive vector field $A_\mu(\mathbf{x}, t) \equiv (A_0, \vec{A})$ is given by

$$S[A_\mu] = \int d^4x\sqrt{-g}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{M^2}{2}A^\mu A_\mu\right], \tag{17}$$

where $F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$ (with ∇_μ the covariant derivative). We note that in the massive case the vector field satisfies the Lorentz gauge condition $\nabla^\mu A_\mu = 0$ and has three independent physical degrees of freedom; in the massless case, instead, to the Lorentz condition one should add $A_0 = 0$, leading to $\vec{\nabla}\cdot\vec{A} = 0$ (radiation gauge) and A_μ reduces to two independent physical components. The final expression one obtains for the action (17), without any gauge fixing, is

$$S[A_\mu] = \int d^4x\frac{a}{2}\left[(\dot{\vec{A}})^2 + (\vec{\nabla}A_0)^2 - 2\dot{\vec{A}}\cdot(\vec{\nabla}A_0) - \left(\frac{\vec{\nabla}\wedge\vec{A}}{a}\right)^2 + M^2(a^2A_0^2 - \vec{A}^2)\right]. \tag{18}$$

For the massive case, the variation of the action with respect to A_0 leads to the Gauss constraint:

$$\frac{1}{a^2}[\vec{\nabla}\cdot\dot{\vec{A}} - \nabla^2 A_0] + M^2 A_0 = 0. \tag{19}$$

If one considers the Fourier transform of the field components

$$A_\mu = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\alpha_\mu(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{20}$$

then the action can be expressed in terms of $\alpha_\mu(\mathbf{k}) \equiv (\alpha_0(\mathbf{k}), \vec{\alpha}(\mathbf{k}))$ as

$$S = \sum_{\mathbf{k}}\int dt\frac{a}{2}\left[\dot{\vec{\alpha}}(\mathbf{k})\cdot\dot{\vec{\alpha}}(-\mathbf{k}) - 2i\alpha_0(\mathbf{k})\dot{\vec{\alpha}}(-\mathbf{k})\cdot\mathbf{k} - \frac{\mathbf{k}\wedge\vec{\alpha}(\mathbf{k})\cdot\mathbf{k}\wedge\vec{\alpha}(-\mathbf{k})}{a^2} + a^2\omega_k^2\alpha_0(\mathbf{k})\alpha_0(-\mathbf{k}) - M^2\vec{\alpha}(\mathbf{k})\cdot\vec{\alpha}(-\mathbf{k})\right] \tag{21}$$

with $\omega_k^2 \equiv M^2 + \frac{\mathbf{k}^2}{a^2}$ and the Gauss equation (19) for each mode is

$$\alpha_0(\mathbf{k}) = -i\frac{\mathbf{k}}{a^2\omega_k^2}\cdot\dot{\vec{\alpha}}(\mathbf{k}). \tag{22}$$

The action (21) can now be written in terms of the components $\vec{\alpha}(\mathbf{k})$. If one decomposes $\vec{\alpha}(\mathbf{k})$ into the sum of a longitudinal vector

$$\vec{\alpha}_L(\mathbf{k}) \equiv (\vec{\alpha}(\mathbf{k})\cdot\hat{k})\hat{k} = \frac{1}{\sqrt{2}}[u_L(\mathbf{k}) + iv_L(\mathbf{k})]\hat{k} \tag{23}$$

with $\hat{k} = \mathbf{k}/\sqrt{\mathbf{k}^2}$, plus a transverse vector

$$\vec{\alpha}_T(\mathbf{k}) = \vec{\alpha}(\mathbf{k}) - \vec{\alpha}_L(\mathbf{k}) = \frac{1}{\sqrt{2}}\sum_{i=1,2}[u_T^{(i)}(\mathbf{k}) + iv_T^{(i)}(\mathbf{k})]\hat{e}_i(\mathbf{k}), \tag{24}$$

where the $\hat{e}_i(\mathbf{k})$ form an orthonormal basis, $\hat{e}_i(\mathbf{k})\cdot\hat{e}_j(\mathbf{k}) = \delta_{ij}$, for the transverse space, $\hat{e}_i(\mathbf{k})\cdot\hat{k} = 0$, the action (21) can finally be written as

$$S = \sum_{\mathbf{k}^+}\left[(S_L[u_L(\mathbf{k})] + S_L[v_L(\mathbf{k})]) + \sum_{i=1,2}(S_T[u_T^{(i)}(\mathbf{k})] + S_T[v_T^{(i)}(\mathbf{k})])\right], \tag{25}$$

where

$$S_L[\varphi(\mathbf{k})] \equiv M^2\int dt\frac{a}{2}\left(\frac{1}{\omega_k^2}\dot{\varphi}(\mathbf{k})^2 - \varphi(\mathbf{k})^2\right) \tag{26}$$

and

$$S_T[\varphi(\mathbf{k})] \equiv \int dt\frac{a}{2}(\dot{\varphi}(\mathbf{k})^2 - \omega_k^2\varphi(\mathbf{k})^2). \tag{27}$$

We note that, owing to the overall M^2 factor in front of (26), one recovers the correct physical limit for the massless case: in fact only the transverse contributions remain when $M \to 0$. In the latter case, the full Hamiltonian vacuum expectation value, in the continuum limit (13), is given by

$$\frac{\langle 0_I|\hat{H}_T|0_I\rangle}{V} \overset{V\to\infty}{\to} \frac{1}{\pi^2}\int_0^\infty k^2 E_0^T(k)dk = \frac{1}{\pi^2}\int_0^\infty k^2\langle 0_I|\left(\frac{\hat{\pi}_T(\mathbf{k})^2}{2a} + \frac{a\omega_k^2\hat{\phi}_T(\mathbf{k})^2}{2}\right)|0_I\rangle dk, \tag{28}$$

where $\hat{\pi}_T(\mathbf{k}) \equiv a\dot{\hat{\phi}}_T(\mathbf{k})$, the factor 2 accounts for the contributions coming from different transverse polarizations, and the $M \to 0$ limit should be taken. In the massive case, one has also the longitudinal part

$$\frac{\langle 0_I|\hat{H}_L|0_I\rangle}{V} \overset{V\to\infty}{\to} \frac{1}{2\pi^2}\int_0^\infty k^2 E_0^L(k)dk = \frac{1}{2\pi^2}\int_0^\infty k^2\langle 0_I|\left(\frac{\omega_k^2\hat{\pi}_L(\mathbf{k})^2}{2aM^2} + \frac{aM^2\hat{\phi}_L(\mathbf{k})^2}{2}\right) \times |0_I\rangle dk \tag{29}$$

with $\hat{\pi}_L(\mathbf{k}) \equiv \frac{aM^2}{\omega_k^2}\dot{\hat{\phi}}_L(\mathbf{k})$.

The Ermakov-Pinney equation (6) can be solved exactly for the transverse part since $F_T = a^{-1}$, $G_T = a\omega_k^2$, and

$$x_T = \sqrt{\frac{\pi H_\nu^{(1)}(z)H_{\nu^*}^{(2)}(z)}{2\mathcal{H}}} \tag{30}$$

with $\nu = \sqrt{\frac{1}{4} - \frac{M^2}{\mathcal{H}^2}}$ is the solution which is associated with the Bunch-Davies vacuum. The vacuum energy, in the large z limit, is given by

$$E_0^T(k) \overset{z\gg1}{=} \frac{k}{2a} + \frac{M^2 a}{4k} - \frac{M^4a^3}{16k^3} + o\left(\frac{1}{z^3}\right). \tag{31}$$

We note that quadratic and logarithmic divergent terms disappear in the massless case. The longitudinal field equations are obtained by setting $F_L = \frac{\omega_k^2}{aM^2}$, $G_L = aM^2$, and cannot be solved exactly; however, one can estimate the vacuum energy of (29) in the adiabatic, $z \gg 1$ limit by expanding the solution of Eq. (8) as a power series in z:

$$x_L \overset{z\gg1}{=} \frac{1}{\sqrt{\mathcal{H}z}}\left[1 + \frac{2\mathcal{H}^2 - M^2}{4\mathcal{H}^2z^2} + \frac{5(M^4 - 12M^2\mathcal{H}^2 + 4\mathcal{H}^4)}{32\mathcal{H}^4z^4} + o\left(\frac{1}{z^4}\right)\right] \tag{32}$$

and then expanding (7) to obtain

$$E_0^L(k) \overset{z\gg1}{=} \frac{k}{2a} + \frac{(\mathcal{H}^2 + M^2)a}{4k} - \frac{M^2(6\mathcal{H}^2 + M^2)a^3}{16k^3} + o\left(\frac{1}{z^3}\right). \tag{33}$$

Interestingly enough, the longitudinal vacuum energy is slightly different from that of a scalar field because of the curved space kinematics.

B. Fermionic field

The action for a massive spin $\frac{1}{2}$ field $\Psi(\mathbf{x}, t)$ is

$$S[\Psi] = \int d^4x\sqrt{-g}\left\{\frac{i}{2}[\bar{\Psi}\tilde{\gamma}^\mu(\nabla_\mu\Psi) - (\overline{\nabla_\mu\Psi})\tilde{\gamma}^\mu\Psi] - \mu\bar{\Psi}\Psi\right\}, \tag{34}$$

where ∇_μ is the gauge covariant derivative for spin 1/2 fields, $\tilde{\gamma}^\mu \equiv e_a{}^\mu(x)\gamma^a$ and $e_a{}^\mu(x)$ are the vierbein. In FRW space-time, on Fourier expanding

$$\Psi(\mathbf{x}, t) = \sqrt{\frac{a^3}{V}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\psi(\mathbf{k}, t), \tag{35}$$

the action (34) can be rewritten in a compact form [14] as

$$S = \sum_{\mathbf{k}}\int dt\psi(\mathbf{k})^\dagger\left[\frac{i}{2}\overleftrightarrow{\partial}_t - \hat{M}(\mathbf{k})\right]\psi(\mathbf{k}), \tag{36}$$

where $\hat{M}(\mathbf{k})$ is the 4×4 matrix

$$\hat{M}(\mathbf{k}) = \begin{pmatrix} \mu & \vec{\sigma}\cdot\frac{\mathbf{k}}{a} \\ \vec{\sigma}\cdot\frac{\mathbf{k}}{a} & -\mu \end{pmatrix}, \tag{37}$$

and $\vec{\sigma}$ are the standard 2×2 Pauli matrices. The Euler-Lagrange equation obtained for a spinor $\psi(\mathbf{k})$ is

$$i\dot{\psi}(\mathbf{k}) = \hat{M}(\mathbf{k})\psi(\mathbf{k}) \tag{38}$$

and the Hamiltonian for the system can be easily calculated in terms of $\psi(\mathbf{k})$ as

$$H_f \equiv \sum_{\mathbf{k}} H_f(\mathbf{k}) = \sum_{\mathbf{k}}\psi(\mathbf{k})^\dagger\hat{M}(\mathbf{k})\psi(\mathbf{k}). \tag{39}$$

The general solution of the equation of motion (38) can be written as a superposition of the 4 independent solutions

$$w_1^{(r)}(\mathbf{k}, t) = \sqrt{\frac{\pi z N_f}{2}}\begin{pmatrix} iH_\nu^{(1)}(z) \\ \frac{\vec{\sigma}\cdot\mathbf{k}}{k}H_{\nu-1}^{(1)}(z)\end{pmatrix}\chi^{(r)} \tag{40}$$

with $r = 1, 2$ and

$$w_2^{(r)}(\mathbf{k}, t) = -\sqrt{\frac{\pi z N_f}{2}}\begin{pmatrix} \frac{\vec{\sigma}\cdot\mathbf{k}}{k}H_{\nu^*-1}^{(2)}(z) \\ iH_{\nu^*}^{(2)}(z)\end{pmatrix}\chi^{(r)} \tag{41}$$

with $r = 3, 4$, where $\nu = \frac{1}{2} - i\frac{\mu}{\mathcal{H}}$, $N_f = \exp[\pi\mu/\mathcal{H}]$, $r = 1, 3$ (2, 4) correspond to spin up (down) for the Pauli spinors $\chi^{(r)}$ and the $w_i^{(r)}$ depend on time through z. Note that the independent solutions (40) and (41) reduce to the usual static solutions in the $z\to+\infty$ limit and they may then be associated with the time-independent creation-annihilation operators $\hat{b}_r(\mathbf{k})$, $\hat{b}_r(\mathbf{k})^\dagger$, $\hat{d}_r(\mathbf{k})$, $\hat{d}_r(\mathbf{k})^\dagger$. One defines the Bunch-Davies vacuum $|0_{\rm BD}\rangle$ by

$$\hat{b}_r(\mathbf{k})|0_{\rm BD}\rangle = \hat{d}_r(\mathbf{k})|0_{\rm BD}\rangle = 0 \tag{42}$$

with

$$\{\hat{b}_r(\mathbf{k}), \hat{b}_r(\mathbf{k})^\dagger\} = \{\hat{d}_r(\mathbf{k}), \hat{d}_r(\mathbf{k})^\dagger\} = 1. \tag{43}$$

In terms of the above operators, the Heisenberg field operator $\hat{\psi}(\mathbf{k}, t)$ can be written as

$$\hat{\psi}(\mathbf{k}, t) = \sum_{r=1,2} [w_1^{(r)}(\mathbf{k}, t)\hat{b}_r(\mathbf{k}) + (-1)^{r+1} w_2^{(r)}(\mathbf{k}, t)\hat{d}_r(-\mathbf{k})^\dagger] \tag{44}$$

and the vacuum expectation value of the Hamiltonian (39) can be evaluated in the continuum limit (13)

$$\frac{\langle 0_{\rm BD}|\hat{H}_F|0_{\rm BD}\rangle}{V} \overset{V\to\infty}{\longrightarrow} \frac{1}{\pi^2}\int_0^\infty k^2 E_0^F(k)dk = \frac{1}{2\pi^2}\int_0^\infty k^2 dk \langle 0_{\rm BD}|\hat{H}_f(\mathbf{k})|0_{\rm BD}\rangle \tag{45}$$

by using (42)–(44). We note that each $\hat{H}_f(\mathbf{k})$ contains the contributions both of a particle and an antiparticle with opposite spin and momentum; in terms of the Hankel functions, one obtains the following exact expression:

$$\begin{aligned}\langle 0_{\rm BD}|\hat{H}_f(\mathbf{k})|0_{\rm BD}\rangle = 2\Big[& m(H^{(1)}_{\nu-1}H^{(2)}_{\nu^*-1} - H^{(1)}_{\nu}H^{(2)}_{\nu^*}) \\ & + i\frac{k}{a}(H^{(1)}_{\nu-1}H^{(2)}_{\nu^*} - H^{(1)}_{\nu}H^{(2)}_{\nu^*-1})\Big] \\ & \times (H^{(1)}_{\nu}H^{(2)}_{\nu^*} + H^{(1)}_{\nu-1}H^{(2)}_{\nu^*-1})^{-1}\end{aligned} \tag{46}$$

and, on counting a single particle/antiparticle degree of freedom, in the large z limit, one finds

$$E_0^F(k) = -\frac{k}{2a} - \frac{\mu^2 a}{4k} + \frac{\mu^2(\mathcal{H}^2 + \mu^2)a^3}{16k^3} + o\left(\frac{1}{z^3}\right). \tag{47}$$

III. SUM RULES AND CANCELLATION OF ULTRAVIOLET DIVERGENCES

In this section we consider the general condition for the cancellation of divergent vacuum contributions to the Einstein equations coming from different free fields using the Källen-Lehmann spectral functions formalism. Some specific models will then be analyzed in detail. For the Minkowski case, one can obtain some general results which will later be generalized in a nontrivial way to the de Sitter case.

A. Spectral function general formalism

Starting from a semiclassical approach, we consider the Friedmann equation for the homogeneous scale factor $a(t)$ in (1):

$$a^3\mathcal{H}^2 = \frac{8\pi G}{3}\sum_j \frac{\langle 0_{\rm BD}|\hat{H}_j|0_{\rm BD}\rangle}{V}, \tag{48}$$

where $\sum_j$ is the sum over all bosonic and fermionic field (representing the particle content evolving on the space-time manifold) Hamiltonians averaged with respect to the corresponding Bunch-Davies vacua. The spectral function $\rho_A(x)$ can be introduced if we replace the above sum by

$$\sum_j \to \frac{1}{2\pi^2}\int_0^\infty k^2 dk\Big[\int dm^2\rho_S(m^2) + \int dM^2(\rho_V(M^2) - v_0\delta(M^2)) + \int dM^2 2\rho_V(M^2) + \int d\mu^2 2\rho_F(\mu^2)\Big], \tag{49}$$

where $\rho_A(x)$ have support for $0 \le x < \infty$, $v_0\delta(M^2)$ is the number density of massless vector particles, and Dirac spinors account for double the contribution of Majorana and Weyl particles. Moreover, the second line of Eq. (49) counts the longitudinal vector degrees of freedom while the third line gives the transverse contributions. On a de Sitter background, setting $m^2 \equiv x$ with Λ being the ultraviolet cutoff ($k \le aH\Lambda$) we adopt to regularize the divergent integral $\int_0^\infty dk$, Eq. (48) can be finally rewritten as

$$\begin{aligned}\frac{3\pi}{4G} &= \mathcal{H}\int_0^\Lambda z^2 dz\int dx[\rho_S(x)E_0^S(x,z) + (\rho_V(x) - v_0\delta(x))E_0^L(x,z) + 2\rho_V(x)E_0^T(x,z) + 2\rho_F(x)E_0^F(x,z)] \\ &= \int dx\Big(\Lambda^4\frac{\mathcal{H}^2}{8}F_1(x) + \frac{\Lambda^2}{8}F_2(x) + \frac{\log\Lambda}{16\mathcal{H}^2}F_3(x) + \mathcal{O}\Big(\frac{1}{\Lambda^2}\Big)\Big),\end{aligned} \tag{50}$$

where

$$F_1(x) = \rho_S(x) + 3\rho_V(x) - v_0\delta(x) - 2\rho_F(x), \tag{51}$$

$$F_2(x) = (\mathcal{H}^2 + x)\rho_S(x) + (3x + \mathcal{H}^2)\rho_V(x) - v_0(x + \mathcal{H}^2)\delta(x) - 2x\rho_F(x), \tag{52}$$

$$F_3(x) = x(2\mathcal{H}^2 - x)\rho_S(x) - x(6\mathcal{H}^2 + 3x)\rho_V(x) + v_0 x(6\mathcal{H}^2 + x)\delta(x) + 2x(\mathcal{H}^2 + x)\rho_F(x). \tag{53}$$

In order to obtain a finite contribution on the right-hand side of Eq. (50), the spectral functions describing the particle content of the universe should be such as to make (51)–(53) vanish simultaneously and this occurs when

$$\int dx F_1(x) = 0 \tag{54}$$

$$\int dx x F_1(x) = \mathcal{H}^2\int dx(v_0\delta(x) - \rho_S(x) - \rho_V(x)) \tag{55}$$

$$\int dx x^2 F_1(x) = 2\mathcal{H}^2 \int dx x(\rho_S(x) - 3\rho_V(x) + 3v_0\delta(x) + \rho_F(x)). \quad (56)$$

In the $\mathcal{H} \to 0$ limit, the conditions (54)–(56) reduce to the very compact form

$$\int dx x^i F_1(x) = 0, \qquad i = 0, 1, 2. \quad (57)$$

We note that the expression (50) is completely general and refers to arbitrary mass distributions. In the following sections we shall analyze models wherein the mass distributions are discrete and the $\rho_i(x)$ are just superpositions of Dirac delta functions multiplied by integer coefficients, $\rho_A = \sum_i n_i(A)\delta(x - x_i)$, and describe diverse particle multiplets.

For $i = 0$ Eq. (57) gives the condition of the equality of numbers of bosonic and fermionic degrees of freedom. For $i = 1, 2$ one has

$$\sum m_s^2 + 3\sum m_V^2 = 2\sum m_F^2$$

$$\sum m_s^4 + 3\sum m_V^4 = 2\sum m_F^4$$

which can be regarded as hyperplane and hypersphere equations, respectively, in term of squared mass variables, provided the bosonic or fermionic field content is fixed.

B. Hypersphere and hyperplane

While the cancellation of the quartic divergences requires the equality of the numbers of fermionic and bosonic degrees of freedom, the conditions for the cancellation of quadratic and logarithmic divergences are more involved. However, *for the Minkowski case*, these conditions can be represented as hyperplane and hypersphere equations, respectively.

Let us consider the hypersphere of dimensionality $n - 1$ embedded into the space of n dimensions, given by the equation

$$\sum_{i=1}^{n} y_i^2 = 1 \quad (58)$$

and the hyperplane given by the equation

$$\sum_{i=1}^{n} y_i = \lambda. \quad (59)$$

We look for the points of intersection between the hypersphere and hyperplane such that all the coordinates y_i have non-negative values. Then the minimum value of the parameter λ is $\lambda = 1$ which corresponds to the case when only one of the coordinates is nonzero, while the maximum value of λ is $\lambda = \sqrt{n}$, when all the coordinates are equal ($y_i = \frac{1}{\sqrt{n}}$, for $\forall i$). Thus

$$1 \le \lambda \le \sqrt{n}. \quad (60)$$

The intersection between the hypersphere (58) and the hyperplane (59) is a $(n - 2)$-dimensional hypersphere, whose center has equal coordinates y_i for all i and radius $\sqrt{1 - \lambda^2/n}$. All the points of the hypersphere of intersection $S^{(n-2)}$ can be represented by the vector

$$\frac{\lambda}{n}\vec{v} + \sqrt{1 - \frac{\lambda^2}{n}}\vec{u}, \quad (61)$$

where

$$\vec{v} \equiv (1, \ldots, 1), \quad (62)$$

$$\vec{u} \cdot \vec{u} = 1, \qquad \vec{u} \cdot \vec{v} = 0. \quad (63)$$

The coordinates of the points, belonging to $S^{(n-2)}$ are

$$y_i = \frac{\lambda}{n} + \sqrt{1 - \frac{\lambda^2}{n}} u_i. \quad (64)$$

Let us find the minimum and maximum possible values of u_i. Without losing generality we will focus on $i = 1$. We shall look for a solution in the form of a vector

$$\vec{u} = (\alpha, \vec{\beta}), \quad (65)$$

where β is a $(n - 1)$-dimensional vector. The conditions (63) give

$$\alpha^2 + \vec{\beta} \cdot \vec{\beta} = 1 \quad (66)$$

and

$$\alpha + \vec{\beta} \cdot \vec{I} = 0, \quad (67)$$

where I is a $(n - 1)$-dimensional vector with all the coordinates equal to 1. Then

$$\vec{\beta} \cdot \vec{\beta}((n - 1)\cos^2\chi + 1) = 1, \quad (68)$$

where χ is an angle between the vectors $\vec{\beta}$ and $\vec{I}$. Apparently, α^2 has maximum value when the angle $\cos\chi = \pm 1$. Then

$$\alpha = \pm\sqrt{\frac{n - 1}{n}}, \quad (69)$$

$$\vec{\beta} = \mp \frac{1}{\sqrt{n(n - 1)}}\vec{I}. \quad (70)$$

Substituting the maximum and minimum values of α from Eq. (69) into Eq. (64), one can find the maximum and minimum values of one of the coordinates y_i:

$$y_{i\,\max} = \frac{\lambda}{n} + \sqrt{1 - \frac{\lambda^2}{n}}\sqrt{\frac{n - 1}{n}}, \quad (71)$$

$$y_{i\,\min} = \frac{\lambda}{n} - \sqrt{1 - \frac{\lambda^2}{n}}\sqrt{\frac{n-1}{n}}. \tag{72}$$

We note that, when one of the coordinates y_i takes an extremum value, all the other coordinates have equal values [see Eq. (70)]. Now, using Eq. (72) we can get the condition for the positivity of all the coordinates y_i of the points lying on the $(n-2)$-dimensional sphere. Requiring $y_{i\,\min} \geq 0$ we have

$$\lambda \geq \sqrt{n-1}. \tag{73}$$

Curiously, when one of the coordinates acquires the maximum value (71) all the other coordinates have value

$$y_j = \frac{\lambda}{n} - \sqrt{1 - \frac{\lambda^2}{n}}\frac{1}{\sqrt{n(n-1)}}, \tag{74}$$

which is non-negative for $\lambda \geq 1$ and hence is always non-negative for our range of λ.

Let us now consider the case $\lambda < \sqrt{n-1}$. In this case some of the points of $S^{(n-2)}$ have negative values of coordinates and should be excluded from consideration. These patches of the hypersphere have the form of n spherical caps, whose pole points are the points where one of the n coordinates has its minimum and, hence, for $\lambda < \sqrt{n-1}$ negative value. The angle characterizing the size of these spherical caps (i.e. the angle between the unit vector pointing to the pole and the vector pointing to the intersection of $S^{(n-2)}$ with the hyperplane $y_i = 0$) is

$$\theta = \arccos\frac{\lambda}{\sqrt{(n-\lambda^2)(n-1)}}. \tag{75}$$

One can find also the angle between two vectors pointing to two different "poles":

$$\chi = \arccos\left(-\frac{1}{n-1}\right). \tag{76}$$

In the case for which $2\theta > \chi$, two spherical caps can intersect each other. A simple calculation shows that it occurs if $\lambda < \sqrt{n-2}$. Let us note that this does not mean that acceptable solutions do not exist. Indeed, as we explained above, when one of the coordinates y_i has the maximum value (71) all the other coordinates have non-negative values (74): for the case $\lambda > 1$ these values are positive.

C. Some simple models

In our case the parameter λ is nothing more than

$$\lambda = \frac{4\sum_{i=1}^{D} m_{iD}^2 + 2\sum_{i=1}^{M} m_{iM}^2}{\sqrt{4\sum_{i=1}^{D} m_{iD}^4 + 2\sum_{i=1}^{M} m_{iM}^4}}, \tag{77}$$

where m_{iD} and m_{iM} are the masses of Dirac and Majorana spinors, respectively, while D and M are the numbers of these spinors. The number n is the total number of fermionic degrees of freedom

$$n = 4D + 2M + 2W, \tag{78}$$

where W is the number of Weyl spinors. The number of bosonic degrees of freedom should be equal to the number of fermionic degrees of freedom n and the boson masses (squared) are given by $y_i\sqrt{4\sum_{i=1}^{D} m_{iD}^4 + 2\sum_{i=1}^{M} m_{iM}^4}$. It is easy to see that the value of λ cannot take the minimum value $\lambda = 1$ [cf. Eq. (60)].

Let us consider the model with 2 degrees of freedom ($n = 2$). In this case we have a Majorana spinor and the value of the parameter λ is $\lambda = \sqrt{2}$ and the circumference and straight line have the only point of intersection, when $y_1 = y_2$. Obviously this situation corresponds to two scalars with the same mass, which coincides with that of the Majorana spinor. Therefore in this case the conditions to have the required cancellations are so severe that the only possible solution is the one which can be obtained by imposing supersymmetry for bosons and fermions. We shall see that for larger systems these constraints are not so stringent. If instead of Majorana one has a Weyl spinor, the bosonic part of the spectrum will be represented by two massless scalar fields or by one electromagnetic field.

The model with the 4 degrees of freedom represented by one Dirac spinor or by two Majorana spinors with identical masses ($n = 4$, $\lambda = 2$) also has only a trivial solution: four scalar fields with the same mass or one scalar field and one massive vector field whose masses coincide. The case with two Majorana spinors with different masses is more interesting. In this case $n = 4$, and

$$\lambda = \sqrt{2}\frac{m_1^2 + m_2^2}{\sqrt{m_1^4 + m_2^4}}. \tag{79}$$

Now on using the results of the preceding section it is easy to see that the prohibited caps do not exist if $\lambda \geq \sqrt{3}$ [see Eq. (73)]. Substituting this condition into Eq. (79), one reexpresses this condition in terms of the relation between two fermion masses:

$$2 + \sqrt{3} \geq m_1^2/m_2^2 \geq 2 - \sqrt{3}. \tag{80}$$

If this condition is violated the prohibited caps exist, but they do not intersect each other.

In the following we shall investigate some specific models by directly studying the conditions (55) and (56) and using a slightly different notation: x_i and y_i will now denote the square of the boson and fermion masses, respectively.

Let us consider a particular case with two fields in the bosonic sector: one scalar and one vector massive fields are present. Now, the sum rules lead to

$$3x_1 + x_2 = 2y_1 + 2y_2, \tag{81}$$

$$3x_1^2 + x_2^2 = 2y_1^2 + 2y_2^2, \tag{82}$$

where x_1 and x_2 are the vector boson and scalar masses squared, while y_1 and y_2 are the Majorana spinor masses squared ($y_2 \geq y_1$). On solving Eqs. (81) and (82) we get

$$x_1 = \frac{y_1 + y_2}{2} \pm \frac{\sqrt{3}}{6}(y_2 - y_1), \tag{83}$$

$$x_2 = \frac{y_1 + y_2}{2} \mp \frac{\sqrt{3}}{2}(y_2 - y_1). \tag{84}$$

Let us note that the first solution for x_2 is negative if $\frac{y_1}{y_2} = \frac{m_1^2}{m_2^2} \leq 2 - \sqrt{3}$. Obviously this condition coincides with the condition (80). This is quite natural, since the case with one scalar and one vector massive fields corresponds to the vector $\vec{u}$, considered in the preceding section, having an extremum value component while all the others have equal values. Thus, when the condition (73) is satisfied there are two solutions for the system of Eqs. (81) and (82), corresponding to the scalar field having maximum and minimum values, while if this condition is not valid only one solution survives.

Let us now consider the model with 6 degrees of freedom, wherein the fermion sector is represented by one Dirac and one Majorana spinors. To begin with, consider the bosonic sector containing two massive vector fields. The sum rules in this case lead to

$$3x_1 + 3x_2 = 4y_1 + 2y_2, \tag{85}$$

$$3x_1^2 + 3x_2^2 = 4y_1^2 + 2y_2^2, \tag{86}$$

where y_1 and y_2 represent the Dirac and Majorana fermion masses squared. The solution of Eqs. (85) and (86) is

$$x_{1,2} = \frac{2y_1 + y_2 \pm \sqrt{2}|y_1 - y_2|}{3}. \tag{87}$$

This solution is positive for both x_1 and x_2 provided the following condition is satisfied:

$$\frac{m_2^2}{m_1^2} = \frac{y_2}{y_1} \leq 4 + 3\sqrt{2}. \tag{88}$$

Let us now consider the bosonic sector which contains a massive vector field, an electromagnetic field, and a scalar field. In this case the sum rules are

$$3x_1 + x_2 = 4y_1 + 2y_2, \tag{89}$$

$$3x_1^2 + x_2^2 = 4y_1^2 + 2y_2^2. \tag{90}$$

The existence of positive solutions for x_1 and x_2 depends on the relation between fermion masses. Namely, if

$$\frac{y_1}{y_2} \leq \frac{3\sqrt{2} - 4}{2}, \tag{91}$$

there is one solution

$$x_1 = y_1 + \frac{y_2}{2} - \sqrt{\frac{y_2^2}{12} - \frac{y_1 y_2}{3}}, \tag{92}$$

$$x_2 = y_1 + \frac{y_2}{2} + 3\sqrt{\frac{y_2^2}{12} - \frac{y_1 y_2}{3}}. \tag{93}$$

If

$$\frac{3\sqrt{2} - 4}{2} \leq \frac{y_1}{y_2} \leq \frac{1}{4}, \tag{94}$$

then two solutions exist. One of them is the solution (92) and (93) and the second solution is

$$x_1 = y_1 + \frac{y_2}{2} + \sqrt{\frac{y_2^2}{12} - \frac{y_1 y_2}{3}}, \tag{95}$$

$$x_2 = y_1 + \frac{y_2}{2} - 3\sqrt{\frac{y_2^2}{12} - \frac{y_1 y_2}{3}}. \tag{96}$$

Finally, if $\frac{y_1}{y_2} > \frac{1}{4}$ solutions do not exist.

D. Simple models in the presence of the cosmological constant

For the case of the cosmological constant being different from zero, the relations between masses of the fields leading to the cancellation of divergences in the vacuum energy become more involved.

Consider the case of 2 degrees of freedom. In this case the fermionic sector is represented by one Majorana spinor with mass squared denoted by y, while the bosonic sector is represented by two scalar fields with masses squared denoted by x_1 and x_2 The sum rules are

$$x_1 + x_2 = 2y - 2\mathcal{H}^2, \tag{97}$$

$$x_1^2 + x_2^2 - 2\mathcal{H}^2(x_1 + x_2) = 2y(y + \mathcal{H}^2). \tag{98}$$

Non-negative solutions of this system of equations are possible provided the Majorana fermion mass satisfies the constraint

$$y \geq \frac{7 + \sqrt{33}}{2}\mathcal{H}^2. \tag{99}$$

In this case the masses of the scalar fields are

$$x_{1,2} = y - \mathcal{H}^2 \pm \sqrt{5y\mathcal{H}^2 - 3\mathcal{H}^4}. \tag{100}$$

For the case of 4 degrees of freedom, one can have as a nontrivial solution for the fermionic sector a Dirac fermion. If the bosonic sector is represented by an electromagnetic field and two scalar fields, the sum rules are

$$x_1 + x_2 = 4y - 2\mathcal{H}^2, \tag{101}$$

$$x_1^2 + x_2^2 - 2\mathcal{H}^2(x_1 + x_2) = 4y(y + \mathcal{H}^2). \quad (102)$$

The system of equations (101) and (102) has nontrivial non-negative solutions and the expected condition $x_1 = x_2 = y = 0$ is recovered as $\mathcal{H}$ tends to zero. In the presence of nonzero $\mathcal{H}$ if

$$2\mathcal{H}^2 \leq y \leq \frac{1}{2}(5 + \sqrt{19})\mathcal{H}^2, \quad (103)$$

one can easily obtain

$$x_{1,2} = 2y - \mathcal{H}^2 \pm \sqrt{10y\mathcal{H}^2 - 2y^2 - 3\mathcal{H}^4}, \quad (104)$$

where the lower bound in (103) is required for $x_{1,2}$ to be positive while the upper bound is needed for the reality of the solutions.

The last case which we treat here is the bosonic sector represented by a scalar and a vector massive fields. The sum rules take the following form:

$$3x_1 + x_2 + 2\mathcal{H}^2 = 4y, \quad (105)$$

$$3x_1^2 + x_2^2 + 2\mathcal{H}^2(3x_1 - x_2) = 4y(y + \mathcal{H}^2). \quad (106)$$

If the condition $y > 2\mathcal{H}^2$ is satisfied there are two non-negative solutions:

$$x_1 = y - \mathcal{H}^2 + \sqrt{\frac{\mathcal{H}^2(\mathcal{H}^2 + y)}{3}}, \quad (107)$$

$$x_2 = y + \mathcal{H}^2 - 3\sqrt{\frac{\mathcal{H}^2(\mathcal{H}^2 + y)}{3}}, \quad (108)$$

and

$$x_1 = y - \mathcal{H}^2 - \sqrt{\frac{\mathcal{H}^2(\mathcal{H}^2 + y)}{3}}, \quad (109)$$

$$x_2 = y + \mathcal{H}^2 + 3\sqrt{\frac{\mathcal{H}^2(\mathcal{H}^2 + y)}{3}}. \quad (110)$$

In the limiting case $y = 2\mathcal{H}^2$ only the solution (107) and (108) is valid because in formula (109) the mass of the massive vector boson vanishes and this breaks the equality of the boson and fermion degrees of freedom.

As a consequence of our approach, it appears that the particle mass spectrum and the Hubble constant of the De Sitter space are related. Of course one may attempt to introduce some dynamics and an evolving mass spectrum, which however we expect to still be connected to the Hubble parameter. We plan to return to this in the future.

IV. CONCLUSIONS

In this paper we considered the problem of vacuum energy in quantum field theory and cosmology. This is often associated with the so-called cosmological constant problem. Indeed, the contribution of quantum vacuum fluctuations to the energy-momentum tensor behaves as a cosmological constant [1]. However, the cosmological constant, in principle, can also be of nonquantum origin and be present in the theory as one of fundamental constants. The real vacuum energy problem consists in the absence of a well established and justified procedure for the renormalization of ultraviolet divergences in the energy-momentum tensor analogous to the theory of renormalization in standard quantum field theory in a flat (absence of gravity) space-time. Indeed, in quantum field theory the ultraviolet divergences are absorbed in the renormalization of a finite number of measurable constants, or, in other words, all the observable quantities become finite due to the introduction of infinite counterterms into the bare Lagrangian (see e.g. [15,16]). As far the very strong ultraviolet divergences in the energy-momentum tensor are concerned, they are set equal to zero by the normal or Wick quantization of quantum fields. The last step could be justified by the fact that one always measures the differences between energy levels and not the absolute values of energy. However, this justification fails in the presence of gravity, because of the very structure of the Einstein equations, which connects the curvature of space-time to the energy-momentum tensor. Thus, one should also take into account the contribution of the vacuum expectation value of the energy-momentum tensor on the right-hand side of Einstein equations. Indeed one may study models wherein vacuum oscillations drive some stage of the expansion of patches of the universe can be studied (see [17]).

It is well known that the naive calculation of the vacuum energy of quantum fluctuations using a cutoff on the Planck scale gives a huge value for it. One has two possible ways out of this situation. The first one is the construction of a consistent renormalization theory for the ultraviolet divergences of the energy-momentum tensor, which is a very difficult task, since we do not have such relatively simple tools as the renormalization of some known physical quantities as, for example, in quantum electrodynamics (see, however, [18], where an attempt to fix the renormalization of an effective cosmological constant based on some self-consistency conditions was undertaken). On the other hand, one may require the exact cancellation of ultraviolet divergences in the energy-momentum tensor. This is the approach attempted in the present paper (a good review of analogous approaches is given in [19], further we also wish to mention a related flat space approach in the context of induced gravity [20]). Our idea can be simply formulated as follows: if you do not know what to do with these ultraviolet divergences, just try to eliminate them by introducing the condition of their cancellation as a quantum consistency condition of the theory. Indeed, the cancellation conditions on the spectral functions mentioned in the Introduction and studied in the third section of this paper is one of the oldest examples of such approach. Other well-known examples of quantum self-consistency are connected with conditions for the cancellation of quantum anomalies. A classical example is the mechanism suggested by Glashow, Iliopoulos, and Maiani [21], who introduced the fourth quark into the theory of electroweak interactions to suppress the chiral anomaly. Later this quark was discovered experimentally and was called "charmed." Another spectacular example is the appearance of critical dimensions in string and superstring theories [22]. The theory of a bosonic string can be formulated consistently in a space-time of 26 dimensions, while the critical dimensionality for superstrings is equal to 10. Analogous relations between the dimensionality of space-time and its matter content arise also in the application of the Batalin-Fradkin-Vilkovisky–Becchi-Rue-Stora-Tyutin (BFV-BRST) quantization mechanism to quantum cosmology [23]. Yet another restriction on the particle spectrum of a theory based on the requirement of the normalizability of the wave function of the universe was obtained in [24].

Thus, our work, which uses the condition of cancellation of ultraviolet divergences in the vacuum energy to arrive to some special conditions on the spectrum of the fundamental theory, is in kinship with a rather fruitful and ramified stream of the development of modern quantum theory. Let us now mention the similarities and the differences between our approach and the supersymmetry one. In the case of exactly supersymmetric models in flat space-time, there is an exact cancellation of the vacuum energy and not only of its divergent part. This cancellation occurs because the fermion and boson contributions to the vacuum energy enter with opposite signs. However, exact supersymmetry cannot be implemented for the construction of a realistic theory of elementary particles. Moreover, there are serious difficulties arising in the formulation of supersymmetry in curved background space-times. Here we share one common feature with supersymmetry: the equal number of fermionic and bosonic degrees of freedom, which is indispensable for the cancellation of quartic ultraviolet divergences. The general conditions for the cancellation of quadratic and logarithmic divergences are much more flexible than the requiring of exact supersymmetry. On analyzing some simple examples we have seen that there are many opportunities of satisfying these conditions. At the same time we do not require the cancellation of the finite part of the vacuum energy. It could be different from zero and be responsible for the observable cosmological constant.

We note that the sum rule constraints may generally allow for theories whose particle spectrum fit the content of the standard model in the low mass sector. It is widely believed that some kind of extension of the standard model is needed and many of them, such as the supersymmetric ones, have indeed been considered. Therefore our approach may be helpful to encode the minimum possible number of constraints on the spectrum of the theory necessary for the cancellation of the divergences of the vacuum energy, even in the presence of an effective cosmological constant. So, one may think of some kind of grand unification theory, where sum rules can be realized. Perhaps, such a theory should contain not only additional supermassive gauge bosons but also some additional families of fermions. A straightforward extension of our analysis can be done for the case of extra spatial dimensions, one has simply to take into account the dimension dependent number of physical boson and fermion degrees of freedom and write the general identity necessary to saturate the corresponding spectral function sum rules. We believe that all these topics deserve further investigation, not to mention the introduction of higher spins.

ACKNOWLEDGMENTS

We wish to thank G. E. Volovik for useful correspondence.

[1] Y. B. Zeldovich, Pis'ma Zh. Eksp. Teor. Fiz. **6**, 883 (1967) [JETP Lett. **6**, 316 (1967)].

[2] S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989).

[3] A. Riess *et al.*, Astron. J. **116**, 1009 (1998); S. J. Perlmutter *et al.*, Astrophys. J. **517**, 565 (1999).

[4] V. Sahni and A. A. Starobinsky, Int. J. Mod. Phys. D **9**, 373 (2000); T. Padmanabhan, Phys. Rep. **380**, 235 (2003); P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. **75**, 559 (2003); E. J. Copeland, M. Sami, and S. Tsujikawa, Int. J. Mod. Phys. D **15**, 1753 (2006).

[5] S. Nobbenhuis, Found. Phys. **36**, 613 (2006).

[6] S. Weinberg, Phys. Rev. Lett. **18**, 507 (1967).

[7] C. W. Bernard, A. Duncan, J. LoSecco, and S. Weinberg, Phys. Rev. D **12**, 792 (1975).

[8] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).

[9] H. R. Lewis, Jr. and W. B. Riesenfeld, J. Math. Phys. (N.Y.) **10**, 1458 (1969).

[10] V. P. Ermakov, Univ. Izv. Kiev III **9**, 1 (1880); E. Pinney, Proc. Am. Math. Soc. **1**, 681 (1950).

[11] X. C. Gao, J. B. Xu, and T. Z. Qian, Phys. Rev. A **44**, 7016 (1991); Ann. Phys. (N.Y.) **204**, 235 (1990).

[12] T. S. Bunch and P. C. W. Davies, Proc. R. Soc. A **360**, 117 (1978).

[13] C. Bertoni, F. Finelli, and G. Venturi, Classical Quantum

Gravity **13**, 2375 (1996); Phys. Lett. A **237**, 331 (1998).
[14] F. Finelli, A. Gruppuso, and G. Venturi, Classical Quantum Gravity **16**, 3923 (1999).
[15] N. N. Bogoliubov and D. V. Shirkov, *Introduction into the Theory of Quantized Fields* (Wiley, New York, 1980).
[16] S. Weinberg, *Quantum Theory of Fields, Vol. I* (Cambridge University Press, Cambridge, England, 1995).
[17] F. Finelli, G. P. Vacca, and G. Venturi, Phys. Rev. D **58**, 103514 (1998).
[18] A. O. Barvinsky and A. Y. Kamenshchik, J. Cosmol. Astropart. Phys. 09 (2006) 014; Phys. Rev. D **74**, 121502 (2006).
[19] M. Visser, Mod. Phys. Lett. A **17**, 977 (2002).
[20] V. P. Frolov and D. Fursaev, Phys. Rev. D **58**, 124009 (1998).
[21] S. L. Glashow, J. Iliopoulos, and L. Maiani, Phys. Rev. D **2**, 1285 (1970).
[22] J. Polchinski, *String Theory, Vols. I, II* (Cambridge University Press, Cambridge, England, 1998).
[23] A. Y. Kamenshchik and S. L. Lyakhovich, Nucl. Phys. **B495**, 309 (1997).
[24] A. O. Barvinsky and A. Y. Kamenshchik, Classical Quantum Gravity **7**, L181 (1990); Phys. Lett. B **332**, 270 (1994).