Non-Gaussian covariance of CMB *B* modes of polarization and parameter degradation

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The *B*-mode polarization lensing signal is a useful probe of the neutrino mass and to a lesser extent the dark energy equation of state as the signal depends on the integrated mass power spectrum between us and the last scattering surface. This lensing *B*-mode signal, however, is non-Gaussian and the resulting non-Gaussian covariance to the power spectrum could impact cosmological parameter measurements, as correlations between *B*-mode bins are at a level of 0.1. On the other hand, for temperature and *E*-mode polarization power spectra, the non-Gaussian covariance is not significant, where we find correlations at the 10^{-5} level even for adjacent bins. When the power spectrum is estimated with roughly 5 uniformly spaced bins from $l = 5$ to $l = 100$ and 13 logarithmic uniformly spaced bins from $l = 100$ to $l = 2000$, the resulting degradation on neutrino mass and dark energy equation of state is about a factor of 2 to 3 when compared to the case where statistics are simply considered to be Gaussian. If we increase the total number of bins between $l = 5$ and $l = 2000$ to be about 100, we find that the non-Gaussianities only make a minor difference with less than a few percent correction to uncertainties of most cosmological parameters determined from the data. For Planck, the resulting constraints on the sum of the neutrino masses is $\sigma_{\Sigma m_\nu} \sim 0.2$ eV and on the dark energy equation of state parameter we find that $\sigma_w \sim 0.5$. A post-Planck experiment can improve the neutrino mass measurement by a factor of 3 to 4.

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I. INTRODUCTION

The applications of cosmic microwave background (CMB) anisotropy measurements are well known [1]; its ability to constrain most, or certain combinations of, parameters that define the currently favorable cold dark matter cosmologies with a cosmological constant is well demonstrated with anisotropy data from Wilkinson Microwave Anisotropy Probe [2]. Furthermore the advent of high sensitivity CMB polarization experiments with increasing sensitivity [3] suggests that we will soon detect the small amplitude *B*-mode polarization signal. While at degree scales one expects a unique *B*-mode polarization signal due to primordial gravitational waves [4], at arcminute angular scales the dominant signal will be related to cosmic shear conversion of *E* modes to *B* modes by the large-scale structure during the photon propagation from the last scattering surface to the observer today [5].

This weak lensing of cosmic microwave background (CMB) polarization by intervening mass fluctuations is now well studied in the literature [6,7], with a significant effort spent on improving the accuracy of analytical and numerical calculations (see a recent review in Ref. [8]). As discussed in recent literature [9], the lensing *B*-mode signal carries important cosmological information on the neutrino mass and possibly the dark energy, such as its equation of state [9], as the lensing signal depends on the integrated mass power spectrum between us and the last scattering surface, weighted by the lensing kernel. The dark energy dependence involves the angular diameter distance projections while the effects related to a nonzero neutrino mass come from suppression of small scale power below the free-streaming scale.

Since the CMB lensing effect is inherently a nonlinear process, the lensing corrections to CMB temperature and polarization are expected to be highly non-Gaussian. This non-Gaussianity at the four-point and higher levels are exploited when reconstructing the integrated mass field via a lensing analysis of CMB temperature and polarization [10]. The four-point correlations are of special interest since they also quantify the sample variance and covariance of two point correlation or power spectrum measurements [11]. A discussion of lensing covariance of the temperature anisotropy power spectrum is available in Ref. [12]. In the case of CMB polarization, the existence of a large sample variance for *B* modes of polarization is already known [13], though the effect on cosmological parameter measurements is yet to be quantified. Various estimates on parameter measurements in the literature ignore the effect of non-Gaussianities and could have overestimated the use of CMB *B* modes to tightly constrain parameters such as a neutrino mass or the dark energy equation of state. To properly understand the extent to which future polarization measurements can constrain these parameters, a proper understanding of non-Gaussian covariance is needed.

Here, we discuss the temperature and polarization covariances due to gravitational lensing. Initial calculations on this topic are available in Refs. [13,14], while detailed calculations on the CMB lensing trispectra are in Ref. [15]. Here, we focus mainly on the covariance and calculate them under the exact all-sky formulation; for flat-sky expressions of the trispectrum, we refer the reader to Ref. [10]. We extend those calculations and also discuss the impact on cosmological parameter estimates. This paper is organized as follows: In Sec. II, we introduce the basic ingredients for the present calculation and present covariances of temperature and polarization spectra. We discuss our results in Sec. III and conclude with a summary in Sec. IV.

II. CALCULATIONAL METHOD

The lensing of the CMB is a remapping of temperature and polarization anisotropies by gravitational angular deflections during the propagation. Since lensing leads to a redistribution of photons, the resulting effect appears only at second order [8]. In weak gravitational lensing, the deflection angle on the sky is given by the angular gradient of the lensing potential, $\delta(\hat{\mathbf{n}}) = \nabla\phi(\hat{\mathbf{n}})$, which is itself a projection of the gravitational potential Φ:

$$\phi(\mathbf{m}) = -2\int_0^{r_0} dr \frac{d_A(r_0 - r)}{d_A(r) d_A(r_0)} \Phi(r, \hat{\mathbf{m}} r), \tag{1}$$

where $r(z)$ is the comoving distance along the line of sight, r_0 is the comoving distance to the surface of last scattering, and $d_A(r)$ is the angular diameter distance. Taking the multipole moments, the power spectrum of lensing potentials is now given through

$$\langle \phi^*_{lm}\phi_{lm}\rangle = \delta_{ll'}\delta_{mm'}C_l^{\phi} \tag{2}$$

as

$$C_l^{\phi} = \frac{2}{\pi}\int k^2 dk P(k) I_l^{\rm len}(k) I_l^{\rm len}(k), \tag{3}$$

where

$$I_l^{\rm len}(k) = \int_0^{r_0} dr W^{\rm len}(k, r) j_l(kr),$$
$$W^{\rm len}(k, r) = -3\Omega_m\left(\frac{H_0}{k}\right)^2 F(r)\frac{d_A(r_0 - r)}{d_A(r) d_A(r_0)}, \tag{4}$$

where $F(r) = G(r)/a(r)$ and $G(r)$ is the growth factor, which describes the growth of large-scale density perturbations. In our calculations we will generate C_l^{ϕ} based on a nonlinear description of the matter power spectrum $P(k)$. In the next three subsections we briefly outline the power spectrum covariances under gravitational lensing for temperature and polarization E and B modes. In the numerical calculations described later, we take a fiducial flat-ΛCDM cosmological model with $\Omega_b = 0.0418$, $\Omega_m = 0.24$, $h = 0.73$, $\tau = 0.092$, $n_s = 0.958$, $A(k_0 = 0.05\ {\rm Mpc}^{-1}) = 2.3\times 10^{-9}$, $m_\nu = 0.05$ eV, and $w = -1$. This model is consistent with recent measurements from WMAP [2].

A. Temperature anisotropy covariance

The trispectrum for the unlensed temperature can be written in terms of the multipole moments of the temperature T_{lm} as [15]

$$\langle T_{l_1m_1}T_{l_2m_2}T_{l_3m_3}T_{l_4m_4}\rangle = C_{l_1}^{TT}C_{l_4}^{TT}(-1)^{m_1+m_4}\delta_{l_1l_3}^{m_1-m_3}\delta_{l_2l_4}^{m_2-m_4} + C_{l_1}^{TT}C_{l_3}^{TT}(-1)^{m_1+m_3}\delta_{l_1l_2}^{m_1-m_2}\delta_{l_3l_4}^{m_3-m_4} + C_{l_1}^{TT}C_{l_2}^{TT}(-1)^{m_1+m_2}\delta_{l_1l_4}^{m_1-m_4}\delta_{l_2l_3}^{m_2-m_3}, \tag{5}$$

where $\delta_{l_1l_2}^{m_1-m_2}$ denotes the Kronecker delta symbol which is nonzero only when $l_1 = l_2$ and $m_1 = -m_2$. It is straight forward to derive the following expression for the multipole moment of lensed T field as a perturbative equation related to the deflection angle [7]:

$$\tilde{T}_{lm} = T_{lm} + \sum_{l_1m_1l_2m_2}\phi_{l_1m_1}T_{l_2m_2}I_{ll_1l_2}^{mm_1m_2} + \frac{1}{2}\sum_{l_1m_1l_2m_2l_3m_3}\phi_{l_1m_1}T_{l_2m_2}\phi^*_{l_3m_3}J_{ll_1l_2l_3}^{mm_1m_2m_3}, \tag{6}$$

where the mode-coupling integrals between the temperature field and the deflection field, $I_{ll_1l_2}^{mm_1m_2}$ and $J_{ll_1l_2l_3}^{mm_1m_2m_3}$, are defined in Ref. [15,16].

As for the covariance of the temperature anisotropy power spectrum, we write

$$\mathrm{Cov}_{TTTT} \equiv \frac{1}{2l_1+1}\frac{1}{2l_2+1}\sum_{m_1m_2}\langle\tilde{T}_{l_1m_1}\tilde{T}^*_{l_1m_1}\tilde{T}_{l_2m_2}\tilde{T}^*_{l_2m_2}\rangle - \tilde{C}_{l_1}^{TT}\tilde{C}_{l_2}^{TT} = \mathcal{O} + \mathcal{P} + (\mathcal{Q} + \mathcal{R})\delta_{l_1l_2}, \tag{7}$$

where the individual terms are

$$\mathcal{O} = \frac{2}{(2l_1+1)(2l_2+1)}\sum_L C_L^{\phi}[(F_{l_1Ll_2}C_{l_2}^{TT})^2 + (F_{l_2Ll_1}C_{l_1}^{TT})^2],$$
$$\mathcal{P} = \frac{4}{(2l_1+1)(2l_2+1)}\sum_L C_L^{\phi}C_{l_1}^{TT}C_{l_2}^{TT}F_{l_1Ll_2}F_{l_2Ll_1},$$
$$\mathcal{Q} = \frac{2}{2l_1+1}(C_{l_1}^{TT})^2 + \frac{4}{(2l_1+1)^2}\sum_{L,l'}C_L^{\phi}C_{l'}^{TT}C_{l_1}^{TT}(F_{l_1Ll'})^2,$$
$$\mathcal{R} = -\frac{(l_1(l_1+1))}{2\pi(2l_1+1)}\sum_L C_L^{\phi}(C_{l_1}^{TT})^2 L(L+1)(2L+1), \tag{8}$$

and the last two terms, which are related to the Gaussian variance, can be written in terms of the lensed temperature anisotropy power spectrum as

$$\mathcal{Q} + \mathcal{R} = \frac{2}{2l_1+1}(\tilde{C}_{l_1}^{TT})^2, \tag{9}$$

where

$$\tilde{C}_l^{TT} = [1 - (l^2 + l)R]C_l^{TT} + \sum_{l_1 l_2} C_{l_1}^{\phi} \frac{(F_{ll_1l_2})^2}{2l+1} C_{l_2}^{TT},$$
$$R = \frac{1}{8\pi}\sum_{l_1} l_1(l_1+1)(2l_1+1)C_{l_1}^{\phi}, \tag{10}$$
$$F_{ll_1l_2} = \frac{1}{2}[l_1(l_1+1) + l_2(l_2+1) - l(l+1)] \times \sqrt{\frac{(2l+1)(2l_1+1)(2l_2+1)}{4\pi}} \begin{pmatrix} l & l_1 & l_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

We note that Eqs. (10) are readily derivable when considering the lensing effect on the temperature anisotropy spectrum as in Ref. [7].

B. *E*-mode polarization covariance

Similar to the case with temperature, the trispectrum for an unlensed E field can be written in terms of the multipole moments of the E mode E_{lm}:

$$\langle E_{l_1m_1}E_{l_2m_2}E_{l_3m_3}E_{l_4m_4}\rangle = C_{l_1}^{EE}C_{l_4}^{EE}(-1)^{m_1+m_4}\delta_{l_1l_3}^{m_1-m_3}\delta_{l_2l_4}^{m_2-m_4} + C_{l_1}^{EE}C_{l_3}^{EE}(-1)^{m_1+m_3}\delta_{l_1l_2}^{m_1-m_2}\delta_{l_3l_4}^{m_3-m_4} + C_{l_1}^{EE}C_{l_2}^{EE}(-1)^{m_1+m_2}\delta_{l_1l_4}^{m_1-m_4}\delta_{l_2l_3}^{m_2-m_3}. \tag{11}$$

To complete the calculation, besides the trispectrum of the unlensed E field in Eq. (11), we also require the expression for the trispectrum of the lensing potentials. Under the Gaussian hypothesis for the primordial E modes and ignoring non-Gaussian corrections to the ϕ field, the lensing trispectra is given by

$$\langle \phi_{l_1m_1}\phi_{l_2m_2}\phi_{l_3m_3}\phi_{l_4m_4}\rangle = C_{l_1}^{\phi}C_{l_4}^{\phi}(-1)^{m_1+m_4}\delta_{l_1l_3}^{m_1-m_3}\delta_{l_2l_4}^{m_2-m_4} + C_{l_1}^{\phi}C_{l_3}^{\phi}(-1)^{m_1+m_3}\delta_{l_1l_2}^{m_1-m_2}\delta_{l_3l_4}^{m_3-m_4} + C_{l_1}^{\phi}C_{l_2}^{\phi}(-1)^{m_1+m_2}\delta_{l_1l_4}^{m_1-m_4}\delta_{l_2l_3}^{m_2-m_3}. \tag{12}$$

For simplicity, we assume that there is no primordial B field such as due to a gravitational wave background and find the following expression for the lensed E field:

$$\tilde{E}_{lm} = E_{lm} + \frac{1}{2}\sum_{l_1m_1l_2m_2}\phi_{l_1m_1}E_{l_2m_2}\,{}_{+2}I_{ll_1l_2}^{mm_1m_2}(1+(-1)^{l+l_1+l_2}) + \frac{1}{4}\sum_{l_1m_1l_2m_2l_3m_3}\phi_{l_1m_1}E_{l_2m_2} \times \phi^*_{l_3m_3}\,{}_{+2}J_{ll_1l_2l_3}^{mm_1m_2m_3}(1+(-1)^{l+l_1+l_2+l_3}), \tag{13}$$

where the expressions for the mode-coupling integrals ${}_{+2}I_{ll_1l_2}^{mm_1m_2}$ and ${}_{+2}J_{ll_1l_2l_3}^{mm_1m_2m_3}$ are described in Refs. [15,16].

As for the covariance of E-mode power spectrum, we write

$$\mathrm{Cov}_{EEEE} \equiv \frac{1}{2l_1+1}\frac{1}{2l_2+1}\sum_{m_1m_2}\langle \tilde{E}_{l_1m_1}\tilde{E}^*_{l_1m_1}\tilde{E}_{l_2m_2}\tilde{E}^*_{l_2m_2}\rangle - \tilde{C}_{l_1}^{EE}\tilde{C}_{l_2}^{EE} = \mathcal{H} + \mathcal{I} + (\mathcal{J}+\mathcal{K})\delta_{l_1l_2}, \tag{14}$$

where

$$\mathcal{H} = \frac{1}{(2l_1+1)(2l_2+1)}\sum_L C_L^{\phi}[({}_2F_{l_1Ll_2}C_{l_2}^{EE})^2 + ({}_2F_{l_2Ll_1}C_{l_1}^{EE})^2](1+(-1)^{l_1+l_2+L}),$$
$$\mathcal{I} = \frac{2}{(2l_1+1)(2l_2+1)}\sum_L C_L^{\phi}C_{l_1}^{EE} \times C_{l_2}^{EE}(1+(-1)^{l_1+L+l_2})\,{}_2F_{l_1Ll_2}\,{}_2F_{l_2Ll_1},$$
$$\mathcal{J} = \frac{2}{2l_1+1}(C_{l_1}^{EE})^2 + \frac{2}{(2l_1+1)^2}\sum_{L,l'}C_L^{\phi}C_{l'}^{EE} \times C_{l_1}^{EE}(1+(-1)^{l_1+L+l'})({}_2F_{l_1Ll'})^2, \tag{15}$$
$$\mathcal{K} = -\frac{(l_1(l_1+1)-4)}{2\pi(2l_1+1)}\sum_L C_L^{\phi}(C_{l_1}^{EE})^2 L(L+1)(2L+1).$$

The last two terms can be written in terms of the lensed power spectrum of E-mode anisotropies as

$$\mathcal{J} + \mathcal{K} = \frac{2}{2l_1+1}(\tilde{C}_{l_1}^{EE})^2, \tag{16}$$

where

$$\tilde{C}_l^{EE} = [1-(l^2+l-4)R]C_l^{EE} + \frac{1}{2}\sum_{l_1l_2}C_{l_1}^{\phi}\frac{({}_2F_{ll_1l_2})^2}{2l+1}C_{l_2}^{EE}(1+(-1)^{l+l_1+l_2}),$$
$$R = \frac{1}{8\pi}\sum_{l_1}l_1(l_1+1)(2l_1+1)C_{l_1}^{\phi}, \tag{17}$$
$${}_2F_{ll_1l_2} = \frac{1}{2}[l_1(l_1+1)+l_2(l_2+1)-l(l+1)] \times \sqrt{\frac{(2l+1)(2l_1+1)(2l_2+1)}{4\pi}}\begin{pmatrix} l & l_1 & l_2 \\ 2 & 0 & -2 \end{pmatrix}.$$

Note that $\tilde{C}_l^{EE}$ is the power spectrum of the lensed E modes.

C. *B*-mode polarization covariance

The calculation related to B-mode power spectrum polarization is similar to the case of the E modes except that we assume that the B-mode polarization is generated solely the lensing of the E-mode polarization. Based on previous work (cf. Ref. [7]), we write the multipole moments of the lensed B modes as

$$i\tilde{B}_{lm} = \frac{1}{2}\sum_{l_1m_1l_2m_2}\phi_{l_1m_1}E_{l_2m_2\,+2}I^{mm_1m_2}_{ll_1l_2}(1-(-1)^{l+l_1+l_2}) + \frac{1}{4}\sum_{l_1m_1l_2m_2l_3m_3}\phi_{l_1m_1}E_{l_2m_2} \times \phi^*_{l_3m_3\,+2}J^{mm_1m_2m_3}_{ll_1l_2l_3}(1-(-1)^{l+l_1+l_2+l_3}). \quad (18)$$

Here, we will only calculate the B-mode trispectrum with terms involving C_l^ϕ since we will make the assumption that corrections to B modes from the bispectrum and higher-order non-Gaussianities of the lensing ϕ field are subdominant. Thus, using the first term of the expansion, we write

$$\langle\tilde{B}_{l_1m_1}\tilde{B}_{l_2m_2}\tilde{B}_{l_3m_3}\tilde{B}_{l_4m_4}\rangle = \frac{1}{16}\sum_{L_1M_1l_1'm_1'}\sum_{L_2M_2l_2'm_2'}\sum_{L_3M_3l_3'm_3'}\sum_{L_4M_4l_4'm_4'}\langle\phi_{L_1M_1}\phi_{L_2M_2}\phi_{L_3M_3}\phi_{L_4M_4}\rangle \times\langle E_{l_1'm_1'}E_{l_2'm_2'}E_{l_3'm_3'}E_{l_4'm_4'}\rangle_{+2}I_{l_1m_1L_1M_1l_1'm_1'\,+2}I_{l_2m_2L_2M_2l_2'm_2'\,+2}I_{l_3m_3L_3M_3l_3'm_3'\,+2}I_{l_4m_4L_4M_4l_4'm_4'} \times(1-(-1)^{l_1+L_1+l_1'})(1-(-1)^{l_2+L_2+l_2'})(1-(-1)^{l_3+L_3+l_3'})(1-(-1)^{l_4+L_4+l_4'}), \quad (19)$$

where

$$_{+2}I_{lml_1m_1l_2m_2} = {}_2F_{ll_1l_2}(-1)^m\begin{pmatrix} l & l_1 & l_2 \\ -m & m_1 & m_2\end{pmatrix}. \quad (20)$$

The covariance of the B-mode angular power spectrum can be now defined as

$$\mathrm{Cov}_{BBBB} \equiv \frac{1}{2l_1+1}\frac{1}{2l_2+1}\sum_{m_1m_2}\langle\tilde{B}_{l_1m_1}\tilde{B}^*_{l_1m_1}\tilde{B}_{l_2m_2}\tilde{B}^*_{l_2m_2}\rangle - \tilde{C}^{BB}_{l_1}\tilde{C}^{BB}_{l_2}. \quad (21)$$

After some straightforward but tedious algebra, we obtain

$$\mathrm{Cov}_{BBBB} = \mathcal{A} + \mathcal{B} + \mathcal{C} + \delta_{l_1l_2}\mathcal{D}, \quad (22)$$

where the terms are given by

$$\mathcal{A} = \frac{2}{4(2l_1+1)(2l_2+1)}\sum_{L=1}^{N_\phi}\bigg[\frac{(C_L^\phi)^2}{2L+1}\bigg(\sum_{l'=|l_1-L|}^{l_1+L}C_{l'}^{EE}(1-(-1)^{l_1+L+l'})({}_2F_{l_1Ll'})^2\bigg) \times\bigg(\sum_{l'=|l_2-L|}^{l_2+L}C_{l'}^{EE}(1-(-1)^{l_2+L+l'})({}_2F_{l_2Ll'})^2\bigg)\bigg],$$

$$\mathcal{B} = \frac{2}{4(2l_1+1)(2l_2+1)}\times\sum_{l'=1}^{N_E}\bigg[\frac{(C_{l'}^{EE})^2}{2l'+1}\bigg(\sum_{L=|l_1-l'|}^{l_1+l'}C_L^\phi(1-(-1)^{l_1+L+l'})({}_2F_{l_1Ll'})^2\bigg) \times\bigg(\sum_{L=|l_2-l'|}^{l_2+l'}C_L^\phi(1-(-1)^{l_2+L+l'})({}_2F_{l_2Ll'})^2\bigg)\bigg],$$

$$\mathcal{C} = \frac{2}{16(2l_1+1)(2l_2+1)}\sum_{L_1=1}^{N_\phi}\sum_{l_1'=|l_1-L_1|}^{l_1+L_1}\sum_{l_2'=|l_2-L_1|}^{l_2+L_1}\sum_{L_2=|l_1-l_2'|}^{l_1+l_2'}C_{L_1}^\phi C_{L_2}^\phi C_{l_1'}^{EE}C_{l_2'}^{EE}[{}_2F_{l_1L_1l_1'}\,{}_2F_{l_1L_2l_2'}\,{}_2F_{l_2L_1l_2'}\,{}_2F_{l_2L_2l_1'}]\begin{Bmatrix} L_1 & l_1' & l_1 \\ L_2 & l_2' & l_2\end{Bmatrix} \times(-1)^{l_1+L_1+l_1'+l_2+L_2+l_2'}(1-(-1)^{l_1+L_1+l_1'})(1-(-1)^{l_1+L_2+l_2'})(1-(-1)^{l_2+L_1+l_2'})(1-(-1)^{l_2+L_2+l_1'}),$$

$$\mathcal{D} = \frac{2}{4(2l_1+1)^3}\bigg(\sum_{Ll'}C_L^\phi C_{l'}^{EE}[1-(-1)^{l_1+L+l'}]({}_2F_{l_1Ll'})^2\bigg)^2 = \frac{2}{2l_1+1}(\tilde{C}^{BB}_{l_1})^2, \quad (23)$$

where

$$\tilde{C}_l^{BB} = \frac{1}{2}\sum_{l_1l_2}C_{l_1}^\phi\frac{({}_2F_{ll_1l_2})^2}{2l+1}C_{l_2}^{EE}(1-(-1)^{l+l_1+l_2}). \quad (24)$$

Unlike the calculation for the covariances of the lensed temperature and polarization E mode, the numerical calculation related to covariance of the B modes is complicated due to the term $\mathcal{C}$, which involves a Wigner-6j symbol. These symbols can be generated using the recursion relation outlines in Appendix of Ref. [15], though we found that such recursions are subject to numerical instabilities when one of the l values is largely different from the others and the l values are large. In these cases, we found that values accurate to better than a 10% of the exact result can be obtained through semiclassical formulae [17]. In any case, we found that $\mathcal{C}$ is no more than 1% of $\mathcal{A}$, $\mathcal{B}$, and these terms are in turn no more than 10% of $\mathcal{D}$. The same situation happens to those expressions in flat-sky approach [18].

D. Other full sky covariances

In addition to covariances of temperature, E and B modes, we also consider covariances of the cross power spectra. This is due to the fact that information is contained in cross correlation power spectra such as TE. Here, we list expressions for these covariances as we will use them for the Fisher matrix analysis described later. To simplify the expression, we make use of the following notation with

$$
\begin{aligned}
{}_T\hat{F}_{l_1 l_2 l_3} &\equiv F_{l_1 l_2 l_3}, \\
{}_E\hat{F}_{l_1 l_2 l_3} &\equiv {}_2F_{l_1 l_2 l_3}\left(\frac{1 + (-1)^{l_1+l_2+l_3}}{2}\right), \\
\text{and} \quad {}_B\hat{F}_{l_1 l_2 l_3} &\equiv {}_2F_{l_1 l_2 l_3}\left(\frac{1 - (-1)^{l_1+l_2+l_3}}{2}\right),
\end{aligned} \tag{25}
$$

denoting mode coupling in temperature and polarization modes. Below X, Y, Z, $W \in \{T, E\}$, and

$$
\begin{aligned}
\mathrm{Cov}^N_{XYZW} = {}& \mathrm{Cov}^N_{XYZW,1} + \mathrm{Cov}^N_{XYZW,2} + \mathrm{Cov}^N_{XYZW,3} \\
&+ \mathrm{Cov}^N_{XYZW,4} + \mathrm{Cov}^N_{XYZW,5} + \mathrm{Cov}^N_{XYZW,6},
\end{aligned} \tag{26}
$$

where

$$
\begin{aligned}
\mathrm{Cov}^N_{XYZW,1} &= \frac{1}{(2l_1+1)(2l_2+1)} \sum_L C^\phi_L (C^{XZ}_{l_2} C^{YW}_{l_2} + C^{XW}_{l_2} C^{YZ}_{l_2})\, {}_X\hat{F}_{l_1 L l_2}\, {}_Y\hat{F}_{l_1 L l_2}, \\
\mathrm{Cov}^N_{XYZW,2} &= \frac{1}{(2l_1+1)(2l_2+1)} \sum_L C^\phi_L (C^{XZ}_{l_1} C^{YW}_{l_1} + C^{XW}_{l_1} C^{YZ}_{l_1})\, {}_Z\hat{F}_{l_2 L l_1}\, {}_W\hat{F}_{l_2 L l_1}, \\
\mathrm{Cov}^N_{XYZW,3} &= \frac{1}{(2l_1+1)(2l_2+1)} \sum_L C^\phi_L C^{YZ}_{l_1} C^{XW}_{l_2}\, {}_X\hat{F}_{l_1 L l_2}\, {}_Z\hat{F}_{l_2 L l_1}, \\
\mathrm{Cov}^N_{XYZW,4} &= \frac{1}{(2l_1+1)(2l_2+1)} \sum_L C^\phi_L C^{YZ}_{l_2} C^{XW}_{l_1}\, {}_Y\hat{F}_{l_1 L l_2}\, {}_W\hat{F}_{l_2 L l_1}, \\
\mathrm{Cov}^N_{XYZW,5} &= \frac{1}{(2l_1+1)(2l_2+1)} \sum_L C^\phi_L C^{YW}_{l_1} C^{XZ}_{l_2}\, {}_X\hat{F}_{l_1 L l_2}\, {}_W\hat{F}_{l_2 L l_1}, \\
\mathrm{Cov}^N_{XYZW,6} &= \frac{1}{(2l_1+1)(2l_2+1)} \sum_L C^\phi_L C^{YW}_{l_2} C^{XZ}_{l_1}\, {}_Y\hat{F}_{l_1 L l_2}\, {}_Z\hat{F}_{l_2 L l_1}.
\end{aligned} \tag{27}
$$

In the case of cross terms involving B modes, the covariances of bandpower XY with BB take a simple form with

$$
\mathrm{Cov}^N_{XYBB} = \frac{2}{(2l_1+1)(2l_2+1)} \sum_L C^\phi_L C^{XE}_{l_1} C^{YE}_{l_2}\, {}_B\hat{F}_{l_1 L l_2}\, {}_B\hat{F}_{l_2 L l_1}. \tag{28}
$$

Finally, the full covariances also include the Gaussian pieces which can be represented by

$$
\mathrm{Cov}^G_{XYZW} = \frac{1}{2l_1+1} (\tilde{C}^{XZ}_{l_1} \tilde{C}^{YW}_{l_1} + \tilde{C}^{XW}_{l_1} \tilde{C}^{YZ}_{l_1}). \tag{29}
$$

Because of parity considerations with $C^{EB}_l = C^{TB}_l = 0$ and one can ignore covariances between these terms and temperature and polarization modes.

III. RESULTS AND DISCUSSION

We begin our discussion on the parameter uncertainties in the presence of non-Gaussian covariance by first establishing that one cannot ignore them for the B-mode power spectrum. In Fig. 1 we show the correlation matrix, which is defined as

$$
r_{ij} \equiv \frac{\mathrm{Cov}_{XYZW}(i, j)}{\sqrt{\mathrm{Cov}_{XYZW}(i, i)\mathrm{Cov}_{XYZW}(j, j)}}, \tag{30}
$$

but considering only terms involving $XYZW = TTTT$, $EEEE$, and $BBBB$. This correlation normalizes the diagonal to unity and displays the off diagonal terms as a value between 0 and 1. This facilitates an easy comparison on the importance of non-Gaussianities between temperature, E, and B modes of polarization. As shown in Fig. 1, the off diagonal entries of temperature and E modes are roughly at

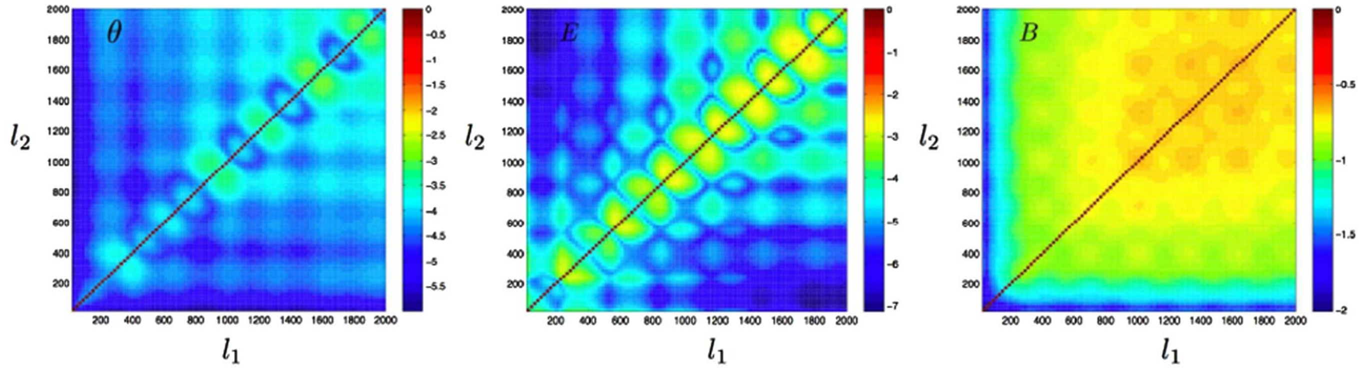

FIG. 1 (color online). The correlation matrix [Eq. (30)] for temperature (left), E -mode (middle), and B-mode (right) power spectra between different l values. The color axis is on a log scale and each scale is different for each panel. As is clear from this figure, the off diagonal correlation is weak for both T- and E-mode power spectra, but is more than 0.1 for most entries for the B-mode power spectrum. This clearly shows that the non-Gaussianities are most pronounced for the B-mode signal and will impact the information extraction from the angular power spectrum of B modes than under the Gaussian variance alone. The B-mode covariance shown in the right panel agrees with Figure 5 of Ref. [14].

the level of 10^{-5} suggesting that non-Gaussian covariance is not a concern for these observations out to multipoles of 2000 [12], while for B modes the correlations are at the level above 0.1 and are significant.

Below when we calculate the signal-to-noise ratio and Fisher matrices, we use the bandpowers as observables with logrithmic bins in the multipole space. Our bandpower estimator for two quantities of X and Y fields involving temperature and polarization maps is

$$\hat{\Delta}^2_{XY,i} = \frac{1}{\alpha_i} \sum_{l=l_{i1}}^{l_{i2}} \sum_{m=-l}^{l} \frac{l}{4\pi} X_{lm} Y^*_{lm}, \tag{31}$$

where $\alpha_i = l_{i2} - l_{i1}$ is an overall normalization factor given by the bin width. The angular power spectra are

$$\Delta^2_i = \langle \hat{\Delta}^2_i \rangle = \frac{1}{4\pi\alpha_i} \sum_l (2l+1) l C_l^{BB,EE,TT,TE}, \tag{32}$$

while the full covariance matrix is

$$\langle (\hat{\Delta}^2_i - \Delta^2_i)(\hat{\Delta}^2_j - \Delta^2_j) \rangle = S^G_{ii}\delta_{ij} + S^N_{ij}, \tag{33}$$

with the Gaussian part

$$S^G_{ii} = \frac{2}{(4\pi)^2\alpha_i^2} \sum_{l_1=l_{i1}}^{l_1=l_{i2}} (2l_1+1) l_1^2 (C_{l_1}^{BB,EE,TT,TE} + N_{l_1})^2,$$
$$N_l = \left(\frac{\Delta_p}{T_{\rm CMB}}\right)^2 e^{l(l+1)\theta^2_{\rm FWHM}/8\ln 2}, \tag{34}$$

and the non-Gaussian part is

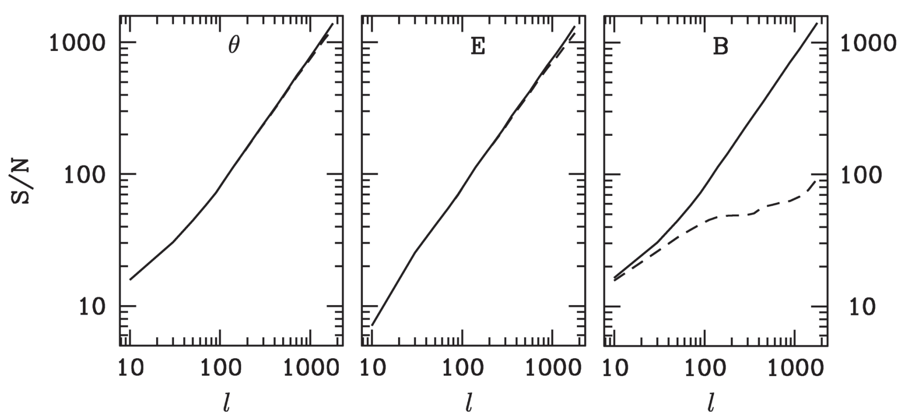

FIG. 2. Here we show the cumulative signal-to-noise ratio for a detection of the power spectrum [Eq. (36)] for temperature (left), E-mode (middle), and B-mode (right) polarization power spectra. The solid line is the case with a Gaussian covariance whereas the dashed line is with a non-Gaussian covariance. We can see that for the case of the temperature and E-mode polarization there is little difference between the Gaussian and non-Gaussian covariance, but for the B-mode polarization there is a difference of a factor of ~10 at large l values.

$$S_{ij}^N = \frac{1}{(4\pi)^2\alpha_i\alpha_j}\sum_{l_1l_2}(2l_1+1)(2l_2+1)l_1l_2(\mathrm{Cov}_{XYZW}^N), \tag{35}$$

where Cov_{XYZW}^N are the corresponding non-Gaussian contributions to the covariance.

To further quantify the importance of non-Gaussianities for B modes, in Fig. 2, we plot the cumulative signal-to-noise ratio for the detection of the power spectra as a function of the bandpowers. These are calculated as

$$(\mathrm{SN})_{XY} = \sum_{\Delta_i\Delta_j} C_{\Delta_i}^{XY}\mathrm{Cov}_{XYXY}^{-1}(\Delta_i, \Delta_j)C_{\Delta_j}^{XY}, \tag{36}$$

by ignoring the instrumental noise contribution to the covariance and only taking terms involving TT, EE, and BB. The result for other nonzero power spectrum involving the cross correlation between temperature and E modes is similar to the case of either temperature or E modes. As shown, there is no difference in the signal-to-noise ratio for the temperature and E-mode power spectra measurement due to non-Gaussian covariances, while there is a sharp reduction in the cumulative signal-to-noise ratio for a detection of the B modes. This reduction is significant and can be explained through the effective reduction in the number of independent modes at each multipole from which clustering measurements can be made. In the case of Gaussian statistics, at each multipole l, there are $2l+1$ modes to make the power spectrum measurements. In the case of non-Gaussian statistics with a covariance, this number is reduced further by the correlations between different modes. If N is the number of independent modes available under Gaussian statistics, a simple calculation shows that the effective number of modes are reduced by $[1+(N-1)r^2]$ when the modes are correlated by an equally distributed correlation coefficient r among all modes. With $N = 2l+1$ and substituting a typical correlation coefficient r of 0.15, we find that the cumulative signal-to-noise ratio should be reduced by a factor of 7 to 8 when compared to the case where only Gaussian statistics are assumed. This is consistent with the signal-to-noise ratio estimates shown in Fig. 2 based on an exact calculation using the full covariance matrix that suggests a slightly larger reduction due to the fact that some of the modes are more strongly correlated than the assumed average value.

To calculate the overall impact on cosmological parameter measurements using temperature and polarization spectra, we make use of the Fisher information matrix given for tow parameters μ and ν as

$$F_{\mu\nu} = \sum_{XY,ZW=BB,EE,TT,TE}\sum_{ij}\frac{\partial(\Delta_i^{XY})^2}{\partial p_\mu}(\mathrm{Cov}_{XYZW}^{-1})\frac{\partial(\Delta_j^{ZW})^2}{\partial p_\nu}, \tag{37}$$

where the summation is over all bins and the covariance matrices are worked out in Sec. II. While this is the full Fisher information matrix, we will divide our results to with and without non-Gaussian covariance as well as to information on parameters present within temperature, and E and B modes of polarization. In this Fisher matrix calculation, terms involving EB and TB cross correlations are not included as the power spectra are exactly zero. Thus, covariance matrices related to these terms are not necessary for the calculation.

Since B modes have been generally described as a probe of neutrino mass and the dark energy equation of state, in Fig. 3, we show $\partial C_l/\partial m_\nu$ and $\partial C_l/\partial w$ to show the extent to which information on these two quantities are present in the spectra. Here, the calculation is done by keeping seven other cosmological parameters fixed at the fiducial values, when the parameter of interest in Fig. 3 is varied (See

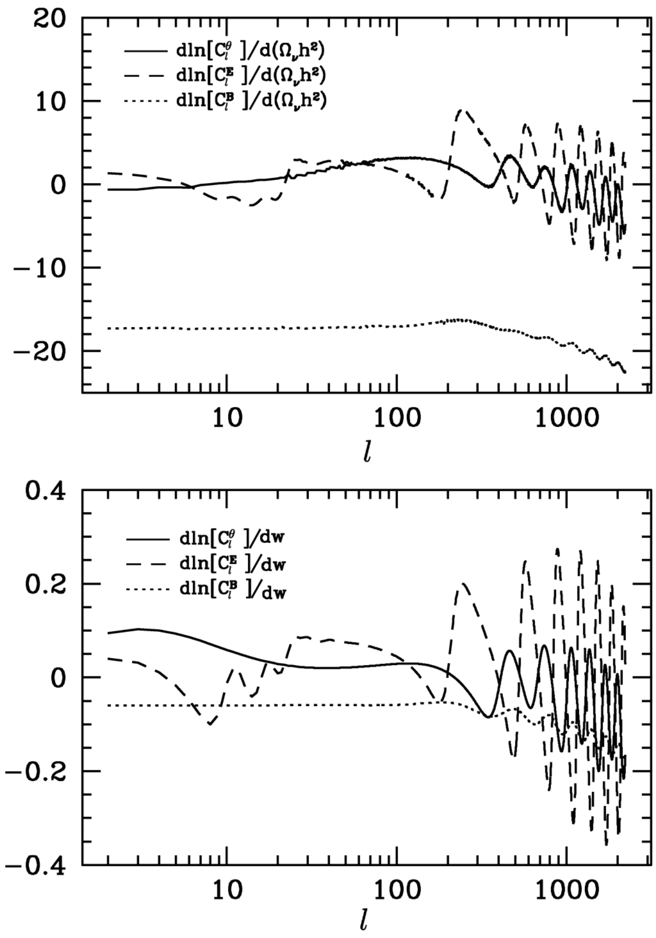

FIG. 3. The derivatives of the temperature (T), E-mode, and B-mode power spectra with respect to the sum of the neutrino masses ($\propto \Omega_\nu h^2$, top panel) and the dark energy equation of state, w (bottom panel). It is clear that in the case of the sum of the neutrino masses the addition of the B-mode polarization greatly increases sensitivity. In both cases we find that large l information also increases sensitivity. We note that the derivative of the temperature power spectrum with respect to neutrino mass agrees with that shown in Fig. 3 of Ref. [19].

Sec. II B). We do, however, assume a flat cosmology, and thus, parameters which are not independent are varied as appropriate to conserve the flatness. The derivatives are taken following the calculation in Ref. [19] and we have verified that we reproduce their results such as the derivatives with respect to neutrino mass. We use those derivatives in forecasting errors from upcoming CMB experiments, such as Planck and a post-Planck mission. As shown in Fig. 3, it is clear that B modes are a strong probe of neutrino mass given that the sensitivity of temperature and E modes are smaller compared to the fractional difference in the B modes. Furthermore, B modes also have some sensitivity to the dark energy equation of state, but fractionally, this sensitivity is smaller compared to the information related to the neutrino mass.

In Fig. 4, we summarize parameter constraints on these two parameters as a function of the instrumental noise for different values of resolution with and without non-Gaussian covariance. In calculating the non-Gaussian covariance, to simplify the calculation, we consider coarsely binned power spectrum estimates where we binned the power spectrum to 5 uniformly spaced bins from $l = 5$ to $l = 100$ and 13 logarithmic uniformly spaced bins from $l = 100$ to $l = 2000$. Later, we will also show results with a fine binning scheme where we divide the multipole range to 100 equal bins. Even with the large bins, the difference between Gaussian and non-Gaussian extraction is marginal low resolution experiments. Non-Gaussianities become more important for high resolution experiments where one probes B modes down to large multipoles. In this case, the parameters extraction is degraded by up to a factor of more than 2.5 for both the neutrino mass and the dark energy equation of state. We have not attempted to calculate the parameter errors for experiments with resolution better than 5 arcminutes. This is due to the fact that such experiments will probe multipoles higher than 2000. Furthermore, we also do not think any of the upcoming B-mode polarization experiments with high sensitivity,

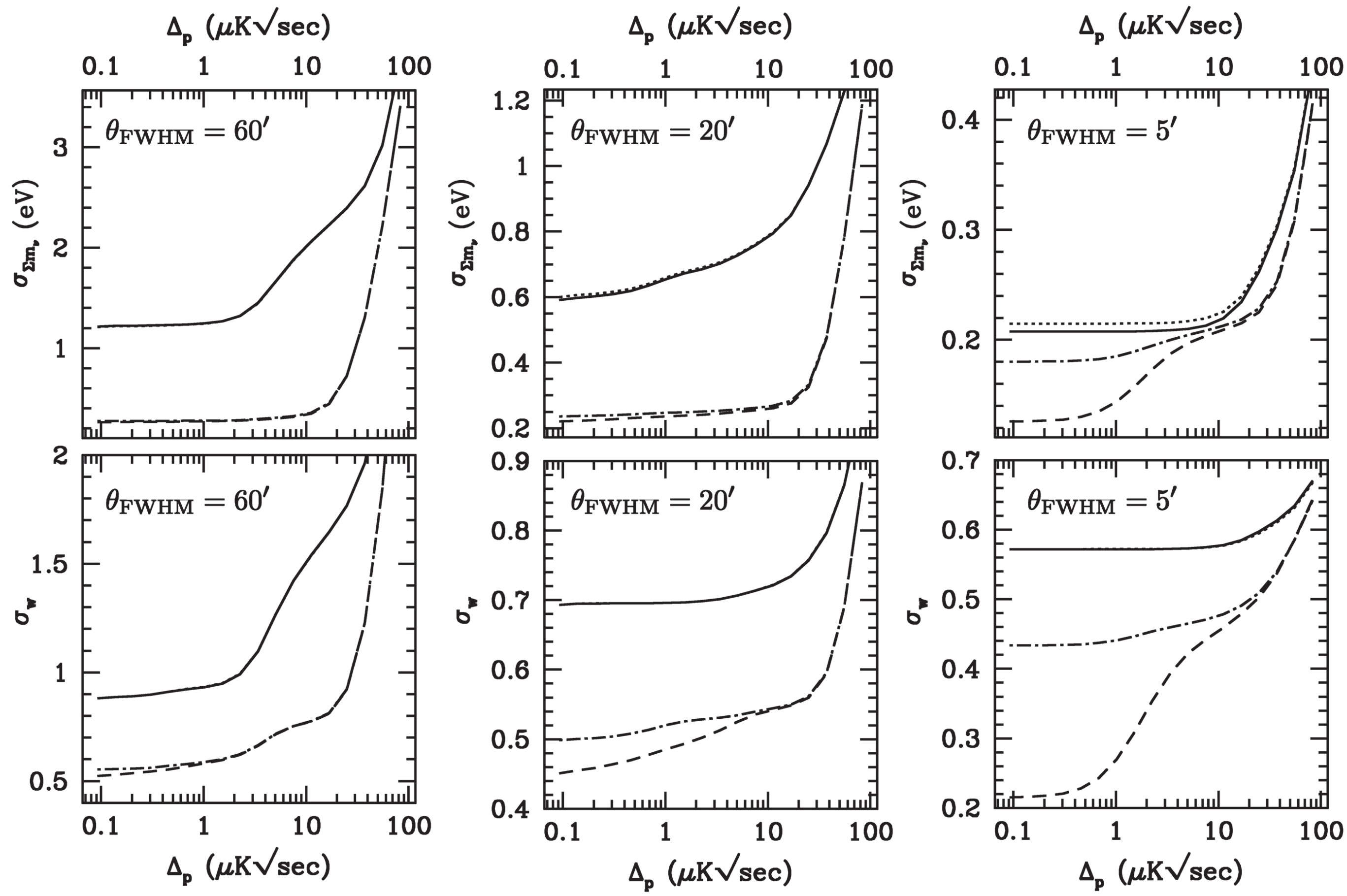

FIG. 4. The expected error on the sum of the neutrino masses (top three panels) and the dark energy equation of state, w (bottom three panels) as a function of experimental noise for three different values of the beam width, $T_{\rm FWHM}$. The solid line considers Gaussian covariance with just temperature information, the dotted line considers non-Gaussian covariance with just temperature information, the dashed line considers Gaussian covariance with both temperature and polarization (E and B mode), and the dashed-dotted line considers non-Gaussian covariance with both temperature and polarization. It is clear that as the beam width is decreased the estimated error on the sum of the neutrino masses and w is increasingly overly optimistic when just the Gaussian covariance is used in the Fisher matrix calculation.

which will be either space-based or balloon-borne, will have large apertures to probe multipoles above 2000.

The value of 2000 where we stop our calculations is also consistent with Planck. Since Planck HFI experiment will have a total focal plane polarization noise of about 25 $\mu\mathrm{K}\sqrt{\mathrm{sec}}$, based on Fig. 4, we find that it will constrain the neutrino mass to be below 0.22 eV and the dark energy equation of state will be determined to an accuracy of 0.5. Note that the combination of Planck noise and resolution is such that one does not find a large difference between Gaussian and non-Gaussian statistics, but on the other hand, experiments that improve the polarization noise well beyond Planck must account for non-Gaussian noise properly. In future, there are plans for a Inflation Probe or a CMBpol mission that will make high sensitive observations in search for a gravitational wave background. If such an experiment reach an effective noise level of 1 $\mu\mathrm{K}\sqrt{\mathrm{sec}}$ and has the same resolution as Planck, the combined polarization observations can constrain the neutrino mass to be about 0.18 while the dark energy equation of state will be known to an accuracy of 0.44. This is well above the suggested constraint from Gaussian noise level. This suggests that while high sensitive *B*-mode measurements are desirable for studies involving the gravitational wave background, they are unlikely to be helpful for increasingly better constraints on the cosmological parameters.

From Fig. 4 we see that as we decrease Δ_p the measurement errors on the parameters asymptote to a constant value. We can understand this in the following way. As we see from Eq. (34), the noise blows up exponentially at large l and therefore sets an effective cutoff l_0. Only the band powers which are smaller than l_0 contribute to parameter estimates. Therefore, if we decrease Δ_p, we increase the number of band powers we can observe and hence obtain better sensitivity with negligible instrumental noise for $l \lesssim l_0$. Therefore, the curves in Fig. 4 become flatter as we decrease Δ_p. The same situation applies to Fig. 5. Figure 4 also shows that as we decrease the beam width, T_{FWHM}, we see the Gaussian covariance becomes more significant. This is a result of the fact that the Gaussian covariance grows in significance with increasing l.

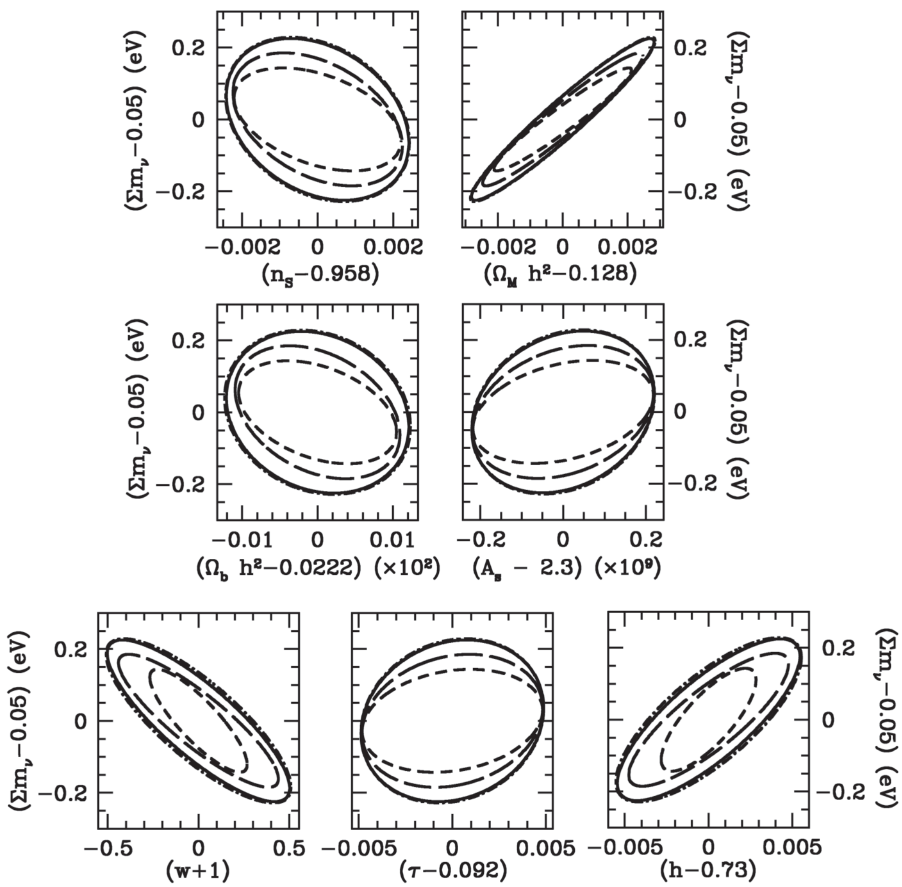

FIG. 5. The error ellipses from our Fisher matrix calculation. We have varied eight parameters, and show the error ellipses for each parameter with Σm_ν. The solid ellipse is the expected error from Planck with just a Gaussian covariance, the dashed-dotted ellipse is same but with a non-Gaussian covariance. The short-dashed ellipse is for an experiment with the same beam width as Planck ($T \sim 5'$) but with decreased noise (1 $\mu\mathrm{K}\sqrt{\mathrm{sec}}$ as opposed to 25 $\mu\mathrm{K}\sqrt{\mathrm{sec}}$) with a Gaussian covariance and the long-dashed ellipse is the same but with a non-Gaussian covariance.

In Fig. 5, to highlight the impact on cosmological parameters beyond the neutrino mass and dark energy equation of state, we also show constraints from the Fisher matrix calculation. We show error ellipses calculated with and without the non-Gaussian lensing covariance for two different experiments: Planck, with $T_{\rm FWHM} = 5'$ and $\Delta_p = 25\ \mu{\rm K}\sqrt{\rm sec}$ and "super-Planck" with $T_{\rm FWHM} = 5'$ and $\Delta_p = 1\ \mu{\rm K}\sqrt{\rm sec}$. This comparison shows that while parameters such as m_ν and w are affected, parameters such as τ, $\Omega_m h^2$ are not affected by non-Gaussian information. This is due to the fact that the cosmological information on these parameters come from temperature and E modes rather than B modes. This highlights the fact that the issues discussed here are primarily a concern for the B-mode measurements and extraction of parameters, especially the parameters that have been recognized to be mostly constrained by the B-mode measurements, and not for temperature and E modes.

To study the extent to which our results are sensitive to the binning scheme, and to compare our results with Smith *et al.* [18] that also performed a calculation similar to ours, but based on the flat-sky expression for lensing covariances, we repeat the Fisher matrix calculation by making use of a binning scheme similar to them (100 bins over the multipole range). In Smith *et al.* [18], non-Gaussian covariances at multipoles below 100 were ignored due to the break down of the flat-sky expression at large angular scales, while the all-sky expression derived here allow us to present a complete formalism which is accurate over the all multipoles of interest. Figure 6 shows the resulting error ellipses from our Fisher matrix calculation for the super-Planck experiment with $T_{\rm FWHM} = 5'$ and $\Delta_p = 1\ \mu{\rm K}\sqrt{\rm sec}$. By Comparing Figs. 5 and 6, we see that the error ellipses with non-Gaussianities included decrease their sizes. The correlations between different band powers also decrease as we decrease the bin sizes in calculating the covariance terms. In Fig. 6 the effect of non-Gaussianity is hardly discernible for the error ellipses. This result is consistent with Figure 6 of Ref. [18]. The resulting effect is that for an experiment like super-Planck mission, the non-Gaussian effect is almost negligible when discussing measurement accuracies of the cosmological parameters and one can reach the original estimates of, for example, neutrino masses (with a few percent difference) under Gaussian variances alone.

The non-Gaussianities in the B modes, while providing information on gravitational lensing, limits accurate parameter estimates from the power spectrum alone (see

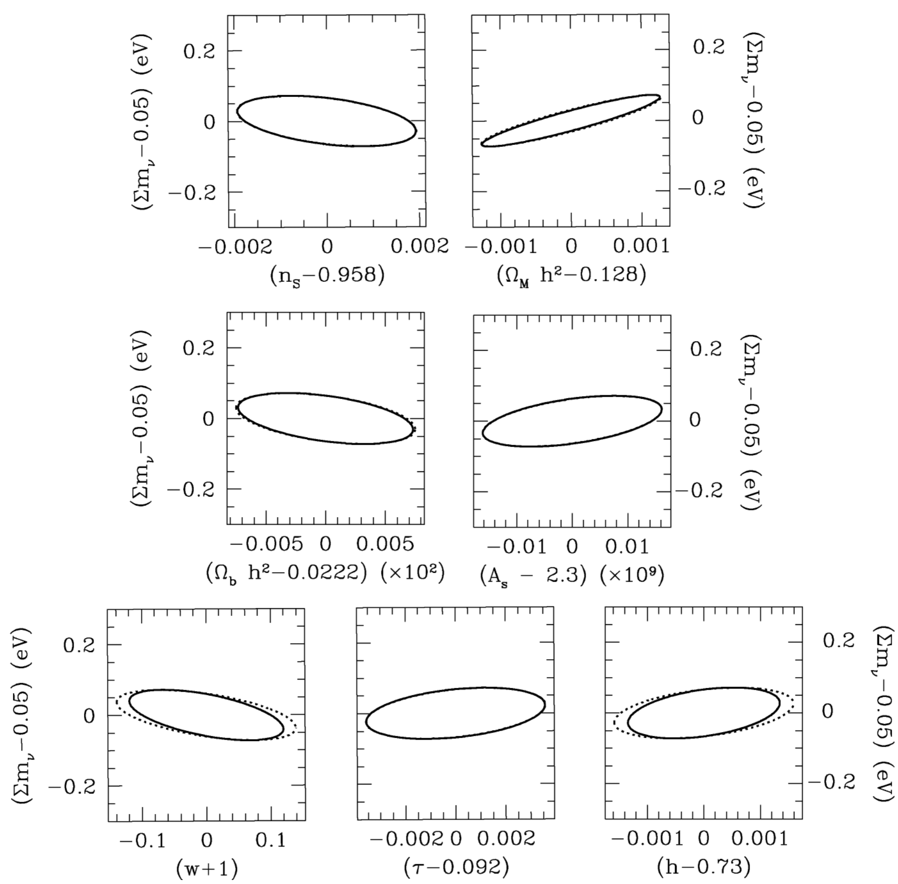

FIG. 6. The same caption as Fig. 5. Here we redo the analysis of super-Planck case, the corrections to our previous analysis are discussed in the end of Sec. III.

Ref. [9] for an analysis beyond the power spectrum). This is contrary to some of the suggestions in the literature that have indicated high precision of measurements on parameters such as the neutrino mass and the dark energy equation of state with CMB *B*-mode power spectrum by ignoring issues related to non-Gaussian correlations. An important issue related to the power spectrum estimated from non-Gaussian fields is the binning of the clustering spectrum measurements. While for Gaussian fields it is well known that the power spectrum analysis is robust to changes in the binning scheme (as long as the bins are narrow compared to any features in the power spectrum), for non-Gaussian fields, this is no longer the case. As we have demonstrated, the impact of non-Gaussian covariance is negligible when one is considering narrow bins as can be achieved with almost all-sky observations.

The non-Gaussian covariance, however, impacts the power spectrum analysis with ground-based experiments that are subjected to partial sky coverage. Since these experiments only probe the power spectrum at some fundamental scale $\Delta\ell$ corresponding to the sky area, the statistics are estimated with larger bins than in the case from space where all-sky observations allow independent estimates with fine bins. The effects of non-Gaussian covariance are increased with wide bins relative to the case where power spectra are evaluated with fine bins which are independent of each other. For partial sky coverage, in addition to non-Gaussian covariance, correlations are also induced by the survey geometry and will also impact the power spectrum analysis. Such correlations between adjacent modes are significantly reduced once the power spectrum is binned at appropriate binning scale corresponding to the survey size. Even with such binning, non-Gaussian covariance from lensing will remain and will impact the parameter estimates as we have discussed here.

IV. SUMMARY

The *B*-mode polarization lensing signal is a useful probe of certain cosmological parameters such as the neutrino mass and the dark energy equation of state as the signal depends on the integrated mass power spectrum between us and the last scattering surface. This lensing *B*-mode signal, however, is non-Gaussian and it could be that the resulting non-Gaussian covariance to the power spectrum cannot be ignored when compared to the case of temperature and polarization *E*-mode anisotropy covariances. Depending on how the power spectrum estimates are binned and used for parameter estimates, we find that the resulting degradation on neutrino mass and dark energy equation of state ranges from about a factor of 2 to a few percent when compared to the case where statistics are simply considered to be Gaussian. The large bins are necessary for ground-based experiments due to lack of whole sky coverage, while for all-sky experiments from space, fine binning will lead to a situation where degradation from non-Gaussianities can be safely ignored.

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