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Gravitational lensing from compact bodies: Analytical results for strong and weak deflection limits

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We develop a nonperturbative method that yields analytical expressions for the deflection angle of light in a general static and spherically symmetric metric. The method works by introducing into the problem an artificial parameter, called δ, and by performing an expansion in this parameter to a given order. The results obtained are analytical and nonperturbative because they do not correspond to a polynomial expression in the physical parameters. Already to first order in δ the analytical formulas obtained using our method provide at the same time accurate approximations both at large distances (weak deflection limit) and at distances close to the photon sphere (strong deflection limit). We have applied our technique to different metrics and verified that the error is at most 0.5% for *all* regimes. We have also proposed an alternative approach which provides simpler formulas, although with larger errors.

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I. INTRODUCTION

The theory of general relativity (GR) predicts that massive bodies deform the spacetime around them; even massless particles, like photons, will therefore feel the gravitational force and their trajectories will necessarily depart from a straight line. Such effect is particularly strong in the proximity of black holes, which are very massive and compact objects: in fact at a certain distance from the black hole (the photon sphere) the deflection angle of the photon becomes infinite. This regime is known as strong deflection limit (SDL), whereas the regime corresponding to deflection at large distances is known as weak deflection limit (WDL).

The WDL has been studied in a series of articles for different metrics: for example, the lensing from a Kerr metric has been considered in [1–3] whereas the lensing from a Reissner-Nordstrom metric has been considered in [4,5].

In recent times there has also been great interest in studying the effects of strong deflection limit, i.e. lensing due to light passing very close to a compact and massive body. For example SDL in a Schwarzschild black hole has been considered by Frittelli, Kling and Newman [6] and by Virbhadra and Ellis [7]; Virbhadra and collaborators have also treated the SDL by naked singularities [8] and in the presence of a scalar field [9]; Eiroa, Romero and Torres [10] have described Reissner-Nordström black hole lensing, while Bhadra has considered the gravitational lensing due to the GMGHS charged black hole [11]; Bozza has studied the gravitational lensing by a spinning black hole [12]; Whisker [13] and Eiroa [14] have considered SDL by a braneworld black hole; still Eiroa [15] has recently considered the gravitational lensing by an Einstein-Born-Infeld black hole; Sarkar and Bhadra have studied the SDL in the Brans-Dicke theory [16]; Konoplya has studied the corrections to the deflection angle and time delay for a black hole immersed in a uniform magnetic field [17]; Gyulchev and Yazadjiev have studied the SDL for a Kerr-Sen dilaton axion black hole [18]; finally Perlick [19] has obtained an exact gravitational lens equation in a spherically symmetric and static spacetime and used it to study lensing by a Barriola-Vilenkin monopole and by an Ellis wormhole. Notice that Bozza and Sereno [20] have also investigated the SDL of gravitational lensing by a Schwarzschild black hole embedded in an external gravitational field.

We can distinguish between two different approaches that have been developed to obtain analytical expressions for the deflection angle in the strong regime: one which looks for improvements of the weak lensing expressions, whose range of validity is therefore extended to distances closer to the photon sphere, without however taking into account the divergence of the deflection angle on the photon sphere, and one which treats exactly the singularity of the photon sphere and whose precision rapidly drops at larger distances[1]. In the first category falls the work of Mutka and Mähönen [22] and of Belorobodov [23] who worked out improved formulas for the deflection angle in a

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[1]For a detailed discussion on the photon surface the reader can refer to Ref. [21]

Schwarzschild metric, and the more systematic approach of Keeton and Petters [5] who have developed a formalism for computing corrections to lensing observables in a static and spherically symmetric metric beyond the WDL. In the second category falls the work of Bozza, who has introduced an analytical method based on a careful description of the logarithmic divergence of the deflection angle which allows one to discriminate among different types of black holes [24]. Recently, Iyer and Petters [25] have also developed an analytic perturbation framework for calculating the bending angle of light rays traversing the field of a Schwarzchild black hole, obtaining accurate expressions even in proximity of the photon sphere.

In a different category falls a method developed by Amore and collaborators [26,27]. This method enables one to convert the integral for the deflection angle in a static and spherically symmetric metric into a series in an artificial parameter δ. Such series has an exponential convergence rate and its terms can be calculated analytically. The method is nonperturbative in the sense that it provides nonpolynomial expressions in terms of the chosen physical parameter and yields sufficiently accurate results even at first order. In our previous works we have tested our formalism on a variety of different metrics, always obtaining very encouraging results.

The purpose of this paper is to improve our method in order to provide an accurate treatment close to the photon sphere, even at first order, but without the customary deterioration of the results at larger distances.

The paper is organized as follows: in Sec. II we illustrate the application of our method by means of the Schwarzschild metric and later we show how to treat a general case. In Sec. III we extend our analysis to a general static and spherically symmetric metric. In Sec. IV we compare our approximations with exact results and other approaches available in the literature. Finally, in the last section we briefly summarize and discuss our results and consider further developments.

II. FORMALISM

Let us first review the method of Amore and collaborators [26,27]. We are interested in the general static and spherically symmetric metric which corresponds to the line element (in the following we set the velocity of light $c = 1$)

$$ds^2 = B(r)dt^2 - A(r)dr^2 - D(r)r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1)$$

and which contains the Schwarzschild metric as a special case. We also assume that the flat spacetime is recovered at infinity, i.e. that $\lim_{r\to\infty} f(r) = 1$, where $f(r) = (A(r), B(r), D(r))$.[2]

[2]This is a sufficient but not necessary condition since it warrants that $f(r)$ is analytic around $r = \infty$;, for example, in [28] we have applied the method to the Weyl metric which is not asymptotically flat.

The angle of deflection of light propagating in this metric can be expressed by means of the integral [9]

$$\Delta\phi = 2\int_{r_0}^{\infty} \sqrt{A(r)/D(r)}\sqrt{\left[\left(\frac{r}{r_0}\right)^2 \frac{D(r)}{D(r_0)}\frac{B(r_0)}{B(r)} - 1\right]^{-1}}\frac{dr}{r} - \pi, \quad (2)$$

where r_0 is the distance of closest approach of the light to the center of the gravitational attraction.

By introducing the variable $z = r_0/r$ one can rewrite the equation for the deflection angle as

$$\Delta\phi = 2\int_0^1 \frac{dz}{\sqrt{V(1) - V(z)}} - \pi, \quad (3)$$

where

$$V(z) \equiv z^2 \frac{D(r_0/z)}{A(r_0/z)} - \frac{D^2(r_0/z)B(r_0)}{A(r_0/z)B(r_0/z)D(r_0)} + \frac{B(r_0)}{D(r_0)} \quad (4)$$

is a sort of "potential" built out of the metric. Notice that the integral in Eq. (3) can be solved analytically only in a limited number of cases, such as for the Schwarzschild metric [29] and for the Reissner-Nordtröm metric [10], where it can be expressed in terms of elliptic integrals. *No analytical formula can can be obtained in the case of a general static and spherically symmetric metric*.

Since an explicit calculation of the integral is not possible, we interpolate the actual potential $V(z)$ with a simpler potential $V_0(z)$, which should be chosen in such a way that the integral (3) can be performed explicitly when $V(z)$ is replaced with $V_0(z)$. Then we write

$$V_\delta(z) \equiv V_0(z) + \delta(V(z) - V_0(z)), \quad (5)$$

where δ is a dummy parameter. In general $V_0(z)$ may depend upon one or more arbitrary parameters; for the time being we simply choose $V_0(z) = \lambda z^2$. We can rewrite the expression for the deflection angle as

$$\Delta\phi_\delta = 2\int_0^1 \frac{dz}{\sqrt{V_0(1) - V_0(z)}}\frac{1}{\sqrt{1 + \delta\Delta(z)}} - \pi, \quad (6)$$

where

$$\Delta(z) \equiv \frac{V(1) - V(z)}{V_0(1) - V_0(z)} - 1. \quad (7)$$

Notice that Eq. (6) *reduces to* Eq. (3) for $\delta = 1$ and therefore is not an approximation. The expansion of Eq. (6) in powers of δ converges at $\delta = 1$ provided that $|\Delta(z)| < 1$ for $0 \leq z \leq 1$. As discussed in earlier papers [26,28,30] this condition requires that λ be greater than a critical value λ_C. In that case one obtains a parameter-dependent series that converges towards the exact result which is however independent of λ. The artificial dependence on λ observed in the partial sums $\Delta\phi^{(N)}$, $N = 1, 2, \ldots$, is minimized by means of the Principle of Minimal Sensitivity (PMS) [31], which corresponds to imposing the condition

$$\frac{\partial}{\partial \lambda}\Delta\phi^{(N)} = 0. \tag{8}$$

A proof of convergence of the series and an estimate of its rate of convergence are given elsewhere [26].

One might be tempted at this point to question our definition of the method as being nonperturbative: after all, the method works by performing a perturbative expansion in δ. However, one should understand that the solution of Eq. (8) is in general a function of the parameters in the problem and when substituted back in the series it provides nonpolynomial expressions in the physical parameters [26,27], whereas the dependence upon δ disappears because it is set to one at the end of the calculation.

We are now ready to generalize this method. The first step is to write the integral as

$$\Delta\phi = 2\int_0^\sigma \frac{dz}{\sqrt{V(1)-V(z)}} + 2\int_\sigma^1 \frac{dz}{\sqrt{V(1)-V(z)}} - \pi \equiv \Delta\phi_a + \Delta\phi_b - \pi, \tag{9}$$

where $\sigma \in (0,1)$ is an arbitrary point in the region of integration. The two integrals in this equation will now be approximated following two different strategies. Clearly, the particular case $\sigma = 1$ corresponds to the method just outlined [26,27].

For clarity, we confine ourselves, for the moment being, to the case of the Schwarzschild metric and later we generalize our results to arbitrary metrics. The Schwarzschild metric is given by

$$B(r) = A^{-1}(r) = \left(1 - \frac{2GM}{r}\right), \qquad D(r) = 1, \tag{10}$$

and the potential $V(z)$ reads

$$V(z) = z^2 - \frac{2}{3\mu}z^3, \tag{11}$$

where $\mu \equiv r_0/3GM \geq 1$. Figure 1 shows the potential $V(z)$ with $\mu = 3/2$. The dashed and dotted lines correspond to the quadratic Taylor polynomials around $z = 0$ and $z = \mu$. Viewed from the perspective of a classical mechanics problem, the points $z = 0$ and $z = \mu$ correspond to a stable and an unstable point of equilibrium, respectively; the nonlinear pendulum, for example, is a simple physical system that displays this behavior. Since the integral that we want to calculate is restricted to $z \leq 1$, the unstable point of equilibrium will not be reached unless $\mu = 1$. In such a case the integral will diverge and it will correspond to the photon sphere.[3]

One expects $\sigma \approx 1$ to be a reasonable choice when $\mu \gg 1$, since it has given accurate results earlier [26,27]. On the other hand, as $\mu \to 1^+$ the second integral will become

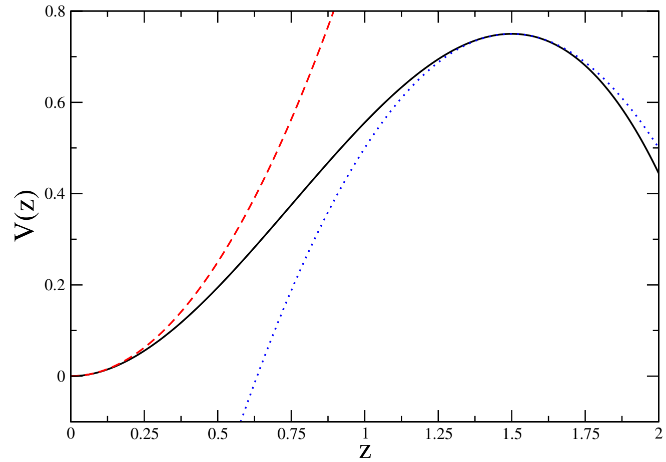

FIG. 1 (color online). The potential $V(z)$ in Eq. (11) with $\mu = 3/2$ (solid line). The dashed and dotted lines correspond to quadratic Taylor expansions around $z = 0$ and $z = \mu$, respectively.

increasingly more important and one expects the optimal value of σ to move to the center of the integration region. We will see that this is the case later on when we discuss a systematic way of partitioning the integral.

Our strategy is simple and consists of treating the integral in each region differently. In the first region we follow essentially the earlier procedure [26,27] with the interpolating potential $V_a(z) = \lambda z^2$, $\lambda > 0$. In the second region, on the other hand, we will interpolate the potential with the inverted parabola $V_b(z) = V(\mu) - \rho(z-\mu)^2$, where $\rho > 0$ is another arbitrary parameter. Notice therefore that we are working with three arbitrary parameters, λ and ρ which enter in the definition of the interpolating potentials, and σ which defines the border between the two regions.

In the first region we write

$$\Delta\phi_a = 2\int_0^\sigma \frac{dz}{\sqrt{V_a(1)-V_a(z)}}\frac{1}{\sqrt{1+\delta\Delta_a(z)}}, \tag{12}$$

where

$$\Delta_a(z) \equiv \frac{V(1)-V(z)}{V_a(1)-V_a(z)} - 1 = -\frac{3(\lambda-1)\mu(z+1) + 2(z^2+z+1)}{3\lambda\mu(z+1)}. \tag{13}$$

After expanding to first order in δ we obtain

$$\Delta\phi_a^{(1)} = \frac{2}{\sqrt{\lambda}}\int_0^\sigma \frac{dz}{\sqrt{1-z^2}}\left[1 - \frac{\Delta_a(z)}{2}\right]. \tag{14}$$

Straightforward integration yields

[3]Notice that $\mu = 1$ yields the well-known result $r_0 = 3GM$.

$$\Delta\phi_a^{(1)} = \frac{-2\sigma(\sqrt{1-\sigma^2}-2) + 3(3\lambda-1)\mu(\sigma+1)\arcsin(\sigma) - 4\sqrt{1-\sigma^2}+4}{3\lambda^{3/2}\mu(\sigma+1)}. \quad (15)$$

The PMS (see Eq. (8)) gives us the optimal value of λ

$$\lambda_{\rm PMS} = \frac{2\sigma(\sqrt{1-\sigma^2}-2) + 4(\sqrt{1-\sigma^2}-1) + 3\mu(\sigma+1)\arcsin(\sigma)}{3\mu(\sigma+1)\arcsin(\sigma)} \quad (16)$$

and

$$\Delta\phi_a^{(1)} = \frac{2\arcsin(\sigma)}{\sqrt{\frac{2[(\sigma+2)\sqrt{1-\sigma^2}-2(\sigma+1)]}{3\mu(\sigma+1)\arcsin(\sigma)}+1}}. \quad (17)$$

In the second region we have

$$\Delta\phi_b = 2\int_\sigma^1 \frac{dz}{\sqrt{V_b(1)-V_b(z)}}\frac{1}{\sqrt{1+\delta\Delta_b(z)}}, \quad (18)$$

where

$$\Delta_b(z) \equiv \frac{V(1)-V(z)}{V_b(1)-V_b(z)} - 1 = \frac{-6\rho\mu^2 + 3(\rho+1)(z+1)\mu - 2(z^2+z+1)}{3\mu\rho(2\mu-z-1)}. \quad (19)$$

After expanding to first order and integrating we obtain

$$\Delta\phi_b^{(1)} = \frac{2(\xi^2+\mu-\mu\sigma+\sigma-1) + 3\mu(3\rho-1)\xi\ln(\frac{1-\mu}{-\mu+\sigma+\xi})}{3\mu\rho^{3/2}\xi}, \quad (20)$$

where we have defined

$$\xi \equiv \sqrt{\sigma^2 - 2\mu\sigma + 2\mu - 1}. \quad (21)$$

It is worth noticing that this simple first-order approximation exhibits the correct logarithmic singularity at $\mu = 1$.

The PMS gives us again the optimal value of ρ,

$$\rho_{\rm PMS} = 1 + \frac{2(-3\mu+\sigma+2)\xi}{3\mu(2\mu-\sigma-1)\ln(\frac{1-\mu}{-\mu+\sigma+\xi})}, \quad (22)$$

and

$$\Delta\phi_b^{(1)} = \frac{2\sqrt{3}\ln(\frac{1-\mu}{-\mu+\sigma+\xi})}{\sqrt{\frac{2(-3\mu+\sigma+2)\xi}{\mu(2\mu-\sigma-1)\ln(\frac{1-\mu}{-\mu+\sigma+\xi})}+3}}. \quad (23)$$

By adding the two expressions we obtain

$$\Delta\phi_{\rm PMS}^{(1)} = \frac{2\sqrt{3}\sqrt{\mu(\sigma+1)}\arcsin^{3/2}(\sigma)}{\sqrt{2\sigma(\sqrt{1-\sigma^2}-2)+4(\sqrt{1-\sigma^2}-1)+3\mu(\sigma+1)\arcsin(\sigma)}} + \frac{2\sqrt{3}\ln(\frac{1-\mu}{-\mu+\sigma+\xi})}{\sqrt{\frac{2(-3\mu+\sigma+2)\xi}{\mu(2\mu-\sigma-1)\ln(\frac{1-\mu}{-\mu+\sigma+\xi})}+3}} - \pi, \quad (24)$$

which still depends on the arbitrary parameter σ.

Figures 2 and 3 show the approximate deflection angle given by Eq. (24) and its percent error $\Xi = 100 \times |(\Delta\phi_{\rm PMS}^{(1)} - \Delta\phi_{\rm exact})/\Delta\phi_{\rm exact}|$ for two values of μ. We appreciate that the arbitrary parameter σ can also be determined by the PMS. In

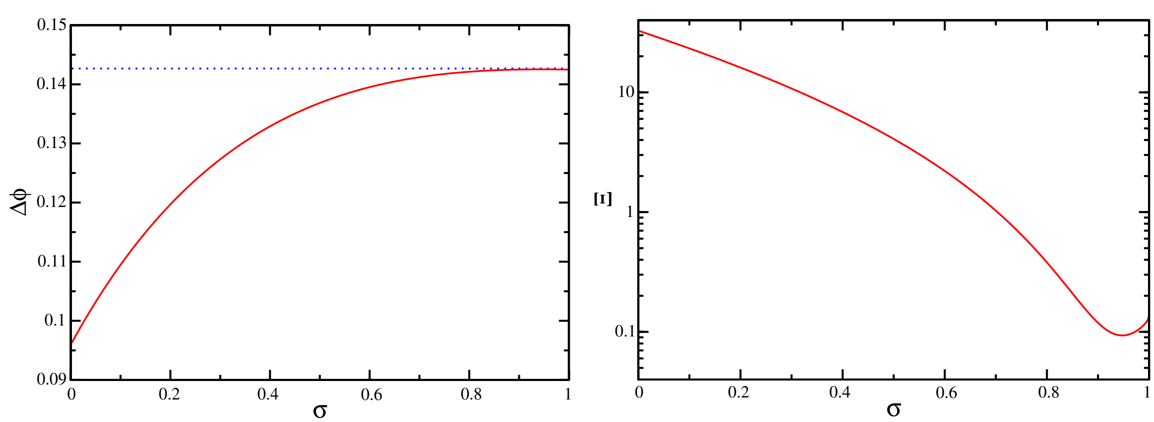

FIG. 2 (color online). Left panel: approximate deflection angle for the Schwarzschild metric (Eq. (25)) with $\mu = 10$ as a function of σ. The horizontal dotted line represents the exact value. Right panel: percent error of the approach as a function of σ.

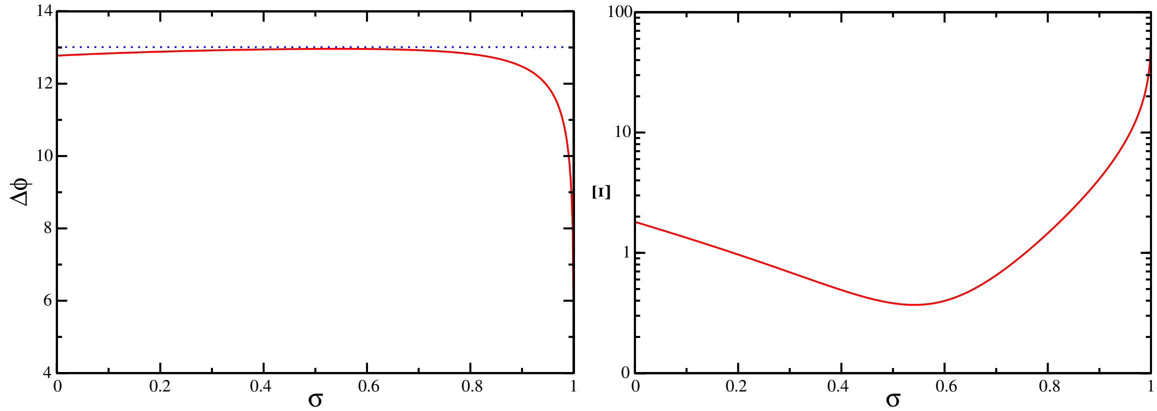

FIG. 3 (color online). Same as in Fig. 2 for $\mu = 1.001$.

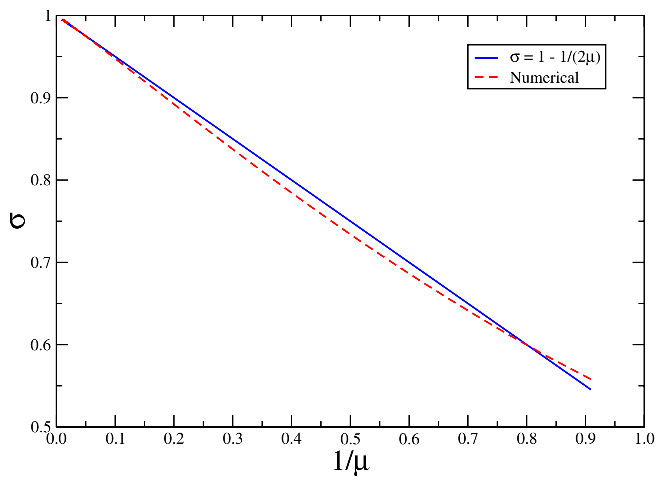

FIG. 4 (color online). $\sigma_{\rm PMS}$ obtained numerically solving the PMS condition (broken line) as a function of $1/\mu$ and the linear approximation $\sigma = 1 - 1/2\mu$ (solid line).

Fig. 2, $\mu = 10$ and the optimal value of σ is close to 1, as expected. On the other hand, if we take $\mu = 1.001$, i.e. close to the photon sphere, the optimal value of σ drops to about $1/2$. It is quite remarkable that in both cases the error made by choosing the optimal value for σ is smaller than 1%. *Another important observation is that the maximum of $\Delta\phi$ is quite flat and, consequently, a slightly imprecise estimation of $\sigma_{\rm PMS}$ will not affect the precision of the approximation drastically.*

For this reason we do not pretend to obtain σ directly by solving the PMS condition, $\partial\Delta\phi^{(1)}/\partial\sigma = 0$ (which is equivalent to finding the maximum of the curve in the left panels of Figs. 2 and 3), since that would certainly be a difficult task and lead to quite involved expressions, but we rather use a simple analytical approximation, which correctly describes the limits $\mu \to \infty$ and $\mu \to 1^+$. As noticed above, since the maximum is quite flat for $\mu \to 1^+$ one expects only a modest loss in precision, while providing much simpler expressions.

In Fig. 4 we have plotted the exact value of σ obtained solving numerically the PMS condition and the reasonable analytical approximation $\sigma_{\rm PMS} \approx 1 - 1/2\mu$. Thus we obtain

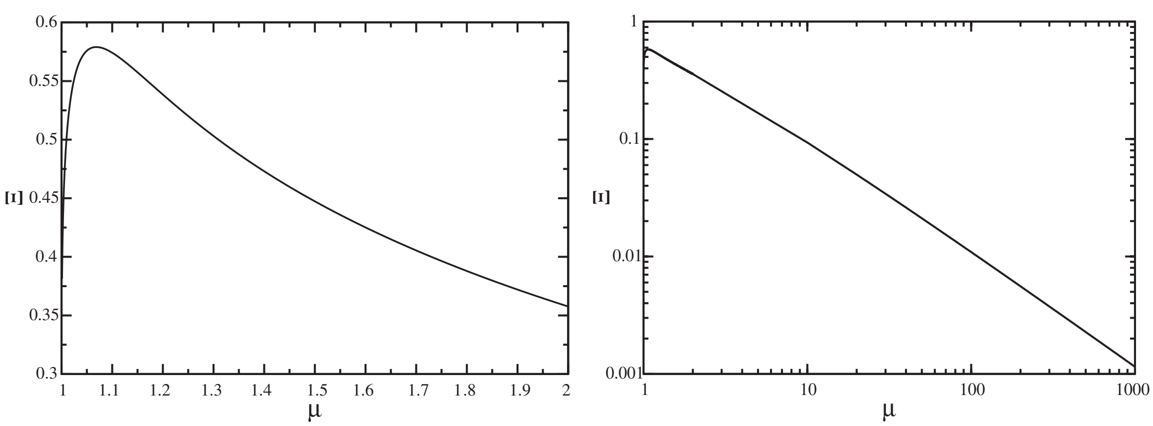

FIG. 5. Percent error of Eq. (25) as a function of μ for the deflection angle in the case of the Schwarzschild metric. Left panel: range $1 \leq \mu \leq 2$; Right panel: range $1 \leq \mu \leq 1000$.

$$\Delta\phi^{(1)}_{\rm PMS} = \frac{2\sqrt{6}\mu[\arcsin(1-\frac{1}{2\mu})]^{3/2}}{\sqrt{6\arcsin(1-\frac{1}{2\mu})\mu^2 - 8\mu + \frac{2(6\mu-1)}{\sqrt{4\mu-1}}}} + \frac{2\sqrt{6}\mu\ln^{3/2}(\frac{\mu}{\mu-1})}{\sqrt{6\ln(\frac{\mu}{\mu-1})\mu^2 - 6\mu + \frac{1}{2\mu-1} + 3}} - \pi, \quad (25)$$

which provides an accuracy better than 1% for *all* values of μ, even arbitrarily close to the photon sphere, as shown in Fig. 5. Later on we will derive an even simpler analytical formula for the deflection angle from Eq. (25).

III. GENERAL METRIC

We will now attack the problem of obtaining a first-order formula for a general static and spherically symmetric metric, in analogy with what has been done in Refs. [26,27]. For a given metric, once the functions $A(r)$, $B(r)$ and $D(r)$ are given (see Eq. (1)), one obtains a potential $V(z)$, as previously explained. We assume that the potential admits two different expansions, one at $z = 0$, and one around $z = \mu$, which is a local maximum

$$V(z) = \sum_{n=1}^{\infty} v_n z^n = \sum_{n=0}^{\infty} \tilde{v}_n (z-\mu)^n. \quad (26)$$

Notice that the first series runs from $n = 1$, since $z = 0$ is not necessarily a local minimum of $V(z)$, although this was the case for the Schwarzschild metric. Clearly, in particular cases, such as the one previously examined, the series coefficients vanish after a certain value of n, thus yielding polynomial potentials.

Following the discussion of the preceding section we split the integral as in Eq. (9) and proceed to the calculation of each part.

A. Region I ($0 \leq z \leq \sigma$)

In this region we expand $V(z)$ around $z = 0$ and use the interpolating potential $V_a(z) = \lambda z^2$. After expanding to first order we obtain Eq. (14) with

$$\Delta_a(z) = \sum_{n=1}^{\infty} \frac{v_n}{\lambda} \sum_{k=0}^{n-1} \frac{z^k}{1+z} - 1. \quad (27)$$

The deflection angle can now be written as

$$\Delta\phi^{(1)}_a = \frac{3}{\sqrt{\lambda}} \arcsin(\sigma) - \frac{1}{\lambda^{3/2}} \sum_{n=1}^{\infty} v_n \sum_{k=0}^{n-1} I_k(\sigma), \quad (28)$$

where we have defined the integrals

$$I_k(\sigma) \equiv \int_0^{\arcsin\sigma} \frac{\sin^k\theta}{1+\sin\theta} d\theta \quad (29)$$

which can be calculated exactly; for example,

$$I_0(\sigma) = 1 - \frac{\sqrt{1-\sigma}}{\sqrt{1+\sigma}} \quad (30a)$$

$$I_1(\sigma) = \arcsin(\sigma) + \frac{\sqrt{1-\sigma}}{\sqrt{1+\sigma}} - 1 \quad (30b)$$

$$I_2(\sigma) = -\arcsin(\sigma) - \sqrt{1-\sigma^2} - \frac{\sqrt{1-\sigma}}{\sqrt{1+\sigma}} + 2 \quad (30c)$$

$$I_3(\sigma) = \frac{3}{2}\arcsin(\sigma) - 2 + \frac{1}{2}(-\sigma^2 + \sigma + 4)\sqrt{\frac{2}{\sigma+1} - 1} \quad (30d)$$

$$I_4(\sigma) = -\frac{3}{2}\arcsin(\sigma) + \frac{8}{3} + \frac{1}{6}\sqrt{\frac{1-\sigma}{\sigma+1}}(-2\sigma^3 + \sigma^2 - 7\sigma - 16). \quad (30e)$$

The PMS yields

$$\lambda_{\rm PMS} = \frac{\sum_{n=1}^{\infty} v_n \sum_{k=0}^{n-1} I_k(\sigma)}{\arcsin(\sigma)}, \quad (31)$$

so that

$$\Delta\phi^{(1)}_a = \frac{2\arcsin^{3/2}\sigma}{\sqrt{\sum_{n=1}^{\infty} v_n \sum_{k=0}^{n-1} I_k(\sigma)}}. \quad (32)$$

Taking into account the form of the integrals in Eq. (30) we can express the deflection angle as

$$\Delta\phi^{(1)}_a = \frac{\arcsin^{3/2}(\sigma)}{\sqrt{F_1(\sigma) + F_2\arcsin\sigma}}. \quad (33)$$

B. Region II ($\sigma \leq z \leq 1$)

We now come to the second region where the potential is expressed in terms of a series around the local maximum at μ. After expanding to first order, and using the same interpolating potential as in the case of the Schwarzchild metric $V_b(z) = V(\mu) - \rho(z-\mu)^2$, we obtain

$$\Delta\phi^{(1)}_b = \frac{2}{\sqrt{\rho}} \int_\sigma^1 \frac{dz}{\sqrt{(z-\mu)^2 - (1-\mu)^2}} \left[1 - \frac{\Delta_b(z)}{2}\right], \quad (34)$$

where

$$\Delta_b(z) \equiv \frac{V(1) - V(z)}{V_b(1) - V_b(z)} - 1 = \sum_{n=2}^{\infty} \frac{\tilde{v}_n(a^n - b^n)}{-\rho(a^2 - b^2)} - 1, \quad (35)$$

and

$$a \equiv 1 - \mu, \qquad b \equiv z - \mu. \quad (36)$$

Since

$$a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k} \quad (37)$$

we have

$$\Delta_b(z) \equiv \frac{V(1) - V(z)}{V_b(1) - V_b(z)} - 1 = \sum_{n=2}^{\infty} \frac{\tilde{v}_n \sum_{k=0}^{n-1} a^k b^{n-1-k}}{-\rho(a+b)} - 1. \quad (38)$$

In terms of the new variable $u = b/a$ one obtains

$$\Delta\phi_b^{(1)} = \frac{2}{\sqrt{\rho}} \int_{u_-}^{u_+} \frac{du}{\sqrt{u^2 - 1}} \left[1 - \frac{\Delta_b(z(u))}{2}\right], \quad (39)$$

where $u_+ \equiv (\mu - \sigma)/(\mu - 1)$ and $u_- \equiv 1$.

Notice that

$$\frac{a^k b^{n-1-k}}{(a+b)} = \frac{u^{n-1-k}}{1+u}(1-\mu)^{n-2}. \quad (40)$$

Therefore we get

$$\Delta\phi_b^{(1)} = \frac{2}{\sqrt{\rho}} \int_{u_-}^{u_+} \frac{du}{\sqrt{u^2-1}} \left[\frac{3}{2} - \frac{1}{2} \sum_{n=2}^{\infty} \frac{\tilde{v}_n}{-\rho(1+u)} \times \sum_{k=0}^{n-1} u^{n-1-k}(1-\mu)^{n-2}\right], \quad (41)$$

where we have defined

$$J_k(\sigma) = \int_{u_-}^{u_+} \frac{u^k}{\sqrt{u^2-1}(u+1)} du. \quad (42)$$

The first integrals are

$$J_0(\sigma) = -\frac{\epsilon - 1}{\epsilon + 1} \quad (43a)$$

$$J_1(\sigma) = \frac{\epsilon - 1}{\epsilon + 1} - \ln(\epsilon) \quad (43b)$$

$$J_2(\sigma) = -\frac{\epsilon - 1}{(\epsilon + 1)2\epsilon}(1 + 4\epsilon + \epsilon^2) + \ln(\epsilon) \quad (43c)$$

$$J_3(\sigma) = \frac{\epsilon - 1}{8\epsilon^2(1+\epsilon)}(-\epsilon^4 + 2\epsilon^3 + 14\epsilon^2 + 2\epsilon - 1) - \frac{3}{2}\ln(\epsilon) \quad (43d)$$

$$J_4(\sigma) = \frac{\epsilon - 1}{24\epsilon^3(\epsilon+1)}(-\epsilon^6 + \epsilon^5 - 17\epsilon^4 - 62\epsilon^3 - 17\epsilon^2 + \epsilon - 1) + \frac{3}{2}\ln(\epsilon), \quad (43e)$$

where

$$\epsilon \equiv \frac{\mu - \sigma + \sqrt{\sigma^2 - 2\mu\sigma + 2\mu - 1}}{\mu - 1}. \quad (44)$$

The PMS yields

$$\rho_{\rm PMS} = \frac{1}{\ln\epsilon} \sum_{n=2}^{\infty} \tilde{v}_n (1-\mu)^{n-2} \sum_{k=0}^{n-1} J_k(\sigma) \quad (45)$$

and we obtain

$$\Delta\phi_b^{(1)} = \frac{2\ln^{3/2}\epsilon}{\sqrt{\sum_{n=2}^{\infty} \tilde{v}_n (1-\mu)^{n-2} \sum_{k=0}^{n-1} J_k(\sigma)}}. \quad (46)$$

Once again, looking at the structure of the integrals (43) we write

$$\Delta\phi_b^{(1)} = \frac{\ln^{3/2}\epsilon}{\sqrt{G_1(\sigma) + G_2 \ln\epsilon}}. \quad (47)$$

Notice that the explicit expression of the coefficients $F_{1,2}$ and $G_{1,2}$ will depend on the metric.

IV. RESULTS

A. Schwarzschild metric

Our first application is to the Schwarzschild metric, which corresponds to

$$B(r) = A^{-1}(r) = \left(1 - \frac{2GM}{r}\right), \qquad D(r) = 1. \quad (48)$$

Here M is the Schwarzschild mass. The angle of deflection of a ray of light reaching a minimal distance r_0 from the black hole can be obtained using Eq. (3). The exact result can be expressed in terms of incomplete elliptic integrals of the first kind [29] and reads

$$\Delta\phi = 4\sqrt{\frac{\bar{r}_0}{\Upsilon}}\left[F\left(\frac{\pi}{2}, \kappa\right) - F(\varphi, \kappa)\right], \quad (49)$$

where $\bar{r}_0 \equiv r_0/GM$ and

$$\Upsilon \equiv \sqrt{\frac{\bar{r}_0 - 2}{\bar{r}_0 + 6}}, \qquad \kappa \equiv \sqrt{(\Upsilon - \bar{r}_0 + 6)/2\Upsilon}, \qquad \varphi \equiv \sqrt{\arcsin\left[\frac{2 + \Upsilon - \bar{r}_0}{6 + \Upsilon - \bar{r}_0}\right]}. \quad (50)$$

We compare our analytical formulas with the exact one and with the approximation proposed by Bozza [24]

$$\Delta\phi_{\rm Bozza} = -2\ln[\tfrac{1}{12}(2 + \sqrt{3})(\mu - 1)] - \pi. \quad (51)$$

For brevity, we shall refer to the approximation developed in the previous section as Method I. In the Appendix we have also derived a simpler analytical expression for the deflection angle using an alternative method, which we refer to as being Method II.

In the former approach, the expression of Eq. (25) can be further simplified maintaining good accuracy by observing that

$$\frac{2\sqrt{6}\mu \arcsin^{3/2}(1 - \frac{1}{2\mu})}{\sqrt{6\arcsin(1 - \frac{1}{2\mu})\mu^2 - 8\mu + \frac{2(6\mu-1)}{\sqrt{4\mu-1}}}} \approx \pi + \frac{12}{-3\sqrt{4\mu - 1} - 4} \quad (52a)$$

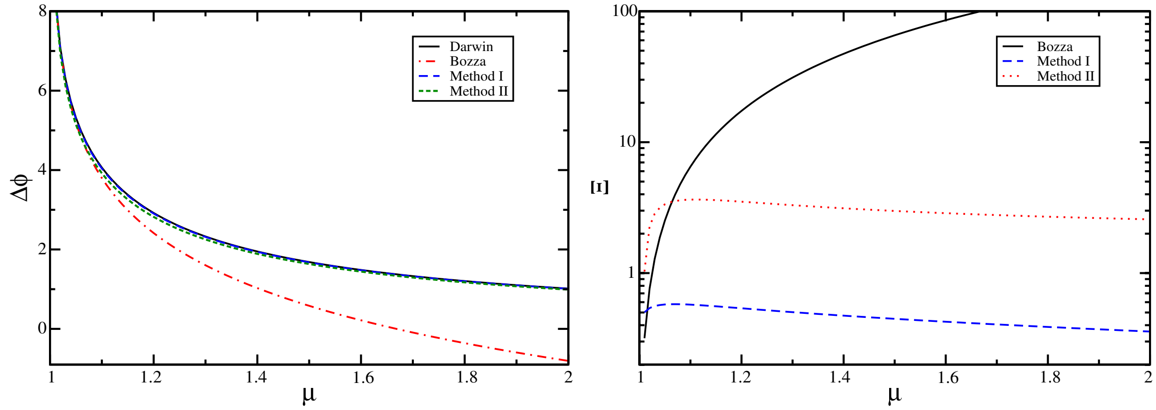

FIG. 6 (color online). Left panel: deflection angle for the Schwarzschild metric as a function of μ using different approximations and the exact result. Right panel: percent error of the approximate solutions as a function of μ.

and

$$\frac{2\sqrt{6}\mu \ln^{3/2}(\frac{\mu}{\mu-1})}{\sqrt{6\ln(\frac{\mu}{\mu-1})\mu^2 - 6\mu + \frac{1}{2\mu-1} + 3}} \approx \sqrt{4\mu + 1/3}\ln\left(\frac{\mu}{\mu-1}\right). \tag{52b}$$

Therefore we get the much simpler expression

$$\Delta\phi^{(1)}_{\mathrm{PMS}} \approx \frac{12}{-3\sqrt{4\mu-1} - 4} + \sqrt{4\mu + 1/3}\ln\left(\frac{\mu}{\mu-1}\right). \tag{53}$$

In Fig. 6 we compare our approximations, the one of Bozza [24], and the exact result of Eq. (49). It is clear that—apart from the region very close to $\mu = 1$—our approximations are more accurate, especially as μ increases: the curves in the left plot corresponding to using our approximations are hardly distinguishable from the exact one. Indeed, in the case of Method I, after having reached a maximum error close to $\mu = 1$, the precision keeps improving, as one can see in Fig. 7. In the case of Method II and of the simplified expression of Eq. (53) the error does not tend to zero as the photon sphere is approached but it appears to remain finite.

B. Reissner-Nordström metric

The Reissner-Nordström (RN) metric describes a black hole with charge and corresponds to

$$B(r) = A^{-1}(r) = \left(1 - \frac{2GM}{r} + \frac{q^2}{r^2}\right), \qquad D(r) = 1. \tag{54}$$

The corresponding potential is found to be

$$V(z) = z^2\left(\frac{q^2}{r_0^2}z^2 - 2\frac{M}{r_0}z + 1\right) = z^2 - \frac{2M}{\mu r_{\mathrm{PS}}}z^3 + \frac{q^2}{\mu^2 r_{\mathrm{PS}}^2}z^4, \tag{55}$$

where $r_{\mathrm{PS}} = 4q^2/(3M - \sqrt{9M^2 - 8q^2})$ is the photon sphere and $\mu = r_0/r_{\mathrm{PS}}$; q is the charge of the black hole. Also in this case we have found that $\sigma = 1 - 1/2\mu$ is a satisfactory analytical approximation to the optimal value σ_{PMS}. Although the exact solution of the PMS condition would in general depend on the charge q, we have verified that the general features discussed in the case of the Schwarzchild metric remain valid and only a quite limited error is introduced by this choice.

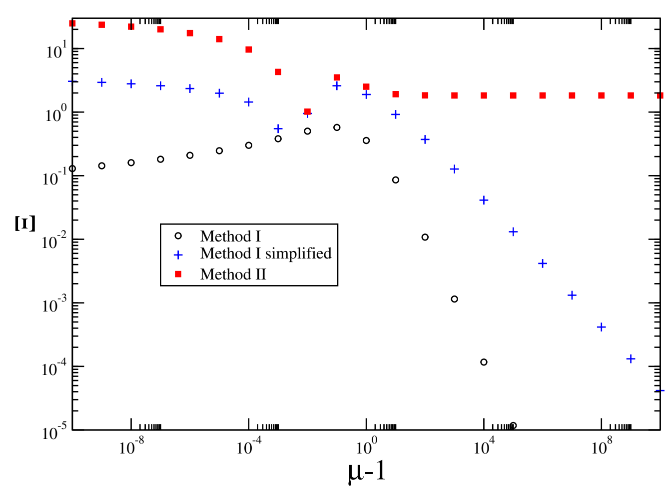

FIG. 7 (color online). Percent error for the deflection angle in the Schwarzschild metric calculated with Method I (Eq. (25)), with the simplified version of Method I (Eq. (53) and with Method II (Eq. (A8)).

Using the former we obtain

$$F_1 = \frac{-4\alpha^3 + 6\alpha^2 - 4\alpha + 2}{3\alpha(\alpha^2+1)^2} + \frac{-2\alpha^7 + 3\alpha^6 - 6\alpha^5 + 4\alpha^4 - 6\alpha^3 + 8\alpha^2 - 2\alpha + 1}{12\alpha(\alpha^2+1)^2} \times \frac{q^2}{r_0^2} \quad (56a)$$

$$F_2 = \frac{1}{4} + \frac{3q^2}{8r_0^2}, \quad (56b)$$

where $\alpha \equiv \sqrt{4\mu - 1}$, and

$$G_1 = \frac{-3\alpha^4 + 6\alpha^2 + 1}{3(\alpha^2-1)(\alpha^2+1)^2} + \frac{(21\alpha^8 + 12\alpha^6 - 110\alpha^4 - 28\alpha^2 - 23)}{96(\alpha^2-1)(\alpha^2+1)^2}\frac{q^2}{r_0^2} \quad (57a)$$

$$G_2 = \frac{1}{4} - \frac{(7\alpha^4 - 10\alpha^2 + 31)}{128}\frac{q^2}{r_0^2}, \quad (57b)$$

where the coefficients $F_{1,2}$ and $G_{1,2}$ were previously introduced in the previous section (see Eqs. (33) and (47)).

Using our expression for the deflection angle we have obtained the coefficients A and B introduced by Eiroa *et al.* in Ref. [10]

$$\Delta\phi \approx -A\log(B\epsilon) - \pi, \quad (58)$$

where $\epsilon = r_0 - r_{ps}$, r_{ps} being the distance corresponding to the photon sphere.

We have found

$$A = -\lim_{\mu\to 1^+}\frac{\Delta\phi^{(1)}_{PMS}}{\log(\mu - 1)} = \frac{4q/M}{\sqrt{8q^2/M^2 + 3\sqrt{9 - 8q^2/M^2} - 9}} \quad (59a)$$

$$B = \frac{4}{\psi + 3}\exp[-0.234 + 0.203/\psi - 1.096\sqrt{\psi/(\psi + 0.826)}], \quad (59b)$$

where $\psi \equiv \sqrt{9 - 8q^2/M^2}$.

Table I shows that our analytical formulas are in remarkable agreement with the numerical results of Eiroa *et al.*

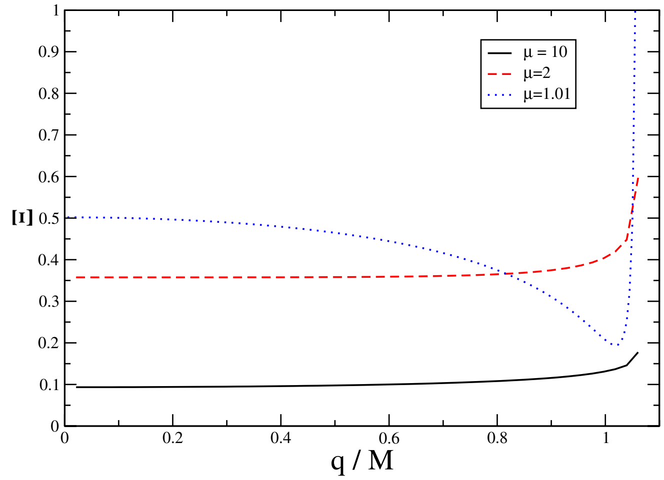

FIG. 8 (color online). Percent error of the deflection angle in the Reissner-Nordström metric calculated with Method I at different values of μ.

[10]. In particular our expression for the coefficient A appears to be exact.

Figure 8 displays the percent error of our approach for different values of μ as a function of q/M. Again, the error is generally below 0.5% and gets smaller as μ increases.

Notice that we do not need to discuss the WDL of our formulas, since in this case one has that $\mu \to \infty$ and the method of [26,27] is recovered. The reader will find a detailed comparison of our method with other methods available in the literature in [26,27].

C. Janis-Newman-Winicour metric

Finally, we consider the spherically symmetric metric solution to the Einstein massless scalar equations [32],

$$A(r) = (1 - b/r)^{-\nu}, \qquad B(r) = (1 - b/r)^{\nu}, \qquad D(r) = (1 - b/r)^{1-\nu}, \quad (60)$$

which reduces to the Schwarzschild metric for $\nu = 1$ and $b = GM$. For this metric we obtain the potential

$$V(z) = -\left(1 - \frac{b}{r_0}\right)^{2\nu-1}\left(1 - \frac{bz}{r_0}\right)^{2-2\nu} + z^2\left(1 - \frac{bz}{r_0}\right) + \left(1 - \frac{b}{r_0}\right)^{2\nu-1}, \quad (61)$$

that can be expanded around $z = 0$ to give

TABLE I. Numerical values of the coefficients A and B: A_{Eiroa} and B_{Eiroa} are taken from Table 1 of Ref. [10], whereas A_{us} and B_{us} are obtained from Eq. (59).

$\lvert q\rvert$	0	$0.1M$	$0.25M$	$0.5M$	$0.75M$	$1M$
A_{Eiroa}	2.00000	2.00224	2.01444	2.06586	2.19737	2.82843
A_{us}	2.00000	2.00224	2.01444	2.06586	2.19737	2.82843
B_{Eiroa}	0.207338	0.207979	0.21147	0.225997	0.262085	0.426782
B_{us}	0.213892	0.214535	0.218032	0.232554	0.268419	0.430856

$$\begin{aligned} V(z) &\approx v_1 z + v_2 z^2 + v_3 z^3 + \ldots \\ &\approx -2(\nu - 1)\left(1 - \tfrac{b}{r_0}\right)^{2\nu-1}\tfrac{b}{r_0} z + \left[1 - (\nu - 1)(2\nu - 1)\right. \\ &\quad \left. \times \left(1 - \tfrac{b}{r_0}\right)^{2\nu-1}\left(\tfrac{b}{r_0}\right)^2\right] z^2 + \left[-\tfrac{2}{3}(\nu - 1)\nu(2\nu - 1)\right. \\ &\quad \left. \times \left(\tfrac{b}{r_0}\right)^3\left(1 - \tfrac{b}{r_0}\right)^{2\nu-1} - \tfrac{b}{r_0}\right] z^3 + O[z^4]. \end{aligned} \tag{62}$$

Notice that the radius of convergence of the series of $V(z)$ around $z = 0$ is $\bar{z} = r_0/b$. We therefore ask that $\bar{z} \geq 1$, i.e. that $r_0 \geq b$. Clearly the accuracy of the expansion above will depend upon the location of $\bar{z}$ and a larger number of terms is expected to be needed when $\bar{z}$ approaches one.

We then find

$$F_1 = \frac{(\alpha - 1)(1 + \alpha^2)}{8\alpha} v_2 + \frac{(\alpha - 1)(\alpha(3\alpha^5 + 9\alpha^3 - 23\alpha + 16) - 13)}{64(\alpha^3 + \alpha)} v_3 \tag{63a}$$

$$F_2 = \frac{v_2}{4} \tag{63b}$$

$$G_1 = \frac{(6(\mu - 1)\mu + 1)}{8\mu(2\mu - 1)} v_3 \tag{63c}$$

$$G_2 = -\frac{v_2 + 3\mu v_3}{4}, \tag{63d}$$

where, again, $\alpha \equiv \sqrt{4\mu - 1}$ and $F_{1,2}$ and $G_{1,2}$ have been introduced in Eqs. (33) and (47). Notice that in the cubic approximation we can express the coefficient v_1 in terms of the other parameters

$$v_1 = -2\mu v_2 - 3\mu^2 v_3. \tag{64}$$

Figure 9 compares our approximation for the deflection angle with the exact result. One can see that also for this metric the accuracy is very good.

V. CONCLUSIONS

We have presented a new method for obtaining analytical expressions for the deflection angle of light in a static and spherically symmetric metric, which is accurate in both the weak and strong regimes. The former corresponding to lensing at arbitrarily large distances from the compact body and the latter to distances arbitrarily close to the photon sphere. Our first-order analytical formulas exhibit errors below 1% at *all* distances from the compact body. For this reason, although our method can be applied to any given order, in a way similar to what has been done in Ref. [27] with the method of Ref. [26], the accuracy of our first-order expressions is certainly sufficient for most physical applications.

Moreover, the method that we have presented in this paper reduces to the previous method discussed in Refs. [26,27] in the weak deflection limit (WDL), since the arbitrary parameter σ tends to one in this limit. For this reason, for the comparison with alternative methods developed to describe the WDL we have relied on the discussion contained in [26,27].

To the best of our knowledge, our method is the only one available that allows one to obtain completely analytical formulas which are valid in both SDL and WDL regimes, regardless of the particular static and spherically symmetric metric used. The results that we obtain are clearly nonperturbative, since they do not correspond to a polynomial expression in any of the physical parameters in the model and provide the correct logarithmic strength of the singularities. Moreover, our analytical expressions never involve special functions and are easy to evaluate. Just to mention one success of our approach, in the case of the Reissner-Nordstrom metric, we have obtained an analytical formula for the coefficients A and B which have been numerically calculated by Eiroa *et al.* in Ref. [10]: in the case of the coefficient A our formula reproduces *all the digits* given by the numerical calculation. We also wish to

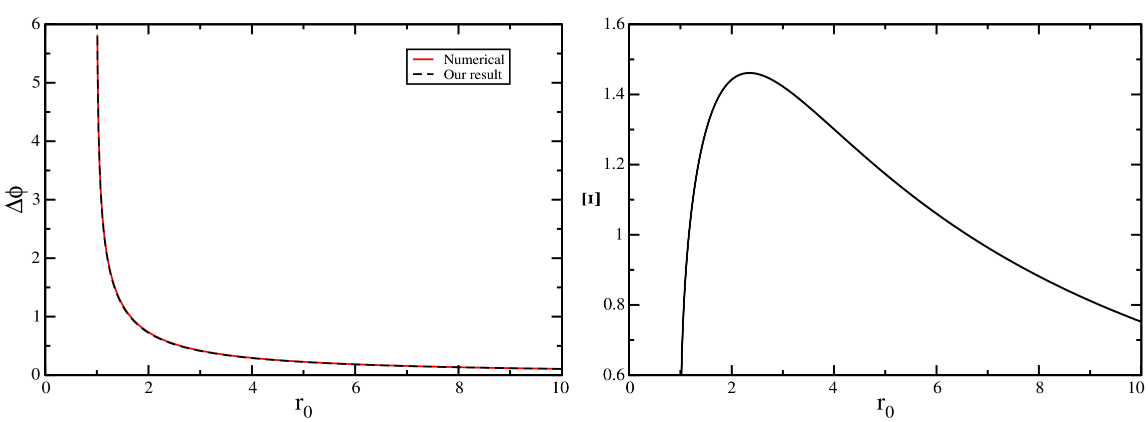

FIG. 9 (color online). Left panel: deflection angle in the Janis-Newman-Winicour metric using $\nu = 1/2$ and $b = 1$; the solid line is the numerical result, whereas the dashed line corresponds to our analytical formula. Right panel: percent error for the deflection angle using our analytical formula.

mention that our method relies on solid mathematical grounds and could be used to obtain an exact series representation for the integrals of the deflection angle. We have not considered this issue because of the highly precise results that are obtained already working to order one.

Finally, we have also discussed—showing an application to the Schwarzschild metric—an alternative method that yields larger errors but even simpler expressions and that can be useful in cases where a somewhat reduced accuracy can be traded for the possibility of more convenient analytical manipulations.

APPENDIX A: ALTERNATIVE FORMULA THROUGH LINEAR INTERPOLATION

In this Appendix we present an alternative method to obtain the deflection angle in a given static and spherically symmetric metric. This method is a generalization of a recently published approach to the period of the simple pendulum [33]. It is our purpose to obtain a simple analytical approximation to an integral of the form

$$I = \int_0^1 \frac{dz}{\sqrt{Q(z)}} \tag{A1}$$

where $Q(1) = 0$. We assume that any other zero of $Q(z)$ is outside the closed interval [0, 1].

We define a reference function

$$Q_0(z) = (1 - z)(z - z_0), \tag{A2}$$

where $z_0 < 0$, and carry out the change of variables

$$z = \frac{1 + z_0}{2} + \frac{1 - z_0}{2} \sin\theta. \tag{A3}$$

We thus obtain

$$I = \int_{\theta_0}^{\pi/2} \frac{d\theta}{\sqrt{F(z(\theta))}}, \qquad \theta_0 = \arcsin\left(\frac{1 + z_0}{z_0 - 1}\right), \qquad F(z) = \frac{Q(z)}{Q_0(z)}. \tag{A4}$$

We next substitute a linear function $\alpha + \beta\theta$ for $\sqrt{F(z(\theta))}$ such that $\sqrt{F(0)} = \alpha + \beta\theta_0$, $\sqrt{F(1)} = \alpha + \beta\pi/2$ and obtain the general approximate expression

$$I \approx \frac{\pi - 2\theta_0}{4(\sqrt{F(1)} - \sqrt{F(0)})} \ln\left(\frac{F(1)}{F(0)}\right). \tag{A5}$$

We have yet to specify the exact location of z_0. Notice that

$$F(1) = \lim_{z\to 1} \frac{Q(z)}{(1 - z)(z - z_0)} = \frac{Q'(1)}{z_0 - 1}. \tag{A6}$$

In the case of the Schwarzschild metric we have $\Delta\phi = \sqrt{6\mu}I - \pi$, where I is given by Eq. (A1), with

$$Q(z) = (z - z_1)(z - z_2)(z - z_3), \qquad z_1 \le 0 \le z_3, \qquad z_2 = 1, \tag{A7}$$

and $z_3 \ge 1$ if $\mu \ge 1$.

We may set the location of z_0 to have either the most accurate analytical expression or the simplest one; in what follows we choose the latter. If $z_0 = -1$, then $\theta_0 = 0$ and

$$\Delta\phi \approx \frac{\sqrt{3\mu}\pi}{2}(\sqrt{3\mu - 2} + \sqrt{3\mu - 3}) \ln\frac{3\mu - 2}{3\mu - 3} - \pi. \tag{A8}$$

Notice the logarithmic singularity at $\mu = 1$ that comes from the fact that $z_3(\mu = 1) = z_2 = 1$ and the integral diverges as $\mu \to 1^+$. This approach is considerably less accurate than the preceding one, but we have decided to include it in this paper for two reasons: first, it provides simple and general expressions; second, its error is quite uniform.

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