Relativistic corrections to J/ψ exclusive and inclusive double charm production at B factories

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In order to clarify the puzzling problems in double-charm production, relativistic corrections at order v^2 to the processes $e^+e^- \rightarrow J/\psi + \eta_c$ and $e^+e^- \rightarrow J/\psi + c\bar{c}$ at B factories are studied in nonrelativistic quantum chromodynamics. The short-distance parts of production cross sections are calculated perturbatively, while the long-distance matrix elements are estimated from J/ψ and η_c decays up to errors of order v^4. Our results show that the relativistic correction to the exclusive process $e^+e^- \rightarrow J/\psi + \eta_c$ is significant, which, when combined together with the next-to-leading-order α_s corrections, could resolve the large discrepancy between theory and experiment; whereas for the inclusive process $e^+e^- \rightarrow J/\psi + c\bar{c}$ the relativistic correction is tiny and negligible. The physical reason for the above difference between exclusive and inclusive processes largely lies in the fact that in the exclusive process the relative momentum between quarks in charmonium substantially reduces the virtuality of the gluon that converts into a charm quark pair, but this is not the case for the inclusive process, in which the charm quark fragmentation $c \rightarrow J/\psi + c$ is significant, and QCD radiative corrections can be more essential.

I. INTRODUCTION

The exclusive and inclusive double-charm production in e^+e^- annihilation at B factories has been one of the most puzzling problems in nonrelativistic QCD (NRQCD) [1] and heavy quarkonium physics [2] for years. The measured cross sections for exclusive double charmonium production $e^+e^- \rightarrow J/\psi + \eta_c$ and inclusive double-charm production $e^+e^- \rightarrow J/\psi + c\bar{c}$ at $\sqrt{s} = 10.6$ GeV are much higher than the leading-order (LO) predictions in NRQCD.

The cross section of $e^+e^- \rightarrow J/\psi + \eta_c$ at $\sqrt{s} = 10.6$ GeV observed by Belle [3,4] is

$$\sigma[e^+e^- \rightarrow J/\psi + \eta_c] \times \mathcal{B}^{\eta_c}[\geq 2] = 25.6 \pm 2.8 \pm 3.4 \text{ fb},$$
(1a)

and by *BABAR* [5] is

$$\sigma[e^+e^- \rightarrow J/\psi + \eta_c] \times \mathcal{B}^{\eta_c}[\geq 2] = 17.6 \pm 2.8 \pm 2.1 \text{ fb},$$
(1b)

where $\mathcal{B}^{\eta_c}[\geq 2]$ is the branching fraction for η_c decay into at least two charged particles. Since $\mathcal{B}^{\eta_c}[\geq 2] < 1$, Eq. (1) is the lower bound of the cross section of the double charmonium production. It is about an order of magnitude larger than the theoretical predictions [6–8], at the leading order of strong coupling constant α_s and quark relative velocity v in the NRQCD factorization approach [1]. In [9] the next-to-leading-order (NLO) QCD result is given and it

is found that the $\mathcal{O}(\alpha_s)$ corrections may greatly reduce the large discrepancy between theory and experiment. Meanwhile, the relativistic corrections are also considered by a number of authors. In [10], relativistic and bound state effects are discussed on the basis of relativistic quark models. In [11–15], relativistic effects are estimated in the framework of the light-cone method, and very large enhancement effects on the cross sections can be found with certain light-cone distribution amplitudes. In [16], the authors calculate the relativistic correction based on NRQCD and the light-cone method with long-distance matrix elements determined from the Cornell potential model, and they point out, however, that after subtracting parts of the light-cone distribution functions that correspond to corrections of relative-order α_s in NRQCD, the enhancement effect due to relativistic corrections is not very large but still substantial. Probably, with both relative-order α_s and relativistic corrections, we may resolve the discrepancy between theory and experiment for $e^+e^- \rightarrow J/\psi + \eta_c$ [16].

The large discrepancies between theory and experiment exist not only in the exclusive double charmonium production processes such as $e^+e^- \rightarrow J/\psi + \eta_c$ but also in the inclusive production process of $e^+e^- \rightarrow J/\psi + c\bar{c}$. The experimental result measured by Belle [3] is

$$\sigma[e^+e^- \rightarrow J/\psi + c\bar{c} + X] = 0.87^{+0.21}_{-0.19} \pm 0.17 \text{ pb}, \quad (2)$$

which is about 5 times larger than the theoretical calculations [17–21] based on NRQCD at leading order of α_s and v^2. In [22] the two-photon contribution and in [21] the color-octet contribution to $J/\psi + c\bar{c}$ production are further considered, but they are too small to resolve the large discrepancy. Other attempts to solve the problem can be

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found in a comprehensive review on heavy quarkonium physics [2]. It is certainly interesting to see whether the large discrepancy can be resolved by inclusion of NLO QCD corrections and relativistic corrections. Recently, in [23], the authors calculate the prompt $J/\psi + c\bar{c} + X$ production at NLO α_s including direct production and feed-down contribution mainly from the $\psi(2S)$, and find the NLO α_s corrections are large and positive, and could be helpful to settle the problem between experiment and theory. But we still need to know how large the relativistic corrections are, and whether they are positive or negative to the solution of the problem.

In this paper, we consider the relativistic corrections to both these exclusive and inclusive double-charm production processes based on NRQCD formulas in the color-singlet sector, since the color-octet contributions are negligible [21]. In order to avoid the model dependence in determining the long-distance matrix elements, differing from [16], we determine the matrix elements of up to dimension-8 four fermion operators from the observed decay rates of J/ψ and η_c. We find that the relativistic effect on the double charmonium production $e^+e^- \rightarrow J/\psi + \eta_c$ is substantial and comparable to the estimate of [16]; whereas for the inclusive production $e^+e^- \rightarrow J/\psi + c\bar{c}$ the relativistic corrections are very small and negligible. The rest of this paper is organized as follows. In Sec. II, we give the general formulas of the production rates in NRQCD at v^2 order. Relativistic corrections to the exclusive process and the inclusive process are studied in Secs. III and IV respectively. A summary for the v^2 order corrections to these two processes will be given in Sec. V. In the Appendix we give some of the analytic results.

II. PRODUCTION CROSS SECTIONS IN NRQCD

In NRQCD the production and decay of charmonia are factorized into two parts, the short-distance part that can be calculated perturbatively, and the long-distance part can be estimated by lattice calculation, phenomenological models, or from other experimental observables. The long-distance parts are related to the four fermion operators, characterized by the velocity v of the charm quark in the meson rest frame. The production cross sections of η_c and J/ψ up to v^2 order are [1]

$$\sigma(\eta_c) = \frac{F_1(^1S_0)}{m_c^2}\langle 0|\mathcal{O}_1(^1S_0^{\eta_c})|0\rangle + \frac{G_1(^1S_0)}{m_c^4}\langle 0|\mathcal{P}_1(^1S_0^{\eta_c})|0\rangle$$
$$+ O(v^4\sigma), \tag{3a}$$

$$\sigma(\psi) = \frac{F_1(^3S_1)}{m_c^2}\langle 0|\mathcal{O}_1(^3S_1^{\psi})|0\rangle + \frac{G_1(^3S_1)}{m_c^4}\langle 0|\mathcal{P}_1(^3S_1^{\psi})|0\rangle$$
$$+ O(v^4\sigma). \tag{3b}$$

The operators are defined as

$$\mathcal{O}_1^{\eta_c}(^1S_0) = \chi^\dagger\psi(a_{\eta_c}^\dagger a_{\eta_c})\psi^\dagger\chi, \tag{4a}$$

$$\mathcal{P}_1^{\eta_c}(^1S_0) = \frac{1}{2}\bigg[\chi^\dagger\psi(a_{\eta_c}^\dagger a_{\eta_c})\psi^\dagger\bigg(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\bigg)^2\chi$$
$$+ \chi^\dagger\bigg(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\bigg)^2\psi(a_{\eta_c}^\dagger a_{\eta_c})\psi^\dagger\chi\bigg], \tag{4b}$$

$$\mathcal{O}_1^{\psi}(^3S_1) = \chi^\dagger\sigma^i\psi(a_{\psi}^\dagger a_{\psi})\psi^\dagger\sigma^i\chi, \tag{4c}$$

$$\mathcal{P}_1^{\psi}(^3S_1) = \frac{1}{2}\bigg[\chi^\dagger\sigma^i\psi(a_{\psi}^\dagger a_{\psi})\psi^\dagger\sigma^i\bigg(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\bigg)^2\chi$$
$$+ \chi^\dagger\sigma^i\bigg(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\bigg)^2\psi(a_{\psi}^\dagger a_{\psi})\psi^\dagger\sigma^i\chi\bigg]. \tag{4d}$$

The short-distance coefficients can be evaluated by the matching condition:

$$\sigma(Q\bar{Q})|_{\text{pert QCD}} = \sum_n \frac{F_n(\Lambda)}{M^{d_n-4}}\langle 0|\mathcal{O}_n^{Q\bar{Q}}(\Lambda)|0\rangle\bigg|_{\text{pert NRQCD}}. \tag{5}$$

The left-hand side of Eq. (5) can be calculated by the spinor projection method [24]. Furthermore (see, e.g., [25,26]), the projection of $v(P/2 - q)\bar{u}(P/2 + q)$ onto a particular angular momentum state can be expressed in a Lorentz covariant form. The momenta of quark and antiquark in an arbitrary frame are, respectively [27],

$$\tfrac{1}{2}P + q = L(\tfrac{1}{2}P_r + \mathbf{q}), \tag{6a}$$
$$\tfrac{1}{2}P - q = L(\tfrac{1}{2}P_r - \mathbf{q}), \tag{6b}$$

where $P_r^\mu = (2E_q, \mathbf{0})$, $E_q = \sqrt{m^2 + \mathbf{q}^2}$, and $2\mathbf{q}$ is the relative momentum between two quarks in the meson rest frame. L_μ^v is the boost tensor from the meson rest frame to an arbitrary frame.

In the meson rest frame, the expression of projection onto a state of $S = 0$ is

$$\sum_{\lambda_1\lambda_2}v(-\mathbf{q}, \lambda_2)\bar{u}(\mathbf{q}, \lambda_1)\Big\langle\frac{1}{2}, \lambda_1, \frac{1}{2}, \lambda_2|0, 0\Big\rangle$$
$$= \frac{1}{\sqrt{2}}(E + m)\bigg(1 - \frac{\boldsymbol{\alpha}\cdot\mathbf{q}}{E + m}\bigg)\gamma_5\frac{1 + \gamma_0}{2}\bigg(1 + \frac{\boldsymbol{\alpha}\cdot\mathbf{q}}{E + m}\bigg)\gamma_0. \tag{7}$$

In an arbitrary frame it becomes

$$\sum_{\lambda_1\lambda_2}v(q, \lambda_2)\bar{u}(q, \lambda_1)\Big\langle\frac{1}{2}, \lambda_1, \frac{1}{2}, \lambda_2|0, 0\Big\rangle$$
$$= -\frac{1}{2\sqrt{2}(E + m)}\bigg(\frac{1}{2}\slashed{P} - \slashed{q} - m\bigg)\gamma_5\frac{\slashed{P} + 2E}{2E}$$
$$\times\bigg(\frac{1}{2}\slashed{P} + \slashed{q} + m\bigg). \tag{8}$$

Similarly, expressions of projection onto a state of $S = 1$ in the rest frame of the meson and an arbitrary frame are

$$\sum_{\lambda_1\lambda_2} v(-\mathbf{q}, \lambda_2)\bar{u}(\mathbf{q}, \lambda_1)\left\langle \frac{1}{2}, \lambda_1; \frac{1}{2}, \lambda_2 | 1, \epsilon \right\rangle = \frac{1}{\sqrt{2}}(E + m)\left(1 - \frac{\boldsymbol{\alpha} \cdot \mathbf{q}}{E + m}\right)\boldsymbol{\alpha} \cdot \boldsymbol{\epsilon} \frac{1 + \gamma_0}{2}\left(1 + \frac{\boldsymbol{\alpha} \cdot \mathbf{q}}{E + m}\right)\gamma_0, \tag{9a}$$

$$\sum_{\lambda_1\lambda_2} v(q, \lambda_2)\bar{u}(q, \lambda_1)\left\langle \frac{1}{2}, \lambda_1; \frac{1}{2}, \lambda_2 | 1, \epsilon \right\rangle = -\frac{1}{2\sqrt{2}(E + m)}\left(\frac{1}{2}\slashed{P} - \slashed{q} - m\right)\epsilon \frac{\slashed{P} + 2E}{2E}\left(\frac{1}{2}\slashed{P} + \slashed{q} + m\right). \tag{9b}$$

In our calculation the Dirac spinors are normalized as $\bar{u}u = -\bar{v}v = 2m_c$. Then the short-distance part of the cross section can be calculated at any order of v.

At order v^2 the cross section of $e^+e^- \rightarrow J/\psi + \eta_c$ and $e^+e^- \rightarrow J/\psi + c\bar{c}$ can be expressed as

$$\sigma(e^+e^- \rightarrow J/\psi + \eta_c) = \langle 0|\mathcal{O}_1(^1S_0^{\eta_c})|0\rangle \frac{\langle 0|\mathcal{O}_1(^3S_1^{\psi})|0\rangle}{3}\frac{1}{2s}\int \bar{M}_0 d\text{LIPS} + \frac{\langle 0|\mathcal{P}_1(^3S_1^{\psi})|0\rangle}{3}\langle 0|\mathcal{O}_1(^1S_0^{\eta_c})|0\rangle\frac{1}{2s}$$

$$\times \int \bar{M}_{1J/\psi} d\text{LIPS} + \langle 0|\mathcal{P}_1(^1S_0^{\eta_c})|0\rangle \frac{\langle 0|\mathcal{O}_1(^3S_1^{\psi})|0\rangle}{3}\frac{1}{2s}\int \bar{M}_{1\eta_c} d\text{LIPS}. \tag{10a}$$

$$\sigma(e^+e^- \rightarrow J/\psi + c\bar{c}) = \frac{\langle 0|\mathcal{O}_1(^3S_1^{\psi})|0\rangle}{3}\frac{1}{2s}\int \bar{N}_0 d\text{LIPS} + \frac{\langle 0|\mathcal{P}_1(^3S_1^{\psi})|0\rangle}{3}\frac{1}{2s}\int \bar{N}_1 d\text{LIPS}. \tag{10b}$$

Here LIPS means the Lorentz invariant phase space, bar means averaging spins over the initial states and summing spins over the final states, and $\bar{M}_0$, $\bar{M}_{1J/\psi}$, $\bar{M}_{1\eta_c}$, $\bar{N}_0$, and $\bar{N}_1$ can be calculated perturbatively.

III. RELATIVISTIC CORRECTIONS TO $e^+e^- \rightarrow J/\psi + \eta_c$

A. Short-distance part

There are four Feynman diagrams in the process $e^+e^- \rightarrow (c\bar{c})_{3S_1} + (c\bar{c})_{1S_0}$, shown in Fig. 1. The amplitude of the process can be expanded in terms of the quark relative momentum in charmonium:

$$M(e^+e^- \rightarrow (c\bar{c})_{3S_1} + (c\bar{c})_{1S_0}) = \left(\frac{m_c}{E_1}\frac{m_c}{E_2}\right)^{1/2}A(q_\psi, q_{\eta_c})$$

$$= \left(\frac{m_c}{E_1}\frac{m_c}{E_2}\right)^{1/2}\left(A(0,0) + q_\psi^\alpha \frac{\partial A}{\partial q_\psi^\alpha}\bigg|_{q_\psi = q_{\eta_c} = 0} + q_{\eta_c}^\alpha \frac{\partial A}{\partial q_{\eta_c}^\alpha}\bigg|_{q_\psi = q_{\eta_c} = 0}\right.$$

$$\left. + \frac{1}{2}q_\psi^\alpha q_\psi^\beta \frac{\partial^2 A}{\partial q_\psi^\alpha \partial q_\psi^\beta}\bigg|_{q_\psi = q_{\eta_c} = 0} + \frac{1}{2}q_{\eta_c}^\alpha q_{\eta_c}^\beta \frac{\partial^2 A}{\partial q_{\eta_c}^\alpha \partial q_{\eta_c}^\beta}\bigg|_{q_\psi = q_{\eta_c} = 0} + \ldots\right), \tag{11}$$

and $A(q_\psi, q_{\eta_c})$ is expressed as

$$A(q_\psi, q_{\eta_c}) = \sum_{\lambda_1\lambda_2\lambda_3\lambda_4}\sum_{ijkl}\left\langle \frac{1}{2}, \lambda_1; \frac{1}{2}, \lambda_2 | 1, S_z \right\rangle\left\langle \frac{1}{2}, \lambda_3; \frac{1}{2}, \lambda_4 | 0, 0 \right\rangle\langle 3, i; \bar{3}, j|1\rangle\langle 3, k; \bar{3}, l|1\rangle$$

$$\times A\left(e^+e^- \rightarrow c_{\lambda1,i}\left(\frac{P1}{2} + q_\psi\right)\bar{c}_{\lambda2,j}\left(\frac{P1}{2} - q_\psi\right) + c_{\lambda3,k}\left(\frac{P2}{2} + q_{\eta_c}\right)\bar{c}_{\lambda4,l}\left(\frac{P2}{2} - q_{\eta_c}\right)\right), \tag{12}$$
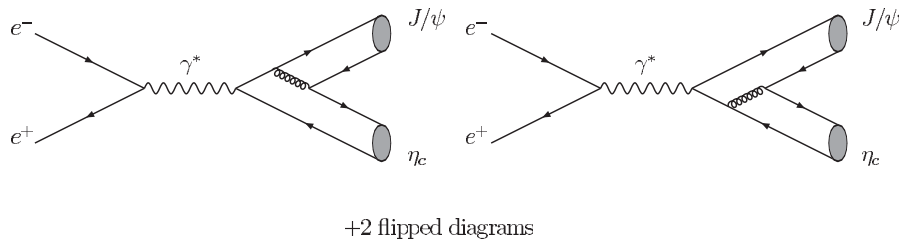

+2 flipped diagrams

FIG. 1. $e^+e^- \rightarrow (c\bar{c})_{3S_1} + (c\bar{c})_{1S_0}$.

where $\langle 3, i; \bar{3}, j|1\rangle = \delta_{ij}/\sqrt{N_c}$ and $\langle 3, k; \bar{3}, l|1\rangle = \delta_{kl}/\sqrt{N_c}$ are the color-SU(3), Clebsch-Gordon coefficients for $Q\bar{Q}$ pairs projecting onto a color-singlet state. Using Eq. (8) and (9b), we can express $A(q_\psi, q_{\eta_c})$ in a covariant form. The factor $(\frac{m_c}{E_1}\frac{m_c}{E_2})^{1/2}$ comes from the relativistic normalization of the $c\bar{c}$ state, and $E_1 = \sqrt{m_c^2 + \mathbf{q}_\psi^2}$, $E_2 = \sqrt{m_c^2 + \mathbf{q}_{\eta_c}^2}$.

For the S wave charmonium $q^\alpha q^\beta = \frac{1}{3}\mathbf{q}^2(-g^{\alpha\beta} + \frac{P^\alpha P^\beta}{P^2})$, where $P^2 = 4E^2$, $P \cdot q = 0$, and the odd-power terms of v vanish. Then at leading order of v^2,

$$|M|^2 = \frac{m_c}{E_1}\frac{m_c}{E_2}A(0,0)A^*(0,0) + \frac{1}{2}q_\psi^\alpha q_\psi^\beta A_\psi^{\alpha\beta}A^*(0,0)$$
$$+ \frac{1}{2}q_{\eta_c}^\alpha q_{\eta_c}^\beta A_{\eta_c}^{\alpha\beta}A^*(0,0) + \frac{1}{2}q_\psi^\alpha q_\psi^\beta A_\psi^{*\alpha\beta}A(0,0)$$
$$+ \frac{1}{2}q_{\eta_c}^\alpha q_{\eta_c}^\beta A_{\eta_c}^{*\alpha\beta}A(0,0), \qquad (13)$$

where $A^{\alpha\beta} = \frac{\partial^2 A}{\partial q^\alpha \partial q^\beta}$ and $A^{*\alpha\beta} = \frac{\partial^2 A^*}{\partial q^\alpha \partial q^\beta}$.

To obtain the short-distance part $E = \sqrt{m_c^2 + \mathbf{q}^2}$ should also be expanded in $v^2 = \frac{\mathbf{q}^2}{m_c^2}$. Then the relation of M_0, $M_{1J/\psi}$, $M_{1\eta_c}$ to $|M|^2$ can be easily written down. Details of how to get the short-distance coefficients from the covariant projection method can be found in e.g. [28]:

$$\bar{M}_0 = \frac{1}{(2N_c m_c)^2}\left(A(0,0)A^*(0,0)\right)\bigg|_{\mathbf{q}_{J/\psi}^2 = \mathbf{q}_{\eta_c}^2 = 0}; \qquad (14a)$$

$$\bar{M}_{1J/\psi} = \frac{1}{(2N_c m_c)^2}\left(\frac{\partial(\frac{m_c^2}{E_1 E_2}A(0,0)A^*(0,0))}{\partial(\mathbf{q}_\psi^2)}\bigg|_{\mathbf{q}_{J/\psi}^2 = \mathbf{q}_{\eta_c}^2 = 0} + \frac{1}{6}(A_\psi^{\alpha\beta}\Pi_{\alpha\beta}(P_{J/\psi})A^*(0,0)\right.$$
$$\left. + A_\psi^{*\alpha\beta}\Pi_{\alpha\beta}(P_{J/\psi})A(0,0))\right)\bigg|_{\mathbf{q}_{J/\psi}^2 = \mathbf{q}_{\eta_c}^2 = 0}; \qquad (14b)$$

$$\bar{M}_{1\eta_c} = \frac{1}{(2N_c m_c)^2}\left(\frac{\partial(\frac{m_c^2}{E_1 E_2}A(0,0)A^*(0,0))}{\partial(\mathbf{q}_{\eta_c}^2)}\bigg|_{\mathbf{q}_{J/\psi}^2 = \mathbf{q}_{\eta_c}^2 = 0} + \frac{1}{6}(A_{\eta_c}^{\alpha\beta}\Pi_{\alpha\beta}(P_{\eta_c})A^*(0,0)\right.$$
$$\left. + A_{\eta_c}^{*\alpha\beta}\Pi_{\alpha\beta}(P_{\eta_c})A(0,0))\right)\bigg|_{\mathbf{q}_{J/\psi}^2 = \mathbf{q}_{\eta_c}^2 = 0}. \qquad (14c)$$

Here $\bar{M}_0$ is just the leading order result, and it agrees with the previous result in Ref. [7]:

$$\bar{M}_0 = \frac{1}{9N_c^2}\frac{2048(s - 16m_c^2)(1 + \cos^2(\theta))(4\pi\alpha)^2(4\pi\alpha_s)^2 e_q^2}{s^4}. \qquad (15)$$

The expressions of $\bar{M}_{1J/\psi}$ and $\bar{M}_{1\eta_c}$ are

$$\bar{M}_{1J/\psi} = \frac{(1 + \cos_\theta^2)(4\pi\alpha)^2(4\pi\alpha_s)^2 e_q^2}{9N_c^2}\left(\frac{512(16m_c^2 - 3s)}{m_c^2 s^4} + \frac{512(2560m_c^4 - 592m_c^2 s + 27s^2)}{3m_c^2 s^5}\right), \qquad (16a)$$

$$\bar{M}_{1\eta_c} = \frac{(1 + \cos_\theta^2)(4\pi\alpha)^2(4\pi\alpha_s)^2 e_q^2}{9N_c^2}\left(\frac{-16384}{s^4} + \frac{1024(11s - 80m_c^2)(s - 16m_c^2)}{3m_c^2 s^5}\right), \qquad (16b)$$

where $Nc = 3$, $e_q = \frac{2}{3}$, and θ is the angle between the J/ψ and the electron.

B. Long-distance part

In this section we present a phenomenological estimate of the color-singlet production matrix elements which are extracted from charmonium decay data. It is known that, up to errors of order v^4, the color-singlet production matrix elements are related to the decay matrix elements through vacuum saturation [1]. There are three independent NRQCD matrix elements at order v^2, $\langle 0|\mathcal{O}_1(^1S_0^{\eta_c})|0\rangle$, $\langle 0|\mathcal{O}_1(^3S_1^\psi)|0\rangle$, and $\langle 0|\mathcal{P}_1(^1S_0^{\eta_c})|0\rangle = \langle 0|\mathcal{P}_1(^3S_1^\psi)|0\rangle/3$ $(1 + O(v^2))$. We can get them through the J/ψ leptonic decay $J/\psi \rightarrow e^+e^-$ and hadronic decay $J/\psi \rightarrow LH$, and the η_c photonic decay $\eta_c \rightarrow \gamma\gamma$. The theoretical results at next-to-leading order of α_s and v^2 for $J/\psi \rightarrow e^+e^-$, $J/\psi \rightarrow LH$,[1] and $\eta_c \rightarrow \gamma\gamma$ are summarized in Ref. [25] as

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[1] We do not include the electromagnetic process of $J/\psi \rightarrow \gamma^* \rightarrow LH$.

$$\Gamma[\eta_c \to \gamma\gamma] = 2e_c^4\pi\alpha^2\left(\left(1 - \frac{(20-\pi^2)\alpha_s}{3\pi}\right)\frac{\langle 0|\mathcal{O}_1(^1S_0^{\eta_c})|0\rangle}{m_c^2} - \frac{4}{3}\frac{\langle 0|\mathcal{P}_1(^1S_0^{\eta_c})|0\rangle}{m_c^4}\right), \tag{17a}$$

$$\Gamma[J/\psi \to e^+e^-] = \frac{2e_c^2\pi\alpha^2}{3}\left(\left(1 - \frac{16\alpha_s}{3\pi}\right)\frac{\langle 0|\mathcal{O}_1(^3S_1^{\psi})|0\rangle/3}{m_c^2} - \frac{4}{3}\frac{\langle 0|\mathcal{P}_1(^3S_1^{\psi})|0\rangle/3}{m_c^4}\right), \tag{17b}$$

$$\Gamma[J/\psi \to LH] = \left(\frac{20\alpha_s^3}{243}(\pi^2-9)\right)\left(\left(1 - 2.55\frac{\alpha_s}{\pi}\right)\frac{\langle 0|\mathcal{O}_1(^3S_1^{\psi})|0\rangle/3}{m_c^2} - \frac{19\pi^2-132}{12\pi^2-108}\frac{\langle 0|\mathcal{P}_1(^3S_1^{\psi})|0\rangle/3}{m_c^4}\right). \tag{17c}$$

Solving these equations at leading order of α_s (QCD radiative corrections not included), we get

$$\langle 0|\mathcal{O}_1(^1S_0^{\eta_c})|0\rangle = 0.286 \text{ GeV}^3, \qquad \langle 0|\mathcal{O}_1(^3S_1^{\psi})|0\rangle/3 = 0.295 \text{ GeV}^3, \tag{18a}$$

$$\frac{\langle 0|\mathcal{P}_1(^1S_0^{\eta_c})|0\rangle}{m_c^2} = \frac{\langle 0|\mathcal{P}_1(^3S_1^{\psi})|0\rangle}{3m_c^2} = 0.321 \times 10^{-1} \text{ GeV}^3, \tag{18b}$$

for $m_c = 1.5$ GeV and $\alpha_s = 0.26$. The experimental data of these decay rates can be found from [29], and we choose their central values $\Gamma[J/\psi \to e^+e^-] = 5.55$ KeV, $\Gamma[J/\psi \to LH] = 69.3$ KeV, and $\Gamma[\eta_c \to \gamma\gamma] = 7.14$ KeV. The matrix elements can be expressed as functions of the charm quark mass,

$$\langle 0|\mathcal{O}_1(^1S_0^{\eta_c})|0\rangle = 0.127m_c^2, \tag{19a}$$

$$\frac{\langle 0|\mathcal{O}_1(^3S_1^{\psi})|0\rangle}{3} = 0.131m_c^2, \tag{19b}$$

$$\frac{\langle 0|\mathcal{P}_1(^1S_0^{\eta_c})|0\rangle}{m_c^2} = \frac{\langle 0|\mathcal{P}_1(^3S_1^{\psi})|0\rangle}{3m_c^2} = 0.014m_c^2 \tag{19c}$$

Including the QCD NLO radiative corrections, and doing the calculation in the same way as above, we have

$$\langle 0|\mathcal{O}_1(^1S_0^{\eta_c})|0\rangle = 0.432 \text{ GeV}^3,$$

$$\langle 0|\mathcal{O}_1(^3S_1^{\psi})|0\rangle/3 = 0.573 \text{ GeV}^3, \tag{20a}$$

$$\frac{\langle 0|\mathcal{P}_1(^1S_0^{\eta_c})|0\rangle}{m_c^2} = \frac{\langle 0|\mathcal{P}_1(^3S_1^{\psi})|0\rangle}{3m_c^2} = 0.514 \times 10^{-1} \text{ GeV}^3, \tag{20b}$$

for $m_c = 1.5$ GeV and $\alpha_s = 0.26$. We then have

$$\langle 0|\mathcal{O}_1(^1S_0^{\eta_c})|0\rangle = 0.192m_c^2, \tag{21a}$$

$$\frac{\langle 0|\mathcal{O}_1(^3S_1^{\psi})|0\rangle}{3} = 0.255m_c^2, \tag{21b}$$

$$\frac{\langle 0|\mathcal{P}_1(^1S_0^{\eta_c})|0\rangle}{m_c^2} = \frac{\langle 0|\mathcal{P}_1(^3S_1^{\psi})|0\rangle}{3m_c^2} = 0.023m_c^2, \tag{21c}$$

as functions of m_c.

In [30], the authors express the next-to-leading-order matrix elements in terms of the quark pole mass and the quarkonium mass, and in [16] the matrix elements are calculated based on the potential model. Differing from their methods, we get the production matrix elements by using the experimentally observed charmonium decay rates. As shown in [1], the difference between the color-singlet production and decay matrix elements are of order v^4. So our method should be valid at order v^2, and our estimates of the production matrix elements should be good numerically, if the high order QCD and v^2 corrections for the decays are small and the experimental errors are not large.

C. Numerical results and discussions

At $\sqrt{s} = 10.6$ GeV, with $m_c = 1.5$ GeV and $\alpha_s = 0.26$, using the matrix elements in Eqs. (18) and (19) determined from charmonium decays with v^2 corrections but without QCD radiative corrections, and making the phase space integral, the leading-order cross section of $e^+e^- \to J/\psi + \eta_c$ (LO means for the short-distance part, since for the long-distance matrix elements the v^2 corrections are already included) is 3.07 fb. The relativistic correction contributes 0.798 fb, which gives about 26% enhancement and results in 3.87 fb for the cross section. The cross sections as
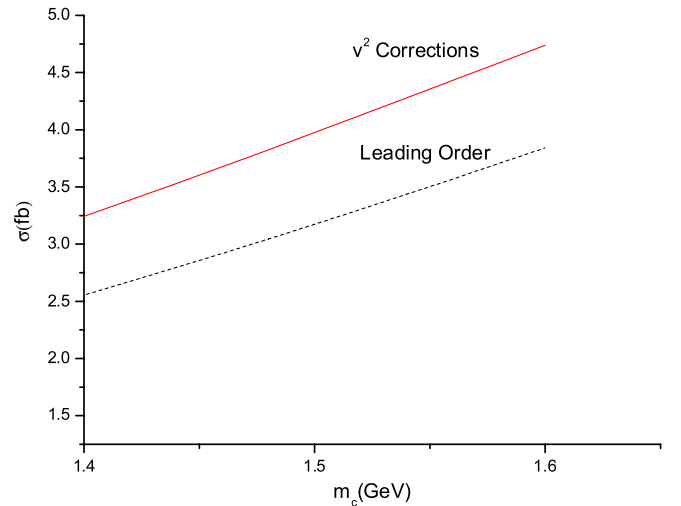

FIG. 2 (color online). $e^+e^- \to J/\psi + \eta_c$ cross sections with relativistic corrections to long-distance matrix elements extracted from charmonium decays (without NLO QCD radiative corrections). The lower line represents the LO result in v, and the upper line represents the result with v^2 corrections to the short-distance coefficients. Here the coupling constant is fixed as $\alpha_s = 0.26$.
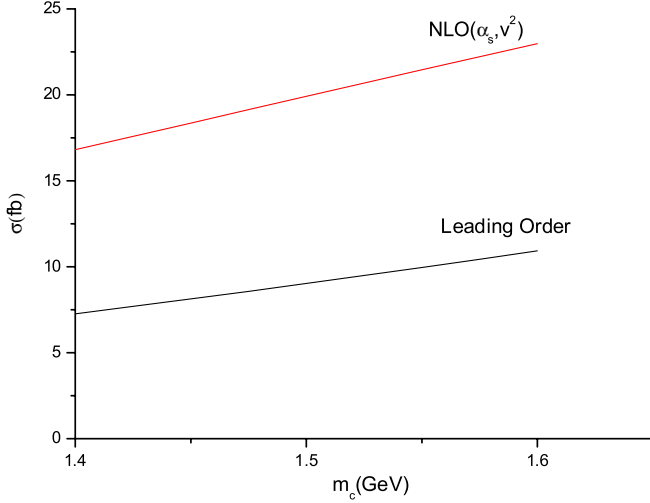
FIG. 3 (color online). $e^+e^- \to J/\psi + \eta_c$ cross sections with relativistic corrections to long-distance matrix elements extracted from charmonium decays (with NLO QCD radiative corrections). The lower line represents the LO result in v, and the upper line represents the result with v^2 corrections to the short-distance coefficients. Here the coupling constant is fixed as $\alpha_s = 0.26$. Note that the QCD radiative corrections to the short-distance coefficients (with $K = 1.8$) are included for the upper line but not the lower line.

functions of m_c are shown in Fig. 2. The lower line represents the LO result in v, and the upper line represents the result with v^2 corrections. Since the long-distance matrix elements are proportional to the squared quark mass, and the short-distance coefficients are found to be quite stable when m_c changes, the cross section goes down as m_c becomes smaller. However, these results are obtained for a fixed value of $\alpha_s = 0.26$, and if the running of $\alpha_s = \alpha_s(2m_c)$ is assumed, the m_c dependence of cross sections will be substantially changed, making the cross section at $m_c = 1.4$ GeV close to that at $m_c = 1.6$ GeV.

If using the matrix elements in Eqs. (20) and (21) determined from charmonium decays with both v^2 and α_s corrections, at order v^0 the cross section is 9.02 fb, and at order v^2 the cross section is 11.26 fb for $\alpha_s = 0.26$

and $m_c = 1.5$ GeV. When m_c varies from 1.4 to 1.6 GeV, the cross section goes up from 9.18 to 13.43 fb.

The experiment result in Eq. (1) is about an order of magnitude larger than the leading-order result [6–8]. The result at order v^2 shows that relativistic corrections can enhance both the short-distance coefficients and the long-distance matrix elements. The next-to-leading-order v^2 coefficient $(\bar{M}_{1J/\psi} + \bar{M}_{1\eta_c})m_c^2$ is about 2.2 times larger than $\bar{M}$ for $m_c = 1.5$ GeV and $\alpha_s = 0.26$, and the matrix elements of $\langle 0|\mathcal{O}_1(^1S_0^{\eta_c})|0\rangle$ and $\langle 0|\mathcal{O}_1(^3S_1^{\psi})|0\rangle$ also become about 1.17 times larger than that in the leading-order calculation. These make the cross section become 1.7 times larger after including the relativistic effect. If we determine the matrix elements including the QCD radiative correction, at v^2 order the theoretical result is 11.26 fb for $\alpha_s = 0.26$ and $m_c = 1.5$ GeV. It shows that the relativistic corrections are significant.

In the above discussions, we have not considered the QCD radiative corrections to the short-distance coefficients as shown in [9]. If we further combine the NLO QCD corrections [9] with the v^2 corrections with a fixed value $\alpha_s = 0.26$, then the cross section will go from 16.8 to 23.0 fb when m_c varies from 1.4 to 1.6 GeV, as shown in Fig. 3. The calculated cross sections of $\sigma[e^+e^- \to J/\psi + \eta_c]$ are listed in Table I, where $\sigma_{LO(\alpha_s,v^2)}$ means the cross section at leading order in both α_s and v^2; $\sigma_{NLO(\alpha_s)}$ means that obtained by using the short-distance part with α_s corrections [9], and the long-distance matrix elements also with α_s corrections extracted from both $J/\psi \to e^+e^-$ and $\eta_c \to 2\gamma$ (not from $J/\psi \to e^+e^-$ alone as was extracted in [9]) with all at leading order in v^2; $\sigma_{NLO(v^2)}$ means that obtained with v^2 corrections but without α_s corrections; and $\sigma_{NLO(\alpha_s,v^2)}$ means that obtained by combining both α_s and v^2 corrections, where the matrix element are taken from Eq. (20).

We see that with both QCD radiative corrections and relativistic corrections, the discrepancy between experiment and theory for $e^+e^- \to J/\psi + \eta_c$ could be largely resolved. Our result is consistent with [16]. In our approach the long-distance matrix elements are extracted from experimental data of J/ψ and η_c decays, and therefore are

TABLE I. Experimental and calculated cross sections of $\sigma[e^+e^- \to J/\psi + \eta_c]$ with $m_c = 1.5$ GeV and $\alpha_s = 0.26$. See text for the definitions of $\sigma_{LO(\alpha_s,v^2)}$, $\sigma_{NLO(\alpha_s)}$, $\sigma_{NLO(v^2)}$, and $\sigma_{NLO(\alpha_s,v^2)}$.

Experimental result					
$\sigma_{Belle}[e^+e^- \to J/\psi + \eta_c] \times \mathcal{B}^{\eta_c}[\geq 2]$ (fb)			$\sigma_{Babar}[e^+e^- \to J/\psi + \eta_c] \times \mathcal{B}^{\eta_c}[\geq 2]$ (fb)		
$25.6 \pm 2.8 \pm 3.4$			$17.6 \pm 2.8 \pm 2.1$		
Theoretical result					
$\langle 0|\mathcal{O}_1(^1S_0^{\eta_c})|0\rangle$	$\frac{\langle 0|\mathcal{O}_1(^3S_1^{\psi})|0\rangle}{3}$	$\frac{\langle 0|\mathcal{P}_1(^1S_0^{\eta_c})|0\rangle}{m_c^2}$	$\frac{\langle 0|\mathcal{P}_1(^3S_1^{\psi})|0\rangle}{3m_c^2}$	α_s	σ (fb)
0.243 GeV^3	0.252 GeV^3	0	0	$\alpha_s = 0.26$	$\sigma_{LO(\alpha_s,v^2)} = 2.26$
0.337 GeV^3	0.450 GeV^3	0	0	$\alpha_s = 0.26$	$\sigma_{NLO(\alpha_s)} = 10.92$
0.286 GeV^3	0.295 GeV^3	0.0321 GeV^3	0.0321 GeV^3	$\alpha_s = 0.26$	$\sigma_{NLO(v^2)} = 3.87$
0.432 GeV^3	0.573 GeV^3	0.0514 GeV^3	0.0514 GeV^3	$\alpha_s = 0.26$	$\sigma_{NLO(\alpha_s,v^2)} = 20.04$

model independent. The main uncertainties may come from the higher order corrections and the errors in the measurements. The relativistic effects on the double charmonium production estimated in our approach are milder than some results obtained by using the light-cone methods.

IV. RELATIVISTIC CORRECTIONS TO $e^+e^- \rightarrow J/\psi + c\bar{c}$

The long-distance matrix elements in process $e^+e^- \rightarrow J/\psi + c\bar{c}$ are the same as in process $e^+e^- \rightarrow J/\psi + \eta_c$, so we just use the result above, and only give the detailed calculation for the short-distance part.

A. Short-distance part

There are four Feynman diagrams in the process $e^+e^- \rightarrow (C\bar{C})_{3S_1} + C\bar{C}$, which are shown in Fig. 4. As in the last section the amplitude can be expanded in terms of the quark relative momentum and reads
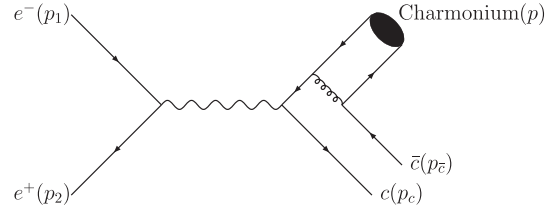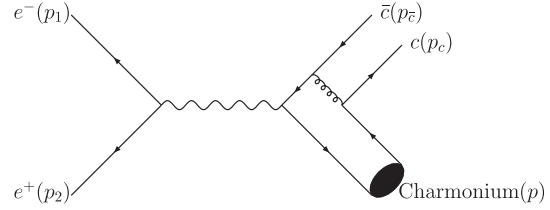

+ 2 flipped graphs

FIG. 4. Feynman diagrams for $e^+ + e^- \rightarrow \gamma^* \rightarrow J/\psi + c\bar{c}$.

$$M(e^+e^- \rightarrow (C\bar{C})_{3S_1} + C\bar{C}) = \left(\frac{m_c}{E}\right)^{1/2} A(q_\psi) = \left(\frac{m_c}{E}\right)^{1/2}\left(A(0) + q_\psi^\alpha \frac{\partial A}{\partial q_\psi^\alpha}\bigg|_{q_\psi=0} + \frac{1}{2} q_\psi^\alpha q_\psi^\beta \frac{\partial^2 A}{\partial q_\psi^\alpha \partial q_\psi^\beta}\bigg|_{q_\psi=0} + \cdots\right), \quad (22)$$

where

$$A(q_\psi) = \sum_{\lambda_1 \lambda_2} \sum_{ij} \left\langle \frac{1}{2}, \lambda_1, \frac{1}{2}, \lambda_2 | 1, S_z \right\rangle \langle 3, i; \bar{3}, j | 1 \rangle A\left(e^+e^- \rightarrow C_{\lambda_1, i}\left(\frac{P}{2} + q_\psi\right) \bar{C}_{\lambda_2, j}\left(\frac{P}{2} - q_\psi\right) + C\bar{C}\right). \quad (23)$$

At leading order of v^2,

$$|M|^2 = \frac{m_c}{E_1} A(0)A^*(0) + \frac{1}{2} q_\psi^\alpha q_\psi^\beta A_{\alpha\beta}A^*(0) + \frac{1}{2} q_\psi^\alpha q_\psi^\beta A^*_{\alpha\beta}A(0), \quad (24)$$

where $A_{\alpha\beta} = \frac{\partial^2 A}{\partial q^\alpha \partial q^\beta}$, and $A^*_{\alpha\beta} = \frac{\partial^2 A^*}{\partial q^\alpha \partial q^\beta}$. Then

$$\bar{N}_0 = \frac{1}{2N_c m_c}(A(0)A^*(0))|_{\mathbf{q}^2_{J/\psi}=0}, \quad (25a)$$

$$\bar{N}_1 = \frac{1}{2N_c m_c}\left(\frac{\partial(\frac{m_c}{E}A(0)A^*(0))}{\partial(\mathbf{q}^2_\psi)}\bigg|_{\mathbf{q}^2_{J/\psi}=0} + \frac{1}{6}(A_\psi^{\alpha\beta} \Pi_{\alpha\beta}(P_{J/\psi})A^*(0) + A_\psi^{*\alpha\beta} \Pi_{\alpha\beta}(P_{J/\psi})A(0))\bigg|_{\mathbf{q}^2_{J/\psi}=0}\right). \quad (25b)$$

The expressions of $A(0)A^*(0)$ and $\Pi^{\alpha\beta}(A_{\alpha\beta}A^*(0) + A^*_{\alpha\beta}A(0))$ can be written in terms of some dimensionless variables $z_i = 2E_i/\sqrt{s}$, $\vec{q}_i = 2\vec{p}_i/\sqrt{s}$, $x_i = \cos\theta_i$, and $\delta_i = 2m_i/\sqrt{s}$. Here $\sqrt{s}$ is the total energy in the center of mass frame, p_3^μ, p_4^μ, p_5^μ are the four-momenta of the final state J/ψ, charm quark and anticharm quark, respectively, $m_i^2 = p_i^2$, and θ_i is the angle between state i and the electron. The scalar productions are expressed as follows:

$$p_1 \cdot p_3 = \frac{s}{4}(z_3 - q_3 x_3); \qquad p_2 \cdot p_3 = \frac{s}{4}(z_3 + q_3 x_3); \qquad p_4 \cdot p_5 = \frac{s}{8}(4 - 4z_3 + \delta_3^2 - \delta_4^2 - \delta_5^2);$$

$$p_1 \cdot p_4 = \frac{s}{4}(z_4 - (q_- x_- - q_3 x_3)/2); \qquad p_2 \cdot p_4 = \frac{s}{4}(z_4 + (q_- x_- - q_3 x_3)/2);$$

$$p_1 \cdot p_5 = \frac{s}{4}(z_5 + (q_- x_- + q_3 x_3)/2); \qquad p_2 \cdot p_5 = \frac{s}{4}(z_5 - (q_- x_- + q_3 x_3)/2); \qquad p_3 \cdot p_4 = \frac{s}{8}(4 - \delta_3^2 - 4z_5);$$

$$p_3 \cdot p_5 = \frac{s}{8}(4 - \delta_3^2 - 4z_4); \qquad p_1 \cdot p_2 = \frac{s}{2}; \quad (26)$$

where
$$z_- = z_4 - z_5, \qquad q_- = |\vec{q}_4 - \vec{q}_5| =$$
$$\sqrt{4 - 4z_3 + z_-^2 + \delta_3^2 - 2\delta_4^2 - 2\delta_5^2}, \qquad q_3 = |\vec{q}_3| =$$
$$\sqrt{z_3 - \delta_3^2}, x_- = \cos\theta_-,$$ and θ_- is the angle between $\vec{q}_-$ and the electron.

B. Numerical result

In principle, the phase space calculation also contains relativistic corrections (e.g., by inclusion of the binding energy in a meson). But for simplicity, we calculate the phase space integration by assuming $m_{J/\psi} = 2m_c$ (the effect due to this simplification is very small). Then the three-body final state phase space expressed in terms of the variables defined above becomes

$$d\Phi_3 = (2\pi)^4 \delta^4(p1 + p2 - p3 - p4 - p5) \prod_{i=3}^{5} \frac{d^3 p_i}{2E_i}$$
$$= \frac{s}{32(2\pi)^4} \frac{dz_3 dx_3 dz_- dw}{\sqrt{(1 - K^2)(1 - x_3^2) - w^2}}, \quad (27)$$

where

$$K = \frac{z_-(2 - z_3)}{q_3 q_-}, \quad (28a)$$

$$w = x_- + Kx_3. \quad (28b)$$

The limits of those variables are

$$\delta \le z_3 \le 1, \quad (29a)$$

$$-1 \le x_3 \le 1, \quad (29b)$$

$$-\sqrt{\frac{(z_3^2 - \delta^2)(4 - 4z_3)}{4 + \delta^2 - 4z_3}} \le z_- \le \sqrt{\frac{(z_3^2 - \delta^2)(4 - 4z_3)}{4 + \delta^2 - 4z_3}}, \quad (29c)$$

$$-\sqrt{(1 - K^2)(1 - x_3^2)} \le w \le \sqrt{(1 - K^2)(1 - x_3^2)}, \quad (29d)$$

where $\delta = 4m_c/\sqrt{s}$.

The expressions of M_0 and M_1 are lengthy, so we only give the expression for the differential cross section. The differential cross section of unpolarized J/ψ production in association with $c\bar{c}$ in $e^+ e^-$ annihilation can be expressed as

$$\frac{d^2\sigma}{dE_3 d\cos\theta_3}(e^+ e^- \to \gamma^* \to \psi + X) = S(E_3)(1 + \alpha\cos^2\theta_3), \quad (30)$$

where E_3 is the energy of J/ψ, and θ_3 is the angle between J/ψ and the electron. When expressing the above equation by using the new parameters, it becomes

$$\frac{d^2\sigma}{dz_3 dx_3}(e^+ e^- \to \gamma^* \to \psi + X) = S(z_3)(1 + \alpha(z_3)x_3^2). \quad (31)$$
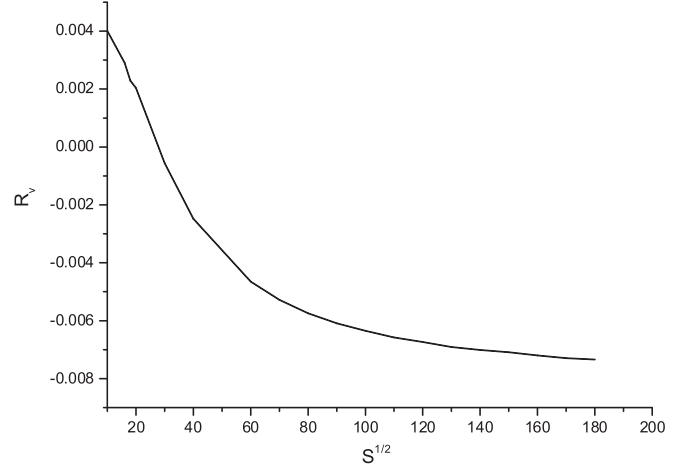

FIG. 5. R_v as a function of $\sqrt{s}$. Here R_v is the ratio of the correction at order v^2 to the leading-order result for the cross section of $e^+ + e^- \to J/\psi + c\bar{c}$.

The leading-order result of $S(z_3)$ and $\alpha(z_3)$ are in agreement with those in [17,21]. We give the next-to-leading-order result of $S(z_3)$ and $\alpha(z_3)$ in the Appendix. Using the long-distance matrix elements in Eq. (18) (without NLO α_s corrections in the charmonium decay widths), the leading-order cross section of $e^+ + e^- \to J/\psi + c\bar{c}$ with $m_c = 1.5$ GeV, $\sqrt{s} = 10.6$ GeV, $\alpha_s = 0.26$ is estimated to be 110 fb,[2] and the next-to-leading-order v^2 correction is only 0.42 fb, which gives about less than a half percent enhancement. As a result, the relativistic correction to $e^+ + e^- \to J/\psi + c\bar{c}$ is found to be very small and negligible, in contrast to the exclusive double charmonium production process $e^+ + e^- \to J/\psi + \eta_c$, in which the relativistic correction is very significant. In fact, we find that in $e^+ + e^- \to J/\psi + c\bar{c}$ the short-distance part $\bar{N}_1 m_c^2$ at order v^2 is much smaller than the leading-order term $\bar{N}_0$, i.e., $\frac{\bar{N}_1 m_c^2}{\bar{N}_0} \ll 1$. Here $\bar{N}_1$ in Eq. (25b) is the sum of two terms, and they both are small and have different signs, so their sum becomes even tiny. So, the tiny effect of relativistic corrections on the rate of $e^+ + e^- \to J/\psi + c\bar{c}$ is due to the smallness of the short-distance coefficient correction, regardless of the long-distance matrix elements.

When $\sqrt{s}$ is larger, the ratio of the correction at order v^2 to the leading-order contribution even changes sign from positive to negative values, but its value is always small. This is shown in Fig. 5, where the values of the parameters are the same as used above.

If we use the enhanced matrix elements in Eq. (20), which are obtained by including NLO α_s corrections in the charmonium decay widths, for $m_c = 1.5$ GeV, $\sqrt{s} = 10.6$ GeV, $\alpha_s = 0.26$, the leading-order result is about

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[2]This value is smaller than 148 fb given in [21], because here a smaller value of the wave function squared at the origin is extracted from the J/ψ data than that used in [21] from potential model calculations.

214 fb, and the relativistic correction at order v^2 is only 067 fb.

C. Discussions

Since for $e^+ + e^- \rightarrow J/\psi + c\bar{c}$ the conflict between experiment and theory at leading order in both α_s and v^2 is quite serious, a number of attempts have been made to solve this problem. In [22] the authors considered the production of $J/\psi + c\bar{c}$ through two photons, and found that the contribution of the two-photon process may be comparable to that of one photon process when $\sqrt{s}$ is large, say, larger than $\sqrt{s} = 20$ GeV, but at $\sqrt{s} = 10.6$ GeV, it is only 15% of the one photon contribution. Authors in [31] considered the process when the mass of the charm quark can be neglected. In [32] the result in the factorization method was compared with the duality method. Other suggestions can also be found in e.g. [33,34]. Despite these efforts, a satisfactory resolution is still needed. Recently, the authors of Ref. [23] have calculated the next-to-leading-order QCD corrections to the direct $J/\psi + c\bar{c} + X$ production in $e^+ + e^-$ annihilation at $\sqrt{s} = 10.6$ GeV, and with $m_c = 1.5$ GeV and $\alpha_s = 0.26$ their result is

$$\sigma_{\text{direct}}(e^+ e^- \rightarrow J/\psi + c\bar{c}) \approx 0.33 \text{ pb.} \qquad (32)$$

It enhances the leading-order result by a factor of 1.8. Further including the contributions from $\psi(2S)$ and other states, the cross section of prompt $J/\psi + c\bar{c}$ production becomes

$$\sigma_{\text{prompt}}(e^+ e^- \rightarrow J/\psi + c\bar{c} + X) \approx 0.51 \text{ pb.} \qquad (33)$$

It is about 60% of the Belle value in Eq. (2). This result shows that the QCD radiative corrections to $e^+ + e^- \rightarrow J/\psi + c\bar{c}$ are essential.

Here, when we further consider the relativistic corrections at order v^2, we find that in contrast to the $J/\psi + \eta_c$ production, the relativistic corrections to $J/\psi + c\bar{c}$ production are very small. The relativistic corrections can only improve the leading-order result by enlarging the long-distance matrix elements, and have little effect on the short-distance part. As a result, the relativistic corrections to $e^+ + e^- \rightarrow J/\psi + c\bar{c}$ are tiny and negligible.

The physical reason for the difference between exclusive and inclusive processes largely lies in the fact that, in the exclusive process $e^+ e^- \rightarrow J/\psi + \eta_c$, the virtuality of the gluon that converts into a charm quark pair takes its maximum value $\frac{s}{4}$ in the nonrelativistic limit (with zero relative momentum between quarks in charmonium), and introducing the relative momentum can substantially reduce the gluon virtuality, and hence enhance the short-distance coefficient and the cross section; whereas for the inclusive process $e^+ + e^- \rightarrow J/\psi + c\bar{c}$, the charm quark fragmentation $c \rightarrow J/\psi + c$ ($\bar{c} \rightarrow J/\psi + \bar{c}$) is significant, in which the quark relative momentum has little effect on the virtuality of the gluon. Therefore, our result for the next-to-leading v^2 corrections may indicate that the rela-

tivistic correction is not a good direction to solve the problem of discrepancy between theory and experiment in $J/\psi + c\bar{c}$ production. Further studies for QCD radiative corrections and other possible new mechanisms related to the double-charm production could be more useful.

V. SUMMARY

In this paper, we have studied the relativistic corrections at order v^2 to the double-charm production processes $e^+ e^- \rightarrow J/\psi + \eta_c$ and $e^+ e^- \rightarrow J/\psi + c\bar{c}$ at B factories in the framework of nonrelativistic quantum chromodynamics. The short-distance parts of production cross sections are calculated perturbatively, while the long-distance matrix elements are extracted from experimental data for J/ψ and η_c decays up to errors of order v^4, and therefore are model independent. The main uncertainties may come from the higher order corrections and the errors in the measurements. Our results show that the relativistic correction to the exclusive process $e^+ e^- \rightarrow J/\psi + \eta_c$ is significant, which, when combined together with the next-to-leading-order α_s corrections, could resolve the large discrepancy between theory and experiment. It can be clearly seen from Table I that for the cross section the relativistic correction alone gives an enhancement factor of 1.7 while the combination of relativistic correction with QCD radiative correction results in a much larger enhancement factor of 9. This conclusion is consistent with [16], in which, however, the long-distance matrix elements are estimated by using potential model calculations. The relativistic effects on the $J/\psi \eta_c$ production estimated in our approach are milder than some results obtained by using the light-cone methods. On the other hand, for the inclusive process $e^+ e^- \rightarrow J/\psi + c\bar{c}$ we find that the relativistic correction is tiny and negligible, and therefore not helpful in resolving the discrepancy between theory and experiment. The physical reason for the above difference between exclusive and inclusive processes largely lies in the fact that in the exclusive process the relative momentum between quarks in charmonium substantially reduces the virtuality of the gluon that converts into a charm quark pair, but this is not the case for the inclusive process, in which the charm quark fragmentation $c \rightarrow J/\psi + c$ ($\bar{c} \rightarrow J/\psi + \bar{c}$) is significant, and QCD radiative corrections can be more essential. Further studies are needed to understand the large ratio of the cross section of $e^+ e^- \rightarrow J/\psi + c\bar{c}$ to the cross section of $e^+ e^- \rightarrow J/\psi + $ anything, which is measured to be in the range of 0.6–0.8 by Belle.

APPENDIX

The formulas of $S(z_3)$ and $\alpha(z_3)$ are given as follows:

$$
S(z_3) = \frac{2e_c^2\pi\alpha^2\alpha_s^2}{3^5 s^3 \delta^4 z_3^4 \sqrt{1-z_3}(2-z_3)^6(z_3^2-\delta^2)} \frac{\langle 0|\mathcal{P}_1(^3S_1^\psi)|0\rangle}{m_c} \Big(-(98\,304z_3^5 + 98\,304z_3^6 + 696\,320z_3^7 - 6\,963\,200z_3^8
$$

$$
+ 14\,895\,104z_3^9 - 13\,801\,472z_3^{10} + 6\,104\,576z_3^{11} - 1\,221\,632z_3^{12} + 93\,696z_3^{13} - 737\,280z_3^3\delta^2
$$

$$
+ 3\,047\,424z_3^4\delta^2 - 6\,987\,776z_3^5\delta^2 + 15\,003\,648z_3^6\delta^2 - 22\,528\,000z_3^7\delta^2 + 17\,085\,440z_3^8\delta^2 - 4\,745\,728z_3^9\delta^2
$$

$$
- 411\,904z_3^{10}\delta^2 + 299\,584z_3^{11}\delta^2 - 26\,176z_3^{12}\delta^2 + 737\,280z_3\delta^4 - 4\,030\,464z_3^2\delta^4 + 7\,950\,336z_3^3\delta^4
$$

$$
- 7\,868\,416z_3^4\delta^4 + 3\,901\,440z_3^5\delta^4 + 2\,526\,208z_3^6\delta^4 - 6\,324\,224z_3^7\delta^4 + 3\,657\,984z_3^8\delta^4 - 540\,096z_3^9\delta^4
$$

$$
- 42\,816z_3^{10}\delta^4 + 3632z_3^{11}\delta^4 + 442\,368\delta^6 - 1\,124\,352z_3\delta^6 + 387\,072z_3^2\delta^6 + 948\,224z_3^3\delta^6 - 1\,466\,368z_3^4\delta^6
$$

$$
+ 1\,291\,008z_3^5\delta^6 - 68\,864z_3^6\delta^6 - 493\,440z_3^7\delta^6 + 99\,136z_3^8\delta^6 + 34\,552z_3^9\delta^6 - 664z_3^{10}\delta^6 + 211\,968\delta^8
$$

$$
- 489\,984z_3\delta^8 + 357\,888z_3^2\delta^8 - 14\,336z_3^3\delta^8 - 365\,056z_3^4\delta^8 + 357\,696z_3^5\delta^8 - 42\,464z_3^6\delta^8 - 23\,104z_3^7\delta^8
$$

$$
- 7356z_3^8\delta^8 - 138z_3^9\delta^8 + 39\,168\delta^{10} - 93\,696z_3\delta^{10} + 95\,232z_3^2\delta^{10} - 20\,224z_3^3\delta^{10} - 52\,576z_3^4\delta^{10}
$$

$$
+ 23\,392z_3^5\delta^{10} + 3904z_3^6\delta^{10} + 432z_3^7\delta^{10} + 45z_3^8\delta^{10} + 3456\delta^{12} - 9216z_3\delta^{12} + 8352z_3^2\delta^{12} + 384z_3^3\delta^{12}
$$

$$
- 2520z_3^4\delta^{12} + 192z_3^5\delta^{12} - 90z_3^6\delta^{12})\frac{4z_3(1-z_3)}{3(2-z_3)^2}\sqrt{\frac{z_3^2-\delta^2}{\delta^2-4z_3+4}} - 192(256 - 512z_3 + 320z_3^2 - 64z_3^3
$$

$$
+ 16z_3^4 - 16z_3^5 + 4z_3^6 - 128\delta^2 + 64z_3\delta^2 + 240z_3^2\delta^2 - 224z_3^3\delta^2 + 20z_3^4\delta^2 + 16z_3^5\delta^2\zeta + 48\delta^4 - 96z_3\delta^4
$$

$$
+ 24z_3^2\delta^4 + 24z_3^3\delta^4 + 3z_3^4\delta^4)\delta^4 z_3^4(1-z_3)\arctan\frac{z_3^2-\delta^2}{\delta^2-4z_3+4}(8192z_3^6 - 102\,400z_3^7 + 262\,144z_3^8 - 282\,624z_3^9
$$

$$
+ 146\,944z_3^{10} - 35\,584z_3^{11} + 3072z_3^{12} + 69\,632z_3^3\delta^2 - 299\,008z_3^4\delta^2 + 542\,720z_3^5\delta^2 - 564\,224z_3^6\delta^2
$$

$$
+ 331\,264z_3^7\delta^2 - 87\,808z_3^8\delta^2 + 1664z_3^9\delta^2 + 1472z_3^{10}\delta^2 + 336z_3^{11}\delta^2 - 61\,440z_3\delta^4 + 299\,008z_3^2\delta^4
$$

$$
- 486\,400z_3^3\delta^4 + 388\,096z_3^4\delta^4 - 180\,480z_3^5\delta^4 + 65\,280z_3^6\delta^4 - 19\,008z_3^7\delta^4 - 2880z_3^8\delta^4 + 3264z_3^9\delta^4
$$

$$
- 288z_3^{10}\delta^4 - 36\,864\delta^6 + 72\,192z_3\delta^6 - 16\,896z_3^2\delta^6 - 64\,256z_3^3\delta^6 + 65\,664z_3^4\delta^6 - 15\,360z_3^5\delta^6 - 6240z_3^6\delta^6
$$

$$
+ 304z_3^7\delta^6 + 2216z_3^8\delta^6 + 6z_3^9\delta^6 - 8448\delta^8 + 14\,336z_3\delta^8 - 6144z_3^2\delta^8 - 4864z_3^3\delta^8 + 1440z_3^4\delta^8 + 2624z_3^5\delta^8
$$

$$
- 1056z_3^6\delta^8 + 15z_3^8\delta^8 - 1152\delta^{10} + 3072z_3\delta^{10} - 2592z_3^2\delta^{10} + 1344z_3^3\delta^{10} - 696z_3^4\delta^{10} + 240z_3^5\delta^{10}
$$

$$
- 30z_3^6\delta^{10})\delta^2\sqrt{1-z_3}\ln\frac{z_3\sqrt{\delta^2-4z_3+4} + 2\sqrt{(1-z_3)(z_3^2-\delta^2)}}{z_3\sqrt{\delta^2-4z_3+4} - 2\sqrt{(1-z_3)(z_3^2-\delta^2)}}\Big), \tag{A1}
$$

$$\alpha(z_3)S(z_3) = \frac{2e_c^2\pi\alpha^2\alpha_s^2}{3^5 s^3 \delta^4 z_3^4 \sqrt{1-z_3}(2-z_3)^6(z_3^2-\delta^2)} \frac{\langle 0|\mathcal{P}_1(^3S_1^\psi)|0\rangle}{m_c}\Big(-(98\,304 z_3^7 + 98\,304 z_3^8 + 696\,320 z_3^9 - 6\,963\,200 z_3^{10}$$

$$+ 14\,895\,104 z_3^{11} - 13\,801\,472 z_3^{12} + 6\,104\,576 z_3^{13} - 1\,221\,632 z_3^{14} + 93\,696 z_3^{15} - 835\,584 z_3^5 \delta^2$$

$$+ 3\,342\,336 z_3^6 \delta^2 - 11\,059\,200 z_3^7 \delta^2 + 31\,518\,720 z_3^8 \delta^2 - 49\,162\,240 z_3^9 \delta^2 + 37\,194\,752 z_3^{10} \delta^2$$

$$- 11\,941\,888 z_3^{11}\delta^2 + 816\,896 z_3^{12}\delta^2 + 153\,152 z_3^{13}\delta^2 - 26\,176 z_3^{14}\delta^2 - 1\,867\,776 z_3^3\delta^4 + 6\,881\,280 z_3^4\delta^4$$

$$- 4\,657\,152 z_3^5\delta^4 - 19\,660\,800 z_3^6\delta^4 + 43\,751\,424 z_3^7\delta^4 - 3\,1881\,216 z_3^8\delta^4 + 6\,889\,984 z_3^9\delta^4 + 366\,080 z_3^{10}\delta^4$$

$$+ 119\,808 z_3^{11}\delta^4 + 31\,744 z_3^{12}\delta^4 - 976 z_3^{13}\delta^4 + 245\,760 z_3\delta^6 - 1\,622\,016 z_3^2\delta^6 + 3\,229\,696 z_3^3\delta^6$$

$$- 1\,411\,072 z_3^4\delta^6 + 2\,289\,664 z_3^5\delta^6 - 12\,958\,720 z_3^6\delta^6 + 16\,642\,816 z_3^7\delta^6 - 7\,645\,696 z_3^8\delta^6 + 1\,814\,848 z_3^9\delta^6$$

$$- 524\,672 z_3^{10}\delta^6 + 16\,520 z_3^{11}\delta^6 - 1240 z_3^{12}\delta^6 + 147\,456\delta^8 - 522\,240 z_3\delta^8 + 967\,680 z_3^2\delta^8 - 2\,955\,776 z_3^3\delta^8$$

$$+ 6\,496\,768 z_3^4\delta^8 - 5\,705\,984 z_3^5\delta^8 + 1\,570\,816 z_3^6\delta^8 - 345\,536 z_3^7\delta^8 + 177\,184 z_3^8\delta^8 + 68\,040 z_3^9\delta^8$$

$$+ 3420 z_3^{10}\delta^8 + 294 z_3^{11}\delta^8 + 46\,080\delta^{10} + 10\,752 z_3\delta^{10} - 254\,208 z_3^2\delta^{10} + 92\,672 z_3^3\delta^{10} + 144\,640 z_3^4\delta^{10}$$

$$+ 86\,976 z_3^5\delta^{10} + 9920 z_3^6\delta^{10} - 81\,632 z_3^7\delta^{10} - 996 z_3^8\delta^{10} - 2070 z_3^9\delta^{10} + 45 z_3^{10}\delta^{10} + 16\,128\delta^{12}$$

$$- 18\,432 z_3\delta^{12} - 15\,744 z_3^2\delta^{12} + 65\,408 z_3^3\delta^{12} - 116\,416 z_3^4\delta^{12} + 61\,312 z_3^5\delta^{12} - 4136 z_3^6\delta^{12} + 1800 z_3^7\delta^{12}$$

$$+ 45 z_3^8\delta^{12} + 3456\delta^{14} - 9216 z_3\delta^{14} + 8352 z_3^2\delta^{14} + 384 z_3^3\delta^{14} - 2520 z_3^4\delta^{14} + 192 z_3^5\delta^{14}$$

$$- 90 z_3^6\delta^{14})\frac{4z_3(1-z_3)}{3(2-z_3)^2}\sqrt{\frac{z_3^2-\delta^2}{\delta^2-4z_3+4}} + 192(256 - 512 z_3 + 320 z_3^2 - 64 z_3^3 + 16 z_3^4 - 16 z_3^5 + 4 z_3^6 - 128\delta^2$$

$$+ 64 z_3\delta^2 + 240 z_3^2\delta^2 - 224 z_3^3\delta^2 + 20 z_3^4\delta^2 + 16 z_3^5\delta^2 + 48\delta^4 - 96 z_3\delta^4 + 24 z_3^2\delta^4 + 24 z_3^3\delta^4$$

$$+ 3 z_3^4\delta^4)\delta^4 z_3^4(1-z_3)\arctan\frac{z_3^2-\delta^2}{\delta^2-4z_3+4} + (32\,768 z_3^7 - 90\,112 z_3^8 + 20\,480 z_3^9 + 180\,224 z_3^{10} - 251\,904 z_3^{11}$$

$$+ 140\,800 z_3^{12} - 35\,072 z_3^{13} + 3072 z_3^{14} - 258\,048 z_3^5\delta^2 + 806\,912 z_3^6\delta^2 - 845\,824 z_3^7\delta^2 + 87\,040 z_3^8\delta^2$$

$$+ 484\,864 z_3^9\delta^2 - 382\,208 z_3^{10}\delta^2 + 116\,096 z_3^{11}\delta^2 - 12\,608 z_3^{12}\delta^2 - 48 z_3^{13}\delta^2 + 49\,152 z_3^3\delta^4 - 286\,720 z_3^4\delta^4$$

$$+ 642\,048 z_3^5\delta^4 - 845\,824 z_3^6\delta^4 + 871\,680 z_3^7\delta^4 - 684\,544 z_3^8\delta^4 + 341\,824 z_3^9\delta^4 - 90\,112 z_3^{10}\delta^4 + 9712 z_3^{11}\delta^4$$

$$- 288 z_3^{12}\delta^4 - 20\,480 z_3\delta^6 + 122\,880 z_3^2\delta^6 - 289\,280 z_3^3\delta^6 + 453\,120 z_3^4\delta^6 - 582\,144 z_3^5\delta^6 + 489\,856 z_3^6\delta^6$$

$$- 242\,112 z_3^7\delta^6 + 64\,864 z_3^8\delta^6 - 6352 z_3^9\delta^6 - 1336 z_3^{10}\delta^6 - 138 z_3^{11}\delta^6 - 12\,288\delta^8 + 36\,352 z_3\delta^8$$

$$- 70\,400 z_3^2\delta^8 + 136\,448 z_3^3\delta^8 - 160\,640 z_3^4\delta^8 + 127\,616 z_3^5\delta^8 - 62\,400 z_3^6\delta^8 + 12\,272 z_3^7\delta^8 + 1256 z_3^8\delta^8$$

$$- 6 z_3^9\delta^8 + 15 z_3^{10}\delta^8 - 768\delta^{10} - 10\,752 z_3\delta^{10} + 29\,568 z_3^2\delta^{10} - 36\,480 z_3^3\delta^{10} + 18\,368 z_3^4\delta^{10} - 2144 z_3^5\delta^{10}$$

$$- 472 z_3^6\delta^{10} - 24 z_3^7\delta^{10} + 15 z_3^8\delta^{10} - 1152\delta^{12} + 3072 z_3\delta^{12} - 2592 z_3^2\delta^{12} + 1344 z_3^3\delta^{12} - 696 z_3^4\delta^{12}$$

$$+ 240 z_3^5\delta^{12} - 30 z_3^6\delta^{12})\delta^2\sqrt{1-z_3}\ln\frac{z_3\sqrt{\delta^2-4z_3+4} + 2\sqrt{(1-z_3)(z_3^2-\delta^2)}}{z_3\sqrt{\delta^2-4z_3+4} - 2\sqrt{(1-z_3)(z_3^2-\delta^2)}}\Big). \tag{A2}$$

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