Pair creation by a photon in a strong magnetic field

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The process of pair creation by a photon in a strong magnetic field is investigated based on the polarization operator in the field. The total probability of the process is calculated in a relatively simple form using an adequate method. The new results are obtained in the scope of the quasiclassical approach in the case when the magnetic field $B \ll B_0$ (B_0 is the critical field). This is the new formulation of the approach which extends its validity to the case when the created particles are not ultrarelativistic. The correction to the standard quasiclassical approximation is found as well, showing the range of applicability of the approach at high photon energy. The very important conclusion is that, for both cases $B \ll B_0$ and $B \geq B_0$, the results of the quasiclassical calculation are very close to averaged probabilities of exact theory in a very wide range of photon energies. The quasiclassical approximation is valid also for the energy distribution if the electron and the positron are created on high enough levels even for $B \geq B_0$.

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I. INTRODUCTION

The study of QED processes in a strong external field is stimulated essentially by the existence of very strong magnetic fields in nature. There is a series of evidences for the existence of neutron stars possessing magnetic fields close to the critical field strength $B_0 = m^2/e = (m^2c^3/e\hbar) = 4.41 \times 10^{13}$ G. The rotating magnetic dipole model, in which the pulsar loses rotational energy through the magnetic dipole radiation, was confirmed with the discovery that the spin-down power predicted for many pulsars is in quite good energetic agreement with the observed radiation, and gives magnetic fields $B \sim 10^{11}$–10^{13} G [1]. There are presently around 1600 spin-powered radio pulsars.

Another class of neutron stars was discovered at x-ray and γ-ray energies and may possess even stronger surface magnetic fields $B \sim 10^{14}$–10^{15} G. Such stars are now referred to as magnetars [2], since they most probably derive their power from their magnetic fields rather than from spin-down energy loss. Two different classes of objects are thought to be magnetars: the soft gamma-ray repeaters (SGR), discovered as sources of short bursts of hard x-rays (>30 keV) with super-Eddington luminosity, and the anomalous x-ray pulsars (APS), discovered as persistent, soft (<10 keV) x-ray sources with pulsations of several seconds and spinning down on time scales of 10^4–10^5 yr. An increasing number of common properties, pointing to a close relationship of these two apparently different classes of objects and leading to their interpretation as magnetars, has been found (see e.g. [3]).

Pair creation by a photon in a strong magnetic field is the basic QED reaction which can play the significant role in processes in the vicinity of a neutron star.

Investigation of pair creation by a photon in a strong magnetic field was started in 1952 independently by Klepikov and Toll [4,5]. In Klepikov's paper [6], which was based on the solution of the Dirac equation in a constant and homogeneous magnetic field, the probabilities of radiation from an electron and e^-e^+ pair creation by a photon were obtained for a magnetic field of arbitrary strength on the mass shell[1] ($k^2 = 0$, k is the 4-momentum of photon). In 1971, Adler [7] calculated the photon polarization operator in the mentioned magnetic field using the proper-time technique developed by Schwinger [8], and Batalin and Shabad [9] calculated the photon polarization operator in a constant and homogeneous electromagnetic field for $k^2 \neq 0$ using the Green function in this field found by Schwinger [8]. In 1975, Strakhovenko and the present authors calculated the contribution of a charged-particles loop with n external photon lines having applied the proper-time method in a constant and homogeneous electromagnetic field [10]. The explicit expression for the photon polarization operator was found. In a purely magnetic field ($\mathbf{E} = 0$), the polarization operator in the general case ($k^2 \neq 0$) was analyzed in detail near the root singularities originating at each threshold of e^-e^+ pair creation in the state with the given Landau quantum numbers. This peculiarity of the pair creation process was observed by Klepikov [6]. Shabad [11] performed an extensive analysis of the photon polarization operator found in [9] in a purely magnetic field. It was shown that its imaginary part in the limit $k^2 \to 0$ reduces to the result of Klepikov's calculation [6]. Daugherty and Harding [12] analyzed in detail the total probabilities (the attenuation coefficients) of the pair creation by a polarized photon in the superstrong field $B \sim B_0$ based on the calculations of Toll and Klepikov and paying special attention to the threshold region $\omega \geq 2m$, where the probabilities exhibit a "sawtooth" pattern due to the mentioned root singularities.

For high Landau levels, Klepikov [6] (and Toll) found the quasiclassical representations for the probabilities of radiation and pair creation in the case $B \ll B_0$. In 1968, the

[1] We use the system of units with $\hbar = c = 1$ and the metric $ab = a^\mu b_\mu = a^0 b^0 - \mathbf{ab}$.

present authors developed the operator quasiclassical method for the consideration of electromagnetic processes in an external field [13] which is valid also in a nonuniform field and (or) depending on time field under conditions: (i) $B \ll B_0$ and (ii) the charged particles have relativistic energies. Both the radiation and pair creation by a photon were considered. The method is given in detail in [14,15]. Tsai and Erber [16] deduced the quasiclassical probability of pair creation using the photon polarization operator in a magnetic field calculated by Adler [7].

In Sec. II, the exact probability of the pair creation by a photon was obtained for the general case $k^2 \neq 0$ starting from the polarization operator in a magnetic field. The probability exhibits a sawtooth pattern because of divergences arising when the electron and the positron are created at the threshold of the Landau energy levels. The pattern will be washed out at averaging over any smooth photon energy distribution. The technical details of the derivation are given in Appendix A. In Sec. III the standard quasiclassical approximation (SQA) is outlined in the limit $r \equiv \omega^2/4m^2 \gg 1$, where ω is the photon energy. The quasiclassical characteristics of the process depend on the parameter $\kappa = 2\mu\sqrt{r}$. The correction to SQA is found. Details of the calculation of the correction starting from the polarization operator in a magnetic field is given in Appendix C. For $\mu \equiv B/B_0 \ll 1$ and not enough high photon energy, where the SQA is not applicable, the new quasiclassical approach for low energy near threshold (th) is developed in Appendix B, valid under the condition $\mu \ll r - 1 \ll \mu^{-2}$. One can expect that the photon energy distribution in the vicinity of a neutron star is wide and smooth. At averaging over this distribution, the sawtooth pattern will be washed out. It is very important that the result of the averaging is very close to SQA even at $\mu \geq 1$. The corresponding analysis has been carried out in Sec. III. The energy spectrum of the created pair is discussed in Sec. IV. Here SQA is very useful also. A remark concerning derivation of the energy spectrum in SQA is given in Appendix D.

II. PROBABILITY OF PAIR CREATION BY A PHOTON: EXACT THEORY

Our analysis is based on the expression for the polarization operator obtained in [10], see Eqs. (3.19), (3.33). For a pure magnetic field ($\mathbf{E} = 0$), this polarization operation can be written in the diagonal form:

$$\Pi^{\mu\nu} = -\sum_i \kappa_i \beta_i^\mu \beta_i^\nu, \qquad \beta_i \beta_j = -\delta_{ij}, \qquad \beta_i k = 0,$$
$$\sum_i \beta_i^\mu \beta_i^\nu = \frac{k^\mu k^\nu}{k^2} - g^{\mu\nu}, \tag{2.1}$$

where

$$\beta_1^\mu = \frac{k^2 k_\perp^\mu + \mathbf{k}_\perp^2 k^\mu}{k_\perp \sqrt{k^2(\omega^2 - k_3^2)}}, \qquad \beta_2^\mu = \frac{F^{\mu\nu}k_\nu}{Bk_\perp},$$
$$\beta_3^\mu = \frac{F^{*\mu\nu}k_\nu}{B\sqrt{\omega^2 - k_3^2}}, \qquad \kappa_1 = \Omega_1(r-q),$$
$$\kappa_2 = \kappa_1 - q\Omega_2, \qquad \kappa_3 = \kappa_1 + r\Omega_3, \tag{2.2}$$
$$r = \frac{\omega^2 - k_3^2}{4m^2}, \qquad q = \frac{k_\perp^2}{4m^2}, \qquad r - q = \frac{k^2}{4m^2},$$

where the axis 3 is directed along the magnetic field $\mathbf{B}$, $\mathbf{k}_\perp \mathbf{B} = 0$, $k_\perp = \sqrt{\mathbf{k}_\perp^2}$, ω is the photon energy, $F^{\mu\nu}$ is the tensor of electromagnetic field, $F^{*\mu\nu}$ is the dual tensor of electromagnetic field, and

$$\Omega_i = -\frac{\alpha m^2}{\pi}\int_{-1}^{1} dv \int_0^{\infty - i0} f_i(v,x)\exp(i\psi(v,x))dx. \tag{2.3}$$

Here

$$f_1(v,x) = \frac{\cos vx}{\sin x} - v\frac{\cos x \sin vx}{\sin^2 x},$$
$$f_2(v,x) = 2\frac{\cos vx - \cos x}{\sin^3 x} - f_1(v,x), \tag{2.4}$$
$$f_3(v,x) = (1 - v^2)\cot x - f_1(v,x),$$
$$\psi(v,x) = z\frac{\cos x - \cos vx}{\sin x} - b(v)x,$$

where

$$z = \frac{2q}{\mu}, \qquad b(v) = \frac{1 - r(1 - v^2)}{\mu}, \qquad \mu = \frac{B}{B_0}. \tag{2.5}$$

Let us note that the integration contour in Eq. (2.3) is turned slightly down, and in the function Ω_1 in the integral over x the subtraction at $\mu = 0$ is implied.

The real part of the polarization operator defines the dispersive properties of the space region with magnetic field. At $r < 1$ ($b > 0$), one can turn the integration contour over x to the negative imaginary axis ($x \to -ix$) so that

$$\Omega_i = \frac{\alpha m^2}{\pi}\int_{-1}^{1} dv \int_0^{\infty} if_i(v,-ix) \times \exp\left(-bx - z\frac{\cosh x - \cosh vx}{\sinh x}\right)dx. \tag{2.6}$$

The functions Ω_i in Eq. (2.6) are real. These functions, studied in [17] at $\omega < 2m$ and arbitrary value of μ, define the index of refraction $n_{2,3}$. On mass shell $k^2 = 0$, one has

$$n_{2,3}^2 = 1 - \frac{\kappa_{2,3}}{\omega^2}. \tag{2.7}$$

For the mode 2, the photon polarization vector $\mathbf{e}_2$ is perpendicular to the plane formed by the vectors $\mathbf{B}$ and $\mathbf{k}$ and for mode 3 the photon polarization vector $\mathbf{e}_3$ lies in this plane ($n_2 \to n_\parallel$ and $n_3 \to n_\perp$ in Adler's notation [7]).

At $r > 1$, the polarization operator eigenvalue κ_i acquires an imaginary part, which determines the probability of the $e^- e^+$ pair creation per unit length W_i by a photon with a given polarization,

$$W = \frac{1}{\omega} \mathrm{Im} e^\mu e^{\nu *} \Pi_{\mu\nu}. \tag{2.8}$$

Details of calculation based on Eqs. (2.2) and (2.3) are given in Appendix A. Using $e_i^\mu = \beta_i^\mu$, we get the explicit expressions for $W_i (i = 1, 2, 3)$ at $k^2 \neq 0$:

$$W_i = -\frac{\mathrm{Im}\kappa_i}{\omega} = \frac{2\alpha m\mu e^{-z}}{\omega\lambda_c\sqrt{r}} \sum_{n,m} \left(1 - \frac{\delta_{n0}}{2}\right) \vartheta(g) \frac{z^n k!}{\sqrt{g} l!} d_i, \tag{2.9}$$

where δ_{n0} is the Kronecker symbol, $\vartheta(g)$ is the Heaviside function: $\vartheta(g) = 1$ for $g \geq 0$, $\vartheta(g) = 0$ for $g < 0$,

$$l = \frac{m+n}{2}, \qquad k = \frac{m-n}{2},$$
$$g = r - 1 - m\mu + \frac{n^2\mu^2}{4r}, \qquad l_{\max} = [d(r)], \tag{2.10}$$
$$d(r) = \frac{2(r - \sqrt{r})}{\mu}.$$

Here $[d(r)]$ is the integer part of $d(r)$,

$$\begin{aligned}
d_1 &= \left(\frac{r}{q} - 1\right) G, \\
d_2 &= \frac{r}{q} G + 4\mu l \vartheta(k-1) L_{k-1}^{n+1}(z) L_k^{n-1}(z), \\
d_3 &= \left(1 + \frac{m\mu}{2} - \frac{n^2\mu^2}{4r}\right) F + 2\mu l \vartheta(k-1) L_k^n(z) L_{k-1}^n(z), \\
G &= \left(\frac{m\mu}{2} - \frac{n^2\mu^2}{4r}\right) F - 2\mu l \vartheta(k-1) L_k^n(z) L_{k-1}^n(z), \\
F &= (L_k^n(z))^2 + \vartheta(k-1) \frac{l}{k} (L_{k-1}^n(z))^2,
\end{aligned} \tag{2.11}$$

where $L_k^n(z)$ is the generalized Laguerre polynomial, z is defined in Eq. (2.5). The obtained probabilities W_i agree with those found by Shabad [11] but are presented in a much more compact form.

On the mass shell ($k^2 = 0$) in Eqs. (2.9), (2.10), and (2.11), one has to put $r = q$. In this case $W_1 = 0$ and only two polarizations $i = 2$ and $i = 3$ remain. The probability of a pair creation averaged over the photon polarizations acquires an especially simple form (in the frame where $k_3 = 0$):

$$\begin{aligned}
W &= \frac{W_2 + W_3}{2} = \frac{\alpha e^{-z}}{\lambda_c} \sum_{n,m} \left(1 - \frac{\delta_{n0}}{2}\right) \vartheta(g) \frac{z^{n-1} k!}{\sqrt{g} l!} \\
&\quad \times \left[\left(1 + m\mu - \frac{n^2\mu}{z}\right) F + 4\mu l \vartheta(k-1) \right. \\
&\quad \left. \times L_{k-1}^{n+1}(z) L_k^{n-1}(z)\right].
\end{aligned} \tag{2.12}$$

The probabilities of pair creation by a photon [Eqs. (2.9) and (2.12)] contain the factor $1/\sqrt{g}$. The function $g = 0$, if

$$r = \left(\frac{\varepsilon(l) + \varepsilon(k)}{2m}\right)^2, \qquad \varepsilon(l) = m\sqrt{1 + 2l\mu}, \tag{2.13}$$

where $\varepsilon(l)$ is the energy of a charged particle on the Landau energy level. So the probability diverges when the electron and the positron are created on the Landau levels with electron and positron momentum $p_3 = 0$, see e.g. [6,10]. The origin of the singularity is due to the properties of the space volume in the lowest order of perturbation theory (infinitesimally narrow level). The reason why this singularity is not a pole but a branch point is that motion along a field is not quantized.

III. PROBABILITY OF PAIR CREATION BY A PHOTON: QUASICLASSICAL APPROXIMATION

When the magnetic field is relatively weak ($\mu \ll 1$) and for large photon energy $r - 1 \gg \mu$ ($l_{\max} \gg 1$), the motion of the created particle at high Landau levels can be considered within the scope of the quasiclassical approximation. In this case, the divergences indicated above are situated very often: $\Delta l \sim \Delta r/\mu \sim 1$. On the photon energy scale, the characteristic distance between the pointed out peaks will be $\Delta\omega \sim \omega_0 \sim \mu m^2/\omega$, where ω_0 is the frequency of motion of the created particle in a magnetic field. This frequency defines the distance between energy levels in the mentioned approximation. So the resonance properties of pair creation probability (a sawtooth pattern) become apparent if the width of the energy levels and the effective spread of photon energy are small compared with the ω_0 value. In the opposite case, the spectrum peculiarities will be washed out and the process probability will be quite adequately described by approximate expressions ultimately connected with the quasiclassical character of the created particle motion.

The quasiclassical approximation was developed in [6,13,16], where the process probabilities were derived using the different approaches, just for the weak magnetic field $\mu \ll 1$ and for high energy photons ($r \gg 1$). The approximate expression, for the exact probability of pair creation by a photon found in [6], was obtained there by using the asymptotic of the Laguerre polynomials and passing from the double sum to the integral. The approach in [13] was created for $\mu \ll 1$ and the particles of pair motion was considered to be quasiclassical (and relativistic) from the very beginning; it is applicable for nonuniform (and nonstationary) fields. The pair creation probabilities in the form [Eq. (C6) (see Appendix C)] were found in [14,16]. Below we will call it the standard quasiclassical approximation (SQA).

In Appendix C the corrections to SQA were calculated. For the probability of pair creation averaged over the photon polarizations, the correction is

$$W^{(1)} = \frac{\alpha m^2 \mu^2}{60\sqrt{3}\pi\omega\kappa} \int_{-1}^{1} [2(1 + v^2 - 27z^2)K_{1/3}(z) + 3(7 - v^2)zK_{2/3}(z)] \frac{dv}{1 - v^2}, \tag{3.1}$$

where

$$z = \frac{8}{3(1 - v^2)\kappa}, \qquad \kappa = 2\sqrt{r}\mu. \tag{3.2}$$

At $\kappa \ll 1$, the main contribution is given by the term $\propto z^2$; it is

$$W^{(1)} = \frac{\sqrt{6}\alpha m^2 \mu^2}{5\omega\kappa^2} \exp\left(-\frac{8}{3\kappa}\right), \qquad \frac{W^{(1)}}{W} = -\frac{32\mu^2}{15\kappa^3}. \tag{3.3}$$

It is seen that at $\kappa \ll 1$ the SQA is applicable if $\mu^2/\kappa^3 \ll 1$.

At $r \leq \mu^{-2/3}$ SQA becomes inapplicable. In this region, one can use the process probability found in Appendix B, which is valid if $\mu \ll r - 1 \ll \mu^{-2}$. Using Eq. (B5), we get the following for the probability of pair creation averaged over the photon polarizations:

$$W^{(\mathrm{th})} = \frac{\alpha m^2 \mu}{4\omega} \frac{3r - 1}{\sqrt{r(r - 1)l(r)\beta(r)}} \exp\left(-\frac{\beta(r)}{\mu}\right), \tag{3.4}$$

where

$$l(r) = \ln\frac{\sqrt{r} + 1}{\sqrt{r} - 1}, \qquad \beta(r) = 2\sqrt{r} - (r - 1)l(r). \tag{3.5}$$

For large $r \gg 1$, limited by the condition $r \ll \mu^{-2}$ ($\kappa \ll 1$), we have

$$W^{(\mathrm{th})} \simeq \frac{3\alpha m^2 \mu}{8\omega} \sqrt{\frac{3r}{2}} \exp\left(-\frac{4}{3\mu\sqrt{r}} - \frac{4}{15\mu r^{3/2}}\right) = W^{(\mathrm{SQA})} \exp\left(-\frac{4}{15\mu r^{3/2}}\right). \tag{3.6}$$

At $\kappa^3 \gg \mu^2$, Eq. (3.6) agrees with Eq. (3.3). The last expression has an essentially wider region of applicability than pure $W^{(\mathrm{SQA})}$.

The behavior of pair creation probability for an unpolarized photon at $\mu = 0.1$ ($1/\mu \gg 1$) as a function of r is illustrated in Fig. 1. The probability in sawtooth form is calculated according to Eq. (2.12). It is seen that the characteristic distance between peaks is of the order $\mu = 0.1$. The result of averaging over this interval $(-\mu/2 + r, \mu/2 + r)$ is given by the dashed line which is smoothed out all of peculiarities of original behavior and which is in a very good agreement with the solid curve calculated according to Eq. (3.4).

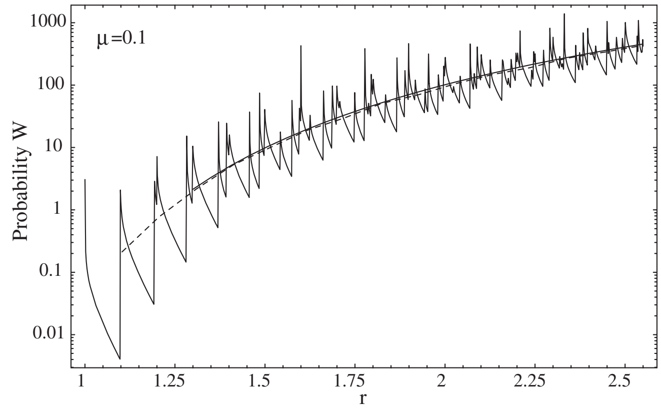

FIG. 1. The pair creation probability for an unpolarized photon at $\mu = 0.1$ as a function of r. The probability in sawtooth form is calculated according to Eq. (2.12). The dashed line is the result of averaging over the interval $(-\mu/2 + r, \mu/2 + r)$. The solid line is calculated according to Eq. (3.4).

At $\kappa \geq 1$ Eq. (3.4) becomes inapplicable. However, in this case SQA is valid. From Eq. (C6), we have the probability of pair creation averaged over the photon polarizations,

$$W^{(\mathrm{SQA})} = \frac{\alpha m^2}{3\sqrt{3}\pi\omega} \int_{-1}^{1} \frac{9 - v^2}{1 - v^2} K_{2/3}(z) dv, \tag{3.7}$$

where z is defined in Eq. (3.2). The correction to this probability is given by Eq. (3.1), its relative value at $\kappa \geq 1$ is $\leq \mu^2$. The ratio $W^{(\mathrm{th})}$ and $W^{(\mathrm{SQA})}$ is shown in Fig. 2. The curves 2 and 3 are for polarized photons; the curve t is for the unpolarized photon [see Eq. (3.4) and (3.7)]. The

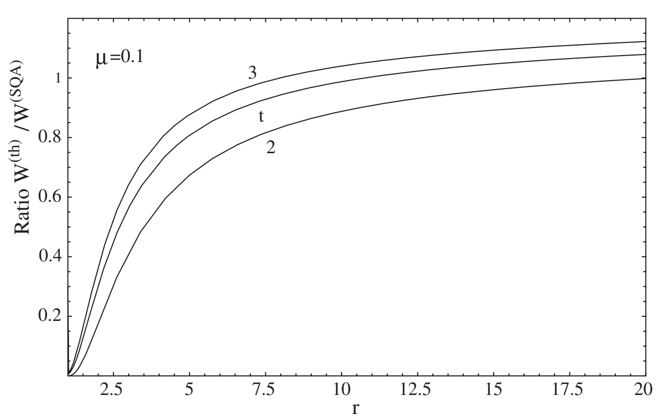

FIG. 2. The ratio of probabilities $W^{(\mathrm{th})}/W^{(\mathrm{SQA})}$. Curves 2 and 3 are for the polarized photons, curve t is for the probability averaged over the photon polarizations [see Eqs. (3.4) and (3.7)].

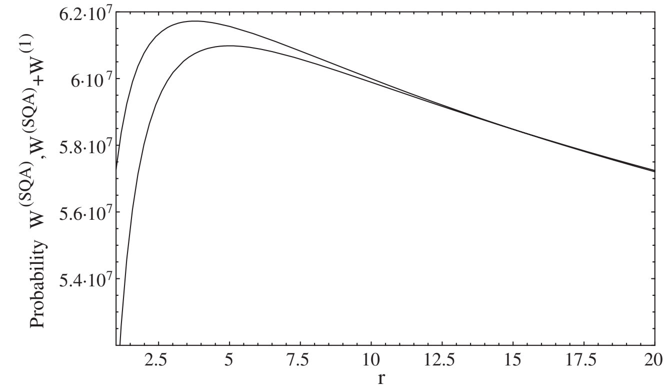

FIG. 3. The pair creation by a photon probability at $\mu = 3$ in the quasiclassical approximation. The upper curve is the probability in SQA [Eq. (3.7)]. The lower curve is the sum $W^{(SQA)} + W^{(1)}$ [Eq. (3.1)].

point where the ratio attains 1 is the boundary of the applicability region of the corresponding approximation.

The contribution of the correction [Eq. (3.1)] to SQA at $\mu = 3$ is shown in Fig. 3. The upper curve is the probability of pair creation by a photon in SQA [Eq. (3.7)], The lower curve is the sum $W^{(SQA)} + W^{(1)}$ (the correction is negative at low r). It is seen that the correction is very small in the region of SQA applicability $r \gg \mu/2 = 1.5$.

It is very important that the quasiclassical approximation appears to be applicable also for superstrong magnetic field $\mu \geq 1$. This is true if the characteristic parameter $l_{\max} \gg 1$. It is assumed in this case that parameter κ is large ($\kappa = 2\sqrt{r}\mu \gg 1$). In the limit $\kappa \gg 1$, the asymptotic expansion of the probability equation (3.7) is

$$W^{(SQA)} = \frac{\alpha m^2}{\pi\omega}\left(B\kappa^{2/3} - \frac{2\pi}{3} + \cdots\right), \qquad B = \frac{3^{7/6}5}{14}\frac{\Gamma^3(2/3)}{\Gamma(1/3)} = 1.1925\ldots. \tag{3.8}$$

The asymptotic expansion of correction equation (3.1) is

$$W^{(1)} = \frac{\alpha m^2}{\pi\omega}\frac{\mu^2}{60\sqrt{3}\kappa}(A\kappa^{1/3} - 6\pi + \cdots), \qquad A = \frac{3^{1/3}2}{5}\frac{\Gamma^3(1/3)}{\Gamma(2/3)} = 8.191\ldots. \tag{3.9}$$

The relative value of the correction defines the boundary of the region where SQA is valid for the superstrong magnetic field $\mu \geq 1$:

$$\frac{W^{(1)}}{W^{(SQA)}} = 3^{-7/3}\frac{7}{125}\frac{\Gamma^4(1/3)}{\Gamma^4(2/3)}\frac{\mu^2}{\kappa^{4/3}}\left(1 - \frac{6\pi}{A\kappa^{1/3}} + \cdots\right) \simeq 0.066\frac{\mu^2}{\kappa^{4/3}}\left(1 - \frac{2.30}{\kappa^{1/3}}\right). \tag{3.10}$$

It is seen from this expression that at $\mu \geq 1$ the applicability of the quasiclassical approximation is controlled by the parameter $(\mu/2r)^{2/3} = l_{\max}^{-2/3} \ll 1$. So, both in the case $\mu \ll 1$ and in the case $\mu \geq 1$, the quasiclassical approximation is valid when the particles of the pair are created in states with large Landau quantum numbers. It should be noted that in Eq. (3.10) the numerical factor is small and the second term ($\propto \kappa^{-1/3}$) contribution is relatively large. This means that the correction is small even at low $l_{\max}$, and one would have to use the exact expression Eq. (3.1). Under these circumstances, one can use Eq. (3.7) with the correction Eq. (3.1) starting from $r \sim \mu$. This is illustrated in Fig. 4 where the probability of pair creation is shown at $\mu = 3$. The probability in sawtooth form is calculated according to Eq. (2.12). It is seen that the characteristic distance between peaks is of the order μ (here $\mu = 3$) just as in the case shown in Fig. 1 where the probability for $\mu = 0.1$ was presented. The result of averaging over this interval $(-\mu/2 + r, \mu/2 + r)$ is given by the dashed line which has smoothed out all peculiarities of original behavior and which is in a very good agreement with the solid curve which is the quasiclassical approximation calculated according to Eqs. (3.1) and (3.7).

The relative contribution of transitions to the ground level into the total probability of pair creation W Eq. (2.12) is of evident interest. It can be found from Eq. (2.12), where $k = 0$, $m = n = l$, $F = 1$. Let us calculate this contribution under conditions $\kappa \gg 1$ for $\mu \ll 1$ and $z \gg 1$ for $\mu \geq 1$. The main contribution in the sum over l is given by $(z - l) \ll z$ [see Eqs. (2.5) and (2.10)]. Conserving the main terms of decomposition, we get

$$\frac{z^l}{l!} \simeq \frac{1}{\sqrt{2\pi z}}\exp\left(z - \frac{(z-l)^2}{2z}\right), \qquad 1 + \mu l - \frac{l^2\mu}{z} \simeq \mu(z - l), \qquad g = \frac{(z-l)^2\mu}{2z} - 1. \tag{3.11}$$

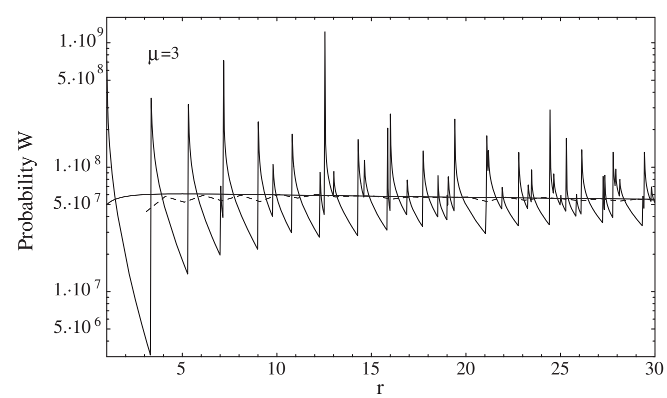

FIG. 4. The probability of pair creation at $\mu = 3$. The probability in sawtooth form is calculated according to Eq. (2.12). The result of averaging over the interval $(-\mu/2 + r, \mu/2 + r)$ is given by the dashed line. The solid curve, which is the quasiclassical approximation, is calculated according to Eqs. (3.1) and (3.7).

Using this result and passing in the probability from the summation to the integration ($\Delta l \gg 1$) and introducing the variable $\sqrt{g}$, we obtain, after evaluation of the Gaussian integral, the probability of the pair creation on the ground level:

$$W_0 = \frac{\alpha m^2}{\omega}\mu e^{-(1/\mu)}(1 - O(1/\kappa, 1/\sqrt{l_{\max}})). \qquad (3.12)$$

Having used Eq. (3.8) for the total probability W for the superstrong field $\mu \geq 1$, we get

$$\frac{W_0}{W} \simeq \frac{\pi}{2^{2/3}B}\left(\frac{\mu}{r}\right)^{1/3} e^{-(1/\mu)} \simeq \frac{2.09}{l_{\max}^{1/3}} e^{-(1/\mu)}. \qquad (3.13)$$

We performed also the numerical calculation for the case $\mu = 10$, $r = 440$ ($\omega = 21$ MeV) of the ratio W_0/W [Eq. (2.12)]. After averaging over the interval $(-\mu/2 + r, \mu/2 + r)$, we found $W_0/W = 0.38$, while from Eq. (3.13) one has $W_0/W \simeq 0.43$. The agreement between these numbers is satisfactory since the parameter of decomposition in Eq. (3.13) is $1/\sqrt{l_{\max}} = 1/\sqrt{83} \simeq 0.11$.

So we conclude that, in the superstrong magnetic field, the transition to the ground level is significant but does not dominate for a large number of levels. For the case just considered, the exact probability [Eq. (2.12)] gives $W = 9.24 \times 10^7$ cm^{-1} after averaging, while the quasiclassical approach [see Eqs. (3.8) and (3.9)] gives $W^{\rm SQA} + W^{(1)} = 9.29 \times 10^7$ cm^{-1}. This means that the quasiclassical approximation is valid in this region.

IV. ENERGY DISTRIBUTION OF THE CREATED PAIR

The energy distribution of the created pair is symmetric with respect to exchange of the electron and the positron, i.e. it is symmetric with respect to the energy $\omega/2$ and one can consider this distribution in the interval $x = \varepsilon_-/\omega \geq 1/2$. In the frame where $k_3 = 0$, the energy of the electron and the positron is

$$\varepsilon_-(l) = \sqrt{p_3^2 + m^2(1 + 2\mu l)},$$
$$\varepsilon_+(k) = \sqrt{p_3^2 + m^2(1 + 2\mu k)}, \qquad (4.1)$$
$$\omega = \varepsilon_-(l) + \varepsilon_+(k).$$

From Eq. (4.1), it follows that $p_3^2/m^2 = g$ [g is defined in Eq. (2.10)]. In the plane (k, l) the allowed states of the electron and the positron are bounded by the axes and the curve $p_3^2 = 0$ which is a parabola with the nearest point to origin $l_n = k_n = [(r-1)/2\mu]$. When $k = 0$ the maximal value $l = l_{\max}$ [$l_{\max}$ is defined in Eq. (2.10)]. Substituting p_3^2/m^2 in ε_-, we get

$$x = \frac{\varepsilon_-}{\omega} = \frac{1}{2}\left(1 + \frac{n\mu}{2r}\right), \qquad n = l - k. \qquad (4.2)$$

So the distribution over the energy of the created particle is of the discrete character. The probability $W(n)$ is defined by Eq. (2.12), where one has to perform the summation at fixed $n = l - k$. The expression for the spectrum contains the factor $1/\sqrt{g}$ discussed above [see Eq. (2.13)].

The energy distribution over the electron energy in SQA [see, e.g. [15], Eq. (3.50)] is

$$\frac{dW^{\rm (SQA)}}{dx} = \frac{\alpha m^2}{\sqrt{3}\pi\omega}\left[\frac{x^2 + (1-x)^2}{x(1-x)}K_{2/3}(\xi) + \int_\xi^\infty K_{1/3}(y)dy\right], \qquad (4.3)$$

where

$$\xi = \frac{2}{3\kappa x(1-x)}. \qquad (4.4)$$

At low values of the parameters μ and κ, the energy distribution of the pair creation probability has a sharp peak when $\varepsilon_- = \varepsilon_+ = \omega/2$ (for details see Sec. 3.2 in [15]). When $\mu \ll 1$ and $\kappa \geq 1$, the SQA is not valid on the edge of the energy spectrum where another created particle is no more relativistic ($1 - x \sim m/\omega$). However, at a smaller energy [$1 - x \ll 1/\kappa = m/(\mu\omega)$], the differential probability is suppressed exponentially, so the contribution of the interval $1 - x \sim m/\omega$, where SQA is not valid, is negligible small.

For the superstrong field $\mu \geq 1$ in the region of applicability of the quasiclassical approximation ($r \gg \mu$), the parameter $\kappa = 2\mu\sqrt{r} \gg 1$. In this case, the upper limit of electron energy $x_b \simeq 1 - 1/(2\sqrt{r})$ is lower (or of the same order) than the electron energy in the maximum of the energy distribution of Eq. (4.3) ($x_m \sim 1 - 1.6/\kappa$). The region of applicability of the quasiclassical approximation near x_b is defined by the quasiclassical character of positron transverse motion ($p_3 = 0$, $k \gg 1$):

$$(1-x)^2 - \frac{1}{4r} \gg \frac{\mu}{2r}; \qquad 1 - x \sim \frac{\Delta n}{n_m} \gg \frac{1}{\sqrt{n_m}}, \qquad (4.5)$$
$$\Delta n \gg \sqrt{n_m},$$

where $n_m = l_{\max}$ [Eq. (2.10)].

In Fig. 5, the spectrum of created particles is shown for $\mu = 3$, $r = 110$ ($n_m = 66$). The spectrum in sawtooth form is calculated according to Eq. (2.12). The dashed curve is the spectrum averaged over the interval $(-2\mu + r, 2\mu + r)$. The solid curve is calculated using Eq. (4.3). It is seen that this curve is in quite good agreement with the result of averaging.

In this case the argument $\xi \ll 1$ in Eq. (4.3), and one can perform the expansion ($k_3 = 0$)

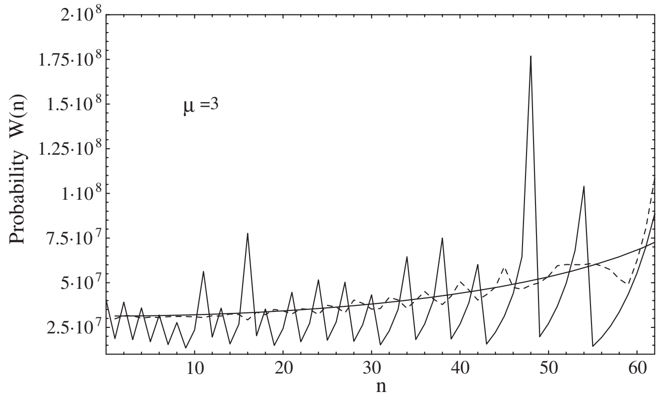

FIG. 5. The spectrum of created particles for $\mu = 3$, $r = 110(n_m = 66)$ vs n [see Eq. (4.2)]. The spectrum in sawtooth form is calculated according to Eq. (2.12). The dashed curve is the spectrum averaged over the interval $(-2\mu + r, 2\mu + r)$. The solid curve is calculated according to Eq. (4.3).

$$\frac{dW^{(\mathrm{SQA})}}{dx} = \alpha m\mu\left[C_1\frac{x^2+(1-x)^2}{(\kappa x(1-x))^{1/3}} + \frac{1}{3\kappa} - C_2\frac{1}{(\kappa x(1-x))^{5/3}} + \cdots\right], \tag{4.6}$$

where

$$C_1 = \frac{3^{1/6}}{2\pi}\Gamma(2/3) = 0.2588\ldots, \quad C_2 = \frac{3^{-1/6}}{4\pi}\Gamma(1/3) = 0.1775\ldots. \tag{4.7}$$

In Fig. 6, the energy distribution of the created electron (positron) at $\mu = 3$ is shown for different r: $r = 30$, $n_m = 16$ (the upper curve), $r = 110$, $n_m = 66$ (the middle curve), $r = 500$, $n_m = 318$ (the lower curve), calculated according to Eq. (4.3) (the solid curve) and according to

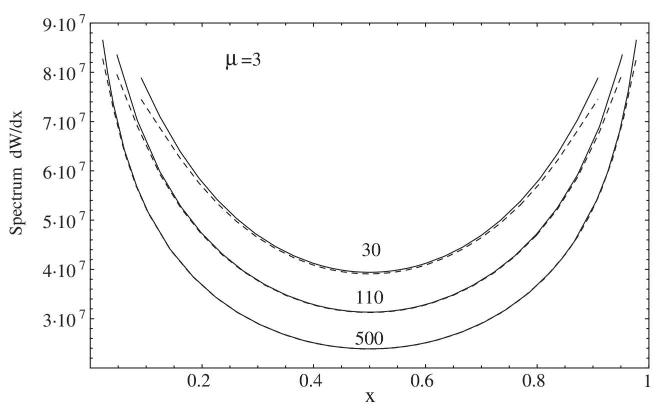

FIG. 6. The energy distribution of the created electron (positron) at $\mu = 3$ for different r: $r = 30$, $n_m = 16$ (the upper curve), $r = 110$, $n_m = 66$ (the middle curve), $r = 500$, $n_m = 318$ (the lower curve), calculated according to Eq. (4.3) (the solid curve) and according to Eq. (4.6) (the dashed curve).

Eq. (4.6) (the dashed curve). Very good agreement of corresponding curves is seen in the region of SQA applicability ($x \geq n_m^{-1/2}$, $(1-x) \geq n_m^{-1/2}$).

V. CONCLUSION

In this paper, we study the process of pair creation by a photon in a strong magnetic field based on the polarization operator in the field calculated by different methods in a set of papers [7,9,10]. By using the decomposition of functions into series containing the Bessel function [see Eq. (A4)], we deduced the imaginary part of the polarization operator (which is the process probability) into the relatively simple combination of the generalized Laguerre polynomial [see Eq. (2.12)]. This probability exhibits a sawtooth pattern because of the factor $1/\sqrt{g}$. This pattern will be washed out at averaging over any smooth photon energy distribution.

The range of applicability of the quasiclassical approximation extended in the photon energy interval where the created particles are no more ultrarelativistic (see Appendix B). The found correction to the standard quasiclassical approximation (SQA) permitted one to extend the range of applicability of the quasiclassical approximation to the region $r \sim \mu$ at $\mu \geq 1$. From the performed analysis follows the remarkable conclusion: for both $\mu \ll 1$ and $\mu \geq 1$, the results of the quasiclassical calculation are very close to the averaged probabilities of exact theory in a very wide interval of photon energies. For the total probability of pair creation, the quasiclassical approach is valid if [see Eq. (2.10)]

$$l_{\max} = \left[\frac{2(r-\sqrt{r})}{\mu}\right] \gg 1. \tag{5.1}$$

In the relatively weak field ($\mu \ll 1$), it is valid not far from threshold: $(r-1)/\mu \gg 1$. For this energy interval, the relative correction to the probability [Eq. (3.4)] is of the order $l_{\max}^{-1/2}$. In the superstrong field $\mu \geq 1$, for relativistic motion of the particles of the created pair and if $l_{\max} \simeq 2r/\mu \gg 1$ the SQA is valid. The expansion parameter in this case is $l_{\max}^{-2/3}$ [see Eq. (C11)].

For such a field and energy, it is helpful to compare the probability of pair creation by a photon W with the radiation length L_{rad} for the photon emission process found in [18]. Conserving the main terms of the decomposition over the Landau energy number l [see Eq. (2.13)] in the SQA as well as in the first correction, we get ($k_3 = 0$)

$$L_{\mathrm{rad}}^{-1} = \frac{a_\gamma}{L}\left[\frac{1}{l^{1/6}} + \frac{b_\gamma}{l^{5/6}}\right], \qquad W = \frac{a_p}{L}\left[\frac{1}{l_{\max}^{1/6}} + \frac{b_p}{l_{\max}^{5/6}}\right]. \tag{5.2}$$

Here

$$\frac{1}{L} = \alpha^{3/2}\left(\frac{B}{e}\right)^{1/2},$$
$$\alpha = \frac{e^2}{\hbar c} = \frac{1}{137},$$
$$a_\gamma = \frac{32}{81}(18)^{-1/6}\Gamma(2/3) = 0.330\,45\ldots,$$
$$a_p = \frac{15}{7}(18)^{-1/6}\frac{\Gamma^2(2/3)}{\Gamma^2(1/3)} = 0.338\,19\ldots, \qquad (5.3)$$
$$b_\gamma = \frac{(12)^{-1/3}}{20}\frac{\Gamma(1/3)}{\Gamma(2/3)} = 0.043\,206\ldots,$$
$$b_p = \frac{7(12)^{-1/3}}{1125}\frac{\Gamma^4(1/3)}{\Gamma^4(2/3)} = 0.041\,633\ldots,$$

where the numbers are taken from Eq. (46) in [18] and Eqs. (3.8) and (3.9) above. It should be noted that the physics characteristics in Eq. (5.2) do not depend on the electron mass. It is important that in the formula not only coefficients a_γ and a_p are very close, but the same properties possess the coefficients b_γ and b_p. This reflects the proximity of details of mechanisms of radiation and pair creation by a photon in a magnetic field. One can expect that this property will be essential for the consideration of an electron-photon shower in a magnetic field. The set of kinetic equations for such a shower with the particle polarization taken into account was investigated in [19,20].

ACKNOWLEDGMENTS

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APPENDIX A: EVALUATION OF DISCONTINUOUS INTEGRALS

Let us consider the integral in Eq. (2.3):

$$T_i = \int_{-1}^{1} dv \int_0^{\infty - i0} f_i(v, x)\exp(i\psi(v, x))dx. \qquad (A1)$$

Using the formula

$$\frac{2}{iz}\cot x \frac{d}{dx} e^{i\psi} = \left(f_2 - f_1 - \frac{2b}{z}\cot x\right)e^{i\psi}, \qquad (A2)$$

and integrating by parts the integral over x in Eq. (A1), we get

$$\int_0^{\infty - i0} f_2 e^{i\psi}dx = \int_0^{\infty - i0} f_1 e^{i\psi}dx + \frac{2}{z}\int_0^{\infty - i0}\left(b\cot x - \frac{i}{\sin^2 x}\right)e^{i\psi}dx. \qquad (A3)$$

Using Eq. 7.2.4.(27) of [21], we have

$$e^{-it\cos vx} = \sum_{n=0}^{\infty}(2 - \delta_{n0})(-i)^n J_n(t)\cos(nvx),$$
$$t = \frac{z}{\sin x},$$
$$e^{-it\cos vx}\cos vx = i\frac{d}{dt}e^{-it\cos vx} = \sum_{n=0}^{\infty}\left(1 - \frac{\delta_{n0}}{2}\right)(-i)^{n+1}(J_{n+1}(t) - J_{n-1}(t)) \times \cos(nvx),$$
$$e^{-it\cos vx}\sin vx = \frac{1}{itv}\frac{d}{dx}e^{-it\cos vx} = \frac{2i}{t}\sum_{n=1}^{\infty}(-i)^n nJ_n(t)\sin(nvx), \qquad (A4)$$

where $J_n(t)$ is the Bessel function. Taking into account that the functions f_i and ψ in the integral over v in Eq. (2.3) are the even functions of v, we can perform the following substitutions in sums in Eq. (A4)

$$\cos(nvx) \to e^{invx}, \qquad \sin(nvx) \to -ie^{invx}. \qquad (A5)$$

So the expression for T_i [Eq. (A1)] can be written as

$$T_i = \sum_n T_i^{(n)},$$
$$T_i^{(n)} = \int_{-1}^{1} dv \int_0^{\infty - i0} F_i^{(n)}(v, x)\exp(ia_n(v)x)dx, \qquad (A6)$$

where

$$F_1^{(n)} = (-i)^n \exp(iz\cot x)\left[\frac{i}{\sin x}\left(1 - \frac{\delta_{n0}}{2}\right) \times (J_{n+1}(t) - J_{n-1}(t)) - \frac{2vn}{z}\cot x J_n(t)\right],$$
$$F_2^{(n)} = F_1^{(n)} + (-i)^n \exp(iz\cot x)\frac{2(2 - \delta_{n0})}{z} \times \left(b\cot x - \frac{i}{\sin^2 x}\right)J_n(t),$$
$$F_3^{(n)} = (-i)^n \exp(iz\cot x)(1 - v^2)\cot x(2 - \delta_{n0})J_n(t) - F_1^{(n)},$$
$$a_n(v) = nv - b, \qquad (A7)$$

and the function $F_i^{(n)}$ is a periodic function of x.

Let us note that at $x \to -i\infty$ the asymptotic of $J_n(z/\sin x)$ is

$$J_n\left(\frac{iz}{\sinh x}\right) \simeq J_n(2ize^{-x}) \simeq \frac{(iz)^n}{n!}e^{-nx}. \qquad (A8)$$

In the case $a_n(v) < n$, the integration contour over x in Eq. (A6) can be unrolled to the lower imaginary semiaxis and the contribution to the imaginary part of $T_i^{(n)}$ vanishes.

One can represent $T_i^{(n)}$ in the form

$$T_i^{(n)} = \int_{-1}^{1} dv \int_0^{2\pi} F_i^{(n)}(v,x) \exp(ia_n(v)x) \sum_{k=0}^{\infty} e^{2\pi i k a_n(v)} dx = \int_{-1}^{1} \frac{dv}{1-e^{2\pi i a_n(v)}+i0} \int_0^{2\pi} F_i^{(n)}(v,x) \times \exp(ia_n(v)x)dx. \tag{A9}$$

We will use in Eq. (A9) the known equality

$$\frac{1}{1-e^{2\pi i a_n(v)}+i0} = \frac{\mathcal{P}}{1-e^{2\pi i a_n(v)}} - i\pi\delta(1-e^{2\pi i a_n(v)}). \tag{A10}$$

By virtue of the above observation,

$$-i\pi\delta(1-e^{2\pi i a_n(v)}) = -i\pi\sum_m \delta(1-e^{2\pi i(a_n(v)-m)}) \to \frac{1}{2}\sum_{m\geq n}\delta(a_n(v)-m). \tag{A11}$$

Substituting Eq. (A10) and (A11) into Eq. (A9) and taking into account that

$$F_i^{(n)}(v, x+\pi) = (-1)^n F_i^{(n)}(v,x), \tag{A12}$$

we have

$$T_i^{(n)} = (-1)^n \frac{i}{2}\mathcal{P}\int_{-1}^{1}\frac{dv}{\sin(\pi a_n(v))}\int_{-\pi}^{\pi} F_i^{(n)}(v,x) \times \exp(ia_n(v)x)dx + \sum_{m\geq n}\sum_{v_{1,2}}\frac{1+(-1)^{n+m}}{2|a_n'(v)|} \times \int_0^{\pi} F_i^{(n)}(v_{1,2},x)\exp(imx)dx, \tag{A13}$$

where

$$v_{1,2} = \frac{n\mu}{2r} \pm \sqrt{\frac{n^2\mu^2}{4r^2}+1-\frac{1+m\mu}{r}}, \quad |a_n'(v)| = \frac{2}{\mu}\sqrt{rg(n,m,r)}, \tag{A14}$$

$$g(n,m,r) = r-1-m\mu+\frac{n^2\mu^2}{4r}.$$

It is seen from Eq. (A13) that $n+m$ is even. Using Eq. (A12), one can represent the imaginary part of $T_i^{(n,m)}$ in the form

$$\mathrm{Im}T_i^{(n,m)} = -\frac{i}{2}\sum_{v_{1,2}}\frac{\mu}{\sqrt{rg(n,m,r)}}\int_{-\pi/2}^{\pi/2} F_i^{(n)}(v_{1,2},x) \times \exp(imx)dx. \tag{A15}$$

Along with integers m and n, we will use also $l=(m+n)/2$ and $k=(m-n)/2$.

We turn now to the calculation of integrals in Eq. (A15). Let us consider the integral

$$C_{n+1}^m(z) = (-i)^{n+1}\int_{-\pi/2}^{\pi/2} J_{n+1}\left(\frac{z}{\sin x}\right)e^{i(mx+z\cot x)}\frac{dx}{\sin x}. \tag{A16}$$

Changing the variable $t=\cot x$, we get

$$C_{n+1}^m(z) = (-i)^{n+1}\int_{-\infty}^{\infty} J_{n+1}(z\sqrt{t^2+1}) \times \left(\frac{t+i}{t-i}\right)^{m/2} e^{izt}\frac{dt}{\sqrt{t^2+1}}. \tag{A17}$$

One can close the integration contour in Eq. (A17) by an infinite half-circle in the upper half-plane and then contract the contour to the point $t=i$. So, we have to calculate the integral in Eq. (A17) over the contour $(i+)$. Changing successive variables $t\to it$, $y=(t-1)/(t+1)$, we get

$$C_{n+1}^m(z) = e^{-z}\oint_{(0+)} I_{n+1}\left(\frac{2z\sqrt{y}}{1-y}\right)y^{-m/2}e^{-2zy/(1-y)} \times \frac{dy}{\sqrt{y}(1-y)}. \tag{A18}$$

Using Eq. 10.12.(20) [21]

$$\frac{1}{1-y}e^{-2zy/(1-y)}(z\sqrt{y})^{-n}I_n\left(\frac{2z\sqrt{y}}{1-y}\right) = \sum_{k=0}^{\infty}\frac{k!}{(k+n)!}(L_k^n(z))^2 y^k, \tag{A19}$$

where $I_n(x)$ is the modified Bessel function, $L_k^n(z)$ is the generalized Laguerre polynomial, we get

$$C_{n+1}^m(z) = 2\pi i e^{-z}z^{n+1}\frac{(k-1)!}{l!}(L_{k-1}^{n+1}(z))^2\vartheta(k-1), \quad n=l-k. \tag{A20}$$

In the same manner, one can calculate the integral:

$$S_n^m(z) = 2(-i)^n\int_{-\pi/2}^{\pi/2}\cot x J_n\left(\frac{z}{\sin x}\right)e^{i(mx+z\cot x)}dx = 2\pi i e^{-z}z^n\frac{k!}{l!}F(l,k,z), \tag{A21}$$

$$F(l,k,z) = (L_k^n(z))^2 + \frac{l}{k}(L_{k-1}^n(z))^2\vartheta(k-1).$$

A somewhat different structure has the integral

$$D_n^m(z) = (-i)^n\int_{-\pi/2}^{\pi/2}J_n\left(\frac{z}{\sin x}\right)e^{i(mx+z\cot x)}\frac{dx}{\sin^2 x} = 2ie^{-z}\oint_{(0+)}I_{n+1}\left(\frac{2z\sqrt{y}}{1-y}\right)y^{-m/2}e^{-2zy/(1-y)}\frac{dy}{(1-y)^2} = 2ie^{-z}z^n\oint_{(0+)}\sum_{s=0}^{\infty}\sum_{j=0}^{\infty}\frac{j!}{(j+n)!}(L_j^n(z))^2y^{j+s-k} = -4\pi e^{-z}z^n\sum_{j=0}^{k-1}\frac{j!}{(j+n)!}(L_j^n(z))^2. \tag{A22}$$

Using Eq. 10.12.(9) of [21],

$$\sum_{j=0}^{k-1} \frac{j!}{(j+n)!} L_j^n(x) L_j^n(y) = \frac{k!}{(k+n-1)!} \frac{1}{x-y} [L_{k-1}^n(x) L_k^n(y) - L_k^n(x) L_{k-1}^n(y)], \tag{A23}$$

one obtains

$$D_n^m(z) = -4\pi e^{-z} z^n \frac{k!}{(l-1)!} \left[L_k^n(z) \frac{d}{dz} L_{k-1}^n(z) - L_{k-1}^n(z) \frac{d}{dz} L_k^n(z) \right] = 4\pi e^{-z} z^n \frac{k!}{l!} l [L_{k-1}^{n+1}(z) L_k^{n-1}(z) - L_k^n(z) L_{k-1}^n(z)]. \tag{A24}$$

In the derivation of the last expression, Eqs. 10.12.(15),(16),(24) of [21] was used. In conclusion of this Appendix, let us note that $T_i^{(n,m)}$ contains the combination

$$\begin{aligned} C_{n+1}^m(z) + C_{n-1}^m(z) &= 2\pi i e^{-z} z^{n-1} \frac{k!}{l!} \left[\frac{z^2}{k} (L_{k-1}^{n+1}(z))^2 + l(L_k^{n-1}(z))^2 \right] \\ &= 2\pi i e^{-z} z^{n-1} \frac{k!}{l!} \left[\frac{1}{k} (l L_{k-1}^n(z) - k L_k^n(z))^2 + l(L_k^n(z) - L_{k-1}^n(z))^2 \right] \\ &= 2\pi i e^{-z} z^{n-1} \frac{k!}{l!} [mF(l,k,z) - 4l L_k^n(z) L_{k-1}^n(z)]. \end{aligned} \tag{A25}$$

It should be noted that the above derivation is the most direct path from the process probability in the form of the integral equations (2.2) and (2.3) to the process probability in the form of Eq. (2.9).

APPENDIX B: QUASICLASSICAL APPROXIMATION AT LOW PHOTON ENERGY ($\omega \sim m, k^2 = 0, r = q$)

In the field which is weak compared to the critical field $B/B_0 = \mu \ll 1$ and at relatively low photon energy ($r \sim 1$), the created particles occupy mainly states with high quantum numbers if the condition $r - 1 \gg \mu$ is fulfilled. In this case, the quasiclassical approach is valid, but created particles are no more ultrarelativistic, so the standard quasiclassical formulas [13–15] are nonapplicable. We will develop here another approach, using the method of the stationary phase at calculation of the imaginary part of the integral over x in Eq. (2.3). Granting that the large parameter $1/\mu$ is the common factor in the phase $\psi(x)$, it is not contained in the equation $\psi'(x) = 0$ which defines the stationary phase point $x_0(r \sim 1) \sim 1$. In this case, the small values of variable v contribute to the integral over v, so that one can extend the integration limits to the infinity. So we get

$$\mathrm{Im}\Omega_i \simeq i \frac{\alpha m^2}{2\pi} \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} f_i(v,x) \times \exp\left\{ -\frac{i}{\mu} [\varphi(x) + v^2 \chi(x)] \right\} dx, \tag{B1}$$

where

$$\varphi(x) = 2r \tan\left(\frac{x}{2}\right) + (1-r)x, \qquad \chi(x) = rx\left(1 - \frac{x}{\sin x}\right). \tag{B2}$$

From the equation $\varphi'(x) = 0$, we find

$$\tan\left(\frac{x_0}{2}\right) = -\frac{i}{\sqrt{r}}, \qquad x_0(r) = -il(r), \qquad l(r) = \ln\frac{\sqrt{r}+1}{\sqrt{r}-1}. \tag{B3}$$

Substituting these results in the expressions which define the integral in Eq. (B1), we have

$$\begin{aligned} &i\varphi(x_0) = \beta(r) = 2\sqrt{r} - (r-1)l(r), \\ &i\varphi''(x_0) = \frac{r-1}{\sqrt{r}}, \qquad i\chi(x_0) = \frac{\sqrt{r}}{2} l(r)\beta(r), \\ &if_2(v=0, x_0(r)) = \frac{r-1}{2r^{3/2}}, \\ &-if_3(v=0, x_0(r)) = \frac{1}{\sqrt{r}}. \end{aligned} \tag{B4}$$

Performing the standard procedure of the method of the stationary phase, one obtains the following for the probability of pair creation by a polarized photon not far from threshold:

$$W_3^{(\mathrm{th})} = \frac{\alpha m^2 \mu}{\omega} \sqrt{\frac{r}{(r-1)l(r)\beta(r)}} \exp\left(-\frac{\beta(r)}{\mu}\right), \qquad W_2^{(\mathrm{th})} = \frac{r-1}{2r} W_3^{(\mathrm{th})}. \tag{B5}$$

In spite of the above assumption $r \sim 1$, Eq. (B5) is valid also at $r \gg 1$ if the condition $\beta(r) \gg \mu$ is fulfilled. This can be traced in the derivation of Eq. (B5). The first two terms of the decomposition of the function $\beta(r)$ over the power of $1/r$ are

$$\frac{\beta(r)}{\mu} \simeq \frac{4}{3\mu\sqrt{r}} + \frac{4}{15\mu r^{3/2}}. \tag{B6}$$

It follows from this formula that applicability of Eq. (B5) is limited by the condition $r \ll \mu^{-2}$. If the second term is much smaller than unity, the exponent with it can be expanded. As a result, we have from Eq. (B5) at $\mu^{-2/3} \ll r \ll \mu^{-2}$

$$W_3 = \frac{\alpha m^2 \mu}{2\omega}\sqrt{\frac{3r}{2}}\exp\left(-\frac{4}{3\mu\sqrt{r}}\right)\left(1 - \frac{4}{15\mu r^{3/2}}\right), \qquad W_2 = \frac{1}{2}W_3. \tag{B7}$$

The main term in the above expression coincides with the probability of pair creation by a photon in the standard quasiclassical method at $\kappa = 2\mu\sqrt{r} \ll 1$. The correction in Eq. (B7) determines the lower boundary over the photon energy of standard approach applicability ($\kappa^3 \gg \mu^2$). So the overlapping region exists where both the formulated here approach and the standard approximation for high energy are valid.

APPENDIX C: CORRECTIONS TO THE STANDARD QUASICLASSICAL APPROXIMATION ($k^2 = 0, r = q$)

The standard quasiclassical approximation, valid for ultrarelativistically created particles ($r \gg 1$), can be derived from Eqs. (2.3) and (2.4) by expanding the functions $f_2(v, x)$, $f_3(v, x)$, and $\psi(v, x)$ over x powers. Taking into account the higher powers of x, one gets

$$\begin{aligned} f_2(v, x) &= \frac{1 - v^2}{12}\left[(3 + v^2)x + \frac{1}{15}(15 - 6v^2 - v^4)x^3\right] \\ f_3(v, x) &= -\frac{1 - v^2}{6}\left[(3 - v^2)x + \frac{1}{60}(15 - 2v^2 + 3v^4)x^3\right] \\ \psi(v, x) &= -\frac{r(1 - v^2)^2}{12\mu}\left(x^3 + \frac{3 - v^2}{30}x^5\right) - \frac{x}{\mu}. \end{aligned} \tag{C1}$$

Here the first terms in the brackets give the known probability of the process in the standard quasiclassical approximation, while the second terms are the corrections. Expanding the term with x^5 in $\exp(i\psi(v, x))$ and making the substitution $x = \mu t$, one finds

$$\begin{aligned} \mathrm{Im}\Omega_i &= i\frac{\alpha m^2 \mu}{2\pi}\int_{-1}^{1} dv \int_{-\infty}^{\infty} g_i(v, t)\exp\left[-i\left(t + \xi\frac{t^3}{3}\right)\right]dt, \\ g_2(v, t) &= \frac{1 - v^2}{12}\mu t\left[(3 + v^2) - i\frac{9 - v^4}{90}\xi\mu^2 t^5 + \frac{\mu^2 t^2}{15}(15 - 6v^2 - v^4)\right], \\ g_3(v, t) &= -\frac{1 - v^2}{6}\mu t\left[(3 - v^2) - i\frac{(3 - v^2)^2}{90}\xi\mu^2 t^5 + \frac{\mu^2 t^2}{60}(15 - 2v^2 + 3v^4)\right], \end{aligned} \tag{C2}$$

where

$$\xi = \frac{[(1 - v^2)\kappa]^2}{16}, \qquad \kappa = 2\sqrt{r}\mu = \frac{\omega}{m}\mu. \tag{C3}$$

We will use the known integrals

$$\int_{-\infty}^{\infty}\cos\left(t + \xi\frac{t^3}{3}\right)dt = \sqrt{3}zK_{1/3}(z), \qquad \int_{-\infty}^{\infty} t\sin\left(t + \xi\frac{t^3}{3}\right)dt = \frac{3\sqrt{3}}{2}z^2K_{2/3}(z), \qquad z = \frac{2}{3\sqrt{\xi}} = \frac{8}{3(1 - v^2)\kappa}. \tag{C4}$$

Differentiating the first integral over ξ, one gets

$$\begin{aligned} \int_{-\infty}^{\infty} t^3\sin\left(t + \xi\frac{t^3}{3}\right)dt &= \frac{27\sqrt{3}}{8}z^3\frac{d}{dz}(zK_{1/3}(z)) = -\frac{27\sqrt{3}}{8}z^3\left(zK_{2/3}(z) - \frac{2}{3}K_{1/3}(z)\right), \\ \xi\int_{-\infty}^{\infty} t^6\cos\left(t + \xi\frac{t^3}{3}\right)dt &= -\frac{81\sqrt{3}}{16z^2}\left(z^3\frac{d}{dz}\right)^2(zK_{1/3}(z)) = \frac{81\sqrt{3}}{16}z^3\left[4zK_{2/3}(z) - \left(\frac{16}{9} + z^2\right)K_{1/3}(z)\right]. \end{aligned} \tag{C5}$$

Substituting the integrals [Eqs. (C3) and (C4)] in Eq. (C2), we obtain the probabilities of pair creation in the standard quasiclassical approximation:

$$W_2 = -\mathrm{Im}\frac{\kappa_2}{\omega} = \frac{\alpha m^2}{3\sqrt{3}\pi\omega}\int_{-1}^{1}\frac{3 + v^2}{1 - v^2}K_{2/3}(z)dv, \qquad W_3 = -\mathrm{Im}\frac{\kappa_3}{\omega} = \frac{2\alpha m^2}{3\sqrt{3}\pi\omega}\int_{-1}^{1}\frac{3 - v^2}{1 - v^2}K_{2/3}(z)dv, \tag{C6}$$

and the corresponding corrections to this approximation,

$$W_i^{(1)} = \frac{\alpha m^2\mu^2}{30\sqrt{3}\pi\omega\kappa}\int_0^1 G_i(v,z)\frac{dv}{1-v^2}, \tag{C7}$$

where

$$\begin{aligned} G_2(v,z) &= (36+4v^2-18z^2)K_{1/3}(z) \\ &\quad + (3v^2-57)zK_{2/3}(z), \\ G_3(v,z) &= -(34+2v^2+36z^2)K_{1/3}(z) \\ &\quad + (78-6v^2)zK_{2/3}(z). \end{aligned} \tag{C8}$$

In the derivation of Eqs. (C7) and (C8), the integration by parts was used as well as the intermediate equalities in Eq. (C5). The obtained representation of probabilities is convenient for calculations asymptotic at $\kappa \gg 1$ which are

$$\begin{aligned} W_i^{(1)} &= \frac{\alpha m^2\mu^2}{30\sqrt{3}\pi\omega\kappa}w_i, \\ w_2 &= 12A\kappa^{1/3} - 90\pi, \\ w_3 &= -11A\kappa^{1/3} + 84\pi, \\ A &= 3^{1/3}\frac{2}{5}\frac{\Gamma^3(1/3)}{\Gamma(2/3)} = 8.191\ldots. \end{aligned} \tag{C9}$$

At $\kappa \ll 1$ the main contribution is given by the terms $\propto z^2$ in G_i [Eq. (C8)]. One has

$$W_2^{(1)} = \frac{\alpha m^2\mu^2}{\omega\kappa^2}\frac{2\sqrt{2}}{5\sqrt{3}}\exp\left(-\frac{8}{3\kappa}\right), \qquad W_3^{(1)} = 2W_2^{(1)}. \tag{C10}$$

The relative magnitude of the corrections at $\kappa \gg 1$ is

$$\frac{W_2^{(1)}}{W_2} \simeq \frac{2\mu^2}{\kappa^{4/3}}\left(1-\frac{2.9}{\kappa^{1/3}}\right), \qquad \frac{W_3^{(1)}}{W_3} \simeq \frac{11\mu^2}{9\kappa^{4/3}}\left(\frac{2.9}{\kappa^{1/3}}-1\right). \tag{C11}$$

At $\kappa \ll 1$, one has

$$\frac{W_i^{(1)}}{W_i} \simeq -\frac{32\mu^2}{15\kappa^3}, \tag{C12}$$

which agrees with Eq. (B7).

APPENDIX D: THE DERIVATION OF THE CREATED PARTICLE SPECTRUM IN SQA

Substituting in Eq. (4.3) $x = (1+v)/2$, one has

$$\frac{dW^{\rm SQA}}{dv} = \frac{\alpha m^2}{2\sqrt{3}\pi\omega}\left[2\frac{1+v^2}{1-v^2}K_{2/3}(z) + \int_z^\infty K_{1/3}(y)dy\right], \tag{D1}$$

where z is defined in Eq. (3.2). If one omits the integration over v in Eq. (3.7), the result of this manipulation disagrees with Eq. (D1). So the described manipulation is erroneous. Only if one performs integration by parts of the second term in Eq. (D1) and use the recursion relation

$$zK_{1/3}(z) = -z\frac{dK_{2/3}(z)}{dz} - \frac{2}{3}K_{2/3}(z), \tag{D2}$$

one obtains Eq. (3.7) for the integral probability.

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