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Conformal gravity from the AdS/CFT mechanism

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We explicitly calculate the induced gravity theory at the boundary of an asymptotically anti–de Sitter five dimensional Einstein gravity. We also display the action that encodes the dynamics of radial diffeomorphisms. It is found that the induced theory is a four dimensional conformal gravity plus a scalar field. This calculation confirms some previous results found by a different approach.

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I. INTRODUCTION

A connection between string theory on $\mathrm{AdS}_5 \times S^5$ and super Yang-Mills theory in four dimensions was proposed by J. Maldacena some years ago [1]. More recently, this result gave rise to what is currently called the AdS/CFT conjecture. Since then, many others results have been reached by using this conjecture. The AdS/CFT conjecture relates the renormalized gravity action induced in the boundary with the expectation value of the stress tensor of the dual CFT as

$$\frac{1}{\sqrt{\gamma}}\frac{\delta S_{\mathrm{ren}}}{\delta \gamma_{ij}} = \langle T_{ij}\rangle_{\mathrm{CFT}}, \tag{1}$$

where γ_{ij} is the metric induced on the boundary.

As stated today, the AdS/CFT conjecture actually represents a realization of holography as proposed 10 years ago by Susskind and 't Hooft [2,3]. This conjecture has been extensively checked in part because the conformal symmetry is strong enough to determine many generic results in a CFT without knowing the details of the particular theory. For instance, one can demonstrate that the thermodynamics of a black hole in an asymptotically (locally) AdS space reproduces the thermodynamics of a CFT. To our knowledge this is true for all the theories of gravity with a single negative cosmological constant (see, for instance, [4]). The main reason is that the thermodynamics of any CFT is almost completely determined by the conformal symmetry.

Furthermore, one can prove that, under certain particular conditions, a gravitational theory can be induced in a lower dimensional surface at the bulk. The brane-worlds proposed in [5] are a realization of these ideas. In [6], using the same underlying idea, it is shown that the Liouville theory arises as the effective theory at the AdS asymptotic boundary in $2+1$ AdS gravity. If the AdS/CFT conjecture is to be understood as a duality relation, then a classical solution in the bulk should rise to a quantum corrected solution at the boundary. This actually was confirmed between $3 + 1/2 + 1$ dimensions in [7].

An asymptotically (locally) AdS space needs to be treated carefully; otherwise, one is usually led to a divergent behavior in the Lagrangian, the conserved charges and/or the variations of the Lagrangian. Therefore, to confirm many of the results of the AdS/CFT conjecture, it is necessary to use some (classical) regularization processes together with a proper set of boundary conditions. The regularization of the conserved charges has been an interesting field by itself where many relevant results have been found (see, for instance, [8–10]). A generic method to deal with the divergent behaviors of the actions appears in [11], where the conjecture is used to build a method to compute anomalies of CFT's. In this work part of these results will be used. In particular, in five dimensions a finite version of the Einstein-Hilbert action [12] reads

$$\begin{aligned} I_{\mathrm{grav}} = {}& \frac{1}{16\pi G}\int_M d^4x d\rho \sqrt{g}(R - 2\Lambda) \\ & - \frac{1}{8\pi G}\int_{\partial M} d^4x\sqrt{\gamma}K - \frac{3}{8\pi G}\int_{\partial M} d^4x\sqrt{\gamma} \\ & - \frac{1}{16\pi G}\int_{\partial M} d^4x\sqrt{\gamma}R[\gamma]. \end{aligned} \tag{2}$$

We would like to underline that, although in this action each term diverges, the addition of all of them turns out to be finite and well behaved. A generalization for this action can be found in Ref. [13].

In this work we prove that an effective conformal gravity theory arises at the boundary when the action displayed in Eq. (2) is used as the bulk's theory. We have extended to five dimensions the work previously done by Carlip [6] in three dimensions.

The theory obtained coincides with the bosonic part from the super conformal gravity that appears in the work of Balasubramanian *et al.* [14] and that of Liu *et al.* [15]; however, the method employed to reach this result is different. There is another approach, independent of the two already mentioned, that reaches the same result [16].

II. THE FOUR DIMENSIONAL CONFORMAL ACTION AND THE ANOMALY

The purpose of this work is to rewrite the action that appears in Eq. (2) and to show that it can be understood as a four dimensional theory under diffeomorphisms that preserve the asymptotically AdS scaling of the metric. To fulfill this program, we begin with a general five dimensional asymptotic AdS metric with a Fefferman-Graham–type expansion near infinity. This yields the following line element:

$$ds^2 = l^2 d\rho^2 + g_{ij}(x,\rho)dx^i dx^j, \tag{3}$$

where the limit $\rho \to \infty$ defines the asymptotically (locally) AdS region. The metric $g_{ij}(x,\rho)$ admits the expansion

$$g_{ij}(x,\rho) = e^{2\rho} g^{(0)}_{ij}(x) + g^{(2)}_{ij}(x) + e^{-2\rho} g^{(4)}_{ij}(x) - 2e^{-2\rho}\rho h_{ij}(x) + \ldots.$$

Next, we set $l = 1$, and thus $\Lambda = -6$. With this expansion the Einstein equations can be solved iteratively. This yields (see Ref. [11])

$$\mathrm{Tr}(g^{(4)}) = \frac{\mathrm{Tr}(g^{(2)2})}{4}, \qquad g^{(2)}_{ij} = -\frac{1}{2}\left(R^{(0)}_{ij} - \frac{1}{6}R^{(0)} g^{(0)}_{ij}\right), \qquad \mathrm{Tr}(h) = 0 \tag{4}$$

where traces are obtained using the metric $g^{(0)}_{ij}$.

The following steps consider a coordinate transformation that must leave invariant the asymptotic form of the metric (3). Using the prescription described in [6], the transformation reads

$$\begin{aligned} \rho &\to \rho + \tfrac{1}{2}\varphi(x) + e^{-2\rho} f^{(2)}(x) + \ldots, \\ x^i &\to x^i + e^{-2\rho} h^{(2)i}(x) + \ldots. \end{aligned} \tag{5}$$

Note that in the new coordinate system ρ and the variable x_i are factorized.

The boundary ($\bar{\rho} \to \infty$) is defined as

$$\rho = \bar{\rho} + \tfrac{1}{2}\varphi(x) + O(e^{-n\bar{\rho}}) = F(x). \tag{6}$$

Therefore the induced metric at the boundary and the unit normal, respectively, read

$$\gamma_{ij} = g_{ij} + \partial_i F \partial_j F, \tag{7}$$

$$n^a = \frac{1}{\sqrt{1 + g^{ij}\partial_i F \partial_j F}}(-1, g^{ij}\partial_j F). \tag{8}$$

In this new system of coordinates (5) one can obtain an expansion in powers of ρ for the determinant $\sqrt{\gamma}$, the extrinsic curvature, and the Ricci scalar near the boundary. The expansions for each one of these geometrical objects are

$$\begin{aligned} \sqrt{\gamma} &= e^{4\rho}\sqrt{g^{(0)}} + \frac{1}{2}\sqrt{g^{(0)}} e^{2\rho}(\mathrm{Tr}(g^{(2)}) + g^{(0)ij}\partial_i F \partial_j F) \\ &\quad + \frac{1}{2}\sqrt{g^{(0)}}\Big(\mathrm{Tr}(g^{(4)}) + \frac{1}{4}\mathrm{Tr}(g^{(2)})^2 - \frac{1}{2}\mathrm{Tr}(g^{(2)2}) \\ &\quad - \frac{1}{4}(g^{(0)ij}\partial_i F\partial_j F)^2 + \frac{1}{2}\mathrm{Tr}(g^{(2)}) g^{(0)ij}\partial_i F \partial_j F \\ &\quad - g^{(0)ai} g^{(0)bj} g^{(2)}_{ij}\partial_a F \partial_b F\Big) + \ldots, \end{aligned} \tag{9}$$

$$\begin{aligned} K &= \frac{1}{2}\mathrm{Tr}(\gamma \mathcal{L}_n g_{\|}) \\ &= -4 + e^{-2\rho}(g^{(0)ij}\partial_i F\partial_j F + g^{(0)ij}\nabla^{(0)}_i \nabla^{(0)}_j F \\ &\quad + \mathrm{Tr}(g^{(2)})) + e^{-4\rho}\Big(2\,\mathrm{Tr}(g^{(4)}) - \mathrm{Tr}(g^{(2)2}) \\ &\quad - \frac{1}{2}\mathrm{Tr}(g^{(2)}) g^{(0)ij}\nabla^{(0)}_i\nabla^{(0)}_j F \\ &\quad - \frac{1}{2}\mathrm{Tr}(g^{(2)}) g^{(0)ij}\partial_i F \partial_j F\Big) + \ldots, \end{aligned} \tag{10}$$

$$\begin{aligned} R[\gamma] &= e^{-2\rho}(R^{(0)} - 6g^{(0)ij}\nabla^{(0)}_i\nabla^{(0)}_j F - 6g^{(0)ij}\partial_i F\partial_j F) \\ &\quad + e^{-4\rho}(-g^{(2)ij}R^{(0)}_{ij} - R^{(0)ij}\partial_i F \partial_j F \\ &\quad + 2g^{(2)ij}\nabla^{(0)}_i\nabla^{(0)}_j F + \mathrm{Tr}(g^{(2)}) g^{(0)ij}\nabla^{(0)}_i \nabla^{(0)}_j F \\ &\quad - 2g^{(2)ij}\partial_i F\partial_j F + 2\,\mathrm{Tr}(g^{(2)}) g^{(0)ij}\partial_i F \partial_j F) + \ldots. \end{aligned} \tag{11}$$

The Ricci scalar is defined up to total derivatives of the order of $O(e^{-4\rho})$. All indices are raised and lowered with respect to $g^{(0)_{ij}}$.

Here we have defined

$$\mathrm{Tr}\,(\gamma\mathcal{L}_n g_{\|}) = \gamma^{ij}\partial_i x^\mu \partial_j x^\nu (\mathcal{L}_n g)_{\mu\nu}, \tag{12}$$

with $\mu = 0\ldots 4$ and $x^4 = \rho$. The derivatives of the coordinates x^μ are given by

$$\partial_i x^\mu = \delta^\mu_i + \partial_i F \delta^\mu_4. \tag{13}$$

We want to use the expansions just described to extract the finite part of the action (2). First we integrate ρ in the five dimensional action (2):

$$
\begin{aligned}
\int_M d^4x d\rho \sqrt{g}(R-2\Lambda)_{\text{on-shell}} &= -8\int_{\partial M} d^4x \int^{\rho=F} d\rho\sqrt{g} \\
&= -8\int_{\partial M} d^4x \int^{\rho=F} d\rho\Big(e^{4\rho}\sqrt{g^{(0)}} + \frac{1}{2}\sqrt{g^{(0)}}e^{2\rho}(\mathrm{Tr}(g^{(2)})) \\
&\quad + \frac{1}{2}\sqrt{g^{(0)}}\Big(\mathrm{Tr}(g^{(4)}) + \frac{1}{4}\mathrm{Tr}(g^{(2)})^2 - \frac{1}{2}\mathrm{Tr}(g^{(2)2})\Big) + \ldots\Big) \\
&= \int_{\partial M} d^4x\Big(-2e^{4F}\sqrt{g^{(0)}} + \frac{1}{3}\sqrt{g^{(0)}}e^{2F}(R^{(0)}) - \frac{1}{4}\sqrt{g^{(0)}}\Big(R^{(0)ij}R^{(0)}_{ij} - \frac{1}{3}R^{(0)2}\Big)F + \ldots\Big). \qquad (14)
\end{aligned}
$$

The term proportional to F has a divergent term that must be eliminated by adding a counterterm. This regularization procedure was first introduced by Skenderis [11].

Finally, evaluating the action on shell we get

$$
\begin{aligned}
I_{\text{grav}} = \frac{1}{16\pi G}\int_{\partial M}\sqrt{g^{(0)}}\Big(&-\frac{1}{16}\Big(R^{(0)ij}R^{(0)}_{ij} - \frac{1}{3}R^{(0)2}\Big) \\
&+ \frac{1}{64}(\partial_i\varphi\partial^i\varphi)^2 + \frac{1}{16}\partial_i\varphi\partial^i\varphi\nabla^{(0)}_j\nabla^{(0)j}\varphi \\
&- \frac{1}{8}\Big(R^{(0)ij}R^{(0)}_{ij} - \frac{1}{3}R^{(0)2}\Big)\varphi + \frac{1}{8}G^{(0)ij}\partial_i\varphi\partial_j\varphi\Big).
\end{aligned} \qquad (15)
$$

This expression can be recognized as the action for a four dimensional conformal gravity plus an anomalous part.

It is worth mentioning that this result has been obtained before by at least two different approaches. For instance, looking for an action that takes care of the anomalous term, in [17] Riegert arrived at the same expression. Different approaches converging toward the same result give a solid confidence to the different methods used.

III. CONCLUSIONS

In this work we have proven that a four dimensional conformal gravity can be obtained through the AdS/CFT mechanism from five dimensional Einstein gravity. We have demonstrated this explicitly using the Fefferman-Graham expansion and regularizing the action. As expected the radial diffeomorphisms induce a Weyl transformation on the boundary metric which in turn produces the anomalous part as demonstrated by Manvelyan *et al.* [18]. The degrees of freedom associated with radial diffeomorphisms are encoded in the dynamics of the scalar field φ. This action was obtained by Riegert [17] as the local form of the action which gives a trace anomaly proportional to $R^{(0)ij}R^{(0)}_{ij} - \frac{1}{3}R^{(0)2}$ and corresponds to the local form of the anomalous part of the effective action associated with the super Yang-Mills theory in $d = 4$; see Refs. [14,15]. Also, from [19] we know that this field encodes part of the degrees of freedom contained in the traceless part of $g^{(4)}$ which, along with $g^{(0)}$, contains all the degrees of freedom of the solutions for pure gravity in five dimensions. This calculation confirms the previous result obtained for the pure gravitational sector by Balasubramanian *et al.* [14] and by Liu *et al.* [15] employing different methods. Our strategy appears to be more direct than used in the two works just mentioned; however, the algebra involved is more complex.

The induced four dimensional action we have found here can be considered as a quantum correction for the Einstein-Hilbert action in $d = 4$. Mottola and Vaulin [20] have considered a similar idea. They considered these terms as deviations from the classical stress tensor coming from quantum corrections. We plan to address this problem in a future work. In a different context, our result may be used as an ansatz for the action proposed in [21] to test Kaluza-Klein corrections in the Randall-Sundrum two-brane system.

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