Duality groups, automorphic forms, and higher derivative corrections

Neil Lambert* and Peter West†

Department of Mathematics, King's College, London WC2R 2LS, United Kingdom

(Received 21 December 2006; published 19 March 2007)

We study the higher derivative corrections that occur in type II superstring theories in ten dimensions or less. Assuming invariance under a discrete duality group $G(\mathbf{Z})$ we show that the generic functions of the scalar fields that occur can be identified with automorphic forms. We then give a systematic method to construct automorphic forms from a given group $G(\mathbf{Z})$ together with a chosen subgroup H and a linear representation of $G(\mathbf{Z})$. This construction is based on the theory of nonlinear realizations and we find that the automorphic forms contain the weights of G. We also carry out the dimensional reduction of the generic higher derivative corrections of the IIB theory to three dimensions and find that the weights of E_8 occur generalizing previous results of the authors on M theory. Since the automorphic forms of this theory contain the weights of E_8 we can interpret the occurrence of weights in the dimensional reduction as evidence for an underlying U-duality symmetry.

DOI: 10.1103/PhysRevD.75.066002 PACS numbers: 11.25.Yb

I. INTRODUCTION

By virtue of the large amount of supersymmetry they possess, the IIA supergravity [1–3] and IIB supergravity [4–6] theories encode all the perturbative and nonperturbative low energy effects of the corresponding string theories. Furthermore, the 11-dimensional supergravity theory [7] is thought to be the low energy effective action for an as yet undefined theory called M theory. The IIB theory possesses an $SL(2, \mathbf{R})$ symmetry [4] while the IIA supergravity and IIB supergravity dimensionally reduced on an $(n-1)$-torus, or equivalently the 11-dimensional supergravity theory on an n-torus, possess an E_n symmetry for $n = 7, 8, 9$ [8–10] and possibly $n = 10$ [11]. For other work on symmetries that appear in dimensional reduction see [12–18] and we note that the E_n symmetries necessarily contain T-duality which is a perturbative symmetry of string theory [19,20]. These theories possess charged states which are rotated by these symmetries and their charges obey the quantization condition [21]. This has lead to the conjecture [22–24] that a discrete version of these groups, denoted by $G(\mathbf{Z})$, are symmetries in string theory, e.g. $SL(2, \mathbf{Z})$ in the case of ten-dimensional IIB string theory.

However, much of the considerations of these discrete symmetries has been within the context of the lowest order effective action, i.e. the maximal supergravity theories, and there has not been much discussion of the role of these symmetries in the higher derivative corrections (however see [25,26]). A particularly notable exception to this are the higher derivative terms of the form $D^{2k}R^4$ that occur in IIB string theory whose coefficients for $k \leq 3$ have been determined exactly [27–33]. These coefficients are functions of $\tau = \chi + ie^{-\phi}$, where χ is the axion and ϕ the dilaton. Under the action of $SL(2, \mathbf{Z})$, τ is acted on by a fractional linear transformation, however the Riemann tensor is inert (in Einstein frame). Imposing that $SL(2, \mathbf{Z})$ is a symmetry one immediately sees that these coefficients must be invariant under $SL(2, \mathbf{Z})$ and hence are given by automorphic forms. The work of [27–33] has identified the automorphic forms for $k \leq 3$ and shown that some of the infinite series of terms are consistent with certain explicit string theory calculations and, perhaps more remarkably, loop calculations in 11-dimensional supergravity. Furthermore the coefficients of the R^4 term that occurs upon compactification on a torus to eight and fewer dimensions have been obtained as automorphic forms of $SL(3, \mathbf{R})$, $SL(5, \mathbf{R})$, and E_n [34–37]. In a similar spirit the coefficients in eight dimensions of $R^4G_3^{4g-4}$ terms, where G_3 is the modified complexified three-form field strength of type IIB string theory, have been given as automorphic forms of $SL(3, \mathbf{R})$ [38].

In a recent paper [39], we explored the dimensional reduction to three dimensions of generic higher derivative terms that arise in 11-dimensional M theory. The main purpose of this work was to see if there are traces of the E_8 symmetry that are present in the low energy effective action in three dimensions, i.e. the $N = 16$ maximal supergravity theory in three spacetime dimensions. Three dimensions is special because it is the first dimension in which all dynamical fields are scalars (after dualizing any vectors modes) and in the low energy effective action these scalar fields can be identified as a nonlinear realization of E_8 with local subgroup $SO(16)$. In Ref. [39] we determined the dependence of arbitrary higher derivatives terms on the diagonal components of the metric associated with the torus, which we may parametrize by $g_{ii} = e^{-c_i\phi_i}$ for some constant c_i. These occur in the action in the form of factors $e^{\sqrt{2}\vec{v}\cdot\vec{\phi}}$ that multiply the derivatives of the scalar fields. The different possible vectors $\vec{v}$ arise as the different possible terms the exponential factor can multiply. For the lowest order effective theory the vectors $\vec{v}$ are just the positive roots of E_8. This is readily understood from the well-known fact that the effective action can be written in

*Electronic address: neil.lambert@kcl.ac.uk
†Electronic address: peter.west@kcl.ac.uk

terms of the Cartan form of the coset $E_8/SO(16)$ which lives in the adjoint representation of E_8 and can be explicitly shown to involve the positive roots. However, in [39] it was found that the dimensional reduction of the higher derivative terms does not lead to the positive roots of E_8. Rather one finds that the various vectors that arise are elements of the weight lattice of E_8. Moreover one only finds weights for the types of higher derivative terms that are expected to arise in M theory [40–57]. Weights have also appeared in the higher derivative effective action in [58,59] within the context of E_{10} and ''cosmological billiards'' [60,61].

While the occurrence of weights of E_8 in the dimensional reduction of the higher derivative terms indicates the presence of some E_8 structure it was unclear what this structure could be since the nonlinear realizations to which the scalars belong are usually constructed from the Cartan forms and these only contain the positive roots of E_8.

In this paper we will show that if one assumes that the higher derivative terms of a type II string theory in ten or less dimensions are invariant under a discrete duality group $G(\mathbf{Z})$ then the generic functions of the scalars that arise in the action transform as automorphic forms. We then give a construction of such automorphic forms and find that they involve the weights of G. As a result, the occurrence of weights in the dimensional reduction of M theory can be thought of as a consequence of the presence of an underlying discrete duality group $G(\mathbf{Z})$ of the string theory in lower dimensions and so interpreted as evidence for such a symmetry.

The systematic method of constructing automorphic forms that we present relies on the ability to construct a nonlinear realization, φ from linear representation ψ of the continuous group G. This construction involves the coset representatives $g(\xi)$ of G/H, where the ξ labels the coset. In the conventions of [39], these are parametrized by

$$g(\xi) = e^{\sum_{\vec{\alpha}>0} E_{\vec{\alpha}}\chi_{\vec{\alpha}}} e^{-(1/\sqrt{2})\vec{\phi}\cdot\vec{H}}, \tag{1.1}$$

where $\vec{H}$ comprises the Cartan subalgebra, $E_{\vec{\alpha}}$ are the generators associated to the positive roots. The automorphic forms, which are generally nonholomorphic, are essentially functions of $\varphi(\xi)$ summed over the representation ψ from which they are constructed. One finds that the automorphic forms contain $g(\xi)$ acting on the representation ψ and so the weights of G corresponding to ψ automatically appear.

The detailed contents of this paper are as follows. In Sec. II we extend the calculation of Ref. [39] to the dimensional reduction of the perturbative contribution to higher derivative terms of the IIB string theory effective action. We will again find weights of E_8. In Sec. III we will examine the consequences of demanding that the higher derivative corrections of string theory in any dimension be invariant under $G(\mathbf{Z})$. Such terms contain functions of the coset fields ξ that parametrize G/H times Riemann tensors, field strengths, and Cartan forms. We calculate how these functions transform under $G(\mathbf{Z})$ and show that, under the natural action of the group on the coset variables ξ, they are ''rotated'' by matrices which belong to a representation of H. In Sec. IV we begin by showing that these transformations are precisely those of nonholomorphic automorphic forms of $G(\mathbf{Z})$ which depend on ξ. We then give a method of constructing automorphic forms once we choose a group G together with a subgroup H and a linear representation ψ of G. In particular, the automorphic form is constructed from the nonlinear representation of G with local subgroup H formed from the linear representation ψ of G. As explained above the group element of Eq. (1.1) enters in this process and in this way the automorphic form will depend on the coset of G/H. As a result of this construction we show that these automorphic forms contain the weights of G associated with the representation ψ and, in particular, the dominant term in the limit of small couplings is of the form $Z_s \sim e^{\sqrt{2}s\vec{w}\cdot\vec{\phi}}$ where $\vec{w}$ is a weight of G. In Sec. V we provide some concluding remarks. Appendices A, B, and C give some details and conventions on nonlinear and induced representations, group representations, and examples of $SL(n)$ automorphic forms, respectively.

II. TYPE IIB HIGHER DERIVATIVE CORRECTIONS AND THEIR REDUCTION

In this section we will evaluate the dimensional reduction to three dimensions of the higher derivative terms that appear in type IIB string theory. Some of these higher derivative terms in ten dimensions involving $D^{2k}R^4$ have been discussed in detail in [27–33]. In particular we will determine vectors $\vec{w}$ that appear in the dimensional reduction as coefficients of the scalar fields $\vec{\phi}$ through the factors $e^{\sqrt{2}\vec{w}\cdot\vec{\phi}}$. This is an extension of the calculation that we performed in [39] for M theory and more details may be found there, although here we will use a slightly more efficient method that we will explain. The higher derivative corrections in the ten-dimensional IIB theory already include automorphic forms of $SL(2,\mathbf{R})$ however we will only include in our calculations the perturbative contribution to the automorphic form. We will find that the general higher derivative correction leads to vectors $\vec{w}$ that are weights of E_8 (more precisely, in the conventions of [39], these are half-weights of E_8).

Since we are going to use a slightly more streamlined method compared to that used in Ref. [39] it will be useful to first consider the dimensional reduction of a generic theory possessing two or more spacetime derivatives involving gravity, gauge fields, and scalars on a n-torus. Our compactification ansatz is given by

$$d\hat{s}^2 = e^{2\alpha\rho}ds^2 + e^{2\beta\rho}G_{ij}(dx^i + A^i_\mu dx^\mu)(dx^j + A^j_\mu dx^\mu), \tag{2.1}$$

where

$$\alpha = \sqrt{\frac{n}{2(n+1)}}, \qquad \beta = -\frac{\alpha}{n} \tag{2.2}$$

which ensures that we remain in Einstein frame in three dimensions. Here $G_{ij} = e_i{}^{\bar{k}} e_j{}^{\bar{l}} \delta_{kl}$ and $e_i{}^{\bar{k}}$ is a vielbein with $\det e = 1$. We adopt the convention that $i, j, k, \ldots$ are world indices and $\bar{i}, \bar{j}, \bar{k}, \ldots$ are tangent indices. We note that this ansatz treats all the directions of the torus on the same footing and as discussed in Ref. [39], we will be able to carry out the dimensional reduction so that the $SL(n, \mathbf{R})$ invariance is manifest. In particular, the degrees of freedom of gravity associated with the torus, apart from any graviphotons enter the lower dimensional theory through a nonlinear realization of $SL(n, \mathbf{R})$ with local subgroup $SO(n)$, i.e. via the group element

$$e(\xi) = e^{\sum_{\underline{\alpha}>0} E_{\underline{\alpha}} \chi_{\underline{\alpha}}} e^{-(1/\sqrt{2})\underline{\phi}\cdot\underline{H}}, \tag{2.3}$$

where $\underline{H}$ forms the Cartan subalgebra, $E_{\underline{\alpha}}$ are positive root generators (when $\underline{\alpha} > 0$) of $SL(n, \mathbf{R})$, respectively, and ξ collectively denotes the fields $\chi_{\underline{\alpha}}$ and $\underline{\phi}$. In fact the terms which contain $e(\xi)$ alone are built out of the Cartan forms $e^{-1}\partial_\mu e = S_\mu + Q_\mu$, where S_μ and Q_μ are symmetric and antisymmetric in $\bar{i}$ and $\bar{j}$, respectively. As this belongs to the Lie algebra of $SL(n, \mathbf{R})$ it does not matter which representation for the generators one takes to evaluate it.

However, the explicit components of the vielbein, $e_i{}^{\bar{k}}$, associated with the torus reduction are given by taking the generators to be in the fundamental representation, with highest weight $\underline{\lambda}^{n-1}$, where $\underline{\lambda}^i$, $i = 1, \ldots, n-1$, are the fundamental weights of $SL(n, \mathbf{R})$. We now explain why this is the case. Given a linear realization of $SL(n, \mathbf{R})$ on a vector space whose vectors have the components ψ_a we can construct a nonlinear realization with components φ_a by[1]

$$\varphi_a(\xi) = D(e(\xi)^{-1})_a{}^b \psi_b, \quad \text{or equivalently} \quad |\varphi(\xi)\rangle = U(e(\xi))|\psi\rangle, \tag{2.4}$$

where $U(e(\xi))$ indicates the action of the generators on the vector space to which $|\psi\rangle = \psi_a|e^a\rangle$ belongs. From Eq. (2.4) we see that $\varphi_a(\xi)$ transforms under $SL(n, \mathbf{R})$ by transforming the parameters of the coset ξ in the usual way and by an $SO(n, \mathbf{R})$ rotation that acts on the index a. In particular if we take $|\psi\rangle = \psi_i|i, \underline{\lambda}^{n-1}\rangle$ to be the representation of $SL(n, \mathbf{R})$, whose highest weight is $\underline{\lambda}^{n-1}$, then $\varphi_{\bar{i}}(\xi)$ will transform as a vector with respect to this $SO(n, \mathbf{R})$ rotation. However the inverse vielbein $(e^{-1})_{\bar{i}}{}^j$ converts world indices to tangent indices and hence converts quantities that transform under $SL(n, \mathbf{R})$ into those that transform under $SO(n, \mathbf{R})$. As such we may identify

$$(e^{-1})_{\bar{i}}{}^j = D(e^{-1}(\xi))_i{}^j. \tag{2.5}$$

Acting on a state $|\psi\rangle = \psi_i|i, \underline{\lambda}^{n-1}\rangle$ with $U(e(\xi))$ we find that $e_{\bar{i}}{}^j$ factors of $e^{-(1/\sqrt{2})\underline{\phi}\cdot[\underline{\lambda}^{n-1}]}$ where $[\underline{\lambda}^{n-1}]$ denotes one of the weights in the $\underline{\lambda}^{n-1}$ representation. The lowest weight in the $\underline{\lambda}^{n-1}$ is just the weight $-\underline{\lambda}^1$ and so we may rewrite this factor as $e^{(1/\sqrt{2})\underline{\phi}\cdot[\underline{\lambda}^1]}$. Thus we find that $e_i{}^{\bar{j}}$ contains factors of $e^{-(1/\sqrt{2})\underline{\phi}\cdot[\underline{\lambda}_1]}$.

The dimensionally reduced theory will involve corrections that contain field strengths of the form $F_{\mu_1\ldots\mu_p i_1\ldots i_k}$, where $i_1, \ldots$ are world volume indices of the torus. The field strength may also carry other internal indices that we neglect for the moment, but we will discuss them below. We can always use the inverse vielbein $e_{\bar{i}}{}^j$ to convert all world volume indices to tangent space indices. Following the same argument we used to the vielbein given above, this can be viewed as the conversion of the linear rank k antisymmetric representation of $SL(n, \mathbf{R})$ into a nonlinear representation whose indices rotate under $SO(n)$. Consequently, $F_{\mu_1\ldots\mu_p\bar{i}_1\ldots\bar{i}_k}$ has a dependence on the metric of the torus that is equivalent to acting with $U(e(\xi)^{-1})$, on the states $|[\underline{\lambda}^{n-k}]\rangle$ where $[\underline{\lambda}^{n-k}]$ are weights in the representation with highest weight $\underline{\lambda}^{n-k}$. Therefore one finds that the fields $\underline{\phi}$ associated with the Cartan subalgebra of $SL(n, \mathbf{R})$ occur in $F_{\mu_1\ldots\mu_p\bar{i}_1\ldots\bar{i}_k}$ through the factor $e^{(1/\sqrt{2})\underline{\phi}\cdot[\underline{\lambda}^k]}$. We recall here that the weights $|[\underline{\lambda}^{n-k}]\rangle$ include the highest weight $\underline{\lambda}^{n-k}$, but also the lowest weight which is $-\underline{\lambda}^k$.

Thus the action after the dimensional reduction contains terms which involve $e(\xi)$ alone and are constructed from $e(\xi)^{-1}\partial_\mu e(\xi)$ (and hence is independent of the representation used) and field strengths, including those generated from the Riemann tensor, which are taken to have tangent space indices. In this way the three-dimensional effective action can be constructed from various building blocks where each one has indices that transform under $SO(n)$. Invariants are constructed using the invariant tensor $\delta_{\bar{i}\bar{j}}$. Consequently, to compute the dependence of the final action on $\underline{\phi}$ one just has to add up the contributions from each building block.

One also finds factors of $e^{\sqrt{2}\rho}$ which are readily computed explicitly from the occurrence of the vielbeins using the metric ansatz of Eq. (2.1) as was done in Ref. [39].

It is also possible to treat any coset symmetries of the original theory in a similar way to the $SL(n, \mathbf{R})$ associated with the torus. We illustrate this for the case of the $SL(2, \mathbf{R})$ symmetry of the IIB theory [4], as this is the case of most interest to us here, but the technique is quite general. Type IIB theory possesses two scalars χ and ϕ which belong to the coset space $SL(2, \mathbf{R})/SO(2, \mathbf{R})$. We may choose our coset representatives of $SL(2, \mathbf{R})/SO(2)$ as

$$g(\tau) = e^{E\chi} e^{-(1/\sqrt{2})\phi H}, \tag{2.6}$$

[1]For further discussion of this construction we refer the reader to Appendix A.

where E and H are the positive root and Cartan subalgebra generators of $SL(2, \mathbf{R})$, respectively. It will be useful to define $\tau = \chi + ie^{-\phi}$; as τ undergoes fractional linear transformations under the action of $SL(2, \mathbf{R})$ on this coset. It also contains two three-form field strengths $F^a_{\mu_1\mu_2\mu_3} = 3\partial_{[\mu_1}A_{\mu_2\mu_3]}$, $a = 1, 2$. The gauge fields must transform as a linear representation of $SL(2, \mathbf{R})$, otherwise, if the gauge fields transformed as a nonlinear representation of $SL(2, \mathbf{R})$, the composite nature of the $SO(2)$ matrix would not preserve the form of the field strength and this in turn would not maintain gauge invariance. Therefore the two three-form field strengths $F^a_{\mu_1\mu_2\mu_3}$ must transform in the doublet representation of $SL(2, \mathbf{R})$. However, given the field strength $F^a_{\mu_1\mu_2\mu_3}$ we can convert it into a three-form $G^a_{\mu_1\mu_2\mu_3}$ that transforms as a nonlinear realization of $SL(2, \mathbf{R})$ using Eq. (A9) and the action of $U(g(\tau)^{-1})$. In particular for the doublet representation the group element of Eq. (2.6) can be written as

$$U(g(\tau)) = \frac{1}{\sqrt{\mathrm{Im}\tau}}\begin{pmatrix} \mathrm{Im}\tau & \mathrm{Re}\tau \\ 0 & 1 \end{pmatrix} \tag{2.7}$$

so that

$$\begin{aligned} G^1_{\mu_1\mu_2\mu_3} &= \frac{1}{\sqrt{\mathrm{Im}\tau}}(F^1_{\mu_1\mu_2\mu_3} - \mathrm{Re}\tau F^2_{\mu_1\mu_2\mu_3}), \\ G^2_{\mu_1\mu_2\mu_3} &= \sqrt{\mathrm{Im}\tau}F^2_{\mu_1\mu_2\mu_3} \end{aligned} \tag{2.8}$$

and hence we can form the complex combination [4]

$$\begin{aligned} G_{\mu_1\mu_2\mu_3} &= G^1_{\mu_1\mu_2\mu_3} - iG^2_{\mu_1\mu_2\mu_3} \\ &= \frac{1}{\sqrt{\mathrm{Im}\tau}}(F^1_{\mu_1\mu_2\mu_3} - \tau F^2_{\mu_1\mu_2\mu_3}). \end{aligned} \tag{2.9}$$

The advantage of working with $G^a_{\mu_1\mu_2\mu_3}$ rather than $F^a_{\mu_1\mu_2\mu_3}$ is that it is simpler to form invariants since they rotate on their a indices as a vector of $SO(2)$. As a result for every factor of $G^a_{\mu_1\mu_2\mu_3}$ that occurs one finds a corresponding factor of $e^{(1/\sqrt{2})\phi[\mu]}$, where

$$[\mu] = \{\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\} \tag{2.10}$$

are the weights that appear in the fundamental representation of $SL(2, \mathbf{R})$.

The above technique also applies to fields that arise from dualization. The computation of the ρ dependence is straightforward and is as explained in [39]. The dualization process changes the position of indices, such as world volume indices, from being upper indices to lower indices and vise versa. However, one can apply the above procedure to the field after dualization and read off the resulting dependence on $\underline{\phi}$. For example when reducing the Riemann tensor one finds graviphoton field strengths which carry a single upper i index. After dualization these become scalar fields with a single lower i index and therefore one finds factors of the form $e^{(1/\sqrt{2})\underline{\phi}\cdot[\underline{\lambda}_1]}$.

Let us now apply this method to the IIB theory dimensionally reduced on a seven torus to three dimensions. We start by giving the form of the ten-dimensional type IIB effective action which has a manifest $SL(2, \mathbf{R})$ symmetry. We will not need to be concerned with fermions or exact coefficients. In Einstein frame we have

$$\begin{aligned} S = \int d^{10}x\sqrt{-\hat{g}}(\hat{R} - (\partial\phi)^2 - e^{2\phi}(\partial\chi)^2 \\ - G^{a\mu_1\mu_2\mu_3}G^a_{\mu_1\mu_2\mu_3} - G_{\mu_1\ldots\mu_5}G^{\mu_1\ldots\mu_5}). \end{aligned} \tag{2.11}$$

The curvature R and five-form field strength are singlets of $SL(2, \mathbf{R})$. As the five-form field strength is self-dual, this condition must be imposed by hand and so the above action only has a limited validity but it is sufficient for our current purposes. The hat on $\hat{R}$ indicates that it is the Riemann tensor of the full higher dimensional metric $\hat{g}$.

We are interested in the dependence on the scalars ϕ, ρ, and $\underline{\phi}$ which we assemble into the 8-vector

$$\vec{\phi} = (\phi, \rho, \underline{\phi}). \tag{2.12}$$

In three dimensions, after the appropriate dualizations, we only have scalars. In addition to $\vec{\phi}$ there are scalars which arise as gauge fields with all internal indices or through dualizing one-form gauge fields in three dimensions. We denote all these additional scalars by $\chi_{\vec{\alpha}}$. The action will contain various terms involving derivatives of these scalars along with a coefficient of the form $e^{\sqrt{2}\vec{w}\cdot\vec{\phi}}$ for some 8-vector $\vec{w}$:

$$\vec{w} = (w, \kappa, \underline{w}). \tag{2.13}$$

The first entry w arises from the behavior of the fields under the $SL(2, \mathbf{R})$. The second entry simply records the power of $e^{\sqrt{2}\rho}$ that accompanies a field after dimensional reduction. The third component $\underline{w}$ corresponds to the $SL(7, \mathbf{R})$ representation of the fields.

It will be instructive to first derive the E_8 symmetry that arises when IIB supergravity is dimensionally reduced to three dimensions, that is the reduction of the action of Eq. (2.11). The reduction of the Einstein-Hilbert term R gives vectors of the form (see [39])

$$\vec{w} = (0, 0, [\underline{\theta}]), \qquad \vec{w} = (0, \sqrt{2}\tfrac{8}{7}\alpha, [\underline{\lambda}^1]), \tag{2.14}$$

where $\underline{\theta} = \underline{\lambda}^1 + \underline{\lambda}^6$ is the highest weight of the adjoint representation of $SL(7, \mathbf{R})$ and $[\underline{\theta}]$ denotes any element in the set of weights that appear in the adjoint representation, i.e. the roots of $SL(7, \mathbf{R})$. Similarly $[\underline{\lambda}^1]$ are the set weights that appear in the fundamental representation of $SL(7, \mathbf{R})$, i.e. $[\underline{\lambda}^1] = \{\underline{\lambda}^1, \ldots, -\underline{\lambda}^6\}$. This last set of vectors arises from the graviphotons that have been dualized and the steps leading to the $[\underline{\lambda}^1]$ part of the vector were outlined as explained above.

Next we can consider the dimensional reduction of

$$\sqrt{-\hat{g}}G^{a}_{\mu\bar{i}_1\bar{i}_2}G^{\mu b}_{\bar{j}_1\bar{j}_2}\delta^{\bar{i}_1\bar{j}_1}\delta^{\bar{i}_2\bar{j}_2}\delta_{ab}. \tag{2.15}$$

The vectors $\underline{\phi}$ that this term contributes are readily found using the discussion above. One finds that as $G^{a}_{\mu\bar{i}_1\bar{i}_2}$ has two $SL(7,\mathbf{R})$ indices associated with its $SL(7,\mathbf{R})$ transformation and hence one finds the contribution $[\underline{\lambda}^2]$ to the vector. Since it only has one index associated with its $SL(2,\mathbf{R})$ transformation this leads to a contribution $[\mu]$ to the part of the vector corresponding to ϕ. Thus one finds that this term gives rise to the series of vectors

$$\vec{w} = ([\mu], \tfrac{2\sqrt{2}}{7}\alpha, [\underline{\lambda}^2]). \tag{2.16}$$

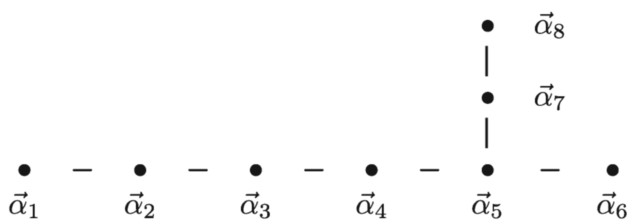
One such vector is

$$\vec{\alpha}_7 = (-\tfrac{1}{\sqrt{2}}, \tfrac{2\sqrt{2}}{7}\alpha, -\underline{\lambda}^5). \tag{2.17}$$

Last we reduce the ten-dimensional axion term $e^{2\phi}(\partial a)^2$ which leads directly to

$$\vec{\alpha}_8 = (\sqrt{2}, 0, 0). \tag{2.18}$$

One can readily verify that $\vec{\alpha}_i = (0, 0, \underline{\alpha}_i)$ with $i = 1, \ldots, 6$, $\vec{\alpha}_7$ and $\vec{\alpha}_8$ are the simple roots of E_8 with the corresponding Dynkin diagram



The bottom line contains the $SL(7,\mathbf{R})$ subalgebra associated to diffeomorphisms of the torus (i.e. the gravity line). The reduction also leads to terms in three dimensions with other vectors $\vec{w}$, in particular, one must reduce the five-form field strength. However the remaining vectors one finds turn out to be nonsimple roots of E_8. This appearance of this Dynkin diagram for type IIB string theory has an elegant origin in terms of E_{11} [62]. This viewpoint allows one to understand in an immediate way how the E_8 algebra arises in the dimensional reduction from the fields of the IIB theory.

Let us now consider the reduction of the possible higher derivative terms that can arise in the IIB string theory. We first consider the terms that involve the polynomials in the Riemann tensor multiplied by functions of the scalar fields τ, $\bar{\tau}$ which have the generic form (in Einstein frame)

$$S_E = \int d^{10}x\sqrt{-\hat{g}}(\hat{R})^{l/2}Z_x(\tau,\bar{\tau}). \tag{2.19}$$

We will take Z_x to behave as a sum of terms of the form $e^{-x\phi}$. In fact Z_x is a nonholomorphic automorphic form and only its leading order terms, corresponding to string perturbation theory, behave in this manner as $\phi \to -\infty$. We will not consider the nonperturbative contributions in the calculation in this section. The vectors $\vec{w}$ that arise from this term are (see [39])

$$\vec{w} = \left(-\frac{x}{\sqrt{2}}, \sqrt{2}\left(1 - \frac{l}{2} + \frac{8}{7}t\right)\alpha, s[\underline{\theta}] + t[\underline{\lambda}^1]\right), \tag{2.20}$$

where s, t are positive integers with $s + t \leq \frac{l}{2}$. In particular $2s$ and $2t$ count the number of S_μ and graviphoton field strengths that are contained in the dimensionally reduced term, respectively. To evaluate whether or not these vectors are weights of E_8 we must show that $\vec{\alpha}_i \cdot \vec{w}$ is an integer for all $i = 1, \ldots, 8$. Calculating away gives

$$\begin{aligned}
\vec{\alpha}_i \cdot \vec{w} &= s[\underline{\theta}]\cdot\underline{\alpha}_i + t[\underline{\lambda}^1]\cdot\underline{\alpha}_i \\
&= m, \qquad \vec{\alpha}_7 \cdot \vec{w} \\
&= \frac{x}{2} + \frac{4}{7}\left(1 - \frac{l}{2} + \frac{8}{7}t\right)\alpha^2 - s[\underline{\theta}]\cdot\lambda^5 - t[\underline{\lambda}^1]\cdot\lambda^5 \\
&= \frac{x}{2} + \frac{1}{4}\left(1 - \frac{l}{2} + \frac{8}{7}t\right) - t\frac{2}{7} + n \\
&= \frac{x}{2} + \frac{1}{4} - \frac{l}{8} + n, \\
\vec{\alpha}_8 \cdot \vec{w} &= -x,
\end{aligned} \tag{2.21}$$

where $m, n \in \mathbf{Z}$. The first expression is automatically an integer because $[\underline{\theta}]$ is a root and $[\underline{\lambda}^1]$ a weight of $SL(7,\mathbf{R})$. In the second expression we have used the facts that $[\underline{\lambda}^1] = \underline{\lambda}^1 - \underline{\alpha}$ where $\underline{\alpha}$ is a positive root of $SL(7,\mathbf{R})$ and $\underline{\lambda}^i \cdot \underline{\lambda}^j = \frac{i(7-j)}{7}$ for $i < j$.

It is instructive to transform this term to string frame by rescaling $g_{\mu\nu} \to e^{-(1/2)\phi}g_{\mu\nu}$. This results in the term

$$\begin{aligned}
S_S &= \int d^{10}x\sqrt{-\hat{g}}e^{((l/4)-(5/2))\phi}(\hat{R})^{l/2}Z(\tau,\bar{\tau}) \\
&\sim \int d^{10}x\sqrt{-\hat{g}}e^{((l/4)-(5/2)-x)\phi}(\hat{R})^{l/2}.
\end{aligned} \tag{2.22}$$

If this term is to arise in string perturbation theory then we require that $\frac{l}{4} - \frac{5}{2} - x = 2g - 2$ for some $g = 0, 1, 2, \ldots$. Thus we find that $\frac{x}{2} + \frac{1}{4} - \frac{l}{8} = -g$ and hence in this case

$$\begin{gathered}
\vec{\alpha}_i \cdot \vec{w} \in \mathbf{Z}, \qquad \vec{\alpha}_7 \cdot \vec{w} = -g + n \in \mathbf{Z}, \\
\vec{\alpha}_8 \cdot \vec{w} = -x.
\end{gathered} \tag{2.23}$$

Note that there is no condition that $x \in \mathbf{Z}$. Rather the condition $\frac{l}{4} - \frac{5}{2} - x = -2g - 2$ for some $g = 0, 1, 2, \ldots$ only implies that x is a half-integer. In Refs. [27–33], IIB higher derivative terms of the form $D^{2k}\hat{R}^4$ have been computed. For our purposes they are equivalent to $\hat{R}^{4+k}$. In particular for $4 + k = \frac{l}{2} = 4$ one finds perturbative corrections at tree level and one loop which have $x = 3/2$ while for $4 + k = \frac{l}{2} = 6$ one finds perturbative corrections at tree level and two loops which have $x = 5/2$. Thus one indeed finds for these and the other known cases that x is half-integer.

More generally we can consider a term of the form (in Einstein frame)

$$S_E = \int d^{10}x\sqrt{-\hat{g}}(\hat{R})^{l_1/2}(G^a_{\mu ij})^{l_2}(F_{\mu ijkl})^{l_3}Z(\tau, \bar{\tau}), \quad (2.24)$$

where $Z_x(\tau, \bar{\tau})$ is a similar function to that used above and, in particular, has the same generic ϕ dependence, i.e. sum of terms of the form $e^{-x\phi}$ in the perturbative limit. Using the analysis of [39], or the quicker method explained above, we can read off the vectors in as

$$\vec{w} = \Big(-\frac{x}{\sqrt{2}} + [\mu]l_2, \sqrt{2}\Big(1 - \frac{l_1}{2} + \frac{8}{7}t - \frac{5}{7}\frac{l_2}{2} - \frac{3}{7}\frac{l_3}{2}\Big)\alpha, s[\underline{\theta}] + t[\underline{\lambda}^1] + \frac{l_2}{2}[\underline{\lambda}_2] + \frac{l_3}{2}[\underline{\lambda}_4]\Big). \quad (2.25)$$

The only nontrivial tests that this is a weight come from $\vec{\alpha}_7 \cdot \vec{w}$ and $\vec{\alpha}_8 \cdot \vec{w}$. In the later case we simply have $\vec{\alpha}_8 \cdot \vec{w} = -x \pm l_2 \in \mathbf{Z}$ whereas

$$\begin{aligned}\vec{\alpha}_7 \cdot \vec{w} &= \frac{x}{2} + \frac{l_2}{2} + \frac{1}{4}\Big(1 - \frac{l_1}{2} + \frac{8}{7}t - \frac{5}{7}\frac{l_2}{2} - \frac{3}{7}\frac{l_3}{2}\Big) \\ &\quad - \frac{2}{7}t - \frac{2}{7}l_2 - \frac{4}{7}l_3 + n \\ &= \frac{x}{2} + \frac{l_2}{2} + \frac{1}{4} - \frac{l_1}{8} - \frac{3l_2}{8} - \frac{5l_3}{8} + n \end{aligned} \quad (2.26)$$

with $n \in \mathbf{Z}$. Again converting to string frame, where the dilaton appears through the factor $e^{2(g-1)\phi}$, tells us that

$$2g - 2 = -x - \frac{5}{2} - \frac{l_1}{4} - \frac{3l_2}{4} - \frac{5l_3}{4} \quad (2.27)$$

and hence

$$\vec{\alpha}_7 \cdot \vec{w} = -g + n + \frac{l_2}{2} \in \mathbf{Z} \quad (2.28)$$

since l_2 must be even. Here we again see that we find weights if $x \in \mathbf{Z}$ but generically x is half an integer.

We note that there are more terms that can be considered. For example, we could include terms involving higher powers of $\partial\phi$ and $\partial\chi$ however these will behave in a similar way to $\partial\rho$ which arises from dimensional reduction of the Riemann tensor. Other terms arise from components of $G^a_{\mu\nu i}$ and $F_{\mu\nu ijk}$ with two spacetime indices in three dimensions. These require dualization into scalar fields but this is complicated by the dilaton (just as was encountered for the Bosonic string in [39]) however we do not expect that these terms will alter the conclusion.

In this section we have examined the possible higher derivative corrections that can arise in the IIB string theory. We have computed the vectors $\vec{w}$ associated with the scalars $\vec{\phi} = (\phi, \rho, \underline{\phi})$. For the lowest order terms of IIB supergravity itself these belong to the root lattice of E_8, in fact they are positive roots of the adjoint representation of E_8. The dilaton dependence is constrained by demanding that the terms arise as a perturbative correction of IIB string theory. Requiring that this is the case one finds that the vectors $\vec{w}$ are half-weights of E_8, using the conventions of [39]. Although we note here that the vectors $\vec{w}$ are weights with respect to $SL(7, \mathbf{R})$, they are only half-weights with respect to the 8th node of E_8 which is associated with the $SL(2, \mathbf{R})$ symmetry of IIB supergravity.

III. AUTOMORPHIC FORMS IN HIGHER DERIVATIVE CORRECTIONS

As mentioned above, the IIA and IIB supergravity theories encode all the low energy effects of IIA and IIB string theories and so must contain all nonperturbative low energy effects including phenomena which are not calculable from our known formulations of string theory. One of the most interesting properties of the IIB supergravity theory is that it possesses an $SL(2, \mathbf{R})$ symmetry [4]. Furthermore, if one dimensionally reduced either the IIA or the IIB supergravity theories on a $n-1$ torus, or the 11-dimensional supergravity theory on a n torus, one finds the same set of supergravity theories and remarkably these possess an E_n symmetry for $n \leq 9$ [8–10].

The dimensionally reduced maximal supergravity theories on a torus are also the low energy effective actions for the type II string theories on a $(n-1)$ torus, or the ill understood M theory on a n torus. As we already mentioned they are invariant under a continuous symmetry group which is nonlinearly realized with respect to a local subgroup. It will prove useful to describe the representations of this symmetry that the fields in these theories belong to and as this discussion applies to many such theories we will denote the nonlinearly realized group by G and the local subgroup by H. For the IIB theory $G = SL(2, \mathbf{R})$ and the local subgroup is $SO(2)$, whereas when dimensionally reduced to four dimensions one finds $G = E_7$ and $H = SU(8)$ and $G = E_8$ and $H = SO(16)$ in three dimensions. It turns out that in all the cases we will consider the local subgroup H is just the Cartan involution invariant subgroup. We recall that the Cartan involution I is an automorphism, i.e. it obeys $I(g_1g_2) = I(g_1)I(g_2)\ \forall\ g_1, g_2 \in G$, such that $I^2 = 1$ and acts on the Chevalley generators as $I(H_a) = -H_a$, $I(E_a) = -F_a$, $I(F_a) = -E_a$.

If one dimensionally reduces to three dimensions one finds, using suitable dualizations, a theory with just scalars which belong to the coset G/H. In this paper we will work with the coset representatives that we denote by $g(\xi)$. These transform under a rigid transformation $g_0 \in G$ as $g(\xi) \rightarrow g(\xi')$ where

$$g_0 g(\xi) = g(\xi')h(g_0, \xi) \quad (3.1)$$

and $h(g_0, \xi) \in H$ is the compensating transformation required to restore the choice of coset representative. This induces a nonlinear realization of G on the parameters ξ which we denote by $\xi' = g_0 \cdot \xi$.

The dynamics of the scalars is constructed from the Cartan form $g^{-1}\partial_\mu g$ which takes values in the Lie algebra of G and is invariant under the rigid transformations $g(x) \rightarrow g_0 g(x)$. The Cartan form can be written as

$$g^{-1}\partial_\mu g = P_\mu + Q_\mu, \tag{3.2}$$

where Q_μ is in the Lie algebra of H. Our choice of local subgroup H is odd under the Cartan involution I ($I(h) = -h$ for $h \in H$) and so $I(Q_\mu) = Q_\mu$ and then $P_\mu = g^{-1}\partial_\mu g - I(g^{-1}\partial_\mu g)$ and so satisfies $I(P_\mu) = -P_\mu$. This implies that the commutators of generators of the Lie algebra of H with the generators which are odd under the Cartan involution leads to generators which are also odd. As such, under the local transformation $g(x) \rightarrow g(x)h(x)$ we find $P_\mu \rightarrow h^{-1}P_\mu h$, while Q_μ transforms as $Q_\mu \rightarrow h^{-1}Q_\mu h + h^{-1}\partial_\mu h$. The invariant low energy Lagrangian for the scalars is then given by $\mathrm{Tr}(P_\mu P^\mu)$.

If one dimensionally reduces on a torus to a dimension above three then one will find Bosonic fields other than scalars, in particular, in addition to gravity one will find gauge fields. As we discussed in the last section, any gauge fields must transform linearly under the rigid transformations g_0 of the group G (see (A8));

$$U(g_0)\psi_a = D(g_0^{-1})_a{}^b\psi_b. \tag{3.3}$$

Consequently the field strengths also transform as in Eq. (3.3). However, as explained at the end of Appendix A, using the scalar fields of the theory, we can always convert a field that transforms under linear representations of G into a field that transforms under the nonlinear representation

$$U(g_0)\varphi_a(\xi) = D(h^{-1}(g_0, \xi))_a{}^b\varphi_b(\xi) \tag{3.4}$$

by taking $\varphi_a(\xi) = D(g^{-1}(\xi))_a{}^b\psi_b$. To respect gauge invariance we must perform this conversion on the field strength and not on the gauge fields.

The scalars by themselves always occur with their derivatives as in Eq. (3.2). However the quantity Q_μ only occurs in the dynamics as a connection for spacetime derivatives acting on fields, such as field strengths, leaving the scalars to appear through P_μ. The fermions also transform as a nonlinear realization. Therefore, all the fields that appear in the dynamics of IIB supergravity theory and IIB supergravity dimensional reduction on a $(n-1)$ torus (or equivalently the IIA supergravity theory on a $n-1$ torus or M theory on a n-torus) can be taken to transform as a nonlinear representation of G with local subgroup H, i.e. as in Eq. (3.4) for some representation D of H.

As mentioned above the continuous groups $SL(2, \mathbf{R})$ and E_n are symmetries of the IIB supergravity theory and this theory dimensionally reduced on a $(n-1)$ torus, respectively. Although these theories are the low energy effective actions for the type IIA and IIB string theories on a $n-1$ torus, these continuous symmetries are not symmetries of the underlying string theories or M theory. The supergravity theories possess solitonic solutions corresponding to strings and branes and the symmetries rotate the field strengths and charges associated with these solitons. However, the latter are subject to quantization conditions [21] and have been conjectured in string theory that the symmetries survive if these groups were restricted to a discrete subgroup which preserves the lattice of charges [22–24]. The precise form of this group is clear for the IIB theory; it is just the one generated by two-by-two matrices with integer entries and the determinant one. This is the so-called U-duality conjecture which can be thought of as a combination of the T-duality, which is known to be a valid symmetry of string theory, combined with the $SL(2, \mathbf{Z})$ symmetry of the IIB theory.

In this section we will consider the higher derivative corrections that can occur in string theories where some of the dimensions are tori. We will assume that they are invariant under the discrete group $G(\mathbf{Z})$ mentioned above and our aim is to discover what are the consequences of demanding such symmetries on the general form of such corrections. We will also assume that the fields in effective actions of such theories transform in the same way that they did in the low energy effective action. In effect this assumes that there is a choice of field variables such that the transformation rules are unaffected by higher derivative terms. That is the fields occur in expressions which involve their spacetime derivatives and transform as in Eq. (3.3), except that now the rigid g_0 transformations will belong to $G(\mathbf{Z})$ rather than the continuous group G. When expressed in Einstein frame the higher derivative terms are of the generic form

$$\int d^d x\sqrt{-g}Z(\xi)X, \tag{3.5}$$

where X is a polynomial in the Riemann curvature, the modified field strengths, and the covariant derivatives of scalar fields. All these quantities will transform as in Eq. (3.3). An important exception to the above statement is the appearance of the function $Z(\xi)$ of the scalar fields ξ which belong to the coset space G/H. Such a function does not contain spacetime derivatives and their appearance signals the fact that we no longer have invariance under the continuous G symmetry, but only under its discrete subgroup $G(\mathbf{Z})$.

Since the objects that make up X transform as in Eq. (3.4), it follows that X itself, will transform as

$$U(g_0)X = D(h^{-1}(g_0, \xi))X, \tag{3.6}$$

where g_0 is a transformation of $G(\mathbf{Z})$ and $h(g_0, \xi)$ is the compensating H transformation required in Eq. (3.1), that is $g(\xi) \rightarrow g_0 g(\xi) = g(\xi')h(g_0, \xi)$ and for suitable representation D. Demanding that the higher derivative term be invariant under $G(\mathbf{Z})$ we find that

$$Z(g_0 \cdot \xi) = D(h(g_0, \xi))Z(\xi). \tag{3.7}$$

When carrying out the variation it is important to note that

$Z(\xi)$ is an explicit function of ξ and so its variation just changes the value of ξ under the action of $g_0 \in G(\mathbf{Z})$. As we will explain in the next section, this last equation is just the transformation property of an automorphic form of $G(\mathbf{Z})$. We note that these automorphic forms are not in general holomorphic and indeed are in most cases nonholomorphic. In Sec. III we will also discuss the additional constraints, such as differential equations as well as growth conditions, that nonholomorphic automorphic forms are expected to also obey.

The simplest case is when X is invariant under the transformations of $G(\mathbf{Z})$, i.e. $D = 1$. It is then obvious that $Z(\xi)$ is inert and so $Z(g_0 \cdot \xi) = Z(\xi)$. Such is the case if X is a polynomial in the Riemannn tensor, or when spacetime derivatives act on a polynomial of Riemann tensors. Such examples have been studied in detail for the IIB theory in Refs. [27–36].

Thus we conclude that if we assume that the higher derivative corrections are invariant under a $G(\mathbf{Z})$ duality symmetry then every possible term will generically contain functions of the scalar fields, which belong to the coset space G/H, that transform as automorphic forms of the group $G(\mathbf{Z})$.

We now illustrate the above discussion in the familiar context of the IIB string theory as this will allow us to make contact with the work of Refs. [27–38]. In the previous section we discussed the $SL(2,\mathbf{R})$ formulation of the IIB supergravity theory where the local subgroup is $SO(2)$. As explicitly derived in Appendix C, under an element

$$g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z}) \tag{3.8}$$

the compensating $SO(2)$ transformation is given by

$$h = \begin{pmatrix} \cos\theta_c & -\sin\theta_c \\ \sin\theta_c & \cos\theta_c \end{pmatrix}, \qquad e^{2i\theta_c} = \frac{c\tau + d}{c\bar{\tau} + d}. \tag{3.9}$$

All of the type IIB fields transforms as Eq. (3.4) with D which is given by

$$D(h^{-1}) = e^{-iq\theta_c} \tag{3.10}$$

for some q. In particular $q = 0$ for the metric and five-form whereas $q = 1, -1$ for the three-form $G_{\mu_1\mu_2\mu_3}$ and its complex conjugate, respectively. The two scalars belong to the coset $SL(2,\mathbf{R})/SO(2)$. The Cartan involution odd part of the Cartan forms, P_μ transform under $SL(2,\mathbf{R})$ by a matrix which is in the doublet representation of $SO(2)$ which is reducible. Writing this representation as P^a_μ, the two irreducible representations are given by $\mathcal{P}_\mu = P^1_\mu + iP^2_\mu$ and its complex conjugate $\bar{\mathcal{P}}_\mu = P^1_\mu - iP^2_\mu$ with $q = 2, -2$, respectively.

The above discussion on higher derivative terms is easy to apply to the IIB supergravity theory. The object X of Eq. (3.4) will have a total charge q_T which is just the sum of the charges of its factors. The corresponding automorphic form $Z(\xi)$ that multiplies X must transform as

$$Z_X(g_0 \cdot \tau) = e^{iq_T\theta_c} Z_X(\tau). \tag{3.11}$$

This agrees with the discussions of a number of terms given in Refs. [27–38].

We can also apply it to higher derivative terms of the IIB theory involving derivatives of scalars. Such a term, which involves only the scalars, will be of the form

$$\int d^{10}x\sqrt{-g} Z_{r,s}(\tau,\bar{\tau}) \mathcal{P}^r_\mu \bar{\mathcal{P}}^{\mu s}. \tag{3.12}$$

As $\mathcal{P}^r_\mu \bar{\mathcal{P}}^{\mu s}$ has a total $U(1)$ weight $2(r-s)$ we find that Z transforms as

$$Z_{r,s}(g_0 \cdot \tau) = \left(\frac{c\tau + d}{c\bar{\tau} + d}\right)^{r-s} Z_{r,s}(\tau). \tag{3.13}$$

It is obvious how to generalize this discussion to terms that involve derivatives of the scalars as well as other objects.

As another example, let us consider the higher derivative terms of superstring theory on a seven torus or M theory on an eight torus. The only dynamical bosonic fields of the low energy theory are scalars and they possess an E_8 symmetry with local subgroup $SO(16)$. As explained above, the scalars arise in the dynamics of the low energy effective action through the Cartan forms of Eq. (3.2) which belong to the Lie algebra of E_8 and so are in the 248-dimensional adjoint representation. The Q_μ which occurs in this equation belongs to the Lie algebra of $SO(16)$ which is the 120-dimensional adjoint representation. Therefore the P_μ belongs to a 128-dimensional representation of $SO(16)$ and must be a Majorana-Weyl spinor $P_{\mu\alpha}$ in 16 dimensions. The kinetic term for the scalars arises in the low energy effective action as $\bar{P}_\mu P^\mu$, the bar now being the Majorana conjugate and we have suppressed the spinor index. The higher derivative corrections are of the form of Eq. (3.5) where X is a polynomial of $P_{\mu\alpha}$ which transforms as in Eq. (3.4) where the specific representation matrix D of $SO(16)$ depends on how the polynomial is constructed. The automorphic form $Z(\xi)$ of the 128 scalars ξ will therefore transform as in Eq. (3.7) with the same D. To be concrete consider the higher order term with $2r$ spacetime derivatives that contains a term of the form

$$\int d^3x\sqrt{-g}(\bar{P}_{\mu_1}\gamma^{a_1}P^{\mu_1})\ldots(\bar{P}_{\mu_r}\gamma^{a_r}P^{\mu_r})Z_{a_1\ldots a_r}(\xi). \tag{3.14}$$

It follows that $Z_{a_1\ldots a_r}(\xi)$ is an automorphic form of E_8 that transforms with a matrix D that is in the rank r symmetric tensor representation of $SO(16)$ and whose argument is the $SO(16)$ compensating transformation.

IV. AUTOMORPHIC FORMS AND INDUCED REPRESENTATIONS

In this section we will show that automorphic forms arise naturally from the theory of induced representations.

As a consequence of adopting this viewpoint we will find that they have precisely the same transformations as do the functions of the scalar fields $Z(\xi)$ that occur in the higher derivative corrections in Eq. (3.6). In this way will be able to identify the $Z(\xi)$ factors as having the transformation properties of automorphic forms. We will also give a procedure for constructing automorphic forms for a general group G with local subgroup H. Much of the mathematics literature on automorphic forms is restricted to the particular case of $SL(2, \mathbf{R})$ with local subgroup $SO(2)$. In this section we will give a limited account of automorphic forms which we expect will cover all the possibilities that occur in the higher derivative corrections of string theory and M theory. Automorphic forms for higher derivative corrections were also discussed in [37], and include their relation to string theory. In particular explicit examples were given for the cases of $SL(n, \mathbf{R})/SO(n, \mathbf{R})$, $SO(d, d)/SO(d) \times SO(d)$, and E_d/H and second order differential equations which these automorphic forms satisfy were given. The examples given in [37] can be constructed by the method that we give below. Another discussion of automorphic forms intended for physicists is given in [63].

For a group G with local subgroup H we consider the coset space G/H whose coset representative is denoted by $g(\xi)$. The group G has natural action on the coset and therefore also on the coset representatives which transform under transformations $g_0 \in G$ through Eq. (3.1). This coset space will be of dimension $\dim G$- $\dim H$ and in general this will not be an even number, as is, for example, the case for $G = SL(n)$ and $H = SO(n)$ if $n \equiv 0$, 3 mod4. Therefore the coset space does not in general have a complex structure and even when it does we will consider nonholomorphic automorphic forms. For the application we have in mind in this paper the coset labels ξ are scalar fields and will depend on spacetime. However this will play no role in the considerations in this section; indeed the dependence of ξ on spacetime is always the same in all equations.

We consider an induced representation of a group G with local subgroup H which consists of map Φ from the coset G/H to a vector space V that has the transformation rule (c.f. Eq. (A4))

$$U(g_0)\Phi_a(\xi) = D(h^{-1}(g_0, \xi))_a{}^b \Phi_b(g_0 \cdot \xi), \tag{4.1}$$

where D is a linear realization of H, and $h(g_0, \xi)$ is the compensation of Eq. (3.1).

Rather than considering the continuous group G to act on G/H we now replace this action by that of a discrete $G(\mathbf{Z})$. For example, instead of $SL(n, \mathbf{R})$ we can consider the discrete group defined from its fundamental representation with integer entries, that is we consider the group of $n \times n$ matrices with integer entries with determinant one. We then consider functions Φ which transform as in Eq. (4.1), but now with $g_0 \in G(\mathbf{Z})$. We note that although Φ transforms under the discrete group $G(\mathbf{Z})$ it depends on the coset G/H associated with the continuous group.

Automorphic forms of $G(\mathbf{Z})$ arise from induced representations if we demand that Φ is invariant under the action of $G(\mathbf{Z})$;

$$U(g_0)\Phi_a(\xi) = \Phi_a(\xi). \tag{4.2}$$

It then follows that

$$\Phi_a(g_0 \cdot \xi) = D(h(g_0, \xi))_a{}^b \Phi_b(\xi). \tag{4.3}$$

The simplest case is when $D(h(g_0, \xi))$ is the identity matrix, in which case the index a takes only one value and the automorphic form is simply invariant. Although this may not be familiar in this form, it is the transformation of an automorphic form. In Appendix C we will show that it does indeed agree with the familiar results for the much studied case of $SL(2, \mathbf{Z})$. Imposing Eq. (4.3) for the continuous group would of course mean that Φ_a is a constant as any two points on the coset are related by a group element of G. However, this is not the case for the discrete group whose fundamental domain is the coset $G(\mathbf{Z}) \setminus G/H$.

The transformation of Eq. (4.3) is the same as the transformation of the coefficients $Z(\xi)$ which appear in the higher derivative terms of string theory discussed in Sec. III. This followed by demanding that these higher derivative terms be $G(\mathbf{Z})$ invariant. Therefore we can identify the coefficients Z as automorphic forms. However, as we are dealing with nonholomorphic modular forms they should also satisfy some additional conditions, such as differential equations, which we will discuss later in this section.

So far we have defined an automorphic form on the coset G/H, however, one can also define them on the group by taking functions Φ_L from the group to the vector space V which are induced representations in the sense of Eq. (A4) under the discrete group $G(\mathbf{Z})$, but also satisfy $U(g_0)\Phi_L(g) = \Phi_L(g)$.

To continue it is useful to compare our treatment of automorphic forms with that which is usually encountered in the mathematics literature for the case of $SL(2, \mathbf{Z})/SO(2)$. The transformation of the automorphic form Φ is often written as

$$\Phi(g_0 \cdot \xi) = J(g_0, \xi)\Phi(\xi), \tag{4.4}$$

where $J(g_0, \xi)$ is called the automorphy factor. This latter factor is usually just a function, but more generally it is a matrix acting on Φ with elements that depend on g_0 and ξ. Evaluating $\Phi(g_0' \cdot (g_0 \cdot \xi)) = \Phi((g_0' g_0) \cdot \xi)$ we conclude that

$$J(g_0' g_0, \xi) = J(g_0', g_0 \cdot \xi) J(g_0, \xi) \tag{4.5}$$

which is consistent with identifying the factors $D(h(g_0, \xi))$ as automorphy factors as a consequence of Eq. (A3).

We now construct some automorphic forms from a linear irreducible representation R, with components ψ_a, of the group G. Given any such representation R we can form a nonlinear representation with components $\varphi_a(\xi)$ which depends on the coset G/H by taking (c.f. Eq. (A9))

$$\varphi_a(\xi) = D(g^{-1}(\xi))_a{}^b \psi_b. \tag{4.6}$$

We are interested in the restriction of this representation to the subgroup $G(\mathbf{Z})$ under which the components $\varphi_a(\xi)$ transform under $g_0 \in G(\mathbf{Z})$ as (see Eq. (A10))

$$\begin{aligned} U(g_0)\varphi_a(\xi) &= D(g^{-1}(\xi))_a{}^b D(g_0^{-1})_b{}^c \psi_c \\ &= D((g_0 g)^{-1}(\xi))_a{}^b \psi_b \\ &= D(h^{-1}(g_0, \xi))_a{}^b \varphi_b(\xi'). \end{aligned} \tag{4.7}$$

Although we started with an irreducible representation of G it will not be an irreducible representation of $G(\mathbf{Z})$. To obtain an irreducible representation we restrict our states to a discrete lattice $\Lambda_R \subset V$. To construct Λ_R one can take a fixed basis of V and then act on it with $G(\mathbf{Z})$.

The automorphic forms are essentially functions of the nonlinear representation $\varphi_a(\xi)$ averaged over the representation ψ_a from which it is constructed, that is functions of the generic form

$$\Phi(\xi) = \sum_{\Lambda_R} f(\varphi_a(\xi)) = \sum_{\Lambda_R} f(D(g^{-1}(\xi))_a{}^b \psi_b), \tag{4.8}$$

where $f\colon V \rightarrow V'$ is a function into some vector space V' and we have suppressed any indices on $\Phi(\xi)$ and f. The sum is over the lattice Λ_R which are the states in the discrete representation R.

Let us first construct automorphic forms that are invariant under $G(\mathbf{Z})$ and so consider taking f of the form

$$f(\varphi_a(\xi)) = K(u(\xi)) \equiv f(\xi), \tag{4.9}$$

for some function $K\colon \mathbf{C} \rightarrow \mathbf{C}$. Here $u(\xi)$ is constructed from the dual and Cartan involution twisted representations introduced in Eqs. (B9) and (B11). In particular, we take $u(\xi)$ to be given by

$$u(\xi) \equiv \varphi_{ID}(\xi)^a \varphi_a(\xi) = \psi^a_{ID} D(\mathcal{M}^{-1}(\xi))_a{}^b \psi_b, \tag{4.10}$$

where $\mathcal{M}(\xi) = g(\xi) g^{\#}(\xi)$. The automorphic form of Eq. (4.8) is given by

$$\Phi(\xi) = \sum_{\Lambda_R} K(u(\xi)). \tag{4.11}$$

Using Eqs. (4.8), (B14), and (B16) we find that under the transformation $g_0 \in G(\mathbf{Z})$ that $\varphi^a_{ID}(\xi)$ transforms as

$$U(g_0)\varphi^a_{ID}(\xi) = \varphi^b_{ID}(\xi') D(h(g_0, \xi))_b{}^a. \tag{4.12}$$

It is clear from Eqs. (4.7) and (4.11) that

$$\begin{aligned} U(g_0)K(u(\xi)) &= K(u(\xi')) \quad \text{and so} \\ U(g_0)\Phi(\xi) &= \Phi(\xi'). \end{aligned} \tag{4.13}$$

We note that $\varphi^a_{ID}(\xi)$ and $\varphi^a_D(\xi)$ transform in the same way as we assumed that the subgroup H is invariant under the Cartan involution I, i.e. $I(h) = h$, $\forall\ h \in H$. However, had we taken the latter instead of the former then $u(\xi)$ would be independent of ξ and so K would be uninteresting.

Last we show that $\Phi(\xi)$ is invariant. We note that $\Phi(\xi)$ is constructed from $\varphi_a(\xi)$ which is in turn given by Eq. (4.6) in terms of ψ_b. Examining the action of $U(g_0)$ on $\varphi_a(\xi)$ given in Eq. (4.7) we see that its effect can also be viewed as replacing ψ_b by $D(g_0^{-1})_b{}^c \psi_c$. However, this just rearranges the states in the lattice Λ_R and as we are summing over all states we conclude that the total is invariant and hence $U(g_0)\Phi(\xi) = \Phi(\xi)$. Together with Eq. (4.13) implies that

$$\Phi(\xi) = \Phi(\xi') \tag{4.14}$$

in other words it transforms as an invariant automorphic form.

A natural choice of $K(u(\xi))$ is to take

$$K(u(\xi)) = \frac{1}{(u(\xi))^s} \tag{4.15}$$

and in Appendix C we will show that this choice along with taking ψ_a to be the vector representation of $SL(2, \mathbf{R})$ leads to the invariant nonholomorphic Eisenstein series of $SL(2, \mathbf{Z})$.

We now construct automorphic forms that transform in a nontrivial way under the action of $G(\mathbf{Z})$. Let us take

$$\begin{aligned} f_a(\xi) &= \varphi_a(\xi) K(u(\xi)), \quad \text{or equivalently} \\ \Phi_a(\xi) &= \sum_{\Lambda_R} \varphi_a(\xi) K(u(\xi)). \end{aligned} \tag{4.16}$$

We note that $\Phi_a(\xi)$ is a map from G/H to the vector space V which carries the representation R.

Using Eqs. (4.7) and (4.17) we find that $f_a(\xi)$ transforms under $g_0 \in G(\mathbf{Z})$ as

$$U(g_0) f_a(\xi) = D(h^{-1}(g_0, \xi))_a{}^b f_b(\xi'). \tag{4.17}$$

Since the matrix factor $D(h^{-1}(g_0, \xi))_a{}^b$ is independent of what is being summed over it follows that

$$U(g_0)\Phi_a(\xi) = D(h^{-1}(g_0, \xi))_a{}^b \Phi_b(\xi'). \tag{4.18}$$

Following the same argument as above which interprets this transformation as a change in the sum over the representation, we conclude that

$$\Phi_a(\xi') = D(h(g_0, \xi))_a{}^b \Phi_b(\xi) \tag{4.19}$$

in other words it transforms as an automorphic form.

The above construction can be generalized in several ways that may be important for the automorphic forms that occur in the higher derivative corrections of string theory. First, one can give a more general construction of $u(\xi)$. An invariant under the transformations of $G(\mathbf{Z})$, apart from the usual transformation of the coset variables, can be found by taking any function of φ_a which is invariant under $\varphi_a \to D(h)_a{}^b\varphi_b$ for all $h \in H$. Although the latter is not a transformation of $G(\mathbf{Z})$, the invariance of $u(\xi)$ under it then ensures that $u(\xi)$ is invariant under $G(\mathbf{Z})$ up to the usual transformation of ξ. This is a consequence of the fact that the composite matrices $D(h^{-1}(g_0, \xi))_b{}^a$ that arise in the $U(g_0)$ transformation of $u(\xi)$ will cancel out. As noted elsewhere, for our special choice of subgroup H, there is a choice of coset representative such that the $U(h_0)$, $h_0 \in H$ transformation of φ_b will be by a matrix which just $D(h_0^{-1})_a{}^b$ and ξ will be a linear representation of H.

We may also generalize the construction by considering automorphic forms which are lattice sums over $\varphi(\xi)_a\varphi(\xi)_b K(u(\xi))$, or more general polynomials. The automorphic forms will then transform by composite matrices belonging to symmetric tensor products of the H-representation that occurs for φ_a. In fact we will use this possibility to construct automorphic forms for $SL(n, \mathbf{Z})$ in Appendix C. One could also use a nonlinear realization that is constructed from a different linear representation to ψ_a for the factors that are outside $K(u(\xi))$.

We note that the automorphic form is constructed from $\varphi_a(\xi)$, which, as shown in Eq. (4.7), has the usual transformation of ξ under the action of the group $G(\mathbf{Z})$ as well as a rotation by a matrix which depends on an, albeit composite, element of H. As such, the most general construction is essentially determined by finding invariants, or other tensors, of the H-representation of $\varphi_a(\xi)$, even though the symmetry group is $G(\mathbf{Z})$. The situation has some similarities to the case of the construction of nonlinear realizations of the continuous group G. These can be constructed from $g^{-1}\partial_\mu g$, or more precisely for the case of scalars alone from $P_\mu = g^{-1}\partial_\mu g - I((g^{-1}\partial_\mu g)$. This transforms under G as $P_\mu(\xi) \to h^{-1}(g_0, \xi)P_\mu(\xi')h(g_0, \xi)$. As a result, $P_\mu(\xi)$ is just a particular instance of a nonlinear representation $\varphi_a(\xi)$. In general what higher order invariants one can construct depends on the invariants that exist in the tensor products of the H-representations that occur in $P_\mu(\xi)$.

There is an essential difference between the construction of nonlinear realization and the construction of automorphic forms which is crucial for this paper. For the continuous groups the effective action for the scalars alone is constructed from $g^{-1}\partial_\mu g$ and this involves the roots of the Lie algebra. However, for the discrete group $G(\mathbf{Z})$ we find that automorphic form depends on the coset fields ξ that are contained in $g(\xi)$ and which can be chosen to be of the form

$$g(\xi) = e^{\sum_{\vec{\alpha}>0} E_{\vec{\alpha}}\chi_{\vec{\alpha}}} e^{-(1/\sqrt{2})\vec{\phi}\cdot\vec{H}}, \tag{4.20}$$

where $\vec{H}$ are the Cartan subalgebra generators and $E_{\vec{\alpha}}$ are the positive root generators of G. In fact, the explicit construction given above actually involves $g(\xi)$ only through

$$\mathcal{M}^{-1} = e^{-\sum_{\vec{\alpha}>0} E_{\vec{\alpha}}\chi_{\vec{\alpha}}} e^{\sqrt{2}\vec{\phi}\cdot\vec{H}} e^{\sum_{\vec{\alpha}>0} E_{\vec{\alpha}}\chi_{\vec{\alpha}}} \tag{4.21}$$

although as discussed more general possibilities may occur. The fields ξ that parametrize the coset are made up of the fields associated with the above generators; we will refer to them as the Cartan subalgebra fields $\vec{\phi}$ and the "axions" $\chi_{\vec{\alpha}}$, respectively. We will now show that in the automorphic forms discussed above one finds that weights of G, rather than the roots, appear as the coefficients of the Cartan subalgebra fields $\vec{\phi}$.

It is particularly instructive to study the perturbative contribution to the automorphic form. In addition to the Cartan subalgebra fields $\vec{\phi}$ the automorphic form depends on the "axion" fields $\chi_{\vec{\alpha}}$. Within the context of string theory these modes arise from components of gauge fields (or in type IIB string theory as Ramond-Ramond 0-form). As such there is a perturbative shift symmetry $\chi_{\vec{\alpha}} \to \chi_{\vec{\alpha}} + \epsilon_{\vec{\alpha}}$ for an arbitrary $\epsilon_{\vec{\alpha}}$. These symmetries typically arise from $U(1)$ gauge transformations that are not single valued on the torus. In the full quantum theory the holonomy of a $U(1)$ gauge field around a circle is required to vanish so that the wave function is single valued. The allowed gauge transformations are therefore restricted and one finds that the continuous shift symmetry is broken to a discrete one. This implies that the corresponding scalar field is periodic. However this discreteness cannot be seen in a perturbative calculation where the gauge fields are taken to be small fluctuations about the trivial configuration. Thus the axions only occur in the nonperturbative contributions to the automorphic form. In fact, the automorphic forms have a sort of periodicity under integer shift in χ_α and so possess a Fourier expansion in χ_α.

Since, the perturbative contribution is independent of χ_α, we can find this contribution by first setting $\chi_\alpha = 0$ and then taking the perturbative limit. In other words, the perturbative part of the automorphic form can be calculated by first restricting $g(\xi)$ to its Cartan subalgebra and then taking the perturbative limit. Thus we make the replacement

$$g(\xi) \to h(\vec{\phi}) = e^{-(1/\sqrt{2})\vec{\phi}\cdot\vec{H}}. \tag{4.22}$$

We note that in this case $\mathcal{M} \to e^{\sqrt{2}\vec{\phi}\cdot\vec{H}}$ and as a result, we find that

$$u(\xi) \to \langle\psi_{ID}|U(\mathcal{M}^{-1})|\psi\rangle = e^{\sqrt{2}\vec{\phi}\cdot[\vec{\Lambda}]}\langle\psi_{ID}|\psi\rangle, \tag{4.23}$$

where $\underline{\Lambda}$ is the highest weight of the representation $|\psi\rangle$. In order for the lattice sum to converge it must be that $K(u) \to 0$ as $u \to \infty$ so let us assume that, at large u, $K = u^{-s}$ with $s > 0$. In the perturbative limit the lattice sum will be dominated by states for which $\vec{\phi}\cdot[\vec{\Lambda}]$ is the most negative[2]

[2]It is possible that more than one weight will contribute but we will ignore this issue here.

$$K \to \sum_{\Lambda_R} \frac{e^{-\sqrt{2}s\vec{\phi}\cdot[\vec{\Lambda}]}}{\langle\psi_{ID}|\psi\rangle^s} \sim e^{-\sqrt{2}s\vec{\phi}\cdot\vec{w}_\Lambda}\sum_{\Lambda'_R}\frac{1}{\langle\psi_{ID}|\psi\rangle^s} \sim N_s e^{-\sqrt{2}s\vec{\phi}\cdot\vec{w}_\Lambda}, \tag{4.24}$$

where $\vec{w}_\Lambda$ is the weight in the representation of $[\vec{\Lambda}]$ for which $\vec{\phi}\cdot[\vec{\Lambda}] \to -\infty$ the most quickly, Λ'_R is the set of states in Λ_R with this weight, and $N_s = \sum_{\Lambda'_R}\langle\psi_{ID}|\psi\rangle^{-s}$ is a constant.

Last we must consider the contribution of φ_a in Eq. (4.16) for the cases where the automorphic form has a nontrivial transformation under H. In the limit that we can set the axions to zero we have that

$$\varphi_a = D(g^{-1})_a{}^b\psi_b \to D(e^{(1/\sqrt{2})\vec{\phi}\cdot\vec{H}})_a{}^b\psi_b = e^{(1/\sqrt{2})\vec{\phi}\cdot\vec{w}_\Lambda}\psi_a \tag{4.25}$$

which has the same form as (4.24). Thus we see that, in the perturbative limit, $\phi_a \sim e^{-\sqrt{2}s'\vec{\phi}\cdot\vec{w}_\Lambda}$ and hence we find weights or half-weights if $s' \in \mathbf{Z}$ or $s' \in \mathbf{Z}+\frac{1}{2}$, respectively.

Even for a given theory there are several ways to take the perturbative limit, depending on which of the components of $\vec{\phi}$ associated with the Cartan subalgebra we choose to take to $-\infty$. Typically one expects that each component can be associated to some coupling constant or physical parameter. For example in Sec. II we saw that the physical radius of the torus is proportional to $e_i{}^{\bar{i}} \sim e^{-\beta\rho}e^{-(1/\sqrt{2})[\underline{\lambda}_i]\cdot\underline{\phi}}$ thus there will be various limits corresponding to which radii become large. Depending on which component of $\vec{\phi}$ one takes large one finds that different weights in Λ_R lead to the dominant behavior in the limit. To give an explicit example, we consider the type IIB string theory on a seven torus, the perturbative limit associated with the string coupling in ten dimensions consists of taking the dilaton $\phi \to -\infty$ large. This implies that the volume modulus ρ and the torus "shape" moduli $\underline{\phi}$ can be kept finite. The explicit form for the roots of E_8 were given in Sec. II. We find the fundamental weights are

$$\begin{aligned}
\vec{\lambda}^i &= (0, 2\sqrt{\tfrac{2}{7}}i, \underline{\lambda}^i), \qquad i = 1,\ldots,5,\\
\vec{\lambda}^6 &= (0, 5\sqrt{\tfrac{2}{7}}, \underline{\lambda}^6),\\
\vec{\lambda}^7 &= (0, \sqrt{14}, \underline{0}),\\
\vec{\lambda}^8 &= (\tfrac{1}{\sqrt{2}}, \sqrt{\tfrac{7}{2}}, \underline{0}).
\end{aligned} \tag{4.26}$$

The first space in the above vectors correspond to the position of the dilaton field ϕ. We see that in the perturbative limit only $\vec{\lambda}^8\cdot\vec{\phi} \to -\infty$. If we express $\vec{w}_\Lambda = n_i\vec{\lambda}^i$ then one sees that the dominate term in the expansion of ϕ comes from a weight $\vec{w}_\Lambda$ with the largest nonvanishing value of n_8. In M theory the weak coupling limit, in so far as it exists, is where the curvatures are small. There is no dilaton but instead the volume modulus must be large, so that $\rho \to -\infty$, with the "shape" moduli $\underline{\phi}^i$ fixed. The explicit weights of E_8 that arise from compactification of M theory (using the anstaz (2.1)) were given in [39] as

$$\begin{aligned}
\vec{\lambda}^i &= (\tfrac{3\sqrt{2}}{4}i, \underline{\lambda}^i), \qquad i = 1,\ldots,4,\\
\vec{\lambda}^i &= (\tfrac{5\sqrt{2}}{4}(8-i), \underline{\lambda}^i), \qquad i = 5,6,7,\\
\vec{\lambda}^8 &= (2\sqrt{2}, \underline{0}),
\end{aligned} \tag{4.27}$$

where $\underline{\lambda}^i$ are the fundamental weighs of the $SL(8,\mathbf{R})$ symmetry associated to the eight torus upon compactification to three dimensions. In this limit we see that $\vec{\lambda}\cdot\vec{\phi} \to -\infty$ for all the fundamental weights but does so most quickly for $\vec{\lambda}^5$.

Let us close this section with some additional comments on automorphic forms. Unlike holomorphic forms, nonholomorphic forms are generally specified by more than just their transformation properties as one cannot use concepts such as analyticity to deduce the full function from a knowledge of its poles or asymptotic behavior. Indeed for the case of $SL(2,\mathbf{R})$ the nonholomorphic automorphic forms are usually defined to transform as in Eq. (4.3) but also to be an eigenvalue of the $SL(2,\mathbf{R})$ invariant Laplacian and behave as $\mathrm{Im}\tau \to \infty$ like $\phi(\tau) \sim (\mathrm{Im}\tau)^N$ for some fixed N. In fact, the $SL(2,\mathbf{R})$ invariant Laplacian is just the Casimir of $SL(2,\mathbf{R})$ when the generators correspond to their natural action on the coset $SL(2,\mathbf{R})/SO(2,\mathbf{R})$. A similar picture is true for the case of $SL(3,\mathbf{R})$ but now the automorphic forms obey two differential equations; indeed they are required to be eigenvalues of the two Casimirs of $SL(3,\mathbf{R})$ [64].

It is natural to consider nonholomorphic automorphic forms of G to satisfy r differential equations where r is the rank of G. In particular one might demand that they be eigenvaulues of the r Casimirs of G whose generators are realized by their natural action on the coset G/H. We note that the perturbative contribution of the automorphic forms constructed above depend on r scalar fields associated with the Cartan subalgebra of G and the values of the r Casimirs will be given in terms of the highest weight of the representation used to construct the automorphic form. Thus it would seem likely that there is an alternative way to characterize these automorphic forms by specifying their transformation rule, as in Eq. (4.3) and a particular highest weight of the representation.

We note that the situation for the automorphic forms that occur in the higher derivative corrections is likely to be more complicated. In particular the invariant automorphic form that occurs for the D^6R^4 term in the IIB theory [27–33] is not an eigenvalue of the $SL(2,\mathbf{R})$ invariant Laplacian, but rather solves the eigenvalue problem in the presence of sources obtained from other automorphic forms that appear at lower order in the effective action. It would be good to understand these differential equations more generally, as was done in [29] for type IIB string theory where they arise as a consequence of the higher order corrections to supersymmetry and also to understand how such differential equations might arise naturally from the mathematical viewpoint.

V. CONCLUSION

In this paper we have given a systematic method of constructing automorphic forms once one specifies a group G and subgroup H, which we took to be the Cartan involution invariant subgroup, as well as a linear representation ψ of $G(\mathbf{Z})$. The automorphic form is built from the nonlinear representation φ constructed from ψ which involves the coset representatives $g(\xi)$ of G/H acting on ψ. In this way the dependence of the automorphic form on the coset G/H appears and it follows from the construction that the automorphic forms involve the weights of G corresponding to the representation ψ.

We also showed that if the higher derivative corrections to the type II strings in any dimension were invariant under a duality group $G(\mathbf{Z})$ then the functions of the scalars that occur could, by considering their transformation properties, be identified with automorphic forms.

Last we found that the dimensional reduction of the higher derivative corrections of the IIB theory to three dimensions on a torus lead to weights of E_8, generalizing the similar result of [39] for M theory. Since, as we just explained above, the type II effective actions must involve automorphic forms and so weights if they are invariant under a $G(\mathbf{Z})$ duality group, we can interpret the appearance of weights upon dimensional reduction as evidence for such an underlying duality symmetry of M theory

In closing we note that there is an important difference between dimensional reduction and compactification. The former discards all the Kaluza-Klein and wrapped brane modes while the latter keeps them. In general the dimensional reduction of a higher derivative term only leads to a part of the corresponding term in the lower dimension. In particular it will not lead to an automorphic form of the full lower dimensional duality group. Rather one can only expect to find the part of the automorphic form that survives the limit where the compact directions are taken to infinite radius. On the other hand one would expect that, given the full higher derivative term calculated in the compactified theory one can obtain the correct higher derivative term in the uncompactified theory by taking the radii to infinity. However compactification of loop amplitudes has been found [27–36] to lead to the full automorphic forms themselves, at least from 11 to nine dimensions.

It has been observed [65] that since E_{11} involves the $SL(2,\mathbf{R})$ symmetry of the IIB theory and this later symmetry is broken to $SL(2,\mathbf{Z})$ then E_{11} itself must be broken to a discrete symmetry. This means, for example, that even Lorentz transformations contained in the E_{11} symmetry are discrete. This paper presents a first step in how one might implement a discrete E_{11} symmetry in M theory and indeed what this could be. One might like to study automorphic forms based on E_{11} and hope that this would encode all, or a large part, of the effective action.

ACKNOWLEDGMENTS

We would like to thank A. Pressley for discussions. This work was supported in part by the PPARC Grant No. PPA/G/O/2000/00451 and P. C. W. is supported by PPARC (PPA/Y/S/2002/001/44).

APPENDIX A: NONLINEAR AND INDUCED REPRESENTATIONS

In this appendix we summarize some basic facts about nonlinear representations [66] and induced representations that will be needed in this paper. A nonlinear realization of a group G with respect to a subgroup H considers group elements $g \in G$, which depend on spacetime, and are taken to transform as

$$g(x) \to g_0 g(x) \quad \text{and} \quad g(x) \to g(x)h(x), \tag{A1}$$

where g_0 is any element of G and is a rigid transformation, that is independent of spacetime, and $h(x)$ is an element of H which depends on spacetime and so is a local transformation. Any theory invariant under the above two transformations can be thought of as the nonlinear realization of G with respect to H. In general the result will not be unique, but if the action has only two spacetime derivatives then it is constrained up to just a few constants. Furthermore if the subgroup H is large enough then the action will indeed be uniquely determined. We note that in this section the spacetime dependence of g and h just goes along for the ride and hence we are just describing the usual transformations on the coset space G/H induced by the natural action of the group.

Associated with the second transformation of Eq. (A1) we see that invariant quantities of the theory will only depend on the coset space G/H. One can use this transformation to fix a set of coset representatives $g(\xi)$ where ξ are the parameters that label the cosets, i.e. the equivalence classes. Once one makes this choice the transformation under g_0 will in general no longer preserve the choice of coset representative and one must make a compensating H transformation

$$g(\xi) \to g_0 g(\xi) = g(g_0 \cdot \xi) h(g_0, \xi). \tag{A2}$$

Here $h(g_0, \xi)$ is the required compensating transformation, which was denoted by h^{-1} in Ref. [39]. We will often denote the action on the coset coordinates by $\xi \to \xi' = g_0 \cdot \xi$. To simplify the notation we have to drop the explicit spacetime dependence of ξ, as it is not relevant in this mathematical account and as the dependence of ξ on spacetime is not changed by any of the steps in this appendix. Evaluating $g_0^1 g_0^2 g(\xi)$ as $(g_0^1 g_0^2) g(\xi)$ or $g_0^1 (g_0^2 g(\xi))$ and comparing the two we find the consistency condition

$$h(g_0^1 g_0^2, \xi) = h(g_0^1, \xi^1) h(g_0^2, \xi), \tag{A3}$$

where $g_0^1 g(\xi) = g(\xi^1) h(g_0^1, \xi)$.

For the groups G and subgroups H of interest to us, the Lie algebra of G can be written as the Lie algebra of H plus an H invariant compliment, denoted $H^\perp$. This means that the generators of $H^\perp$ possess commutators with the elements in the Lie algebra of $H^\perp$ that are again in $H^\perp$. This is guaranteed if the algebra G possesses an automorphism which squares to one such that the generators of H and those of $H^\perp$ transform into themselves with a minus and plus sign, respectively. For the groups we have in mind the subgroups H are by definition those that are preserved by the Cartan involution I and as a result the generators of H and $H^\perp$ transform in the required way, i.e. $I(H) = H, I(H^\perp) = -H^\perp$. In this case the coset representatives can be chosen to be constructed from the generators of $H^\perp$ and then they obey $h_0 g(\xi) = g(\xi') h_0$ for $h_0 \in H$. Consequently ξ transforms linearly under H and $h(h_0, \xi) = h_0$.

An induced representation of a group G with respect to a subgroup H consists of a set of functions φ which map G to some vector space V which carries a linear representation $D(h)$ of H where $h \in H$. If φ_a are the components of φ, they are required to satisfy the condition

$$\varphi(gh)_a = D(h^{-1})_a{}^b \varphi_b(g), \quad \forall\ g \in G, \qquad h \in H. \tag{A4}$$

The transformation of the group G is defined by

$$U(g_0)\varphi(g) = \varphi(g_0 g), \quad \forall\ g, \qquad g_0 \in G. \tag{A5}$$

In fact φ does not really depend on the full group G, but only on the coset G/H as by Eq. (A5) the value of φ at two points in the same coset is the same up to the matrix factor $D(h^{-1})$. As such, we can define a function on the coset G/H by

$$\varphi_a(\xi) = \varphi_a(g(\xi)), \tag{A6}$$

where $g(\xi)$ are the above discussed coset representatives. The transformation of Eq. (A5) then becomes

$$\begin{aligned} U(g_0)\varphi_a(\xi) &= \varphi_a(g_0 g(\xi)) \\ &= \varphi_a(g(g_0 \cdot \xi) h(g_0, \xi)) \\ &= D(h^{-1}(g_0, \xi))_a{}^b \varphi_b(g_0 \cdot \xi). \end{aligned} \tag{A7}$$

One can verify that it is indeed a representation using Eq. (A3). As noted above, for the subgroups H of interest to us one can make a choice of coset representative such $D(h(h_0, \xi)) = D(h_0)$ if $h_0 \in H$ and so for these transformations $D(h)$ is independent of ξ and is just the usual representation matrix and the action of h_0 on ξ is just a linear realization. In this sense φ just transforms linearly under the subgroup H.

Given any linear representation of G carried by an element $\psi \in V$

$$U(g_0)\psi_b = D(g_0^{-1})_a{}^b \psi_b, \tag{A8}$$

we can convert it into a nonlinear representation of the form discussed above. To do this we define

$$\varphi_a(\xi) = D((g^{-1}(\xi))_a{}^b \psi_b, \tag{A9}$$

whereupon it transforms as

$$\begin{aligned} U(g_0)\varphi_a(\xi) &= D(g^{-1}(\xi))_a{}^b D(g_0^{-1})_b{}^c \psi_c \\ &= D((g_0 g(\xi))^{-1})_a{}^b \psi_b \\ &= D((g(\xi') h(g_0, \xi))^{-1})_a{}^b \psi_b \\ &= D(h^{-1}(g_0, \xi))_a{}^b \varphi(\xi')_b, \end{aligned} \tag{A10}$$

where $h(g_0, \xi)$ are the H group elements of Eq. (A2). We note that φ transforms under a representation of G, but the matrix D has an argument that only involves the group element $h(g_0, \xi)$ which belongs to H. We will refer to this as a nonlinear representation.

It can happen that one finds that φ transforms under more that one irreducible representation of G as the matrix D is not an irreducible representation of H. Nevertheless we find that we can always convert the linear realization of Eq. (A8) to the nonlinear realization of G given in Eq. (A8).

In the above we have used the passive interpretation of transformations. For example, for a linear realization of Eq. (A8) it means that

$$\begin{aligned} U(g_0^1)U(g_0^2)\psi_a &= U(g_0^1) D((g_0^2)^{-1})_a{}^b \psi_b \\ &= D((g_0^2)^{-1})_a{}^b D((g_0^1)^{-1})_b{}^c \psi_c \\ &= D((g_0^1 g_0^2)^{-1})_a{}^b \psi_b = U(g_0^1 g_0^2)_a{}^b \psi_b \end{aligned} \tag{A11}$$

i.e. $U(g_0^1)U(g_0^2) = U(g_0^1 g_0^2)$ as it should for a representation.

APPENDIX B

In this appendix we will give an account of certain aspects of the theory of group representations that are required in this paper. This appendix is similar to that of Ref. [39], but we will explicitly use the passive interpretation of transformations and give the expressions in terms of components. We recall that a linear representation R of a group G consists of a vector space V and a set of operators $U(g)$, $\forall\ g \in G$ which act on V, namely $|\psi\rangle \to U(g)|\psi\rangle$ such that $U(g_1 g_2) = U(g_1)U(g_2)$. If the vector space has a basis $|e^a\rangle$ we can write $|\psi\rangle = \psi_a |e^a\rangle$ where we use the repeated index summation convention. The action of the group is given by

$$U(g)|\psi\rangle = \psi_a(U(g)|e^a\rangle) = (U(g)\psi_a)|e^a\rangle, \tag{B1}$$

where

$$U(g)\psi_a = D(g^{-1})_a{}^b\psi_b \quad \text{and so} \quad U(g)|e^a\rangle = |e^b\rangle D(g^{-1})_b{}^a. \tag{B2}$$

We note that while the components ψ_a transform with argument g^{-1}, the vectors $|e^a\rangle$ transform with $(g^{-1})^T$.

In this paper we will take the algebra G to be finite dimensional semisimple and simply laced. The states in the representation can be chosen so as to be eigenstates of $\vec{H}$. The eigenvalues are called weights. It can be shown that the weights of G belong to the dual lattice to the lattice of roots, i.e. a weight $\vec{w}$ satisfies

$$\vec{w}\cdot\vec{\alpha}_a \in \mathbf{Z} \tag{B3}$$

for the simple roots $\vec{\alpha}_a$. The representations of interest to us are finite dimensional and so must have a highest weight $\vec{\lambda}$ which is the one such that $\vec{\lambda}+\vec{\alpha}_a$ is not a weight for all simple roots $\vec{\alpha}_a$. The representations will also have a lowest root denoted $\vec{\mu}$. Of particular interest are the fundamental representations which are those whose highest weights $\vec{\lambda}^a$ obey the relation

$$\vec{\lambda}^a\cdot\vec{\alpha}_b = \delta^a_b \tag{B4}$$

for all simple roots $\vec{\alpha}_a$. The roots are themselves weights and these correspond to the adjoint representation, whose highest weight we will denote by $\vec{\theta}$.

For $SL(n)$, i.e. A_{n-1}, the fundamental weights $\vec{\lambda}^a$ satisfy

$$\vec{\lambda}^a\cdot\vec{\lambda}^b = a(n-b)/n \tag{B5}$$

for $b \geq a$. The representation with highest weight $\vec{\lambda}^{n-k}$ is realized on a tensor with k totally antisymmetrized superscript indices, i.e. $T^{i_1\ldots i_k} = T^{[i_1\ldots i_k]}$. Using the group invariant epsilon symbol $\epsilon^{i_1\ldots i_n}$, this representation is equivalent to taking a tensor with $n-k$ lowered indices.

Given any simple root one may carry out its Weyl reflection on any weight

$$S_\alpha(w) = \vec{w} - (\vec{\alpha}\cdot\vec{w})\vec{\alpha}. \tag{B6}$$

The collection of all such reflections is called the Weyl group and it can be shown that any member of it can be written in terms of a product of Weyl reflections in the simple roots. Although the precise decomposition of a given element of the Weyl group is not unique its length is defined to be the smallest number of simple root reflections required. However, there does exist a unique Weyl reflection, denoted W_0, that has the longest length. This element obeys $W_0^2 = 1$, takes the positive simple roots to negative simple roots, and its length is the same as the number of positive roots. As a result, $-W_0$ exchanges the positive simple roots with each other and, as Weyl transformations preserve the scalar product, it must also preserve the Cartan matrix. Consequently, it must lead to an automorphism of the Dynkin diagram. Given any representation of G the highest and lowest weights are related by

$$\vec{\mu} = W_0\vec{\lambda}. \tag{B7}$$

Given the definition of the fundamental weights and carrying out a Weyl transformation W_0, we may conclude that the negative of the highest and lowest weights of a given fundamental representation are the lowest and highest representation of one of the other fundamental representations. It is always the case that the two representations have the same dimension. However it can happen that a fundamental representation is self-dual.

For $SL(n)$

$$W_0 = (S_{\vec{\alpha}_1}\ldots S_{\vec{\alpha}_{n-1}})(S_{\vec{\alpha}_1}\ldots S_{\vec{\alpha}_{n-2}})\ldots(S_{\vec{\alpha}_1}S_{\vec{\alpha}_2})S_{\vec{\alpha}_1}$$

and one finds that, in this case,

$$W_0\vec{\lambda}_{n-k} = \vec{\mu}_{n-k} = -\vec{\lambda}_k \Leftrightarrow W_0\vec{\mu}_{n-k} = \vec{\lambda}_{n-k} = -\vec{\mu}_k. \tag{B8}$$

This result also follows from the above remarks on W_0 as it must take a fundamental representation to a fundamental representation and correspond to an automorphism of the Dynkin diagram which in this case just takes the nodes k to $n-k$.

Given a linear representation R acting on $|\psi\rangle \in V$ we may consider the dual representation R_D that is carried by the space of linear functionals, denoted V^*, acting on V. The group action is defined by

$$\langle\psi_D| \to \langle U(g)\psi_D| = \langle\psi_D|U(g^{-1}), \quad \forall\ g\in G, \quad \langle\psi_D| \in V^*. \tag{B9}$$

We note that $\langle\psi_D|\psi\rangle$ is G-invariant. If we introduce a dual basis e^*_a for V^* such that $e^*_a(e^b) \equiv \langle e^*_a|e^b\rangle = \delta^b_a$ we can express $\langle\psi_D| = e^*_a\psi^a_D$. From the invariance of the scalar product and Eq. (B2) we find that the transformation in terms of the components is given by

$$\psi^a_D \to U(g)\psi^b_D = \psi^a_D D(g)_a{}^b. \tag{B10}$$

Since the linear functionals carry a representation we may also choose a basis for them that is labeled by the weights. It is easy to see that a linear functional with a weight $\vec{w}$ only has a nonzero result on a state with weight $-\vec{w}$. A little further thought allows one to conclude that if the representation R has highest and lowest weight $\vec{\lambda}$ and $\vec{\mu}$, respectively, then the dual representation has a highest weight $-\vec{\mu}$ and lowest weight $-\vec{\lambda}$. Indeed the dual representation has the same dimension as the original representation. For the case of $SL(n)$, i.e. A_{n-1}, if the representation R is the fundamental representation with highest weight $\vec{\lambda}^k$ then it follows from Eq. (B9) that the dual representation is the fundamental representation with highest weight $\vec{\lambda}^{n-k}$. Thus the representation carried by $T^{i_1\ldots i_{(n-k)}}$ is dual to the representation carried by $T^{i_1\dots i_k}$ or equivalently carried by $T_{i_1\dots i_{(n-k)}}$ if we lower the indices with epsilon.

Given a representation R and any automorphism τ of the group τ (i.e. $\tau(g_1g_2) = \tau(g_1)\tau(g_2)$) we may also define a twisted representation R_τ on the same vector space V as follows. If $|\psi_\tau\rangle$ are the states of the twisted representation we may write $|\psi_\tau\rangle = \psi_{\tau a}|e^a\rangle$ then the components transform as

$$\psi_{\tau a} \to U(g)\psi_{\tau a} = D(\tau(g^{-1}))_a{}^b\psi_{\tau b} \quad \forall\ g \quad \in G. \tag{B11}$$

In this paper we will take the automorphism to be the Cartan involution which we also denoted by I. It is easy to see that if the representation R has highest and lowest weight $\vec{\lambda}$ and $\vec{\mu}$, respectively, then the dual representation has a highest weight $-\vec{\mu}$ and lowest weight $-\vec{\lambda}$ and so the Cartan involution twisted representation is isomorphic to the dual representation.

In Appendix A we showed, using Eq. (A9), how we can convert a linear representation, with components ψ_a, into a nonlinear representation with components $\varphi_a(\xi)$ which transform as in Eq. (A10) under the group element g_0 as

$$U(g_0)\varphi_a(\xi) = D(h^{-1}(g_0, \xi))_a{}^b\varphi(\xi')_b. \tag{B12}$$

Given the dual representation we can also construct an analogous nonlinear representation if we define the component fields by

$$\varphi_D^a(\xi) = \psi_D^b D(g(\xi))_b{}^a. \tag{B13}$$

One verifies that it transforms as

$$U(g_0)\varphi_D^a(\xi) = \varphi_D^b(\xi')D(h(g_0, \xi)_b{}^a. \tag{B14}$$

Taking the automorphism to be the Cartan involution I we can similarly construct a nonlinear representation from the twisted linear representation of Eq. (B11) by taking the components

$$\varphi_{Ia} = D(g^{\#}(\xi))_a{}^b\psi_{Ib}, \tag{B15}$$

where $g^{\#} = (I(g))^{-1}$. This representation transforms as

$$\varphi_{Ia}(\xi) \to U(g_0)(\varphi_{Ia}(\xi)) = D(h(g_0, \xi)^{-1})_a{}^b\varphi_{Ib}(\xi'). \tag{B16}$$

We note that $h^{\#} = h^{-1}$ as by definition $I(h) = h$.

Examining the above transformations we observe that

$$\begin{aligned}\varphi_{ID}^a(\xi)\varphi_a &= \psi_{ID}^b D(I(g(\xi)))_b{}^a D((g(\xi)^{-1})_a{}^c\psi_c \\ &= \psi_{ID}^b D(\mathcal{M}(\xi)^{-1})_b{}^c\psi_c,\end{aligned} \tag{B17}$$

where $\mathcal{M}(\xi) = gg^{\#}$, is invariant under the nonlinear realization of G, using Eqs. (B12) and (B14). We note that the twisted dual representations ψ_{ID}^a and the original representation ψ_a are isomorphic to each other. In particular, for A_n if ψ_a is the representation with highest weight λ_k so is ψ_{ID}^a. The expression $\varphi_D^a\varphi_{Ia}$ is also invariant under nonlinear transformations of G, however, if we consider all representations of G in the expression of Eq. (B17) we do not gain any new invariant quantities by considering this latter quantity.

APPENDIX C: EXAMPLES OF AUTOMORPHIC FORMS OF $SL(n, \mathbf{Z})$

In Sec. IV we have given a general procedure for constructing automorphic forms which may be unfamiliar to the reader. In this appendix we apply this formalism first to the case of $SL(2, \mathbf{Z})$ and recover some of the well-known automorphic forms and then to the case of $SL(n, \mathbf{Z})$.

1. $SL(2, \mathbf{Z})$

Let us start by recalling the well-known properties of the coset $SL(2, \mathbf{R})/SO(2, \mathbf{R})$. The local subgroup is the Cartan involution invariant subgroup of $SL(2, \mathbf{R})$ which is just $SO(2, \mathbf{R})$. It consists of the matrices

$$h(g_0, \tau) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{C1}$$

Using such a local transformation in Eq. (A1) we may choose our coset representatives $g(\xi) \in SL(2, \mathbf{R})/SO(2, \mathbf{R})$ to have the upper triangular form

$$g(\chi, \rho) = \frac{1}{\sqrt{\rho}}\begin{pmatrix} \rho & \chi \\ 0 & 1 \end{pmatrix} \tag{C2}$$

with $\rho > 0$. Thus the pair $\xi = (\chi, \rho)$ parametrize the coset space G/H and it will be helpful to introduce $\tau = \chi + i\rho$. Under a discrete $SL(2, \mathbf{Z})$ transformation of the form

$$g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{C3}$$

one finds that $g_0 g$ is no longer of the form of Eq. (C1) as it does not preserve the choice of coset representative. However if we also consider a local compensating $SO(2, \mathbf{R})$ transformation as in Eq. (A1) we find that

$$g_0 g(\tau) = g(\tau)h(g_0, \tau) = \frac{1}{\sqrt{\rho'}}\begin{pmatrix} \rho' & \chi' \\ 0 & 1 \end{pmatrix} h(g_0, \tau) \tag{C4}$$

with

$$e^{2i\theta} = \frac{c\tau + d}{c\bar{\tau} + d} \tag{C5}$$

and

$$\tau' = \frac{a\tau + b}{c\tau + d}. \tag{C6}$$

Note that even though g_0 is a discrete transformation we require h to be a local transformation since it depends on τ in addition to g_0. This is the well-known action of $SL(2, \mathbf{Z})$ on the coset which one can denote by $\tau' = g_0 \cdot \tau$.

We now construct automorphic forms using the method given in Sec. IV. We must choose a representation ψ of

$G = SL(2, \mathbf{R})$ which we take to be the vector representation. This is just the column vector

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

The dual Cartan involution twisted representation is just the transpose, that is $\langle\psi_{ID}| = (\psi_1, \psi_2)$. Next we consider $G(\mathbf{Z}) = SL(2, \mathbf{Z})$ and to obtain an irreducible representation we restrict the states to the lattice $\Lambda_R = \mathbf{Z}^2 - \{(0, 0)\}$ with elements

$$|\psi\rangle = \begin{pmatrix} m \\ -n \end{pmatrix},$$

$m, n \in \mathbf{Z}$ and similarly for $\langle\psi_{ID}|$. For $SL(n, \mathbf{R})$ # is just the transpose and hence, in the vector representation,

$$D(\mathcal{M}^{-1})_a{}^b = \frac{1}{\rho}\begin{pmatrix} 1 & -\chi \\ -\chi & \rho^2 + \chi^2 \end{pmatrix} \tag{C7}$$

and therefore

$$u(\tau) = \psi^a_{ID} D((\mathcal{M}^{-1})_a{}^b \psi_b = \frac{|m + n\tau|^2}{Im\tau}. \tag{C8}$$

An invariant automorphic form is then given by Eq. (4.11) with the choice of $K(u)$ of Eq. (4.15) and it is given by

$$\phi(\tau) = \sum_{(m,n)\in\mathbf{Z}^2-\{0,0\}} \frac{1}{u(\tau)^s} = \sum_{(m,n)\in\mathbf{Z}^2-\{0,0\}} \frac{(\mathrm{Im}\tau)^s}{|m + n\tau|^{2s}}. \tag{C9}$$

We recognize these as the well-known invariant nonholomorphic Eisenstein series.

We now construct the automorphic forms that transform nontrivially. From Eq. (4.7) we see that

$$\varphi = \frac{1}{\sqrt{\rho}}\begin{pmatrix} m + n\chi \\ -n\rho \end{pmatrix}. \tag{C10}$$

The irreducible representations of $SL(2, \mathbf{Z})$ are $\varphi_\pm = \varphi_1 \pm i\varphi_2$ where

$$\varphi_+ = \frac{(m + n\bar{\tau})}{\sqrt{\mathrm{Im}\tau}} \quad \text{and} \quad \varphi_- = \frac{(m + n\tau)}{\sqrt{\mathrm{Im}\tau}}. \tag{C11}$$

An automorphic form is given in Eq. (4.18) and taking into account the possible modification discussed below Eq. (4.21) we consider

$$\begin{aligned}\phi_w(\tau) &= \sum_{\Lambda_R} \frac{(\varphi_-)^w}{(u(\tau))^s} \\ &= \sum_{(m,n)\in\mathbf{Z}^2-\{0,0\}} \frac{(\mathrm{Im}\tau)^s}{|m + n\tau|^{2s}} \frac{(m + n\tau)^w}{\mathrm{Im}\tau^{w/2}} \\ &= \sum_{(m,n)\in\mathbf{Z}^2-\{0,0\}} \frac{(\mathrm{Im}\tau)^{s-w/2}}{|m + n\tau|^{2s-w}} \left(\frac{m + n\tau}{m + n\bar{\tau}}\right)^{w/2}.\end{aligned} \tag{C12}$$

It follows from its construction that this automorphic form transforms nontrivially with a $D(h(g_0, \tau)) = e^{iw\theta_c}$.

Let us now consider the pertubative limit as $\rho \to \infty$. One readily sees from (C12) that the dominant terms from $n = 0$. These are just the states

$$|\psi\rangle = \begin{pmatrix} m \\ 0 \end{pmatrix}$$

in the lattice Λ_R with weight $w_\Lambda = 1/\sqrt{2}$. Thus we see that

$$\phi \to \sum_{m\in\mathbf{Z}-0} \frac{\rho^{s-w/2}}{|m|^{2s-w}} = 2\zeta(2s - w)\rho^{s-w/2} \tag{C13}$$

and indeed we see that this term is independent of χ.

2. $SL(n, \mathbf{Z})$

Let us now consider automorphic forms for $SL(n, \mathbf{Z})$. Again we consider the vector representation and we can generalize the previous discussion by using the local $SO(n, \mathbf{R})$ invariance to write the coset representatives $g \in SL(n, \mathbf{R})/SO(n, \mathbf{R})$ as

$$g(\rho, \chi) = \frac{1}{(\rho_1 \ldots \rho_{n-1})^{1/n}} \begin{pmatrix} \rho_1 & \rho_2\chi_{12} & \rho_3\chi_{13} & \cdots & \chi_{1n} \\ & \rho_2 & \rho_3\chi_{23} & \cdots & \chi_{2n} \\ & & \rho_3 & \cdots & \chi_{3n} \\ & & & \ddots & \vdots \\ & & & & 1 \end{pmatrix}. \tag{C14}$$

This is just the product of a matrix involving the χ's multiplied by the diagonal matrix diag(ρ_i) which is of the form of the group element of Eq. (1.1). One also finds that the inverse element takes the form

$$g^{-1}(\rho,\chi) = (\rho_1\dots\rho_{n-1})^{1/n}\begin{pmatrix} \rho_1^{-1} & -\rho_1^{-1}\tilde\chi_{12} & -\rho_1^{-1}\tilde\chi_{13} & \dots & -\rho_1^{-1}\tilde\chi_{1n} \\ & \rho_2^{-1} & -\rho_2^{-1}\tilde\chi_{23} & \dots & -\rho_2^{-1}\chi_{2n} \\ & & \rho_3^{-1} & \dots & -\rho_3^{-1}\tilde\chi_{3n} \\ & & & \ddots & \vdots \\ & & & & 1 \end{pmatrix}, \tag{C15}$$

where $\tilde\chi_{ij} = \chi_{ij} + \mathcal{O}(\chi^2)$ are polynomials in χ_{ij}.

Acting with a discrete $g_0 \in SL(n, \mathbf{Z})$ transformation acting on $g(\rho,\chi)$ will change this form, however it can then be put back into upper triangular form by a local $h \in SO(n, \mathbf{R})$ transformation. This will generate a nonlinear realization $\xi \to g_0 \cdot \xi$ where now ξ collectively labels the fields ρ_i and χ_{ij} for $i < j = 1, \dots, n-1$.

To construct automorphic forms we start with the vector representation of $SL(n, \mathbf{R})$ where $|\psi\rangle \in \mathbf{R}^n$. We then restrict attention to $SL(n, \mathbf{Z})$ and take the states $|\psi\rangle \in \Lambda_R = \mathbf{Z}^n - \{0, \dots, 0\}$. Thus if we take

$$|\psi\rangle = \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{pmatrix} \tag{C16}$$

we find that

$$|\varphi\rangle = (\rho_1\dots\rho_{n-1})^{1/n}\begin{pmatrix} m_1\rho_1^{-1} - m_2\rho_1^{-1}\tilde\chi_{12} - \dots \\ m_2\rho_2^{-1} - \dots \\ \vdots \\ m_n \end{pmatrix} \tag{C17}$$

and

$$\begin{aligned} u(\xi) &= \varphi^a\varphi_a \\ &= (\rho_1\dots\rho_{n-1})^{2/n}(\rho_1^{-2}(m_1 - m_2\tilde\chi_{12} - \dots)^2 \\ &\quad + \rho_2^{-2}(m_2 - \dots)^2 + \dots + m_n^2). \end{aligned} \tag{C18}$$

We can then find automorphic forms by taking

$$\Phi(\xi) = \sum_{\vec m \in \mathbf{Z}^n - \vec 0} \frac{1}{(u(\xi))^s} \tag{C19}$$

which are invariant under $SL(n, \mathbf{Z})$, or

$$\Phi_{a_1\dots a_r}(\xi) = \sum_{\vec m \in \mathbf{Z}^n - \vec 0} \frac{\varphi_{a_1}(\xi)\dots\varphi_{a_r}(\xi)}{(u(\xi))^s} \tag{C20}$$

which will transform in the symmetric r-tensor representation of $SO(n)$ under an $SL(n, \mathbf{Z})$ transformation.

These expressions are clearly somewhat complicated. However we can consider the limit where $\rho_i \to 0$. In this case we find the automorphic forms are dominated by states with $m_1 = \dots = m_{n-1} = 0$ and hence we can set $\chi_{ij} = 0$. In this limit we find

$$\begin{aligned} \Phi(\xi) &\to (\rho_1\dots\rho_{n-1})^{-(2s/n)} \sum_{m_n \in \mathbf{Z} - 0} \frac{1}{|m_n|^{2s}} \\ &= 2\zeta(2s)(\rho_1\dots\rho_{n-1})^{-(2s/n)} \end{aligned} \tag{C21}$$

and

$$\begin{aligned} \Phi_{a_1\dots a_r}(\xi) &\to (\rho_1\dots\rho_{n-1})^{-((2s-r)/n)} \sum_{m_n \in \mathbf{Z} - 0} \frac{1}{|m_n|^{2s-r}} \\ &= 2\zeta(2s-r)(\rho_1\dots\rho_{n-1})^{-((2s-r)/n)}. \end{aligned} \tag{C22}$$

Our last step is to show that this limit does indeed have the form of Eq. (4.24) in terms of a weight of $SL(n, \mathbf{R})$. To this end we consider a decomposition of $SL(n, \mathbf{R})$ in terms of $SL(n-1, \mathbf{R})$. In particular we will work with the fundamental representation where we can choose a Cartan basis such that

$$\begin{aligned} H_i &= \begin{pmatrix} h_i & 0 \\ 0 & 0 \end{pmatrix}, \qquad i = 1, \dots, n-2, \\ H_{n-1} &= \frac{1}{\sqrt{n^2 - n}}\begin{pmatrix} 1 & 0 \\ 0 & -(n-1) \end{pmatrix}, \end{aligned} \tag{C23}$$

where h_i are the Cartan matrices for $SL(n-1, \mathbf{R})$. The generators $E_{\vec\alpha}$ for $\vec\alpha > 0$ can then be chosen to have zeros everywhere except for a single entry above the diagonal that is equal to one. A straightforward calculation shows that the simple roots take the form

$$\begin{aligned} \vec\alpha_i &= (\underline{\alpha}_i, 0), \qquad i = 1, \dots, n-2, \\ \vec\alpha_{n-1} &= \left(-\underline{\lambda}^{n-2}, \sqrt{\frac{n}{n-1}}\right), \end{aligned} \tag{C24}$$

where $\underline{\alpha}_i$ and $\underline{\lambda}^i$, $i = 1, \dots, n-2$ are the simple roots and fundamental weights of $SL(n-1, \mathbf{R})$. The states $|\psi\rangle$ that dominated the sum are of the form

$$|\psi\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ m_n \end{pmatrix} \tag{C25}$$

and hence their $\vec H$ eigenvalue is $\vec w_\Lambda = (\underline{0}, -\sqrt{\frac{n-1}{n}}) = -\vec\lambda^{n-1}$. Comparing (1.1) and (C14) we see that

$$\begin{aligned} (\rho_1\dots\rho_{n-1})^{-(1/n)} &= (e^{\sum_{\vec\alpha>0} E_{\vec\alpha}\chi_{\vec\alpha}} e^{-(1/\sqrt{2})\vec\phi\cdot\vec H})_{nn} \\ &= e^{(1/\sqrt{2})\sqrt{(n-1)/n}\,\phi_n} = e^{-(1/\sqrt{2})\vec\phi\cdot\vec\lambda^{n-1}}, \end{aligned} \tag{C26}$$

where the subscript nn denotes the nnth component of the matrix representative of $g(\xi)$. Hence we see that, in the limit $\rho_i \to 0$,

$$\Phi(\xi) \to 2\zeta(2s)e^{-\sqrt{2}s\vec{\phi}\cdot\vec{\lambda}^{n-1}} \tag{C27}$$

and

$$\Phi_{a_1\dots a_r}(\xi) \to 2\zeta(2s-r)e^{-\sqrt{2}(s-r/2)\vec{\phi}\cdot\vec{\lambda}^{n-1}}. \tag{C28}$$

[1] I. C. G. Campbell and P. C. West, Nucl. Phys. **B243**, 112 (1984).
[2] F. Giani and M. Pernici, Phys. Rev. D **30**, 325 (1984).
[3] M. Huq and M. A. Namazie, Classical Quantum Gravity **2**, 293 (1985); **2**, 597(E) (1985).
[4] J. H. Schwarz and P. C. West, Phys. Lett. B **126**, 301 (1983).
[5] P. S. Howe and P. C. West, Nucl. Phys. **B238**, 181 (1984).
[6] J. H. Schwarz, Nucl. Phys. **B226**, 269 (1983).
[7] E. Cremmer, B. Julia, and J. Scherk, Phys. Lett. B **76**, 409 (1978).
[8] E. Cremmer and B. Julia, Phys. Lett. B **80**, 48 (1978).
[9] N. Marcus and J. H. Schwarz, Nucl. Phys. **B228**, 145 (1983).
[10] B. Julia and H. Nicolai, Nucl. Phys. **B482**, 431 (1996).
[11] B. Julia, in *Vertex Operators and Mathematical Physics*, Publications of the Mathematical Sciences Research Institute, Vol. 3 (Springer-Verlag, Berlin, 1984); in *Superspace and Supergravity*, edited by S. W. Hawking and M. Rocek (Cambridge University Press, Cambridge, England, 1981).
[12] S. Mizoguchi, Nucl. Phys. **B528**, 238 (1998).
[13] J. Ehlers, Ph.D. thesis, Hamburg University, 1957.
[14] R. Geroch, J. Math. Phys. (N.Y.) **13**, 394 (1972).
[15] P. Breitenlohner, D. Maison, and G. W. Gibbons, Commun. Math. Phys. **120**, 295 (1988).
[16] E. Cremmer, B. Julia, H. Lu, and C. N. Pope, hep-th/9909099.
[17] N. D. Lambert and P. C. West, Nucl. Phys. **B615**, 117 (2001).
[18] K. Kikkawa and M. Yamasaki, Phys. Lett. B **149**, 357 (1984).
[19] T. H. Buscher, Phys. Lett. B **194**, 59 (1987); **201**, 466 (1988).
[20] M. Rocek and E. P. Verlinde, Nucl. Phys. **B373**, 630 (1992).
[21] M. Henneaux and C. Teitelboim, Phys. Rev. Lett. **56**, 689 (1986).
[22] A. Sen, Nucl. Phys. **B404**, 109 (1993).
[23] A. Font, L. E. Ibanez, D. Lust, and F. Quevedo, Phys. Lett. B **249**, 35 (1990).
[24] C. M. Hull and P. K. Townsend, Nucl. Phys. **B438**, 109 (1995).
[25] K. A. Meissner, Phys. Lett. B **392**, 298 (1997).
[26] N. Kaloper and K. A. Meissner, Phys. Rev. D **56**, 7940 (1997).
[27] M. B. Green and M. Gutperle, Nucl. Phys. **B498**, 195 (1997).
[28] M. B. Green, M. Gutperle, and P. Vanhove, Phys. Lett. B **409**, 177 (1997).
[29] M. B. Green and S. Sethi, Phys. Rev. D **59**, 046006 (1999).
[30] M. B. Green, H. h. Kwon, and P. Vanhove, Phys. Rev. D **61**, 104010 (2000).
[31] M. B. Green and P. Vanhove, J. High Energy Phys. 01 (2006) 093.
[32] M. B. Green, J. G. Russo, and P. Vanhove, hep-th/0610299.
[33] A. Basu, hep-th/0610335.
[34] J. A. Harvey and G. W. Moore, Phys. Rev. D **57**, 2323 (1998).
[35] E. Kiritsis and B. Pioline, Nucl. Phys. **B508**, 509 (1997).
[36] B. Pioline, H. Nicolai, J. Plefka, and A. Waldron, J. High Energy Phys. 03 (2001) 036.
[37] N. A. Obers and B. Pioline, Commun. Math. Phys. **209**, 275 (2000).
[38] N. Berkovits and C. Vafa, Nucl. Phys. **B533**, 181 (1998).
[39] N. Lambert and P. West, Phys. Rev. D **74**, 065002 (2006).
[40] M. B. Green and J. H. Schwarz, Nucl. Phys. **B198**, 441 (1982).
[41] D. J. Gross and E. Witten, Nucl. Phys. **B277**, 1 (1986).
[42] M. T. Grisaru and D. Zanon, Phys. Lett. B **177**, 347 (1986).
[43] N. Sakai and Y. Tanii, Nucl. Phys. **B287**, 457 (1987).
[44] R. C. Myers, Nucl. Phys. **B289**, 701 (1987).
[45] D. J. Gross and J. H. Sloan, Nucl. Phys. **B291**, 41 (1987).
[46] M. J. Duff, J. T. Liu, and R. Minasian, Nucl. Phys. **B452**, 261 (1995).
[47] J. G. Russo and A. A. Tseytlin, Nucl. Phys. **B508**, 245 (1997).
[48] M. B. Green, H. Kwon, and M. Gutperle, J. High Energy Phys. 08 (1999) 012.
[49] S. Deser and D. Seminara, Phys. Rev. Lett. **82**, 2435 (1999); Phys. Rev. D **62**, 084010 (2000).
[50] A. Dasgupta, H. Nicolai, and J. Plefka, J. High Energy Phys. 05 (2000) 007.
[51] A. A. Tseytlin, Nucl. Phys. **B584**, 233 (2000).
[52] M. Cederwall, U. Gran, M. Nielsen, and B. E. W. Nilsson, J. High Energy Phys. 10 (2000) 041.
[53] J. Plefka, Int. J. Mod. Phys. A **16**, 660 (2001).
[54] K. Peeters, P. Vanhove, and A. Westerberg, Classical Quantum Gravity **18**, 843 (2001).
[55] P. S. Howe and D. Tsimpis, J. High Energy Phys. 09 (2003) 038.
[56] P. S. Howe, hep-th/0408177.
[57] G. Policastro and D. Tsimpis, Classical Quantum Gravity **23**, 4753 (2006).
[58] T. Damour and H. Nicolai, Classical Quantum Gravity **22**, 2849 (2005).

[59] T. Damour, A. Hanany, M. Henneaux, A. Kleinschmidt, and H. Nicolai, Gen. Relativ. Gravit. **38**, 1507 (2006).

[60] T. Damour and M. Henneaux, Phys. Rev. Lett. **86**, 4749 (2001).

[61] T. Damour, M. Henneaux, and H. Nicolai, Phys. Rev. Lett. **89**, 221601 (2002).

[62] I. Schnakenburg and P. C. West, Phys. Lett. B **517**, 421 (2001).

[63] B. Pioline and A. Waldron, hep-th/0312068.

[64] D. Bump, *Automorphic Forms in GL(3,R)*, Lecture Notes in Mathematics (Springer-verlag, Berlin, 1984).

[65] P. C. West, Classical Quantum Gravity **18**, 4443 (2001).

[66] S. R. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1969); C. G. Callan, S. R. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2247 (1969).