PHYSICAL REVIEW D **75,** 065024 (2007)

Constraints on string networks with junctions

E. J. Copeland*

School of Physics and Astronomy, University of Nottingham, University Park, Nottingham NG7 2RD, United Kingdom

T. W. B. Kibble†

Blackett Laboratory, Imperial College, London SW7 2AZ, United Kingdom

D. A. Steer‡

APC, UMR 7164, 10 rue Alice Domon et Léonie Duquet, 75205 Paris Cedex 13, France, and LPT, Université de Paris-Sud, Bat. 210, 91405 Orsay Cedex, France

(Received 27 November 2006; published 22 March 2007)

We consider the constraints on string networks with junctions in which the strings may all be different, as may be found, for example, in a network of (p, q) cosmic superstrings. We concentrate on three aspects of junction dynamics. First we consider the propagation of small-amplitude waves across a static three-string junction. Then, generalizing our earlier work, we determine the kinematic constraints on two colliding strings with different tensions. As before, the important conclusion is that strings do not always reconnect with a third string; they can pass straight through one another (or in the case of non-Abelian strings become stuck in an X configuration), the constraint depending on the angle at which the strings meet, on their relative velocity, and on the ratios of the string tensions. For example, if the two colliding strings have equal tensions, then for ultrarelativistic initial velocities they pass through one another. However, if their tensions are sufficiently different they can reconnect. Finally, we consider the global properties of junctions and strings in a network. Assuming that, in a network, the incoming waves at a junction are independently randomly distributed, we determine the root-mean-square (r.m.s.) velocities of strings and calculate the average speed at which a junction moves along each of the three strings from which it is formed. Our findings suggest that junction dynamics may be such as to preferentially remove the heavy strings from the network leaving a network of predominantly light strings. Furthermore the r.m.s. velocity of strings in a network with junctions is smaller than $1/\sqrt{2}$, the result for conventional Nambu-Goto strings without junctions in Minkowski space-time.

DOI: 10.1103/PhysRevD.75.065024 PACS numbers: 11.27.+d, 98.80.Cq

I. INTRODUCTION

The evolution of cosmic string networks and their observational consequences are attracting a great deal of interest, particularly since they may lead to indirect observational tests of superstring theory. The annihilation of two branes at the end of brane inflation [1–5] is thought to lead to the formation of cosmic superstrings which can be either fundamental F-strings, Dirichlet D-strings, or (p, q)-strings, bound states of the two [6–9]. Most importantly, the predicted tensions of these strings are not only compatible with current observational bounds ($G\mu \lesssim 2.3 \times 10^{-7}$ using the third year Wilkinson Microwave Anisotropy Probe (WMAP) data [10]), but they also lie in a window which may be testable with the future LISA gravitational-wave detector [11–13].

Whilst cosmic superstrings share many of the properties of standard grand unified (GUT) cosmic strings [14,15], they differ in some important respects. First, when 2 F-strings (or 2 D-strings) intersect, they do not necessarily "intercommute"—or exchange partners—with probability $P = 1$ [16,17]. (An exception for local cosmic strings is when they intercommute at speeds very close to the speed of light [18].) The effect of reducing P is to increase the density of strings in the final scaling solution and hence the gravity wave signal, though the exact manner in which this happens is under debate [11–13,19,20]. Second, the different kinds of strings in a cosmic superstring network can meet at junctions [21]. Thus when an F and D-string intersect they cannot intercommute, but rather two junctions are formed and the original strings become connected by a third (p, q) bound state. Junctions also exist in non-Abelian string networks, for which the fundamental group $\pi_1(\mathcal{M})$ of the vacuum manifold $\mathcal{M}$ is non-Abelian.

Even though several authors have addressed both analytically and numerically the cosmological evolution of string networks with junctions [19,22–28], the late-time behavior of the network is still not fully understood, particularly when there are different types of string. One possible outcome is that the junctions play a minor role and the network reaches a scaling solution similar to that of a GUT string network. In that case it is possible to make predictions for cosmic superstring networks by extracting the information from the GUT string simulations, with suitable allowance for $P \neq 1$ and in principle for different tensions. A second possibility is that the presence of junc-

*Electronic address: ed.copeland@nottingham.ac.uk
†Electronic address: kibble@imperial.ac.uk
‡Electronic address: steer@apc.univ-paris7.fr

tions makes the network freeze and eventually dominates the energy density of the Universe. Similar questions have been addressed for networks of domain walls with junctions [29–34].

In a previous paper [35] we studied the dynamics of junctions in a local string network in which the individual strings have no long-range interactions and are well described by the Nambu-Goto action. We set up the equations of motion for three strings of tensions μ_j ($j = 1, 2, 3$) meeting at a junction at position $\mathbf{X}(t)$, and were able to solve for the dynamics of $\mathbf{X}(t)$ as well as to determine how the junction moved along each of the strings. A related approach has also been developed in the context of representations of baryons as pieces of open string connected at one common point [36,37]. Having constructed this formalism, we initially presented a simple highly symmetric exact oscillating loop solution. As in the case of ordinary cosmic strings, the existence of exact loop solutions may be important in analyzing the likely behavior of loops in general. Loops of strings with junctions generally evolve rather differently to standard string loops; they do not oscillate periodically and initial analysis indicates that the number of cusps on the loops may be rather different than for conventional Nambu-Goto loops. We intend to return to these points in a future publication.

Our second result concerned the intersection of two strings with equal tensions μ_1, meeting at an angle α, but with equal and opposite velocities v. When these strings collide, for them to exchange partners and become linked by a third string of tension μ_3 requires a strong kinematical constraint. We also discussed the case of non-Abelian strings, which may become joined by a new string lying along the direction of motion of the two initial strings, and we found the kinematic constraint for this case too.

We expect that these kinematical constraints could have significant consequences on the evolution of string networks with junctions: if the relative velocity of the strings is large, no junction will form and the strings will pass through each other. This in turn means that fewer loops will be formed, and hence that the network would radiate less energy. It may thus be important to include these constraints in analytic and numerical models of network evolution with junctions.

In this paper we extend the work of [35], and focus on three aspects of junction dynamics which we believe will be important to understand the global properties of the network. After a review of the equations of motion describing junction evolution in Sec. II, we first consider the propagation of small-amplitude waves across a static junction formed of three strings with generally different tensions (Sec. III). Such small waves will inevitably be present on strings in a network, and will radiate gravitationally. We determine the fraction of energy transmitted and reflected across the junction. Then, in Secs. IV and V, we generalize the kinematic constraints of [35] to the case in which the two colliding strings have different tensions μ_1 and μ_2. Since the initial velocity of strings is crucial to determining the importance of the kinematical constraints, in Secs. VI and VII we consider the global properties of junctions and strings in a network. Assuming that, in a network, the incoming waves at a junction are randomly distributed, we determine the r.m.s. velocities of strings and calculate the average speed at which a junction moves along each of the three strings from which it is formed. Our findings suggest that junction dynamics may be such as to preferentially remove the heavy strings from the network. Finally we conclude in Sec. VIII.

II. REVIEW OF THE BASICS

We begin by briefly reviewing the equations of motion for strings that form Y junctions obtained in [35].

We choose the temporal world sheet coordinate to be the global time, $\tau = t$, and use the conformal gauge in which the spatial coordinates $\mathbf{x}(\sigma, t)$ satisfy

$$\dot{\mathbf{x}} \cdot \mathbf{x}' = 0, \qquad \dot{\mathbf{x}}^2 + \mathbf{x}'^2 = 1, \tag{1}$$

where $\dot{\mathbf{x}} = \partial_t \mathbf{x}$ and $\mathbf{x}' = \partial_\sigma \mathbf{x}$. The action for three strings of tensions μ_j ($j = 1, 2, 3$) meeting at a junction is

$$S = -\sum_j \mu_j \int dt \int d\sigma \theta(s_j(t) - \sigma)\sqrt{\mathbf{x}_j'^2(1 - \dot{\mathbf{x}}_j^2)} + \sum_j \int dt \mathbf{f}_j(t) \cdot [\mathbf{x}_j(s_j(t), t) - \mathbf{X}(t)], \tag{2}$$

where $\mathbf{X}(t)$ is the position of the vertex, $\mathbf{f}_j$ are Lagrange multipliers, and the $s_j(t)$ are the values of the spatial world sheet coordinate at the vertex. It is straightforward to generalize this action to include other vertices, present, for example, in loop configurations.

Varying $\mathbf{x}_j$ yields the wave equation, with solution

$$\mathbf{x}_j(\sigma, t) = \tfrac{1}{2}[\mathbf{a}_j(\sigma + t) + \mathbf{b}_j(\sigma - t)], \tag{3}$$

where, in order to satisfy the gauge conditions (1),

$$\mathbf{a}_j'^2 = \mathbf{b}_j'^2 = 1. \tag{4}$$

The Lagrange multipliers impose boundary conditions that may be written

$$\mathbf{a}_j(s_j + t) + \mathbf{b}_j(s_j - t) = 2\mathbf{X}(t), \tag{5}$$

while varying $\mathbf{X}$ gives a relation between the Lagrange multipliers which can be reduced to

$$\sum_j \mu_j[(1 + \dot{s}_j)\mathbf{a}_j' + (1 - \dot{s}_j)\mathbf{b}_j'] = \mathbf{0}. \tag{6}$$

Eliminating the $\mathbf{a}_j'$ using the time derivative of (5) then yields

$$\dot{\mathbf{X}} = -\frac{1}{\mu}\sum_j \mu_j(1-\dot{s}_j)\mathbf{b}'_j, \tag{7}$$

where

$$\mu = \mu_1 + \mu_2 + \mu_3. \tag{8}$$

At the junction, the amplitudes $\mathbf{b}'_j$ of the incoming waves are known from the initial conditions; those of the outgoing waves, $\mathbf{a}'_j$, together with the position of the vertex, are given by (5) and (6), provided the $\dot{s}_j$ are known. Finally, the latter may be found by solving the constraint equations $\mathbf{a}'^2_j = 1$. The result depends on the string tensions and on the angles between the unit vectors $\mathbf{b}'_j$. Relative to [35], we introduce a slight change of notation, writing for example $c_1 = \mathbf{b}'_2 \cdot \mathbf{b}'_3$ (denoted by c_{23} in [35]). It will be useful to introduce the quantities ν_j defined, for example, by

$$\nu_1 = \mu_2 + \mu_3 - \mu_1. \tag{9}$$

Then the value of $\dot{s}_k$ is given by

$$1 - \dot{s}_k = \frac{\mu M_k(1-c_k)}{\mu_k \mathcal{M}}, \tag{10}$$

where the M_j are defined by $M_1 = \nu_2\nu_3$ along with two similar equations, and

$$\mathcal{M} = \sum_j M_j(1-c_j). \tag{11}$$

Notice that since the $\dot{s}_k \leq 1$, it follows that the string tensions satisfy triangle inequalities,

$$\nu_j \geq 0. \tag{12}$$

Finally, it follows from (10) that the $\dot{s}_j$ satisfy the energy conservation condition

$$\sum_j \mu_j \dot{s}_j = 0. \tag{13}$$

III. SMALL-AMPLITUDE WAVES

A first interesting application of these results is to the reflection and transmission of small-amplitude waves at a junction. Small-amplitude waves are expected to build up on strings as a result of self-intersections and interactions with other strings, and they will radiate gravitationally. It is important to understand how they propagate across a junction.

Consider three straight, static strings, for which

$$\mathbf{a}'_j = \mathbf{b}'_j = (\cos\theta_j, \sin\theta_j, 0) \equiv \mathbf{e}_j, \tag{14}$$

with the junction at $\mathbf{X} = 0$. The equilibrium condition $\dot{s}_j = 0$ determines the angles between them through (10); for example, we have

$$c_1 = \cos(\theta_3 - \theta_2) = \frac{\mu_1^2 - \mu_2^2 - \mu_3^2}{2\mu_2\mu_3}, \qquad \sin(\theta_3 - \theta_2) = \frac{\Delta}{2\mu_2\mu_3}, \tag{15}$$

where

$$\Delta = \sqrt{\mu\nu_1\nu_2\nu_3}. \tag{16}$$

Note that as a consequence of (12), Δ is real.

Now suppose there is a small incoming perturbation on the first string, so that $\mathbf{b}'_1 = \mathbf{e}_1 + \delta\mathbf{b}'_1$, with

$$\delta\mathbf{b}'_1(s) = \epsilon\mathbf{f}_1 \cos ks, \tag{17}$$

where ϵ is a small dimensionless parameter and $\mathbf{f}_1$ is a unit vector satisfying $\mathbf{e}_1 \cdot \mathbf{f}_1 = 0$; we set $\delta\mathbf{b}'_2 = \delta\mathbf{b}'_3 = \mathbf{0}$.

The simplest case is where $\mathbf{f}_1 = (0, 0, 1)$ since then all $\delta c_j = 0$, and therefore $\delta\dot{s}_j = 0$ so that the junction remains at the same value of σ on all strings. In that case, the outgoing waves are also in the z-direction. From (7) we find

$$\delta\dot{\mathbf{X}}(t) = -\frac{\mu_1}{\mu}\epsilon\mathbf{f}_1 \cos kt \tag{18}$$

and hence

$$\delta\mathbf{a}'_1(s) = \frac{\nu_1}{\mu}\epsilon\mathbf{f}_1 \cos ks, \qquad \delta\mathbf{a}'_2(s) = \delta\mathbf{a}'_3(s) = -\frac{2\mu_1}{\mu}\epsilon\mathbf{f}_1 \cos ks. \tag{19}$$

It is then straightforward to find the fraction of energy transmitted along the strings of tension $\mu_{2,3}$ and reflected along the string of tension μ_1. Since

$$\delta E_j = \frac{\mu_j}{4}\int d\sigma(\delta\mathbf{a}'^2_j + \delta\mathbf{b}'^2_j), \tag{20}$$

we find

$$R_1 = \frac{\nu_1^2}{\mu^2}, \qquad T_2 = 4\frac{\mu_1\mu_2}{\mu^2}, \qquad T_3 = 4\frac{\mu_1\mu_3}{\mu^2}. \tag{21}$$

For instance, suppose that $\mu_1 \ll \mu_2 \sim \mu_3$, so that the initial perturbation is along the light string. Then, as expected, almost all the energy is essentially reflected off the junction.

The results are more interesting if we take $\mathbf{f}_1$ in the xy-plane, because then the values of s_j do oscillate and the junction no longer stays at a fixed position on each string. Let us set

$$\mathbf{f}_j = (-\sin\theta_j, \cos\theta_j, 0), \tag{22}$$

so that $\mathbf{e}_j \cdot \mathbf{f}_j = 0$ for each j, and again take $\delta\mathbf{b}'_1$ of the form (17), with $\delta\mathbf{b}'_2 = \delta\mathbf{b}'_3 = \mathbf{0}$. Then we have

$$\delta c_1 = 0, \qquad \delta c_2 = -\frac{\Delta}{2\mu_1\mu_3}\epsilon \cos kt, \qquad \delta c_3 = \frac{\Delta}{2\mu_1\mu_2}\epsilon \cos kt. \tag{23}$$

It follows that

$$\delta \dot{s}_1 = \frac{(\mu_2 - \mu_3)\nu_1}{\Delta}\epsilon \cos kt, \qquad \delta \dot{s}_2 = -\frac{\mu_1\nu_1}{\Delta}\epsilon \cos kt = -\delta \dot{s}_3. \tag{24}$$

Despite this difference, the expressions for the outgoing-wave amplitudes are very similar, except that they are no longer all in the same direction. We find

$$\delta \dot{\mathbf{X}}(t) = \epsilon \cos kt \left(\frac{(\mu_2 - \mu_3)\nu_1}{\Delta}\mathbf{e}_1 - \frac{\mu_2 + \mu_3}{\mu}\mathbf{f}_1 \right) \tag{25}$$

from which it follows that

$$\delta \mathbf{a}_1'(s) = -\frac{\nu_1}{\mu}\epsilon \mathbf{f}_1 \cos ks, \qquad \delta \mathbf{a}_2'(s) = \frac{2\mu_1}{\mu}\epsilon \mathbf{f}_2 \cos ks, \qquad \delta \mathbf{a}_3'(s) = \frac{2\mu_1}{\mu}\epsilon \mathbf{f}_3 \cos ks. \tag{26}$$

The fact that the amplitudes of the outgoing waves are the same in magnitude independent of the orientation of $\mathbf{f}_1$ is at first sight remarkable, but actually it follows from the fact that we can regard the waves as representing massless particles propagating along the strings. The amplitudes of the outgoing waves can be derived from conservation of energy and momentum. It follows that the reflection and transmission coefficients are as given in (21), a result that has also been obtained in the more general setting of multivolume junctions by [38].

Comparing (24) with (26), one interesting point emerges: in the first case, $\mathbf{a}_1'$ is in phase with $\mathbf{b}_1'$, whereas in the second case it is in antiphase. This means that if the incoming wave is at an intermediate angle, the reflected wave is tilted in the opposite direction.

IV. COLLISIONS OF STRAIGHT STRINGS

As summarized in the introduction, in [35] we discussed the collision of two straight strings of equal tension, $\mu_1 = \mu_2$, and derived kinematical constraints on such a process. Here we wish to extend this discussion to the case of unequal tension, and hence it will be useful to define

$$\mu_+ \equiv \mu_1 + \mu_2, \qquad \mu_- \equiv \mu_1 - \mu_2. \tag{27}$$

Since $\nu_i \geq 0$, μ_3 is bounded by

$$\mu_- \leq \mu_3 \leq \mu_+. \tag{28}$$

This extension is nontrivial as the problem now lacks the symmetry associated with the equal-tension case, and as a first step one must determine the orientation and velocity of the joining string after the collision. This is the aim of the present section.

Consider two strings of tension μ_1 and μ_2 parallel to the xy-plane but at angles $\pm\alpha$ to the x-axis, and approaching each other with velocities $\pm v$ in the z-direction. Before the collision (for $t < 0$, see Fig. 1) we take

$$\mathbf{x}_{1,2}(\sigma, t) = (-\gamma^{-1}\sigma \cos\alpha, \mp\gamma^{-1}\sigma \sin\alpha, \pm vt), \tag{29}$$

where $\gamma^{-1} = \sqrt{1 - v^2}$. Thus

$$\mathbf{a}_{1,2}' = (-\gamma^{-1}\cos\alpha, \mp\gamma^{-1}\sin\alpha, \pm v), \qquad \mathbf{b}_{1,2}' = (-\gamma^{-1}\cos\alpha, \mp\gamma^{-1}\sin\alpha, \mp v). \tag{30}$$

At the collision, $t = 0$, we suppose that the strings interchange partners. Now, since $\mu_- \neq 0$, we can no longer conclude that the third—joining—string formed after the two original strings exchange partners, will be static and that it will lie either along the x- or y-axis. It is still true that this third string must be parallel to the xy-plane and that there are two orientations for it, depending on which ends of the original strings are joined to each other. To be specific, let us assume that the two ends in the positive-x region are joined at a new vertex, $\mathbf{X}$, while those in the

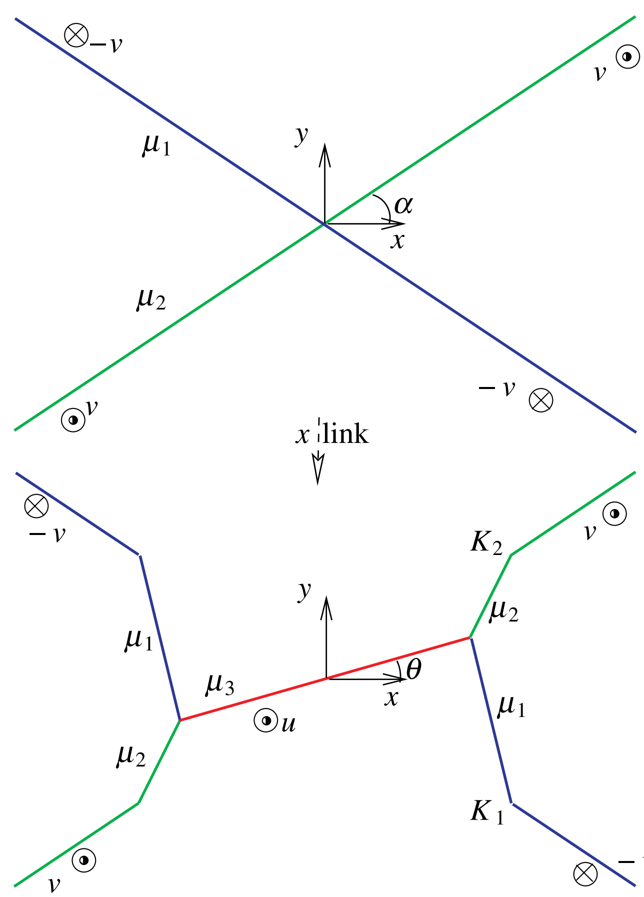

FIG. 1 (color online). Two colliding strings, of unequal tension, joined by a third string (an x-link).

negative-x region are joined to the other end of the new string which has tension μ_3. We call this an x-link—see Fig. 1. (The corresponding y-link is obtained by replacing $\alpha \to (\pi/2 - \alpha)$ and will be discussed below.)

The new string of tension μ_3 thus lies at an angle θ to the x axis, and moves in the z-direction with some velocity u; thus

$$\mathbf{x}_3(\sigma, t) = (\gamma_u^{-1}\sigma\cos\theta, \gamma_u^{-1}\sigma\sin\theta, ut), \tag{31}$$

where $\gamma_u = 1/\sqrt{1-u^2}$. We now determine θ and u. It follows from (31) that

$$\mathbf{b}'_3 = (\gamma_u^{-1}\cos\theta, \gamma_u^{-1}\sin\theta, -u), \tag{32}$$

and thus

$$\begin{aligned} c_1 &= -\gamma_u^{-1}\gamma^{-1}\cos(\alpha+\theta) - uv, \\ c_2 &= -\gamma_u^{-1}\gamma^{-1}\cos(\alpha-\theta) + uv, \\ c_3 &= 2\gamma^{-2}\cos^2\alpha - 1. \end{aligned} \tag{33}$$

The position of the vertex is of course $\mathbf{X}(t) = \mathbf{x}_3(s_3(t), t)$, so from (31),

$$\dot{\mathbf{X}} = (\dot{s}_3\gamma_u^{-1}\cos\theta, \dot{s}_3\gamma_u^{-1}\sin\theta, u). \tag{34}$$

Thus, from (7) and (10), we obtain

$$\mathcal{M}\dot{\mathbf{X}} + M_3(1-c_3)\mathbf{b}'_3 = -M_1(1-c_1)\mathbf{b}'_1 - M_2(1-c_2)\mathbf{b}'_2, \tag{35}$$

whose three components are

$$\begin{aligned} &[\mathcal{M}\dot{s}_3 + M_3(1-c_3)]\gamma_u^{-1}\cos\theta \\ &\quad = [M_1(1-c_1) + M_2(1-c_2)]\gamma^{-1}\cos\alpha, \\ &[\mathcal{M}\dot{s}_3 + M_3(1-c_3)]\gamma_u^{-1}\sin\theta \\ &\quad = [M_1(1-c_1) - M_2(1-c_2)]\gamma^{-1}\sin\alpha, \\ &[\mathcal{M} - M_3(1-c_3)]u \\ &\quad = [M_1(1-c_1) - M_2(1-c_2)]v. \end{aligned} \tag{36}$$

Dividing the y-component of (36) by the x-component, and comparing with the z-component gives

$$\frac{\tan\theta}{\tan\alpha} = \frac{u}{v} = \frac{M_1(1-c_1) - M_2(1-c_2)}{M_1(1-c_1) + M_2(1-c_2)}. \tag{37}$$

The first equality here allows us to eliminate θ. The second, combined with (33), then allows us to obtain an equation for u, which simplifies to a quadratic for u^2 (see Appendix A):

$$\mu_-^2(\sin^2\alpha)u^4 + [\mu_3^2(1-v^2) + \mu_-^2(v^2\cos^2\alpha - \sin^2\alpha)]u^2 - \mu_-^2 v^2\cos^2\alpha = 0. \tag{38}$$

This equation always has one positive root which then determines θ from (37). Notice that when $\mu_- = 0$ as in [35], then $u = 0$ and $\theta = 0$: this is the symmetric case. On the other hand, in the limit when the bound in (28) is saturated, i.e., $\mu_-^2 \to \mu_3^2$, then $u \to v$ and $\theta \to \alpha$. Also, when $v \to 1$ for $\mu_- > 0$, then $u \to 1$ and again $\theta \to \alpha$.

The intercommutation produces kinks on the original strings, moving along them at the speed of light; they are in fact at the same positions as in the equal-tension case, namely, at

$$K_{1,2} = (\gamma^{-1}\cos\alpha, \pm\gamma^{-1}\sin\alpha, \pm v)t. \tag{39}$$

V. KINEMATIC CONSTRAINTS

Having found the orientation and velocity of the connecting string, the physical condition that the junctions must move apart imposes the requirement

$$\dot{s}_3 > 0 \tag{40}$$

where $\dot{s}_3$ is given in (10). This in turn implies kinematic constraints on the values of v and α of the colliding strings for which an x-link can be formed. Replacing α by $\pi/2 - \alpha$ gives a similar condition for the formation of a y-link.

When $\mu_- = 0$, constraint (40) reproduces the results of [35], namely

$$\alpha < \arccos\left(\frac{\mu_3\gamma}{2\mu_1}\right) \quad (x\text{-link}), \tag{41}$$

$$\alpha > \arcsin\left(\frac{\mu_3\gamma}{2\mu_1}\right) \quad (y\text{-link}). \tag{42}$$

If these bounds are not satisfied, then Abelian strings presumably pass through one another. Essentially this same result, in the special case of all tensions equal $\mu_1 = \mu_2 = \mu_3$, was obtained earlier in the context of hadronic strings [39], and also when $\mu_- = 0$ from a different starting point in the context of Type-I strings [40].

For non-Abelian strings, as discussed in [35], there is a third possibility, namely, that a link forms in the direction of motion of the two initial strings (a z-link). Indeed if the string fluxes on strings 1 and 2 do not commute, then for topological reasons the strings cannot simply pass through one another: they may only do so with the formation of a third string. Note that in this case we do not have strings with three different tensions meeting at a vertex: at one end of string 3 it is attached to two segments of string 1, and at the other to two segments of string 2. Hence the linking string in this case *does* lie along the z-axis, simplifying the analysis. In effect, we have two vertices of the type discussed in [35]. The triangle inequalities (12) now require that both $\mu_3 \leq 2\mu_1$ and $\mu_3 \leq 2\mu_2$. Furthermore the kinematic constraint in this case is that the *total* length of string 3 must increase; we must add the rates from the two ends. It is not absolutely necessary that $\dot{s}_3$ at one particular vertex should be positive. The required condition is

$$\frac{2\mu_1 v - \mu_3}{2\mu_1 - \mu_3 v} + \frac{2\mu_2 v - \mu_3}{2\mu_2 - \mu_3 v} > 0. \tag{43}$$

Solving for v gives the limit

$$v > \frac{4\mu_1\mu_2 + \mu_3^2 - \sqrt{(4\mu_1^2 - \mu_3^2)(4\mu_2^2 - \mu_3^2)}}{2(\mu_1 + \mu_2)\mu_3}. \tag{44}$$

These different constraints are summarized in Figs. 2 and 3 showing the (α, v) plane for $\mu = 3$. Figure 2 shows the three constraints for the case $\mu_- = 0$ (Eqs. (41), (42), and (44)) and various values of μ_3. For non-Abelian strings there are three different regimes: (i) if $\mu_3 > \sqrt{2}\mu_1$ there is a kinematically forbidden region for all α; (ii) for $2\mu_1/\sqrt{3} > \mu_3 > \sqrt{2}\mu_1$ only a restricted range of α are forbidden; (iii) for $\mu_3 < 2\mu_1/\sqrt{3}$ the whole (α, v) plane is included in at least one of the allowed regions. Allowed regions for the formation of an x-link are to the left of the solid lines, those for a y-link to the right of the dashed lines, while for a z-link in the non-Abelian case they are above the horizontal dotted lines.

Figure 3 shows the allowed regions for a number of cases with $\mu_- > 0$. (The sign of μ_- does not affect the limits.) The allowed x-link regions are to the left of the curves, and those for a z-link above the horizontal lines. (The y-link regions can be found by the substitution $\alpha \to \pi/2 - \alpha$.)

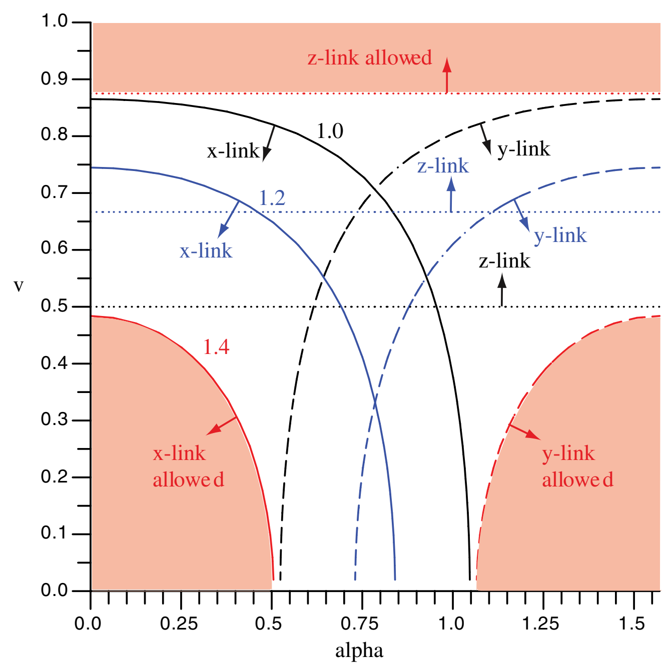

FIG. 2 (color online). Kinematic constraints for $\mu_- = 0$. Allowed regions for x-links are to the left of the full curves; for y-links to the right of the dashed curves; and, for z-links in the non-Abelian case, above the horizontal dotted lines. The values of μ_3 are 1.4, 1.2, 1.0. Allowed regions are shaded for the $\mu_3 = 1.4$ case.

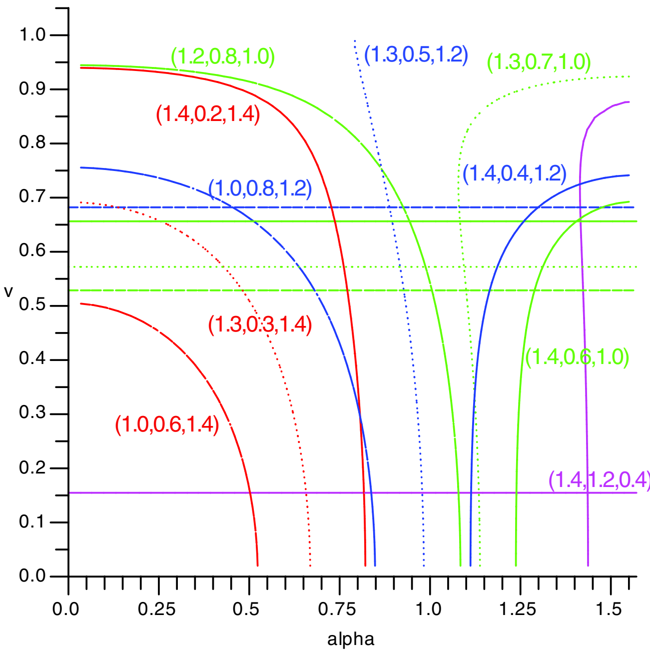

FIG. 3 (color online). Kinematic constraints for strings of unequal tensions normalized to $\mu = 3$ for the formation of x- and z-links. The allowed region is to the left of the x-link curves. For z-links, it is above the horizontal line (which is absent if $\mu_3 > 2\mu_2$). The values of (μ_1, μ_2, μ_3) shown are: (1.0, 0.6, 1.4), dashed line; (1.3, 0.3, 1.4), dotted line; (1.4, 0.2, 1.4), solid line; (1.0, 0.8, 1.2), dashed line; (1.3, 0.5, 1.2), dotted line; (1.4, 0.4, 1.2), solid line; (1.2, 0.8, 1.0), dashed line; (1.3, 0.7, 1.0), dotted line; (1.4, 0.6, 1.0), solid line; (1.4, 1.2, 0.4), solid line.

When $\mu_- \neq 0$, it does not seem to be possible to solve (40) analytically for all α. We note, however, that there are certain limiting cases for which an analytic solution is possible. First, in the low-velocity limit, $v \to 0$, it is clear from (38) that also $u \to 0$. In this limit, $\sin\theta = (\mu_-/\mu_3) \times \sin\alpha$. It is then straightforward to show that an x-link is only possible if

$$\sin^2\alpha_{(v=0)} < \frac{\mu_+^2 - \mu_3^2}{\mu_+^2 - \mu_-^2}. \tag{45}$$

Note that the value of α increases if μ_3 decreases, or if μ_- increases. [Recall that the triangle inequalities impose the restrictions (28).]

We can also find a solution for $\alpha = 0$. Here u is readily obtainable from (38) and then (40) yields

$$v^2_{(\alpha=0)} < \frac{\mu_3^2(\mu_+^2 - \mu_3^2)}{\mu_+^2(\mu_3^2 - \mu_-^2)} \equiv v_c^2. \tag{46}$$

Thus v_c defined in (46) increases with μ_-^2. Both these limits indicate that in general the kinematic constraints exclude a smaller region of the (α, v) plane as the string tensions become more different. This behavior is readily seen in Fig. 3.

In the equal-tension case, $\mu_- = 0$, v_c is always less than 1. Here, however, $v_c^2 \leq 1$ only if

$$\mu_3^2 \geq \mu_+\mu_- = |\mu_1^2 - \mu_2^2|. \tag{47}$$

If this condition is satisfied by the tension of the potentially linking string, there is a velocity v_c above which Abelian strings will necessarily pass through each other rather than intercommuting. However, when the condition is violated, the entire high-velocity region $v \to 1$ for all α is included in the allowed region. The bounding curve in this case bends to the right, and reaches $\alpha = \pi/2$ at a finite velocity, above which an x-link is kinematically allowed for any angle α. This velocity constraint for $\alpha = \pi/2$ is

$$v^2_{(\alpha=\pi/2)} > \frac{\mu_+^2(\mu_3^2 - \mu_-^2)}{\mu_3^2(\mu_+^2 - \mu_3^2)}. \tag{48}$$

Note that this limit is equal to $1/v_c^2$.

VI. RATE OF CHANGE OF STRING LENGTHS

One of the important reasons for studying the kinematics of string collisions is that the results may throw some light on the question of how a network of such strings would evolve in the early universe. If we ignore the Hubble expansion and any energy loss mechanisms, then the energy in the string network is fixed, but some strings will shorten and others will grow. We may ask how fast, on average, is this growth or shortening.

It is reasonable to assume that at any string junction, the unit vectors $\mathbf{b}'_j$ representing the ingoing waves are randomly distributed on the unit sphere, and mutually independent. (This might not be true if, for example, two of the strings come from the same other vertex, but that is presumably not a common situation.) If the strings are all of the same tension, then because of energy conservation it is clear that $\langle \dot{s}_j \rangle$ must vanish for each j, but this is not necessarily so if the tensions are different. And as we shall see, even for the equal-tension case, the zero mean does not mean that the distribution is symmetrical; it is actually not true that strings are as likely to grow as to shrink.

Let us start by looking at the distribution of the variables c_j, assuming that the unit vectors $\mathbf{b}'_j$ are randomly distributed on the unit sphere, and mutually independent. Let us choose the z-axis along the direction of $\mathbf{b}'_3$, and $\mathbf{b}'_2$ in the (x, z) plane; $\mathbf{b}'_2 = (\sin\theta, 0, \cos\theta)$. Then $\mathbf{b}'_1 = (\sin\beta\cos\phi, \sin\beta\sin\phi, \cos\beta)$ and we may assume a uniform distribution in the variables c_1, c_2, and ϕ where

$$c_3 = c_1c_2 + \sqrt{1 - c_1^2}\sqrt{1 - c_2^2}\cos\phi. \tag{49}$$

Our aim is to calculate the probability distribution for, say, the rate $\dot{s}_1$ at which the first string grows. Specifically, let $P(\dot{s}_1)d\dot{s}_1$ be the probability that $\dot{s}_1$ lies between $\dot{s}_1$ and $\dot{s}_1 + d\dot{s}_1$. Clearly,

$$P(\dot{s}_1) = \frac{1}{8\pi}\int_{-1}^{1} dc_1 \int_{-1}^{1} dc_2 \int_0^{2\pi} d\phi\, \delta\Big(\dot{s}_1 - 1 + \frac{\mu M_1(1 - c_1)}{\mu_1 \mathcal{M}}\Big), \tag{50}$$

where we have used (10), and $\mathcal{M} = \mathcal{M}(c_1, c_2, c_3)$ is defined in (11).

The resulting calculation is fairly lengthy, so we relegate the details to Appendix B. The distribution $P(\dot{s}_1)$ takes three different analytic forms in three intervals, namely

$$\begin{aligned} &1. \quad -1 < \dot{s}_1 < \frac{\mu_3^2 - \mu_1^2 - \mu_2^2}{2\mu_1\mu_2};\\ &2. \quad \frac{\mu_3^2 - \mu_1^2 - \mu_2^2}{2\mu_1\mu_2} < \dot{s}_1 < \frac{\mu_2^2 - \mu_1^2 - \mu_3^2}{2\mu_1\mu_3};\\ &3. \quad \frac{\mu_2^2 - \mu_1^2 - \mu_3^2}{2\mu_1\mu_3} < \dot{s}_1 < 1. \end{aligned} \tag{51}$$

(Without loss of generality, we have chosen $\mu_2 \geq \mu_3$.) To write the final expressions for the probability distribution concisely, it is useful to introduce the constant

$$Q = \frac{\mu}{3\nu_1\nu_2\nu_3}. \tag{52}$$

Then the expressions for $P(\dot{s}_1)$ in the three regions are

$$\begin{aligned} &1. \quad P(\dot{s}_1) = Q\mu^2\mu_1^2\Big(\frac{1 + \dot{s}_1}{1 - \dot{s}_1}\Big)^2 \frac{(2\mu_2 + 2\mu_3 - \mu_1) + (2\mu_1 - \mu_2 - \mu_3)\dot{s}_1}{(\mu_1\dot{s}_1 + \mu_2 + \mu_3)^3};\\ &2. \quad P(\dot{s}_1) = Q\nu_2^2\frac{(2\mu_1 + \mu_2 - \mu_3)}{\mu_1(1 - \dot{s}_1)^2};\\ &3. \quad P(\dot{s}_1) = Q\mu_1^2\nu_1^2\frac{(\mu_1 + 2\mu_2 + 2\mu_3) + (2\mu_1 + \mu_2 + \mu_3)\dot{s}_1}{(\mu_1\dot{s}_1 + \mu_2 + \mu_3)^3}. \end{aligned} \tag{53}$$

The probability distribution has kinks at the boundaries between these regions, given by (53). The form of the distribution is illustrated for various cases in Fig. 4.

In the particular case where the tensions are all equal, the distribution has a single kink, at $\dot{s}_1 = -\frac{1}{2}$. Its form is

$$\begin{aligned} &1. \quad P(\dot{s}_1) = \frac{27}{(\dot{s}_1 + 2)^3}\Big(\frac{1 + \dot{s}_1}{1 - \dot{s}_1}\Big)^2 \quad \Big(\dot{s}_1 < -\frac{1}{2}\Big),\\ &3. \quad P(\dot{s}_1) = \frac{4\dot{s}_1 + 5}{(\dot{s}_1 + 2)^3} \quad \Big(\dot{s}_1 > -\frac{1}{2}\Big). \end{aligned}$$

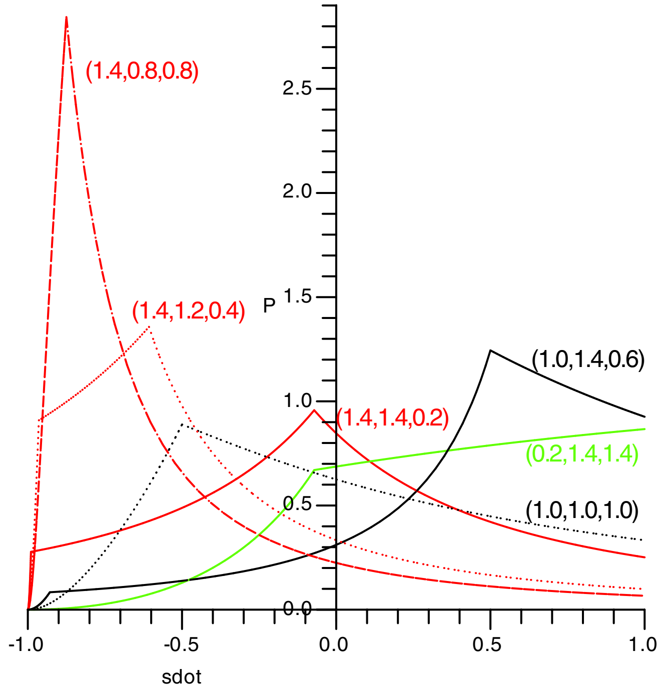

FIG. 4 (color online). The distribution of $P(\dot{s}_1)$ plotted against $\dot{s}_1$. The curves shown are for $(\mu_1, \mu_2, \mu_3) = (1.4, 0.8, 0.8)$, dashed line; (1.4, 1.2, 0.4), dotted line; (1.4, 1.4, 0.2), solid line; (1.0, 1.0, 1.0), dotted line; (1.0, 1.4, 0.6), solid line; (0.2, 1.4, 1.4), solid line.

It is interesting that although the mean value of $\dot{s}_1$ in this distribution is zero, as it must be, the distribution is not symmetrical—the dotted curve in Fig. 4. At any particular time, it is most probable that one of the three legs is growing, while the other two are shrinking (of course at a slower rate).

More generally, if $\mu_2 = \mu_3$, there is only a single kink; three of the curves in Fig. 4 are examples of this case, with $\mu_1 = 1.4$, 1.0, and 0.2.

It is now straightforward (if tedious) to compute the average value of $\dot{s}_1$. We find

$$\langle \dot{s}_1 \rangle = \frac{3\mu_1 - \mu}{3\mu_1} + \frac{Q}{\mu_1}\bigg[-(\mu_1 + \mu)\nu_1^2 \ln\frac{\mu\nu_1}{4\mu_2\mu_3} + (\mu_1 + \nu_3)\nu_2^2 \ln\frac{\mu\nu_2}{4\mu_1\mu_3} + (\mu_1 + \nu_2)\nu_3^2 \ln\frac{\mu\nu_3}{4\mu_1\mu_2}\bigg]. \tag{54}$$

Note that the expression (54) is symmetrical under the interchange $\mu_2 \leftrightarrow \mu_3$, as it should be. It is also easy to check that it satisfies the consistency condition

$$\mu_1\langle \dot{s}_1\rangle + \mu_2\langle \dot{s}_2\rangle + \mu_3\langle \dot{s}_3\rangle = 0. \tag{55}$$

Finally it is interesting to note that in general $\langle \dot{s}_1 \rangle$ is more likely to be positive if μ_1 is small, or if the other two tensions are very different. In Fig. 5, the average value is plotted against $|\mu_2 - \mu_3|$ for various values of μ_1. Notice

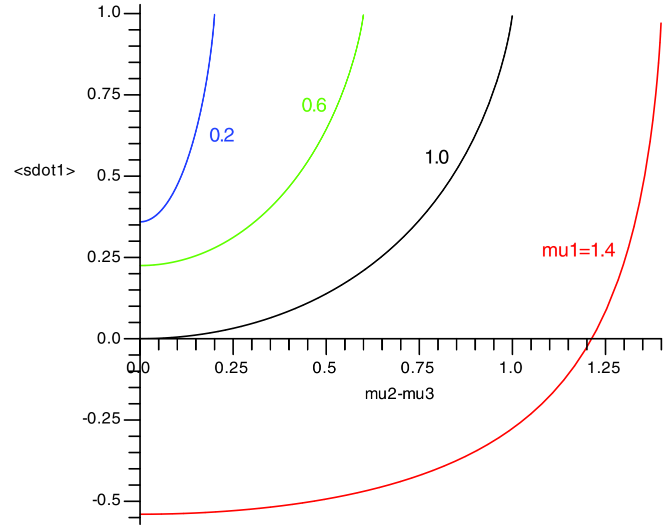

FIG. 5 (color online). Average value of $\dot{s}_1$ plotted against $|\mu_2 - \mu_3|$ for $\mu_1 = 0.2$, 0.6, 1.0, and 1.4.

that for $\mu_1 \leq 1$, $\langle \dot{s}_1 \rangle$ is *always* positive. It appears that in a network of strings there may be a tendency for the lighter strings to grow at the expense of the heavier ones. This seems to be consistent with results obtained by studying the statistical mechanics of such networks of strings [41].

VII. AVERAGE STRING VELOCITY

For ordinary Nambu-Goto strings in flat space-time, the r.m.s. transverse string velocity in a random tangle of strings is $1/\sqrt{2}$. It is interesting to ask whether this figure would be different for junction-forming strings. It is easy in principle to answer this question. For example the square of the velocity of the first string adjacent to the vertex is

$$\dot{\mathbf{x}}_1^2 = \tfrac{1}{2}(1 - \mathbf{a}_1' \cdot \mathbf{b}_1'), \tag{56}$$

and using (5) and (7) this can be expressed in terms of the c_j. We have

$$(1 + \dot{s}_1)\mathbf{a}_1' \cdot \mathbf{b}_1' = \frac{\nu_1}{\mu}(1 - \dot{s}_1) - \frac{2\mu_2}{\mu}(1 - \dot{s}_2)c_3 - \frac{2\mu_3}{\mu}(1 - \dot{s}_3)c_2. \tag{57}$$

Using (10) this becomes

$$\mathbf{a}_1' \cdot \mathbf{b}_1' = \frac{\nu_1 M_1(1-c_1) - 2\mu_1 M_2(1-c_2)c_3 - 2\mu_1 M_3(1-c_3)c_2}{-\nu_1 M_1(1-c_1) + 2\mu_1 M_2(1-c_2) + 2\mu_1 M_3(1-c_3)}. \tag{58}$$

We can then compute the average over the probability distribution already obtained by plugging this expression into an integral of the same form as (50). In this case, there does not seem to be any obvious way of obtaining an analytic result, but it is possible to make some progress.

It seems to be slightly easier to compute the average of $\mathbf{x}'^2$ rather than $\dot{\mathbf{x}}^2$:

$$\langle \mathbf{x}_1'^2 \rangle = \frac{1}{2}\langle 1 + \mathbf{a}_1' \cdot \mathbf{b}_1' \rangle = \left\langle \frac{2\mu_1^2(1-c_2)(1-c_3)}{-M_1(1-c_1) + 2\mu_1[\nu_3(1-c_2) + \nu_2(1-c_3)]} \right\rangle. \tag{59}$$

Then if we use as independent variables c_2, c_3, ψ with now

$$c_1 = c_2 c_3 + \sqrt{1-c_2^2}\sqrt{1-c_3^2}\cos\psi, \tag{60}$$

we can perform the ψ integration, to yield

$$\langle \mathbf{x}_1'^2 \rangle = \frac{\mu_1^2}{2}\int_{-1}^{1} dc_2 \int_{-1}^{1} dc_3 \frac{(1-c_2)(1-c_3)}{\sqrt{F^2 - G^2}}, \tag{61}$$

where

$$F = 2\mu_1[\nu_3(1-c_2) + \nu_2(1-c_3)] - \nu_2\nu_3(1-c_2c_3), \tag{62}$$

and

$$G = \nu_2\nu_3\sqrt{1-c_2^2}\sqrt{1-c_3^2}. \tag{63}$$

Remarkably (61) can be evaluated to give the general result

$$\langle \dot{\mathbf{x}}_1^2 \rangle = \langle (1 - \mathbf{x}_1'^2) \rangle = \frac{\mu_1^2 - 13\bar{\mu}^2}{15(\mu_1^2 - \bar{\mu}^2)} + \frac{(4\mu_1 - \bar{\mu})(\mu_1 + \bar{\mu})^2}{15\mu_1(\mu_1 - \bar{\mu})^2} \ln\frac{2\mu_1}{\mu_1 + \bar{\mu}} + \frac{(4\mu_1 + \bar{\mu})(\mu_1 - \bar{\mu})^2}{15\mu_1(\mu_1 + \bar{\mu})^2} \ln\frac{2\mu_1}{\mu_1 - \bar{\mu}}, \tag{64}$$

where we have introduced $\bar{\mu} = \mu_2 - \mu_3$. A number of interesting features can be seen. First, the result is independent of $\mu_2 + \mu_3$, depending only on the single ratio $\bar{\mu}/\mu_1$. Second, the limit $\bar{\mu} = 0$ gives

$$\langle \dot{\mathbf{x}}_1^2 \rangle = \tfrac{1}{15}[1 + 8\ln 2] \simeq 0.436, \tag{65}$$

independently of μ_1. In particular we note that, even for strings of equal tension, the r.m.s. velocity is not $1/\sqrt{2}$ as it is for ordinary Nambu-Goto strings without junctions. Surprisingly, on the contrary, this value is obtained when $\bar{\mu} = \mu_1$, a limit in which the triangle inequality is only just satisfied.

For completeness, in Fig. 6 we have plotted $v_{\rm rms} = \sqrt{\langle \dot{\mathbf{x}}_1^2 \rangle}$, for various choices of μ_1, a function of $|\mu_2 - \mu_3|$.

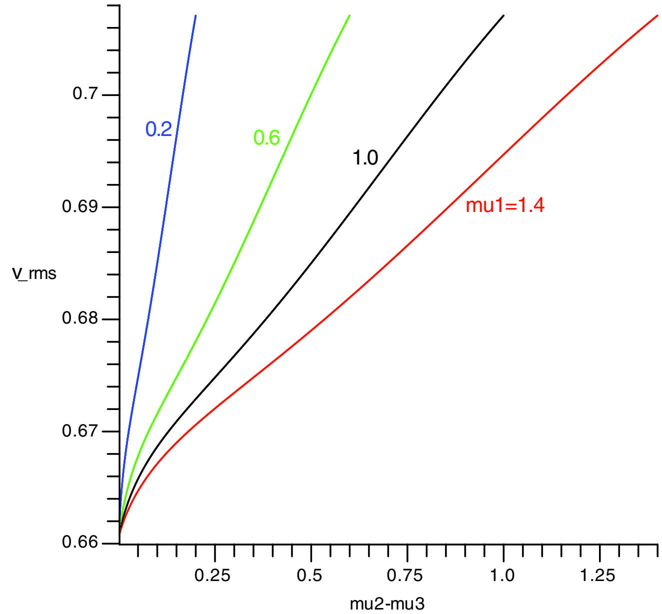

FIG. 6 (color online). R.m.s. value of the string velocity v plotted against $|\mu_2 - \mu_3|$ for several values of μ_1.

VIII. CONCLUSIONS

In an earlier paper [35], we considered the kinematic constraints on the possibility of intercommuting of strings that form junctions. Concentrating on the case where the approaching strings have equal tensions, we showed that if the relative velocity with which they meet is too large, no exchange can take place and the strings will pass through one another (or, for non-Abelian strings, become joined by a string in the direction of the relative velocity or form a linked X configuration). In this paper we have extended the analysis to the more realistic case where the approaching strings have different tensions. For some values of the string tensions, it may happen that ultrarelativistic strings *can* exchange partners.

We first studied the reflection and transmission of small-amplitude waves at string junctions—something that may be important in predicting the gravitational radiation from strings. We determined the fraction of energy transmitted and reflected across the junction, and showed how the reflected wave is generally tilted in the opposite direction to the incoming wave.

In Secs. IV and V, we generalized the kinematic constraints (41) and (42) of I to the case in which the two colliding strings have different tensions μ_1 and μ_2. The question of whether strings always intercommute is vital in understanding the evolution of a network of strings; in particular, it affects the final density of strings found in the scaling regime of a network, if indeed a scaling solution is reached. We have established the criteria required for intercommuting in terms of the incoming velocity and angle of approach, as a function of the string tensions. A particularly important combination of tensions is that given in the inequality (47). If this inequality is satisfied, for instance when $\mu_1 = \mu_2$, then ultrarelativistic strings cannot intercommute. If it is violated, then ultrarelativistic strings may intercommute. Our result that intercommuting does not always happen appears at first sight to be in contrast to the claim in [42], but we believe the regime where their results are applicable corresponds to low velocities of approach as the moduli approximation they use breaks down for high velocities. It is in the high-velocity regime that intercommutation may break down.

Since the initial velocity of colliding strings is crucial to determining the importance of the kinematical constraints, in Secs. VI and VII we considered the global properties of junctions and strings in a network. A plausible assumption for a string network is that the incoming waves at a junction are randomly distributed. This has allowed us to make progress in determining the r.m.s. velocities of strings. In particular (ignoring energy loss mechanisms such as expansion of the universe) we have calculated the average speed at which a junction moves along each of the three strings from which it is formed. Our results are intriguing. For example, even for the case of equal-tension strings, although the average velocity of the junctions is zero as expected, the distribution of the velocities is not peaked around zero, but around a negative velocity (actually around $\dot{s}_i = -1/2$) indicating that even in this apparently symmetric case, it is most probable that at any particular time, one of the three legs is growing, while the other two are shrinking, all be it at a slower rate. The case of unequal tensions can also be solved analytically and our results suggest that junction dynamics may be such as to preferentially remove the heavy strings from the network. It thus seems likely that the system will evolve to one where the lightest strings are dominating the dynamics, though of course junctions are still present. Regarding the r.m.s. velocities of the strings themselves, we showed that they are generically smaller than the $1/\sqrt{2}$ characteristic of Nambu-Goto networks, even in the case when the strings all have equal tensions.

In a future publication we intend to present some exact solutions for loops containing junctions. Given the new features we have uncovered for the dynamics of string networks with junctions, we should not be surprised to find important results concerning the distribution of kinks and cusps in these more complicated configurations. This in turn could have a bearing on the gravitational radiation emitted from loops of strings with junctions.

ACKNOWLEDGMENTS

The work reported here was assisted by the ESF COSLAB Programme. D. A. S. is grateful to the Physics Department at Case Western for hospitality while some of this work was done. We are grateful to Kepa Sousa for a useful comment, and also Xavier Artru for drawing Ref. [39] to our attention.

APPENDIX A

Here we outline the calculation of Eq. (38). Let $\mu_\pm = \mu_1 \pm \mu_2$, so that

$$M_{1,2} = (\mu_+ - \mu_3)(\mu_3 \pm \mu_-), \qquad M_3 = \mu_3^2 - \mu_-^2.$$

Then the second equality in (37) yields

$$\frac{u}{v} = \frac{\mu_-(1 + \gamma^{-1}\gamma_u^{-1}\cos\alpha\cos\theta) + \mu_3(uv - \gamma^{-1}\gamma_u^{-1}\sin\alpha\sin\theta)}{\mu_3(1 + \gamma^{-1}\gamma_u^{-1}\cos\alpha\cos\theta) + \mu_-(uv - \gamma^{-1}\gamma_u^{-1}\sin\alpha\sin\theta)}.$$

Next we can eliminate $\cos\theta$ and $\sin\theta$. From the first equality in (37),

$$\cos\theta = \frac{v\cos\alpha}{\sqrt{v^2\cos^2\alpha + u^2\sin^2\alpha}},$$

$$\sin\theta = \frac{u\sin\alpha}{\sqrt{v^2\cos^2\alpha + u^2\sin^2\alpha}}.$$

Thus multiplying up and grouping all the terms involving γ^{-1} on the right, gives

$$\begin{aligned}(\mu_3 u - \mu_- v) &+ (\mu_- u - \mu_3 v)uv \\ &= \frac{\gamma^{-1}\gamma_u^{-1}}{\sqrt{v^2\cos^2\alpha + u^2\sin^2\alpha}}[(\mu_- v - \mu_3 u)v\cos^2\alpha \\ &\quad + (\mu_- u - \mu_3 v)u\sin^2\alpha].\end{aligned}$$

Now square to obtain

$$\begin{aligned}(v^2\cos^2\alpha &+ u^2\sin^2\alpha)[\mu_3 u(1 - v^2) - \mu_- v(1 - u^2)]^2 \\ &= -(1 - v^2)(1 - u^2)[\mu_-(v^2\cos^2\alpha + u^2\sin^2\alpha) \\ &\quad - \mu_3 uv]^2.\end{aligned}$$

On expansion, the terms in uv, which come from the cross terms in each square bracket, cancel, and we are left with

$$F[\mu_3^2 u^2(1 - v^2) - \mu_-^2(1 - u^2)(v^2\cos^2\alpha + u^2\sin^2\alpha)] = 0,$$

where

$$F = (1 - v^2)(v^2\cos^2\alpha + u^2\sin^2\alpha) - v^2(1 - u^2)$$

which vanishes only if $v = \pm u$. Thus we finally get the quadratic (38) for u^2:

$$\mu_-^2(\sin^2\alpha)u^4 + [\mu_3^2(1 - v^2) + \mu_-^2(v^2\cos^2\alpha - \sin^2\alpha)]u^2 - \mu_-^2 v^2\cos^2\alpha = 0.$$

This always has one positive root as the discriminant is positive.

APPENDIX B

To carry out the integral in Eq. (50), it is convenient to go over to a more symmetrical form by changing variable from ϕ to c_3, using (49). This introduces a Jacobian factor, which is the inverse of

$$\frac{dc_3}{d\phi} = -\sqrt{1-c_1^2}\sqrt{1-c_2^2}\sin\phi = -\sqrt{J},$$

where

$$J = (1-c_1^2)(1-c_2^2) - (c_3 - c_1c_2)^2$$
$$= 1 - c_1^2 - c_2^2 - c_3^2 + 2c_1c_2c_3.$$

Note that the physically allowed region in (c_1, c_2, c_3) space is characterized by $J \geq 0$. There is also a factor of 2 because there are two values of ϕ for each c_3. Letting $w_1 = 1 - \dot{s}_1$, we arrive at

$$P(1-w_1) = \frac{1}{4\pi}\int_{-1}^{1} dc_1 \int_{-1}^{1} dc_2 \int_{-1}^{1} dc_3 \frac{\theta(J)}{\sqrt{J}} \times \delta\left(w_1 - \frac{\mu M_1(1-c_1)}{\mu_1 \mathcal{M}}\right). \tag{B1}$$

It is straightforward to perform the c_3 integral using the delta function. This gives

$$P(1-w_1) = \frac{1}{4\pi}\frac{\mu}{\mu_1}\frac{M_1}{M_3}\frac{1}{w_1^2}\int_{-1}^{1} dc_1(1-c_1) \times \int_{-1}^{1} dc_2 \frac{\theta(J)}{\sqrt{J}}, \tag{B2}$$

where now

$$J = (1-c_1^2)(1-c_2^2) - [c_3(w_1) - c_1c_2]^2. \tag{B3}$$

Here $c_3(w_1)$ is obtained by equating to zero the argument of the delta function in (B2) and solving for c_3:

$$c_3(w_1) = 1 + \frac{M_1}{M_3}(1-c_1)\left(1 - \frac{\mu}{\mu_1 w_1}\right) + \frac{M_2}{M_3}(1-c_2). \tag{B4}$$

Now since, by (B3), J is a quadratic function of c_2, we can go on to perform the c_2 integral in (B2). It is clear from (B3) that $J \leq 0$ when $c_2 = \pm 1$, so the effective limits of integration for the c_2 integral are given by the roots of $J = 0$. If there are no real roots, the contribution is zero. Specifically, J has the form

$$J = \frac{1}{M_3^2}(A + 2Bc_2 - Kc_2^2), \tag{B5}$$

where

$$A = (1-c_1^2)M_3^2 - H^2, \qquad B = (M_2 + c_1M_3)H, \qquad K = M_2^2 + M_3^2 + 2c_1M_2M_3, \tag{B6}$$

with

$$H = M_2 + M_3 - (1-c_1)Y, \tag{B7}$$

where

$$Y = M_1\left(\frac{\mu}{\mu_1 w_1} - 1\right). \tag{B8}$$

Note that by the triangle inequality, Y is positive. The discriminant, which determines whether real roots exist, is

$$B^2 + AK = (1-c_1^2)(K - H^2) = (1-c_1^2)D, \tag{B9}$$

say. If the roots are $c_{2\pm}$, then the c_2 integral reduces to

$$\int_{c_{2-}}^{c_{2+}} \frac{dc_2}{\sqrt{J}} = \frac{\pi M_3}{\sqrt{K}}. \tag{B10}$$

Thus

$$P(1-w_1) = \frac{\mu M_1}{4\mu_1 w_1^2}\int_{-1}^{1} dc_1 \frac{(1-c_1)}{\sqrt{K}}\theta(D). \tag{B11}$$

Now from (B6)–(B8),

$$D = (M_2+M_3)^2 - 2(1-c_1)M_2M_3 - [M_2+M_3-(1-c_1)Y]^2$$
$$= (1-c_1)[-2M_2M_3 + 2(M_2+M_3)Y - (1-c_1)Y^2]. \tag{B12}$$

It is useful to change the variable from c_1 to ξ defined by

$$\xi^2 = K = (M_2+M_3)^2 - 2(1-c_1)M_2M_3. \tag{B13}$$

The range of values of ξ corresponding to $-1 < c_1 < 1$ is then

$$|M_2 - M_3| < \xi < M_2 + M_3. \tag{B14}$$

For convenience, we shall assume in what follows that $\mu_2 \geq \mu_3$, so the lower limit is $M_2 - M_3$. The effect of the condition that the square bracket in (B12) be positive is to require that

$$\xi > \xi_c = \frac{|(M_2+M_3)Y - 2M_2M_3|}{Y}. \tag{B15}$$

Hence

$$P(1-w_1) = \frac{\mu M_1}{4\mu_1 w_1^2}\int_{\xi_-}^{\xi_+} \frac{\xi d\xi}{M_2M_3}\frac{(M_2+M_3)^2 - \xi^2}{2M_2M_3\xi}, \tag{B16}$$

where

$$\xi_+ = M_2 + M_3, \qquad \xi_- = \max(\xi_c, M_2 - M_3). \tag{B17}$$

The integral (B16) can be rewritten as

$$P(1-w_1) = \frac{\mu M_1}{8\mu_1 M_2^2 M_3^2 w_1^2}\int_{\xi_-}^{\xi_+}(\xi_+^2 - \xi^2)d\xi, \tag{B18}$$

and hence evaluated to yield the final result

$$P(1-w_1) = \frac{\mu M_1}{24\mu_1 M_2^2 M_3^2}\frac{(\xi_+ - \xi_-)^2(2\xi_+ + \xi_-)}{w_1^2}. \tag{B19}$$

We must distinguish three different ranges of values of w_1 corresponding to the following ranges of Y:

$$\begin{aligned} &1.\quad 0 < Y < M_3, \qquad \xi_- = \frac{2M_2M_3}{Y} - (M_2 + M_3); \\ &2.\quad M_3 < Y < M_2, \qquad \xi_- = M_2 - M_3; \\ &3.\quad M_2 < Y, \qquad \xi_- = (M_2 + M_3) - \frac{2M_2M_3}{Y}. \end{aligned} \tag{B20}$$

It is straightforward to rewrite the distribution in terms of the string tensions μ_j and the variable $\dot{s}_1$. The values of $\dot{s}_1$ corresponding to the three ranges are easily seen to be those in (51), and the corresponding expressions for $P(\dot{s}_1)$ are given in (53).

[1] C. P. Burgess, M. Majumdar, D. Nolte, F. Quevedo, G. Rajesh, and R. J. Zhang, J. High Energy Phys. 07 (2001) 047.
[2] G. R. Dvali, Q. Shafi, and S. Solganik, hep-th/0105203.
[3] N. T. Jones, H. Stoica, and S. H. H. Tye, J. High Energy Phys. 07 (2002) 051.
[4] S. Kachru, R. Kallosh, A. Linde, and S. P. Trivedi, Phys. Rev. D **68**, 046005 (2003).
[5] S. Kachru, R. Kallosh, A. Linde, J. M. Maldacena, L. McAllister, and S. P. Trivedi, J. Cosmol. Astropart. Phys. 10 (2003) 013.
[6] G. Dvali and A. Vilenkin, J. Cosmol. Astropart. Phys. 03 (2004) 010.
[7] E. J. Copeland, R. C. Myers, and J. Polchinski, J. High Energy Phys. 06 (2004) 013.
[8] J. Polchinski, AIP Conf. Proc. **743**, 331 (2005); Int. J. Mod. Phys. A **20**, 3413 (2005).
[9] H. Firouzjahi, L. Leblond, and S. H. Henry Tye, J. High Energy Phys. 05 (2006) 047.
[10] U. Seljak, A. Slosar, and P. McDonald, J. Cosmol. Astropart. Phys. 10 (2006) 014.
[11] T. Damour and A. Vilenkin, Phys. Rev. D **64**, 064008 (2001).
[12] T. Damour and A. Vilenkin, Phys. Rev. D **71**, 063510 (2005).
[13] X. Siemens, J. Creighton, I. Maor, S. Ray Majumder, K. Cannon, and J. Read, Phys. Rev. D **73**, 105001 (2006).
[14] M. B. Hindmarsh and T. W. B. Kibble, Rep. Prog. Phys. **58**, 477 (1995).
[15] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and other Topological Defects* (Cambridge University Press, Cambridge, England, 1994).
[16] M. G. Jackson, N. T. Jones, and J. Polchinski, J. High Energy Phys. 10 (2005) 013.
[17] A. Hanany and K. Hashimoto, J. High Energy Phys. 06 (2005) 021.
[18] A. Achucarro and R. de Putter, Phys. Rev. D **74**, 121701 (2006).
[19] M. Sakellariadou, J. Cosmol. Astropart. Phys. 04 (2005) 003.
[20] A. Avgoustidis and E. P. S. Shellard, Phys. Rev. D **73**, 041301 (2006).
[21] K. Hashimoto and D. Tong, J. Cosmol. Astropart. Phys. 09 (2005) 004.
[22] T. Vachaspati and A. Vilenkin, Phys. Rev. D **35**, 1131 (1987).
[23] D. Spergel and U. L. Pen, Astrophys. J. **491**, L67 (1997).
[24] C. J. A. Martins, Phys. Rev. D **70**, 107302 (2004).
[25] S. H. Henry Tye, I. Wasserman, and M. Wyman, Phys. Rev. D **71**, 103508 (2005); **71**, 129906(E) (2005).
[26] E. J. Copeland and P. M. Saffin, J. High Energy Phys. 11 (2005) 023.
[27] P. M. Saffin, J. High Energy Phys. 09 (2005) 011.
[28] M. Hindmarsh and P. M. Saffin, J. High Energy Phys. 08 (2006) 066.
[29] M. Bucher and D. N. Spergel, Phys. Rev. D **60**, 043505 (1999).
[30] R. A. Battye, M. Bucher, and D. Spergel, astro-ph/9908047.
[31] R. A. Battye, B. Carter, E. Chachoua, and A. Moss, Phys. Rev. D **72**, 023503 (2005).
[32] P. Pina Avelino, C. J. A. Martins, J. Menezes, R. Menezes, and J. C. R. Oliveira, Phys. Rev. D **73**, 123519 (2006).
[33] P. P. Avelino, C. J. A. Martins, J. Menezes, R. Menezes, and J. C. R. Oliveira, Phys. Rev. D **73**, 123520 (2006).
[34] B. Carter, hep-ph/0605029.
[35] E. J. Copeland, T. W. B. Kibble, and D. A. Steer, Phys. Rev. Lett. **97**, 021602 (2006).
[36] G. S. Sharov, hep-ph/9809465.
[37] G. 't Hooft, hep-th/0408148.
[38] P. I. Fomin and Yu. V. Shtanov, Classical Quantum Gravity **19**, 3139 (2002).
[39] X. Artru, Nucl. Phys. **B85**, 442 (1975).
[40] L. M. A. Bettencourt and T. W. B. Kibble, Phys. Lett. B **332**, 297 (1994).
[41] R. J. Rivers and D. A. Steer (unpublished).
[42] M. Eto, K. Hashimoto, G. Marmorini, M. Nitta, K. Ohashi, and W. Vinci, Phys. Rev. Lett. **98**, 091602 (2007).