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Noncommutative spaces, the quantum of time, and Lorentz symmetry

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We introduce three space-times that are discrete in time and compatible with the Lorentz symmetry. We show that these spaces are not commutative, with commutation relations similar to the relations of the Snyder and Yang spaces. Furthermore, using a reparametrized relativistic particle we obtain a realization of the Snyder type spaces and we construct an action for them.

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I. INTRODUCTION

Because of several interesting results in string theory [1], noncommutative spaces have recently attracted attention. In this context, the noncommutativity parameter is a constant, which leads to inconsistencies with the usual Lorentz symmetry [2]. In that case a deformed symmetry, namely, the so-called twisted Poincaré invariance holds instead [3]. It should be stressed, however, that not all proposals of noncommutative spaces are incompatible with the Lorentz symmetry. In a remarkable work, H. Snyder constructed a noncommutative space-time compatible with the Lorentz symmetry which is discrete in the spatial coordinates [4]. Moreover, G.'t Hooft showed that by considering quantum gravity in $(2 + 1)$ dimensions a Snyder-like space-time can be obtained [5]. This result suggests that quantum gravity in other dimensions may imply a noncommutative space of this kind. Another feature that makes the Snyder space-time (SST) interesting is the fact that it can be mapped to the k-Minkowski space-time [6]; which is the arena for the so-called Doubly Special Relativity theory. It has been shown [7] that this theory coincides in some aspects with quantum gravity in $(2 + 1)$ dimensions. All these features make it worth studying Snyder-like noncommutative space-times and their realizations.

It is worth mentioning that C. N. Yang [8] based in the Snyder construction obtain another space-time that is compatible with the Lorentz symmetry and also it is invariant under translations. This space-time is discrete in the spatial coordinates.

In this work we construct three space-times that are discrete in time and compatibles with the Lorentz symmetry, and we show that these models are noncommutative. The first space has commutation relations that looks very similar to the SST, the others follow a similar pattern to the Yang space. In the case of the Snyder type space we are able to obtain a realization of the model using the action of a reparametrized relativistic particle, and from this result we construct a general action for a particle in this kind of noncommutative spaces. We remark that this result is applicable not only to spaces that are discrete in time, since is valid also for Snyder-like theories with discrete space.

The manuscript is organized as follows: In Sec. II we construct the proposed models and show that they are noncommutative. Section III is composed of three parts. In Subsec. III A we review the parametrized free particle, whereas in Subsec. III B we provide a realization of the Snyder-type spaces. Furthermore, a general form for the action of a particle in a Snyder-like noncommutative space with arbitrary Hamiltonian is presented in Subsec. III C. Finally, we summarize our results in Sec. IV.

II. THE QUANTUM OF TIME

We construct here three examples of space-times that are discrete in time and compatible with the Lorentz symmetry. First we introduce the space with commutation relations similar to the SST and following a similar approach we obtain two spaces that resemble the Yang space.

A. Noncommutative Snyder-like space-time

Let us start by considering a $(D + 2)$-dimensional space with $\zeta^A = (\zeta^0, \zeta^{0'}, \zeta^1, \cdots, \zeta^D)$ a vector in this space and a metric η with components given by

$$\begin{gathered}\eta_{00} = 1, \qquad \eta_{0'0'} = 1,\\ \eta_{00'} = \eta_{0'0} = 0, \qquad \eta_{i0'} = \eta_{0'i} = 0,\\ \eta_{i0} = \eta_{0i} = 0, \qquad \eta_{ij} = -\delta_{ij}, \qquad i, j = 1, \cdots, D.\end{gathered} \tag{1}$$

If the transformations Λ let the quadratic form $\tilde{S}^2 = \zeta^T \eta \zeta = -(\zeta^{0'})^2 - (\zeta^0)^2 + (\zeta^1)^2 + \cdots + (\zeta^D)^2$ invariant, then they satisfy the identity

$$\Lambda^T \eta \Lambda = \eta. \tag{2}$$

For the infinitesimal transformations close to the identity, $\Lambda = I + \epsilon M$, with I the identity matrix, Eq. (2) implies

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$$(I + \epsilon M)^T \eta (I + \epsilon M) = \eta, \tag{3}$$

and thus $M^T \eta = -\eta M$. That is

$$M_{AB} = -M_{BA}. \tag{4}$$

From this matrix we define the infinitesimal transformation $\delta\zeta^A = \epsilon M^A{}_B \zeta^B$ with the generators of the group given by

$$V = \epsilon M^A{}_B \zeta^B \frac{\partial}{\partial \zeta^A} = \epsilon M_{AB} \zeta^B \frac{\partial}{\partial \zeta_A}$$
$$= \epsilon M_{00'}\left(\zeta^{0'} \frac{\partial}{\partial \zeta_0} - \zeta^0 \frac{\partial}{\partial \zeta_{0'}}\right) + \epsilon M_{0'i}\left(\zeta^i \frac{\partial}{\partial \zeta_{0'}} - \zeta^{0'} \frac{\partial}{\partial \zeta_i}\right) + \epsilon M_{0i}\left(\zeta^i \frac{\partial}{\partial \zeta_0} - \zeta^0 \frac{\partial}{\partial \zeta_i}\right) + \epsilon \frac{1}{2} M_{ji}\left(\zeta^i \frac{\partial}{\partial \zeta_j} - \zeta^j \frac{\partial}{\partial \zeta_i}\right). \tag{5}$$

Then, we define the operators

$$l^{00'} = \zeta^{0'} \frac{\partial}{\partial \zeta_0} - \zeta^0 \frac{\partial}{\partial \zeta_{0'}}, \tag{6}$$

$$l^{i0'} = \zeta^{0'} \frac{\partial}{\partial \zeta_i} - \zeta^i \frac{\partial}{\partial \zeta_{0'}}, \tag{7}$$

$$l^{0i} = \zeta^i \frac{\partial}{\partial \zeta_0} - \zeta^0 \frac{\partial}{\partial \zeta_i}, \tag{8}$$

$$l^{ji} = \zeta^i \frac{\partial}{\partial \zeta_j} - \zeta^j \frac{\partial}{\partial \zeta_i}. \tag{9}$$

We go now onto considering the reduced space of dimension $(D+1)$, $\zeta^\mu = (\zeta^0, \zeta^1, \cdots, \zeta^D)$ a vector in this space and the Lorentz transformation in it. As the variable $\zeta^{0'}$ is invariant under the Lorentz transformation in this reduced space, then $R^\mu = l^{\mu 0'}$ is a contravariant vector.

Therefore, we define the Hermitian operator

$$\hat{X}^\mu = -ia\left(\zeta^{0'} \frac{\partial}{\partial \zeta_\mu} - \zeta^\mu \frac{\partial}{\partial \zeta_{0'}}\right), \qquad \mu = 0, 1, \cdots, D, \tag{10}$$

where a is a constant of unit length. By employing this, we construct the Hermitian operator invariant under the Lorentz transformation

$$\hat{S}^2 = \hat{X}_\mu \hat{X}^\mu. \tag{11}$$

From (10) we find the commutation rules

$$[\hat{X}_\mu, \hat{X}_\nu] = \frac{ia^2}{\hbar} \hat{L}_{\mu\nu}, \tag{12}$$

$$[\hat{X}_\mu, \hat{P}_\nu] = i\hbar\left(\eta_{\mu\nu} + \frac{a^2}{\hbar^2} \hat{P}_\mu \hat{P}_\nu\right), \tag{13}$$

$$[\hat{P}_\mu, \hat{P}_\nu] = 0, \tag{14}$$

where

$$\hat{L}_{\mu\nu} = \hat{X}_\mu \hat{P}_\nu - \hat{X}_\nu \hat{P}_\mu, \tag{15}$$

$$\hat{P}_\mu = \frac{-\hbar}{a} \frac{\zeta_\mu}{\zeta^{0'}}. \tag{16}$$

Therefore, this space is noncommutative and has the Lorentz symmetry.

Now, it can be shown that $\psi_i = e^{-i\tilde{L}\varphi_i}$, with $\varphi_i = \operatorname{arctanh}(\zeta^i/\zeta^{0'})$, is eigenfunction of $\hat{X}^i$,

$$\hat{X}^i \psi_i = a\tilde{L}\psi_i. \tag{17}$$

In this case $\tilde{L}$ can take any arbitrary value. Analogously $\psi_0 = e^{iL\varphi_0}$, but now with $\varphi_0 = \arctan(\zeta^0/\zeta^{0'})$, is eigenfunction of $\hat{X}^0$,

$$\hat{X}^0 \psi_0 = aL\psi_0. \tag{18}$$

As the tangent is 2π-periodic, $\varphi_0 + 2\pi = \arctan(\zeta^0/\zeta^{0'})$. Therefore, in order to avoid ψ_0 from being a multivalued function, one must constrain the values of L to be integers. Thus, the time is quantized:

$$t_N = N\tfrac{a}{c}, \tag{19}$$

with N an integer and c the speed of light. Thus, this spacetime is discrete in time and consistent with the Lorentz symmetry. Notice that the Lorentz symmetry is generated by (15) and that these generators are included in the generators of the conformal group $SO(D, 2)$ (6)–(9). So, the Lorentz symmetry of this space is a contraction of the conformal group.

Notice that we can also take the reduced space $\zeta^\alpha = (\zeta^0, \zeta^1, \cdots, \zeta^d)$, $d = D - 1$ and define $V^\alpha = l^{\alpha D}$, $(\alpha = 0, 1, \cdots, d)$. In the $(d+1)$-dimensional Minkowski space, where ζ^D is an invariant and V^α is a contravariant vector. So, we can define the Hermitian operator

$$\hat{X}^\alpha = -ia\left(\zeta^D \frac{\partial}{\partial \zeta_\alpha} - \zeta^\alpha \frac{\partial}{\partial \zeta_D}\right). \tag{20}$$

This case yields the commutation rules

$$[\hat{X}_\mu, \hat{X}_\nu] = \frac{-ia^2}{\hbar} \hat{L}_{\mu\nu}, \tag{21}$$

$$[\hat{X}_\mu, \hat{P}_\nu] = i\hbar\left(\eta_{\mu\nu} - \tfrac{a^2}{\hbar^2} \hat{P}_\mu \hat{P}_\nu\right), \tag{22}$$

$$[\hat{P}_\mu, \hat{P}_\nu] = 0, \tag{23}$$

which are those of the SST [4]. The difference with the space we constructed here is that in the SST the time is continuous and the spatial coordinates are discrete.

B. Noncommutative Yang-like space-times

The Snyder space-time lacks translational invariance, implying in principle the nonconservation of the energy-momentum in field theory. Based in this observation Yang [8] proposed another version of a noncommutative space-time. This space is invariant under Lorentz transformations and infinitesimal translations. The Yang construction is similar to the Snyder construction, but requires an extra dimension. In this subsection we modify the Yang construction to consider space-times that are continuous in space but discrete in time.

Assume that we have a flat space-time of dimension $D+3$ with coordinates $(\zeta^0, \zeta^1, \ldots, \zeta^D, \zeta^r, \zeta^{r'})$. We define the labels s, s', with $s=0$ $(s'=0)$ iff ζ^r $(\zeta^{r'})$ is a temporal coordinate and $s=1$ $(s'=1)$ iff ζ^r $(\zeta^{r'})$ is a spatial coordinate. In this context, the Hermitic differential operators,

$$\hat{X}^\mu = -ia\left(\zeta^r \frac{\partial}{\partial \zeta_\mu} - \zeta^\mu \frac{\partial}{\partial \zeta_r}\right), \qquad \mu = 0, 1, \cdots, D, \tag{24}$$

$$\hat{P}^\mu = -i\frac{\hbar}{b}\left(\zeta^{r'} \frac{\partial}{\partial \zeta_\mu} - \zeta^\mu \frac{\partial}{\partial \zeta_{r'}}\right), \tag{25}$$

are contravariant vectors under the transformations that leave invariant the components ζ^r, $\zeta^{r'}$. Using these operators we obtain the algebra,

$$[\hat{X}^\mu, \hat{X}^\nu] = i\tfrac{a^2}{\hbar}(-)^s \hat{l}^{\mu\nu} \tag{26}$$

$$[\hat{X}^\mu, \hat{P}^\nu] = \frac{i\hbar\hat{\epsilon}}{b}\eta^{\mu\nu}, \tag{27}$$

$$[\hat{P}^\mu, \hat{P}^\nu] = i\tfrac{\hbar}{b^2}(-)^{s'} \hat{l}^{\mu\nu}, \tag{28}$$

where,

$$\hat{l}^{\mu\nu} = -i\hbar\left(\zeta^\mu \frac{\partial}{\partial \zeta_\nu} - \zeta^\mu \frac{\partial}{\partial \zeta_\nu}\right), \tag{29}$$

$$\hat{\epsilon} = -ia\left(\zeta^r \frac{\partial}{\partial \zeta_{r'}} - \zeta^{r'} \frac{\partial}{\partial \zeta_r}\right). \tag{30}$$

By considering the operator $\hat{L}_{\mu\nu}$ from (15), the commutation rules can be written as

$$\hat{L}^{\mu\nu} = \frac{\hat{\epsilon}}{b}\hat{l}^{\mu\nu} = \frac{i(-)^{s+1}\hbar\hat{\epsilon}}{a^2 b}[\hat{X}^\mu, \hat{X}^\nu] = \frac{i(-)^{s'+1} b\hat{\epsilon}}{\hbar}(-)^{s'}[\hat{P}^\mu, \hat{P}^\nu], \tag{31}$$

and also we have

$$[\hat{X}^\mu, \hat{\epsilon}] = i\frac{a^2 b}{\hbar}(-)^{s+1}\hat{P}^\mu, \tag{32}$$

$$[\hat{P}^\mu, \hat{\epsilon}] = i\tfrac{\hbar}{b}(-)^{s'}\hat{X}^\mu, \tag{33}$$

$$[\hat{X}^\beta, \hat{l}^{\mu\nu}] = i\hbar(\hat{X}^\mu\eta^{\beta\nu} - \hat{X}^\nu\eta^{\beta\mu}), \tag{34}$$

$$[\hat{P}^\beta, \hat{l}^{\mu\nu}] = i\hbar(\hat{P}^\mu\eta^{\beta\nu} - \hat{P}^\nu\eta^{\beta\mu}). \tag{35}$$

Clearly the commutation relations (26)–(28) are compatible with the Lorentz symmetry and from (34) and (35) we observe that $\hat{l}^{\mu\nu}$ is the generator of this group. Now, taking into account that the translation operator is given by

$$U(\alpha) = e^{-i\alpha_\mu \hat{P}^\mu} \approx 1 - i\alpha_\mu \hat{P}^\mu, \qquad \alpha_\mu = \text{const}, \tag{36}$$

and using Eqs. (32)–(35), we get the transformation rules for the operators

$$U^{-1}(\alpha)\hat{X}^\mu U(\alpha) \approx \hat{X}^\mu + \tfrac{\hbar}{b}\alpha^\mu\hat{\epsilon}, \tag{37}$$

$$U^{-1}(\alpha)\hat{P}^\mu U(\alpha) \approx \hat{P}^\mu + \tfrac{\hbar}{b^2}(-)^{s'+1}\alpha_\nu\hat{l}^{\nu\mu}, \tag{38}$$

$$U^{-1}(\alpha)\hat{l}^{\mu\nu}U(\alpha) \approx \hat{l}^{\mu\nu} + \hbar(\alpha^\mu\hat{P}^\nu - \alpha^\nu\hat{P}^\mu). \tag{39}$$

Applying these transformation rules we can show that the commutation relations (26)–(28) are invariant under translations. As a consequence for each value of s and s', we get a noncommutative space compatible the translations and Lorentz symmetries. These space-times are discrete in time or in the spatial coordinates. For example, in the case that $s=1$, $s'=1$, i.e. when both extra coordinates are spatial we get the usual Yang space [8]. In this case the Lorentz group is obtained as a subgroup of $SO(D+2,1)$. Whereas, if $s=1$, $s'=0$, we get also discrete spatial coordinates. For this case the Lorentz group corresponds to a subgroup of $SO(D+1,2)$, this space was found in [9]. Now, for $s=0$, $s'=1$ the temporal coordinate is discrete and the spatial ones are continuous, for this situation the Lorentz group is obtained as a subgroup of $SO(D+1,2)$. The last case corresponds to $s=0$, $s'=0$, here the temporal coordinate is discrete and the spatial coordinates are continuous, the Lorentz group is in this case a subgroup of $SO(D,3)$.

Other discrete-time models can be found in [10–12]. Reference [10] is particularly remarkable as from quantum gravity in $(2+1)$ dimensions the authors obtain a momenta space having two time-coordinates. The spectrum of the space-time is similar to the one obtained here, but the commutation rules are not the same. It is worth mentioning that it was shown recently that different physical systems can be unified by a two time-coordinates model [13]. A proposal with two time-coordinates at the level of string theory can be seen in [14].

In the literature exists some proposals where space-times are analyzed with noncommutativity in time, these spaces present to the level of field theory problems with causality and unitarity [15]. However, in these examples the Lorentz symmetry is broken, or they have a twisted Poincaré symmetry. Furthermore, in all these cases the parameter of noncommutativity is a constant, implying a noncommutative product of Moyal type. Whereas in our case we have a nonconstant noncommutative parameter implying a Konsevich product [16] and in consequence the above results are not directly applicable to our spaces.

Next we go on to constructing an explicit realization of the space-time having the commutation rules (12)–(14).

III. REALIZATIONS OF NONCOMMUTATIVE SPACES

A way to obtain realizations of noncommutative spaces comes from mechanical systems [17]. In particular, several authors have recently reported realizations of the SST; one of them within the so-called two-times physics [18]. Other realizations have been obtained by considering the dynamics of a free particle and remarkable references on that can be found in [19]. It is worth pointing out that in the realizations based on the free particle, the parameter for noncommutativity depends on the particle mass as $\theta \sim 1/m$, and thus the model loses meaning for massless particles.

From a parametrized relativistic particle we obtain in this section a realization of Snyder-like noncommutative spaces; i.e. spaces having the commutation rules (12)–(14) or (21)–(23). We show, in addition, that such a realization remains meaningful even for massless particles. A general form for the action of a particle in this kind of space-time having an arbitrary Hamiltonian is also provided.

A. Parametrized relativistic particle

In this part we briefly review the parametrized relativistic particle. We start by showing that an action of the form

$$S = K\int d\tau\sqrt{\mathcal{L}}, \qquad K = \text{const}, \tag{40}$$

is equivalent to

$$S = \frac{1}{2}\int d\tau\left[\frac{\mathcal{L}}{\lambda} + \lambda K^2\right]. \tag{41}$$

This can be seen by obtaining the equation of motion for λ from action (41),

$$\lambda = \frac{\sqrt{\mathcal{L}}}{K}, \tag{42}$$

and then substituting it back into (41) to directly obtain (40). Notice, however, that the $K = 0$ case can be considered from action (41) but not from (40).

Now, the action of the free particle is

$$S = -mc\int_{\tau_1}^{\tau_2} d\tau\sqrt{\dot{X}^\mu\dot{X}_\mu}, \tag{43}$$

which is, then, equivalent to

$$S_* = \frac{1}{2}\int_{\tau_1}^{\tau_2} d\tau\left[\frac{\dot{X}^2}{\lambda} + \lambda m^2c^2\right]. \tag{44}$$

In this case $K = -mc$, so for $m = 0$ we are dealing with a massless particle such as the photon. Some relevant applications of action (44) at the level of field theory can be found in [20].

The equations of motion for action (44) are

$$\frac{d}{d\tau}\left(\frac{\dot{X}^\mu}{\lambda}\right) = 0, \tag{45}$$

$$-\frac{\dot{X}^2}{\lambda^2} + m^2c^2 = 0. \tag{46}$$

From the second we obtain

$$\lambda = \frac{\sqrt{\dot{X}^2}}{mc}. \tag{47}$$

Thus, by taking $\tau = \tau_p$ with τ_p being the proper time, one gets to

$$\lambda = \frac{1}{m}. \tag{48}$$

In this case action (44) becomes

$$S_* = \frac{1}{2}\int_{\tau_1}^{\tau_2} d\tau(m\dot{X}^2 + mc^2). \tag{49}$$

If $m = 0$ no definition of proper time exists. However, we can take the condition $\lambda = 1/m_\nu$, with $m_\nu = h\nu/c^2$ the equations of motion (45) and (46) become

$$\frac{d^2X^\mu}{ds^2} = 0, \qquad \dot{X}^2 = 0, \tag{50}$$

which are consistent with the equations of motion of a massless free particle. For this case action (44) takes the form

$$S_* = \frac{m_\nu}{2}\int_{\tau_1}^{\tau_2} d\tau(\dot{X}^2). \tag{51}$$

Now, by defining $m_\gamma = m$ if $m \neq 0$ and $m_\gamma = m_\nu$ if $m = 0$, actions (49) and (51) can be rewritten as

$$S_* = \frac{1}{2}\int_{\tau_1}^{\tau_2} d\tau(m_\gamma\dot{X}^2 + mc^2). \tag{52}$$

We point out that, contrary to action (43), action (52) is not invariant under reparametrizations. If we want (52) to have this invariance, we must introduce an extra parameter ζ; which we assume a relativistic invariant. Thus, the action invariant under reparametrizations is

$$S = \frac{1}{2}\int_{\tau_1}^{\tau_2} d\tau\left(\frac{m_\gamma \dot{X}^2}{\dot{\zeta}} + mc^2\dot{\zeta}\right). \quad (53)$$

This action is a generalization of the relativistic particle, it is clear that only for the case $\dot{\zeta} = 1$ we recover the usual case on the proper time gauge. To analyze the difference between this system and the usual relativistic particle, we consider the Hamiltonian analysis of the action (53). The canonical momenta that one obtains from (53) are

$$P_\mu = m_\gamma \frac{\dot{X}_\mu}{\dot{\zeta}}, \quad (54)$$

$$P_\zeta = \frac{1}{2}\left(-m_\gamma \frac{\dot{X}_\mu \dot{X}^\mu}{\dot{\zeta}^2} + mc^2\right), \quad (55)$$

and the equations of motion can be written as

$$\dot{P}_\mu = 0, \qquad \dot{P}_\zeta = 0. \quad (56)$$

Now, from the canonical momenta (54) and (55) one obtains the constraint

$$\phi = P_\zeta + \frac{1}{2m_\gamma}(P_\mu P^\mu - m^2c^2) \approx 0, \quad (57)$$

which implies that, if $P_\zeta \neq 0$ then $P_\mu P^\mu - m^2c^2 \neq 0$. That is, the dispersion relation is changed. Moreover, by taking $\tau = \tau_p$, with τ_p being the proper time, from (57) one gets to

$$\tau_p = \zeta\left(1 - \frac{2}{m_\gamma c^2}P_\zeta\right)^{1/2}. \quad (58)$$

Therefore, there exists a relation between ζ and τ_p. Notice that this relationship implies

$$P_\zeta \leq \frac{m_\gamma c^2}{2}. \quad (59)$$

For the $P_\zeta < 0$ case, Eq. (57) can be written as

$$P_\mu P^\mu - m_{\rm eff}^2 c^2 = 0, \quad (60)$$

with

$$m_{\rm eff}^2 = m^2 + \frac{2m_\gamma |P_\zeta|}{c^2}. \quad (61)$$

Thus, the mass of the particle gets modified.

It can be shown that the canonical Hamiltonian is zero and thus the action of the system in terms of the phase-space variables is

$$S = \frac{1}{2}\int d\tau\Big[P_\zeta\dot{\zeta} + P_\mu \dot{X}^\mu - \lambda\left(P_\zeta + \frac{1}{2m_\gamma}(P_\mu P^\mu - m^2c^2)\right)\Big]. \quad (62)$$

Notice that, as ϕ is the only constraint, it is of first class [21]. Then, according to Dirac's method, the physical states of the quantum system are those satisfying

$$\left[\hat{P}_\zeta + \frac{1}{2m_\gamma}(\hat{P}_\mu \hat{P}^\mu - m^2c^2)\right]|\psi\rangle = 0. \quad (63)$$

By assigning operators as

$$\hat{P}_\mu = -i\hbar\partial_\mu \quad \text{and} \quad \hat{P}_\zeta = -i\hbar\partial_\zeta, \quad (64)$$

the Eq. (63) is a generalization of the Klein-Gordon equation. An interesting property of this is that integration of its propagator over ζ yields the usual propagator from the Klein-Gordon equation [22], in consequence the usual Klein-Gordon is an effective version of (63). Notice that defining the mass operator as $\hat{m}^2 = m^2 - 2m_\gamma \hat{P}_\zeta / c^2$, the Eq. (63) takes the form

$$[\hat{P}_\mu \hat{P}^\mu - \hat{m}^2 c^2]|\psi\rangle = 0. \quad (65)$$

We can consider that this Klein-Gordon equation corresponds to a particle whose mass depends on the physical state. A more detailed analysis of this equation can be found in [23].

Equation (63) was originally proposed by V. Fock [24] and was later considered by Stueckelberg and Nambu [25]. For the $m \neq 0$ case, a derivation of action (53) can be found in [26]. In the next subsection we will use the action (53) to obtain a realization of the Snyder-like noncommutative spaces.

B. Snyder space-time

The action (53) is invariant under reparametrizations that imply that the system has gauge freedom. The arbitrariness of the gauge is essentially the freedom to choose the time. Let us now see what happens by fixing a gauge on this system. It can be shown that by imposing

$$\chi = \zeta - \tau \approx 0, \quad (66)$$

the equations of motion (56) can be written as

$$\ddot{X}_\mu = 0, \quad (67)$$

$$\dot{X}_\mu \dot{X}^\mu = lc^2, \qquad l = \text{const}. \quad (68)$$

For $l = 0$, these are the equations of motion of a massless relativistic particle. For $l = 1$, they are those of a relativistic particle with mass and for $l = -1$, they are the equations of motion of a tachyon.

Considering now the gauge condition

$$\chi_1 = A\tau + B\zeta + C\zeta P_\zeta + X^\mu P_\mu, \qquad A, B, C = \text{const}, \quad (69)$$

which is an appropriate choice as by defining $\chi_2 = \phi$, one gets

$$\{\chi_1, \chi_2\} = B + CP_\zeta + \frac{1}{m_\gamma} P_\mu P^\mu = B + \frac{P_\mu P^\mu}{m_\gamma}\left(1 - \frac{C}{2}\right) + C\frac{mc^2}{2} \neq 0 \quad (70)$$

and therefore (χ_1, χ_2) forms a second-class constraint set [21]. Thus, matrix $C_{\alpha\beta} = \{\chi_\alpha, \chi_\beta\}$ and its inverse $C^{\alpha\beta}$ are well defined. Notice that this gauge remains valid even for the $m = 0$ case. Now, by fixing the gauge, the constraint (57) and the gauge condition (69) are a pair of second-class constraints. In consequence, we need to change the Poisson's into Dirac's brackets [21]. If F and G are functions from phase space, Dirac's brackets are defined as

$$\{F, G\}^* = \{F, G\} - \{F, \chi_\alpha\} C^{\alpha\beta}\{\chi_\beta, G\}. \quad (71)$$

In particular, for X_μ and P_ν one obtains

$$\{X_\mu, X_\nu\}^* = -\frac{d}{\hbar^2} L_{\mu\nu}, \qquad L_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu, \quad (72)$$

$$\{X_\mu, P_\nu\}^* = \eta_{\mu\nu} - \frac{d}{\hbar^2} P_\mu P_\nu, \quad (73)$$

$$\{P_\mu, P_\nu\}^* = 0, \quad (74)$$

where

$$d = \frac{\hbar^2}{Bm_\gamma + (1 - \frac{C}{2})P_\mu P^\mu + \frac{C}{2} m^2 c^2}. \quad (75)$$

To quantize this system we promote the Dirac's brackets to commutators and then by quantizing this theory within the canonical formalism one obtains a noncommutative space-time. Notice that d depends on the momentum P_μ, and this in principle appears to imply a problem of ordering in the commutation rules. However, as

$$\{P_\alpha P^\alpha, L_{\mu\nu}\}^* = \{P_\alpha P^\alpha, P_\mu P_\nu\}^* = 0, \quad (76)$$

the problem does not actually exist.

Now, if $C = 2$ then d is a constant. By taking $C = 2$ and $B = -2m_\gamma c^2$, then $d = -a^2$ is negative for both the particle with or without mass. In such a case the Dirac brackets (72)–(74) become

$$\{X_\mu, X_\nu\}^* = \frac{a^2}{\hbar^2} L_{\mu\nu}, \quad (77)$$

$$\{X_\mu, P_\nu\}^* = \eta_{\mu\nu} + \frac{a^2}{\hbar^2} P_\mu P_\nu, \quad (78)$$

$$\{P_\mu, P_\nu\}^* = 0, \quad (79)$$

and so a realization of the noncommutative space-time defined by the commutation rules (12)–(14) holds. For this the time is quantized in units of

$$\frac{a}{c} = \frac{\hbar}{c^2\sqrt{2m_\gamma^2 - m^2}}. \quad (80)$$

On the other hand, if $C = 2$ and $B = m_\gamma c^2$ then $d > 0$ and the SST holds. In this case the space gets discretized [4] in quanta of

$$\sqrt{d} = a = \frac{\hbar}{\sqrt{m_\gamma^2 c^2 + m^2 c^2}}. \quad (81)$$

For $m \neq 0$ this length-scale is proportional to Compton's length; just as Snyder conjectured [4]. Some realizations of the SST lose meaning in the $m = 0$ case [19], but notice that this does not happen in this model.

C. Boundary conditions and general action

Boundary conditions are an important element to quantize a system [27]. For this reason we look for boundary conditions consistent with this system.

Clearly in this case it is not possible to fix variables X^μ at the boundary because they do not commute. It can be shown from Eqs. (72)–(74) that $\{P_\zeta, P_\mu\}^* = 0$. Thus, (P_ζ, P_μ) forms a complete set of commuting variables, which indicates that these can be fixed at the action boundaries. The corresponding action in this case is

$$S_{sp} = \int_{\tau_1}^{\tau_2} d\tau\left(-\zeta\dot{P}_\zeta - X^\mu \dot{P}_\mu - \lambda\left(P_\zeta + \frac{1}{2m_\gamma}(P_\mu P^\mu - m^2c^2)\right)\right). \quad (82)$$

By introducing the constraints χ_1 and χ_2 into this, one gets to

$$S_{rsp} = -\int_{\tau_1}^{\tau_2} d\tau\left(X^\mu + \frac{P^\mu}{m_\gamma}\left(\frac{A + X^\alpha P_\alpha}{B - Ch}\right)\right)\dot{P}_\mu, \quad (83)$$

with $h = \frac{1}{2m_\gamma}(P_\mu P^\mu - m^2c^2)$. For this action the boundary conditions are

$$P_\mu(\tau_1) = P_{\mu 1}, \qquad P_\mu(\tau_2) = P_{\mu 2}. \quad (84)$$

Notice that by taking $A = 0$ in the constraint χ_1, Dirac's brackets remain unchanged. Considering this and using (75) we can define

$$g^{\alpha\beta} = \eta^{\alpha\beta} + \frac{P^\alpha P^\beta}{m_\gamma(B - Ch)} = \eta^{\alpha\beta} + \frac{P^\alpha P^\beta d}{\hbar^2 - P_\mu P^\mu d},$$

with which (83) becomes

$$S_{rsp} = -\int_{\tau_1}^{\tau_2} d\tau g^{\alpha\beta}(P) X_\alpha \dot{P}_\beta. \quad (85)$$

This is an action with a metric depending on the momenta. Now, the Hamiltonian action of a system in a curved space-time with metric $G_{\alpha\beta}(X)$ and Hamiltonian H can be written in the form

$$S = \int_{\tau_1}^{\tau_2} (d\tau G_{\alpha\beta}(X)\dot{X}^\alpha P^\beta - H(X, P)). \quad (86)$$

By analogy we propose the action of a particle in the

Snyder-like space-time and Hamiltonian H as

$$S_S = \int_{\tau_1}^{\tau_2} d\tau(-g^{\alpha\beta}(P)X_\alpha \dot{P}_\beta - H(X, P)). \quad (87)$$

By a direct calculation it can be shown that the dynamics produced by this action are consistent with the symplectic structure (72)–(74) and so the quantum version of this system has the Snyder-like space-time as its background.

Another interesting point of the reduced action (83) is that by choosing

$$\tilde{X}^\alpha = g^{\alpha\beta}X_\beta \quad (88)$$

$$\tilde{P}_\alpha = P_\alpha, \quad (89)$$

as a new set of phase-space coordinates, this corresponds to a local Darboux map [28] that transforms the symplectic structure of Snyder-like (72)–(74) into the usual one. Hence, in this set of coordinates one obtains the classical dynamics of an ordinary particle. We point out, however, that this map is not canonical and therefore quantum theory will not be equivalent.

IV. SUMMARY

We have presented three space-times discrete in time which are compatible with the Lorentz symmetry, with two of these spaces also compatible with the invariance under translations. It was shown that all these spaces are noncommutative, one of them has commutation rules similar to the SST and the other two similar to the Yang space. Moreover, by using a parametrized relativistic particle we obtain a realization of the Snyder-like spaces. Contrary to other realizations reported, the SST realization remains meaningful even for the massless particle. Finally, a general form for the action of a particle in this kind of noncommutative spaces with arbitrary Hamiltonian is proposed.

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[1] N. Seiberg and E. Witten, J. High Energy Phys. 09 (1999) 032.

[2] S. M. Carroll, J. A. Harvey, V. A. Kostelecky, C. D. Lane, and T. Okamoto, Phys. Rev. Lett. **87**, 141601 (2001).

[3] M. Chaichian, P. P. Kulish, K. Nishijima, and A. Tureanu, Phys. Lett. B **604**, 98 (2004); M. Chaichian, P. Presnajder, and A. Tureanu, Phys. Rev. Lett. **94**, 151602 (2005).

[4] H. S. Snyder, Phys. Rev. **71**, 38 (1947).

[5] G. 't Hooft, Class. Quant. Grav. **13**, 1023 (1996).

[6] J. Kowalski-Glikman and S. Nowak, Int. J. Mod. Phys. D **12**, 299 (2003).

[7] L. Freidel, J. Kowalski-Glikman, and L. Smolin, Phys. Rev. D **69**, 044001 (2004).

[8] C. N. Yang, Phys. Rev. **72**, 874 (1947).

[9] S. Tanaka, hep-th/0303105.

[10] H. Matschull and M. Welling, Class. Quant. Grav. **15**, 2981 (1998).

[11] G. 't Hooft, Class. Quant. Grav. **16**, 395 (1999).

[12] M. Bojowald, Class. Quant. Grav. **18**, 1071 (2001); A. P. Balachandran, T. R. Govindarajan, A. G. Martins, and P. Teotonio-Sobrinho, J. High Energy Phys. 11 (2004) 068.

[13] I. Bars, Class. Quant. Grav. **18**, 3113 (2001); Phys. Rev. D **74**, 085019 (2006).

[14] C. Vafa, Nucl. Phys. **B469**, 403 (1996).

[15] N. Seiberg, L. Susskind, and N. Toumbas, J. High Energy Phys. 06 (2000) 044; M. Chaichian, A. Demichev, P. Presnajder, and A. Tureanu, Phys. Lett. B **515**, 426 (2001).

[16] M. Kontsevich, Lett. Math. Phys. **66**, 157 (2003).

[17] A. Pinzul and A. Stern, Phys. Lett. B **593**, 279 (2004); R. Banerjee, B. Chakraborty, and S. Gangopadhyay, J. Phys. A **38**, 957 (2005); R. P. Malik, J. Phys. A **37**, 12077 (2004); W. Chagas-Filho, hep-th/0601117.

[18] J. M. Romero and A. Zamora, Phys. Rev. D **70**, 105006 (2004).

[19] S. Ghosh and P. Pal, Phys. Lett. B **618**, 243 (2005); S. Ghosh, Phys. Lett. B **623**, 251 (2005); F. Girelli, T. Konopka, J. Kowalski-Glikman, and E. R. Livine, Phys. Rev. D **73**, 045009 (2006); R. Banerjee, S. Kulkarni, and S. Samanta, J. High Energy Phys. 05 (2006) 077; S. Ghosh, Phys. Rev. D **74**, 084019 (2006).

[20] M. J. Strassler, Nucl. Phys. **B385**, 145 (1992).

[21] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Dover, New York, 2001); M. Henneaux and C. Teitelboim, *Quantization of Gauge System* (Princeton University Press, USA, 1992).

[22] R. P. Feynman, Phys. Rev. **80**, 440 (1950).

[23] J. R. Fanchi, *Parametrized Relativistic Quantum Theory* (Kluwer, Netherlands, 1993).

[24] V. Fock, Phys. Zeit. Sow. Un. **12**, 404 (1937).

[25] E. C. G. Stueckelberg, Helv. Phys. Acta **14**, 372 (1941); Y. Nambu, Prog. Teor. Phys. **5**, 82 (1950).

[26] J. L. Lucio-M, J. A. Nieto, and J. D. Vergara, Phys. Lett. A **219**, 150 (1996).

[27] M. Henneaux, C. Teitelboim, and J. D. Vergara, Nucl. Phys. **B387**, 391 (1992).

[28] J. E. Marsden and T. Ratiu, *Introduction to Mechanics and Symmetry* (Springer-Verlag, New York, 2003).