Mass zeros of the fermionic determinant in four-dimensional quantum electrodynamics

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The Euclidean fermionic determinant in four-dimensional quantum electrodynamics is considered as a function of the fermionic mass for a class of $O(2) \times O(3)$ symmetric background gauge fields. These fields result in a determinant free of all cutoffs. Consider the one-loop effective action, the logarithm of the determinant, and subtract off the renormalization dependent second-order term. Suppose the small-mass behavior of this remainder is fully determined by the chiral anomaly. Then either the remainder vanishes at least once as the fermionic mass is varied in the interval $0 < m < \infty$ or it reduces to its fourth-order value in which case the new remainder, obtained after subtracting the fourth-order term, vanishes at least once. Which possibility is chosen depends on the sign of simple integrnals involving the field strength tensor and its dual.

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I. INTRODUCTION

Within the standard model fermionic determinants are required for the calculation of every physical process. These determinants produce an effective functional measure for the gauge fields when the fermionic fields are integrated. They are the means by which virtual fermion loops are incorporated into a calculation. Without them color screening, quark fragmentation into hadrons and unitarity would be lost. In quantum electrodynamics the biggest barrier to understanding its nonperturbative structure is its fermionic determinant. These determinants are therefore fundamental.

They are also hard to calculate and physicists by and large lost interest in them during the 1980s. With the advent of large machines lattice QCD physicists are now starting to include the determinant in their calculations. Analytic results for QCD and QED determinants are very scarce, especially in four dimensions. Such results as they become available will serve as benchmarks for determinant algorithms, including the various lattice discretizations of the Dirac operator in use, and hence a means of reliably estimating computational error, a major problem in lattice QCD at present.

Most analytic nonperturbative results obtained so far deal with the dependence of the determinant on the coupling constant. Little attention has been given to their dependence on the fermion's mass. One notable exception is the work of Dunne *et al.* [1], which gives a semianalytic calculation of the QCD determinant's mass dependence in an instanton background.

In two-dimensional Euclidean QED the author has shown that mass can have a profound effect on its determinant. Namely, for a large class of centrally symmetric, finite-range background gauge fields the growth of the determinant in the limit $mR \ll 1$ followed by $|e\Phi| \gg 1$ is

$$\ln \det \sim -\frac{|e\Phi|}{4\pi} \ln\left(\frac{|e\Phi|}{(mR)^2}\right), \tag{1}$$

where det denotes the determinant, R is the field strength's range, and Φ is the background field's flux [2]. In the massless case, the Schwinger model, the determinant is quadratic in the field strength.

The second example of the nontrivial mass dependence of det in Euclidean QED_2 is the presence of mass zeros. Let det be written as $\ln \det = \Pi_2 + \ln \det_3$, where Π_2 is the second-order vacuum polarization graph and $\ln \det_3$ is a technical term, defined in Sec. II, for the remainder after the conditionally convergent second-order term has been isolated and made gauge invariant by some regularization procedure. Then there is at least one real value of m at which $\ln \det_3 = 0$ when $0 < |e\Phi| < 2\pi$, subject to some mild restrictions on the field strength [3]. There may be other mass zeros. Now recall Schwinger's result [4] that $\ln \det_3 = 0$ when $m = 0$. For fields with $\Phi = 0$ then it is also true that $\lim_{m=0} \ln \det_3 = 0$; otherwise not [5]. So the result is this: when $0 < |e\Phi| < 2\pi$ the zero in m of $\ln \det_3$ moves up from $m = 0$ to some finite value $m > 0$. Beyond $|e\Phi| > 2\pi$ we can say nothing definite yet.

The obvious question to ask is whether there are mass zero(s) in the remainder term of ln det in QED_4 , denoted by $\ln \det_{\mathrm{ren}}$. The background gauge fields $A_\mu(x)$ considered in two dimensions have a slow $1/|x|$ falloff resulting in a nonvanishing chiral anomaly $\Phi/2\pi$. Here we will consider a large class of $O(2) \times O(3)$ symmetric background gauge fields that also have a $1/|x|$ falloff with a nonvanishing chiral anomaly. If the small-mass behavior of the remainder is fully determined by the chiral anomaly, as in two dimensions, then there are circumstances in which mass zeros are present in the remainder. The idea of the proof is extremely simple: show that for $m \to 0$ the remainder is negative and that as $m \to \infty$ it becomes positive. The demonstration that the chiral anomaly determines the small-mass behavior of the remainder turns out to be non-

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trivial, and we are not able to settle this matter here. Evidence is presented that it does, but it is not conclusive.

At this point it may be asked why these mass zeros for a special class of background gauge fields are of interest. First and foremost they are a truly nonperturbative result for the exact QED_4 determinant. As such, they would serve as a benchmark result that lattice theorists could aim to reproduce. As discussed in Sec. II, once the second- and fourth-order contributions to $\det_{\mathrm{ren}}$ are isolated the remainder of $\det_{\mathrm{ren}}$ is determined by the distribution of its complex zeros in the coupling constant plane. Little is known about how these zeros distribute themselves. The presence of mass zeros in the remainder terms in $\ln\det_{\mathrm{ren}}$ must place a strong constraint on their distribution which future work could deal with.

In Sec. II $\det_{\mathrm{ren}}$ is defined and some of its properties are reviewed. Section III introduces the background gauge fields used in the calculation and is an introduction to the zero-mass limit of the remainder and some of the subtleties involved. Section IV establishes that all of the square-integrable zero modes of the Dirac operator $\not{D}$ have positive chirality. In addition it is necessary to know the scattering states and low-energy phase shifts associated with the background gauge field, and this is done in Sec. V. It also gives an analysis of the low-energy behavior of the exact negative chirality propagator and seeks to justify a particular approach to proving that the chiral anomaly is sufficient to describe the small-mass limit of the remainder. Section VI demonstrates that the remainder can become positive as $m \to \infty$. Finally, Sec. VII summarizes our conclusions.

II. QED_4 DETERMINANT

We begin by reviewing some established results for the QED_4 determinant [6,7]. By fermionic determinant we mean the ratio of determinants of the interacting and free Euclidean Dirac operators, $\det(\not{P} - e\not{A} + m)/\det(\not{P} + m)$, defined by the renormalized determinant on $\mathbb{R}^4$, namely

$$\det{}_{\mathrm{ren}} = \exp(\Pi_2 + \Pi_3 + \Pi_4)\det{}_5(1 - eS\not{A}), \quad (2)$$

where

$$\ln\det{}_5 = \mathrm{Tr}\left[\ln(1 - eS\not{A}) + \sum_{n=1}^{4}\frac{(eS\not{A})^n}{n}\right], \quad (3)$$

and $S = (\not{P} + m)^{-1}$; $\Pi_{2,3,4}$ are the second, third and fourth-order contributions to the one-loop effective action defined by some consistent regularization procedure together with a charge renormalization subtraction in Π_2. The regularization should also result in $\Pi_3 = 0$ by C-invariance, and it should give a gauge-invariant result for Π_4. The remainder, $\det_5$, after these subtractions is gauge invariant and has a well-defined power series expansion without regularization. The remainder $\ln\det_3$ in Sec. I is given by (3) with the restriction $n = 1, 2$.

The operator $S\not{A}$ is a bounded operator on the Hilbert space $L^2(\mathbb{R}^4, \sqrt{k^2 + m^2}d^4k)$ for $A_\mu \in \bigcap_{n>4} L^n(\mathbb{R}^4)$, in which case it belongs to the trace ideal $\mathcal{C}_n$ for $n > 4$ [$\mathcal{C}_n = \{K|\mathrm{Tr}(K^\dagger K)^{n/2} < \infty\}$] [6–9]. This includes the case when $A_\mu(x)$ falls off as $1/|x|$ as $|x| \to \infty$. As a result $\det_5$ is an entire function of the coupling e, and it can be represented in terms of the discrete complex eigenvalues $1/e_n$ of the non-Hermitian compact operator $S\not{A}$ [10]:

$$\det{}_5(1 - eS\not{A}) = \prod_n\left[\left(1 - \frac{e}{e_n}\right)\exp\left(\sum_{k=1}^{4}\frac{(e/e_n)^k}{k}\right)\right]. \quad (4)$$

By C-invariance and the reality of $\det_5$ these eigenvalues appear in quartets $\pm e_n, \pm\bar{e}_n$ or as imaginary pairs. Because $\det_{\mathrm{ren}}$ has no zeros for real e when $m \neq 0$ [11] and $\det_{\mathrm{ren}}(e = 0) = 1$, it is positive for real e. Because $S\not{A} \in \mathcal{C}_n$, $n > 4$, it is of order 4. This means that for suitable positive constants $A(\epsilon)$, $K(\epsilon)$ and any complex value of e, $|\det_{\mathrm{ren}}| < A(\epsilon)\exp(K(\epsilon)|e|^{4+\epsilon})$ for any $\epsilon > 0$. The first paper to show that $\det_{\mathrm{ren}}$ is of order 4 was that in Ref. [12].

The regularization procedure used here is Schwinger's heat kernel representation [13]:

$$\begin{aligned}\ln\det{}_{\mathrm{ren}} &= \tfrac{1}{2}\int_0^\infty\frac{dt}{t}\int d^4x\Big\{\mathrm{tr}\langle x|e^{-P^2t} - e^{-(D^2+(1/2)\sigma F)t}|x\rangle \\ &\quad + \frac{1}{24\pi^2}F_{\mu\nu}^2(x)\Big\}e^{-tm^2} \\ &= \frac{1}{8\pi^2}\int\frac{d^4k}{(2\pi)^4}|\hat{F}_{\mu\nu}(k)|^2\int_0^1 dz z(1-z) \\ &\quad \times\ln\left(\frac{z(1-z)k^2 + m^2}{m^2}\right) + \Pi_4 \\ &\quad + \ln\det{}_5(1 - S\not{A}). \end{aligned} \quad (5)$$

Here e has been absorbed into A_μ, $D^2 = (P - A)^2$, $\sigma_{\mu\nu} = [\gamma_\mu, \gamma_\nu]/2i$, $\gamma_\mu^\dagger = -\gamma_\mu$, $\hat{F}_{\mu\nu}$ denotes the Fourier transform of $F_{\mu\nu}$, and m is the fermionic mass. A second-order on-shell charge renormalization subtraction has been incorporated. All terms appearing on the right-hand side of (5) follow from the heat kernel expression on the left-hand side. The requirement that $A_\mu \in \bigcap_{n>4} L^n(\mathbb{R}^4)$ and certain differentiability conditions on A_μ introduced later are sufficient to ensure that (5) makes mathematical sense.

In the representation

$$\gamma_0 = -i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \gamma_k = \begin{pmatrix}0 & \sigma_k\\ -\sigma_k & 0\end{pmatrix}, \qquad \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad (6)$$

one gets

$$D^2 + \tfrac{1}{2}\sigma F = \begin{pmatrix}H_+ & 0\\ 0 & H_-\end{pmatrix}, \quad (7)$$

where $H_\pm = (P - A)^2 - \boldsymbol{\sigma}\cdot(\mathbf{B} \pm \mathbf{E})$.

Denote the remainder in Eq. (5) after removing the renormalization dependent second-order term by

$$\mathcal{R} = \Pi_4 + \ln \det_5(1 - S\not{A}). \tag{8}$$

It will be shown in Sec. IV that all the zero modes of the Dirac operator are confined to the positive chirality sector for the class of $O(2) \times O(3)$ symmetric background fields used to calculate $\det_{\text{ren}}$. Differentiating Eq. (5) with respect to m^2 allows one to isolate H_+. After some rearrangement of terms there follows:

$$\begin{aligned} m^2 \frac{\partial \mathcal{R}}{\partial m^2} &= \tfrac{1}{2} m^2 \,\mathrm{Tr}[(H_+ + m^2)^{-1} - (H_- + m^2)^{-1}] \\ &\quad + m^2 \int_0^\infty dt e^{-tm^2} \int d^4k \Big\{ \mathrm{tr}\langle k| e^{-tH_-} - e^{-tP^2} |k\rangle \\ &\quad - \frac{1}{128\pi^6} |\hat{F}_{\mu\nu}(k)|^2 \int_0^1 dz z(1-z) e^{-k^2 z(1-z)t} \Big\}, \end{aligned} \tag{9}$$

where the spin traces are now over 2×2 matrices. Equation (9) expanded as a power series defines the conditionally convergent fourth-order term in $\mathcal{R}$. This requires iterating the second term in Eq. (9) 4 times using the operator identity

$$e^{-t(P^2+V)} - e^{-tP^2} = -\int_0^t ds e^{-(t-s)(P^2+V)} V e^{-sP^2}, \tag{10}$$

with $V = -AP - PA + A^2 + \boldsymbol{\sigma} \cdot (\mathbf{E} - \mathbf{B})$. The result is

$$\begin{aligned} m^2 \frac{\partial \mathcal{R}}{\partial m^2} &= \tfrac{1}{2} m^2 \,\mathrm{Tr}[(H_+ + m^2)^{-1} - (H_- + m^2)^{-1}] \\ &\quad - \frac{m^2}{16\pi^2} \int \frac{d^4k}{(2\pi)^4} \frac{\hat{\mathbf{E}}(k) \cdot \hat{\mathbf{B}}(-k) + \hat{\mathbf{E}}(-k) \cdot \hat{\mathbf{B}}(k)}{z(1-z)k^2 + m^2} \\ &\quad + m^2 \frac{\partial}{\partial m^2} \Pi_4^- - m^2 \,\mathrm{Tr}[\Delta_- V\Delta V\Delta V\Delta V\Delta V\Delta \\ &\quad - (\Delta A^2 \Delta V\Delta V\Delta V\Delta + \text{ all perms. of } A^2, V) \\ &\quad + \Delta A^2 \Delta A^2 \Delta A^2 \Delta]. \end{aligned} \tag{11}$$

The second term in Eq. (11) is the remainder after adding the second-order contribution from the second term in Eq. (9) to the last term. This remainder would be canceled by the first term in Eq. (11) were it expanded in a power series. It, as well as Π_4^-, are calculated from the regulated expansion of the second term in Eq. (9) using Eq. (10).

The quantity Π_4^- is obtained by adding all the fourth-order terms in the expansion. It is the contribution of H_- to the photon-photon scattering graph. Its structure is $\Pi_4^- = \Pi_4^{\text{scalar}} + \Pi_4^{\boldsymbol{\sigma}\cdot(\mathbf{B}-\mathbf{E})}$, where Π_4^{scalar} is the contribution to Π_4^- neglecting the $\boldsymbol{\sigma} \cdot (\mathbf{B} - \mathbf{E})$ term in V. That is, Π_4^{scalar} is the contribution to Π_4 in scalar QED_4 multiplied by 2. The factor 2, and not 4, is due to the factor of $\frac{1}{2}$ in the definition (5) of the spinor determinant. The remainder is the contribution to Π_4^- from the $\boldsymbol{\sigma} \cdot (\mathbf{B} - \mathbf{E})$ term in V. Additional information on Π_4^- may be found in Ref. [14].

Continuing with our discussion of Eq. (11), Δ_- in the last trace is the exact negative chirality propagator $\langle x|(H_- + m^2)^{-1}|y\rangle$ and Δ is the scalar propagator $\langle x|(P^2 + m^2)^{-1}|y\rangle$. The regulating exponentials have been removed as the terms in the trace are fifth order and higher, and so the implicit loop integral is unambiguous. We leave the discussion of Δ_- to Sec. V. In order to discuss the $m = 0$ limit of Eq. (11) we must be more specific about the background gauge field.

III. BACKGROUND GAUGE FIELDS

QED determinants in constant-field backgrounds have volume divergences and so are not defined on noncompact manifolds. Instead one considers the associated effective Lagrangians, which do make sense. In the simplest case of the Euclidean QED_2 determinant there is just a constant magnetic field, and the volume divergence arises from the degeneracy of the Landau levels. In constant-field QED_4, by making two rotations (a Lorentz boost plus a rotation in Minkowski space) the operator $(P - A)^2$ can be transformed into the sum of two two-dimensional harmonic oscillator Hamiltonians, leading to a degeneracy factor that grows as a four-volume [15]. The lesson is that constant fields have too much degeneracy to define the determinant on a noncompact manifold.

We have found that $O(2) \times O(3)$ symmetric background fields allow a satisfactory definition of the QED_4 determinant and that they are sufficiently tractable to permit substantial analytic analysis. Such fields were first explicitly considered in QED_4 by Adler [12,16]. In this paper these fields take the form [17–19].

$$A_\mu(x) = M_{\mu\nu} x_\nu a(r^2), \tag{12}$$

where $M_{\mu\nu}$ is chosen to be antiself-dual and is given by

$$M_{\mu\nu} = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}. \tag{13}$$

This field has an $O(2) \times O(3)$ invariance, subgroups present in the reduction of $O(4)$ to $O(3) \times O(3)$. It is further assumed that $a(r^2)$ is smooth, well-behaved at the origin, and satisfies

$$a(r^2) = \frac{\nu}{r^2}, \qquad r > R, \tag{14}$$

where ν is a dimensionless constant. Without loss of generality assume $\nu > 0$.

The orbital angular momentum operators of the first and second $O(3)$ subgroups of $O(4)$ satisfy $[L_i^{(p)}, L_j^{(q)}] = \delta_{pq} i\epsilon_{ijk} L_k^{(p)}$, $p, q, = 1, 2$. The spin angular momentum operators in the representation (6) are

$$S_k^{(1)} = \frac{1}{2}\begin{pmatrix} \sigma_k & 0 \\ 0 & 0 \end{pmatrix}, \qquad S_k^{(2)} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_k \end{pmatrix}. \tag{15}$$

The total angular momentum operator relative to the second subgroup, $J_k^{(2)} = L_k^{(2)} + S_k^{(2)}$, commutes with $\not{A}$, $[J_k^{(2)}, \not{A}] = 0$, $k = 1, 2, 3$, while $\not{A}$ is invariant only with respect to rotations about the third axis of the first subgroup: $[J_3^{(1)}, \not{A}] = 0$. We adopt the conventions of Ref. [19] for the four-dimensional rotation matrices $D^l_{m_1 m_2}$. They are normalized so that

$$\int d\Omega_4 D^{l_1 *}_{m_1 m_2}(x) D^{l_2}_{m_3 m_4}(x) = \frac{2\pi^2}{2l_1 + 1}\delta_{l_1 l_2}\delta_{m_1 m_3}\delta_{m_2 m_4}(r^2)^{2l_1}, \tag{16}$$

where Ω_4 is the surface element in four dimensions. Additional properties of these matrices appear in Ref. [19] and in Appendix A of Ref. [14].

Following Ref. [19] we construct eigenstates of $J^{(1)} \cdot J^{(1)}$, $J_3^{(1)}$ (eigenvalues $j \pm \frac{1}{2}$, M) and $J^{(2)} \cdot J^{(2)}$, $J_3^{(2)}$ (eigenvalues j, m). In the positive chirality sector these are

$$\varphi^{j\pm(1/2),M}_{j,m}(x) = \begin{pmatrix} \mp\left(j \pm M + \frac{1}{2}\right)^{1/2} D^j_{M-(1/2),m}(x) \\ \left(j \mp M + \frac{1}{2}\right)^{1/2} D^j_{M+(1/2),m}(x) \\ 0 \\ 0 \end{pmatrix}, \tag{17}$$

and in the negative chirality sector they are

$$\psi^{j+(1/2),M}_{j,m}(x) = \begin{pmatrix} 0 \\ 0 \\ -(j - m + 1)^{1/2} D^{j+(1/2)}_{M,m-(1/2)}(x) \\ (j + m + 1)^{1/2} D^{j+(1/2)}_{M,m+(1/2)}(x) \end{pmatrix}, \tag{18}$$

$$\psi^{j-(1/2),M}_{j,m}(x) = \begin{pmatrix} 0 \\ 0 \\ (j + m)^{1/2} D^{j-(1/2)}_{M,m-(1/2)}(x) \\ (j - m)^{1/2} D^{j-(1/2)}_{M,m+(1/2)}(x) \end{pmatrix}. \tag{19}$$

Since $\not{A}$ commutes with $J_3^{(1)}$ and $J^{(2)}$, eigenstates of $\not{D} = \not{P} - \not{A}$ are of the form [19]

$$\psi^+_{EjMm}(x) = F(r^2)\varphi^{j-(1/2),M}_{j,m}(x) + G(r^2)\varphi^{j+(1/2),M}_{j,m}(x), \tag{20}$$

$$\psi^-_{EjMm}(x) = f(r^2)\psi^{j-(1/2),M}_{j,m}(x) + g(r^2)\psi^{j+(1/2),M}_{j,m}(x), \tag{21}$$

where the superscripts on $\psi^\pm_{EjMm}$ denote chirality and E is the energy eigenvalue. In the following we will write $\psi^\pm_{EjMm}$ as two-component spinors.

From ${}^*F_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ and (12) it follows that

$${}^*F_{\mu\nu}F_{\mu\nu} = -16a^2 - 16r^2 a a', \tag{22}$$

$$F_{\mu\nu}F_{\mu\nu} = 8r^4 a'^2 - {}^*F_{\mu\nu}F_{\mu\nu}, \tag{23}$$

where the prime denotes differentiation with respect to r^2. From (14) and (22) the chiral anomaly is

$$-\frac{1}{16\pi^2}\int d^4x\, {}^*F_{\mu\nu}F_{\mu\nu} = \frac{\nu^2}{2}, \tag{24}$$

provided $\lim_{r\to 0} r^2 a = 0$. Note, as expected, that $F_{\mu\nu}$ is not square-integrable. But this does not matter as far as the remainder $\mathcal{R}$ in Eq. (8) is concerned. Recall that it is only required that $A_\mu \in \bigcap_{n>4} L^n(\mathbb{R}^4)$, which it does here. Furthermore, because we have chosen on-shell charge renormalization the $1/k^2$ behavior of $\hat{F}_{\mu\nu}$ for small k in the first term on the right-hand side of (5) is regulated by the vanishing logarithm as $k \to 0$. So everything in Eq. (5) is finite.

We will now discuss in a preliminary way the limit of (11) as $m \to 0$. Consider the first term. A working definition of the chiral anomaly for $\not{D}$ on noncompact manifolds is [20]

$$\lim_{m\to 0} m^2 \operatorname{Tr}[(H_+ + m^2)^{-1} - (H_- + m^2)^{-1}] = -\frac{1}{16\pi^2}\int d^4x\, {}^*F_{\mu\nu}F_{\mu\nu}. \tag{25}$$

Because the manifold is a noncompact Euclidean one the right-hand side of Eq. (25) need not be the difference between numbers $n_+ - n_-$ of positive and negative chirality L^2 zero modes. The remainder, if any, is related to the zero-energy phase shifts associated with $H_\pm$ [20]. More will be said about this in Sec. V. If the remaining terms in (11) vanish in the $m = 0$ limit then Eqs. (24) and (25) indicate that $\mathcal{R}$ in (8) behaves as

$$\mathcal{R} \underset{m\to 0}{\sim} \frac{\nu^2}{4}\ln m^2 + \text{less singular in } m^2. \tag{26}$$

Thus, $\mathcal{R}$ would become negative as $m \to 0$.

A necessary condition for the vanishing of the remaining terms is that there be no L^2 zero modes in the negative chirality sector. It will be shown in Sec. IV that this is true for our choice of gauge fields. Otherwise, Δ_- in Eq. (11) would develop a simple pole at $m = 0$ and (26) would contain more terms varying as $\ln m^2$ for $m \to 0$. But this is not a sufficient condition for the remaining terms in Eq. (11) to vanish at $m = 0$. One can see already from the second term in (11) some of the subtleties involved. If $\mathbf{B}(x)$ and $\mathbf{E}(x)$ fall off as $1/r^2$, as our fields do, without any particular symmetry constraint then their Fourier transforms will be such that $\hat{\mathbf{B}}(k)$, $\hat{\mathbf{E}}(k)$ behave as $1/k^2$ as $k \to 0$. In this case the integral will have an infrared divergence even when $m \neq 0$. But this does not happen due to the $O(2) \times O(3)$ symmetry of the gauge fields.

Define the Fourier transform of $F_{\mu\nu}(x)$ for $|x| > R$ by $\hat{F}^{>}_{\mu\nu}(k)$. Then

$$\hat{F}^{>}_{\mu\nu}(k) = \frac{8\pi^2\nu}{k^2}\bigg[M_{\mu\nu}J_2(kR) + \frac{M_{\nu\alpha}k_\alpha k_\mu - M_{\mu\alpha}k_\alpha k_\nu}{k^2} \times (J_0(kR) + 2J_2(kR))\bigg], \tag{27}$$

and

$$\hat{\mathbf{B}}^{>}(k) \cdot \hat{\mathbf{E}}^{>}(-k) = -\frac{(8\pi^2\nu)^2}{k^4}(J_0(kR)J_2(kR) + J_2^2(kR)). \tag{28}$$

Thus, $\hat{\mathbf{B}}^{>}(k) \cdot \hat{\mathbf{E}}^{>}(-k)$ behaves as R^2/k^2 instead of $1/k^4$ as $k \to 0$. For large k, $\hat{F}^{<}_{\mu\nu}(k)$, calculated for $|x| < R$, behaves as $\sin(kR - 3\pi/4)/k^{5/2}$ for any reasonable behavior of $a(r^2)$ near $r = 0$, such as $a \underset{r\to 0}{\sim} \mathrm{C}r^\beta$ with $\beta > -\frac{1}{2}$ or $-\frac{1}{3}$ as required in Sec. VI. Therefore, the integral in (11) is absolutely convergent in the ultraviolet and its small-mass limit varies as $(\ln m^2)^2$, allowing us to conclude that the second term in Eq. (11) vanishes in the limit $m = 0$.

Now consider the third term in Eq. (11), $m^2\partial\Pi_4^-/\partial m^2$. Simple power counting of momenta suggests that the integrals defining Π_4^- have a logarithmic mass singularity of the form $(\ln m^2)^n$ with $n \geq 1$. If so, then the $m = 0$ limit of $m^2\partial\Pi_4^-/\partial m^2$ would be nonvanishing, thereby falsifying (26). It is encouraging that there is no immediate infrared divergence for $m \neq 0$ that has to be canceled by the symmetry of $F_{\mu\nu}$, as in the second-order term of (11). The confluence of singularities in $\hat{\mathbf{E}}$ and $\hat{\mathbf{B}}$ is no longer present in fourth order. Nor are they present in higher orders due to the result cited in Sec. II that $\det_5$ is well-defined for any $A_\mu \in_{n>4} \cap L^n(\mathbb{R}^4)$.

The fact that power counting does not ensure finiteness in the $m = 0$ limit of Π_4^- indicates that the low momentum symmetry properties of $\hat{F}_{\mu\nu}(k)$ will be required to give a finite limit. Because of this reliance on symmetry, theorems on mass singularities of Feynman amplitudes known to the author are inapplicable here. The analysis required to give a definitive answer one way or another is beyond the scope of this paper. All we are able to do here is to present evidence for a finite limit of Π_4^- as $m \to 0$. We note that Adler's stereographic mapping to the surface of a 5-dimensional unit hypersphere [12,16] cannot help here due to the slow $1/r$ falloff of the vector potential.

A representative term appearing in the expression for Π_4^- is

$$\begin{aligned}\int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\int_0^1 dz_1 z_1\int_0^{1-z_1} dz_2 \\ \times \int_0^{1-z_1-z_2} dz_3[z_1k^2 + z_2(p+k)^2 + z_3q^2 \\ - (z_1k + z_2(p+k) - z_3q)^2 + m^2]^{-2}(1-2z_1) \\ \times (1-2z_2)(1-2z_3)(1-2z_1-2z_2-2z_3) \\ \times \hat{F}_{\alpha\beta}(p)\hat{F}_{\alpha\beta}(-k-p-q)\hat{F}_{\mu\nu}(q)\hat{F}_{\mu\nu}(k)\end{aligned} \tag{29}$$

The $1/k^2$ behavior of $\hat{F}_{\mu\nu}(k)$ arises from the $J_0(kR)$ term in (27). Fixing on the most singular terms we see that

$$\hat{F}_{\mu\nu}(q)\hat{F}_{\mu\nu}(k) \underset{k,q\to 0}{=} \frac{128\pi^4\nu^2}{k^4q^4}[(q\cdot k)^2 - (q_0k_3 - q_1k_2 + q_2k_1 - q_3k_0)^2]. \tag{30}$$

To isolate the leading singularity in Eq. (29) when $k, q \to 0$ we neglect the denominator when integrating over the angles defining q_μ. Using $\int d\Omega_q(k\cdot q)^2 = \frac{\pi^2}{2}k^2q^2$,

$$\int d\Omega_q(q_0k_3 - q_1k_2 + q_2k_1 - q_3k_0)^2 = \frac{\pi^2}{2}k^2q^2, \tag{31}$$

we see that the leading singularity for small k and q cancels. For the other factor, $F_{\alpha\beta}(p)F_{\alpha\beta}(-k-p-q)$, the case $k, p, q \to 0$ with $p \ll k, q$ reduces to the case just considered when the angles defining p_μ are integrated over. In place of these terms in (29) are additional terms of the form $\hat{\mathbf{B}}(-k-p-q)\cdot\hat{\mathbf{E}}(k)$ and $\hat{\mathbf{B}}(k)\cdot\hat{\mathbf{E}}(-k-p-q)$. Referring to Eq. (28) their singularity at $p, q = 0$ is R^2/k^2, not $1/k^4$.

A nonvanishing chiral anomaly requires the $1/r$ falloff of A_μ. A faster falloff results in a zero-mass limit of Π_4^- that is finite by power counting. A falloff of A_μ slower than $1/r$ would not guarantee a well-defined determinant even for nonzero mass. So a $1/r$ falloff of A_μ is a marginal case that will require further study.

IV. ZERO MODES

In the representation (6) $\not{D}$ has the supersymmetric structure

$$\not{D} = \begin{pmatrix} 0 & D \\ -D^\dagger & 0 \end{pmatrix}, \tag{32}$$

and hence positive chirality zero modes are square-integrable solutions of

$$D^\dagger\psi^+ = 0, \tag{33}$$

where all subscripts on ψ^+ have been dropped. From Eqs. (17) and (20) $\psi^+ \in L^2$ provided

$$\int_0^\infty dr r^{4j+3}(F^2 + G^2) < \infty. \tag{34}$$

Inserting Eqs. (17) and (20) in (33) results in

$$G' + \frac{a}{2j+1}\left(\sqrt{\left(j+\tfrac{1}{2}\right)^2 - M^2}F - MG\right) = 0, \tag{35}$$

$$r^2F' + (2j+1)F + \frac{ar^2}{2j+1}\left(MF + \sqrt{\left(j+\tfrac{1}{2}\right)^2 - M^2}G\right) = 0. \tag{36}$$

Here $j = 0, \frac{1}{2}, \ldots$ and $-j - \frac{1}{2} \leq M \leq j + \frac{1}{2}$. Eqs. (35) and (36) appear in Refs. [18,19] in a different notation, although the authors are considering an entirely different problem. There are three cases to consider.

Case 1 $M = -j - \frac{1}{2}$. Then

$$\psi^+ = \sqrt{2j+1}D^j_{-j,m}(x)G(r^2)\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{37}$$

with $\frac{dG}{dr^2} = -\frac{a}{2}G$, and so

$$G(r^2) = G(r_0^2)e^{-(1/2)\int_{r_0^2}^{r^2} ds a(s)}. \tag{38}$$

Since $a = \nu/r^2$ for $r > R$, $\psi^+ \in L^2$ for $j = 0, \frac{1}{2}, \ldots, j_{\max}$, where $j_{\max}$ is the largest value of j for which $\nu > 2j + 2$ is satisfied.Case 2 $M = j + \frac{1}{2}$. By inspection of Eq. (35) the sign of a is reversed and $G \notin L^2$ for any $\nu > 0$.Case 3 $|M| < j + \frac{1}{2}$. We claim that there are no L^2 zero modes in this case. To show this let $z = r^2$, $\triangle = (2j+1)F$,

$$\Gamma = -2MF - 2\sqrt{\left(j+\tfrac{1}{2}\right)^2 - M^2}G, \tag{39}$$

in Eqs. (35) and (36). Then these become

$$z\frac{d\triangle}{dz} + (2j+1)\triangle = \tfrac{1}{2}za\Gamma, \tag{40}$$

$$z\frac{d\Gamma}{dz} = \left(2M + \tfrac{1}{2}za\right)\triangle. \tag{41}$$

Assume a has a power series expansion about $z = 0$. Then it is straightforward to show that Eqs. (40) and (41) have a solution that is finite at $z = 0$.

It will now be shown that the solution $\triangle, \Gamma$ that is finite at $r = 0$ does not converge fast enough to make $\psi^+ \in L^2$ for $|M| < j + \frac{1}{2}$. Let $t = \ln z$, $\Gamma = \gamma e^{-(j+(1/2))t}$, $\triangle = \delta e^{-(j+(1/2))t}$ and $a = \alpha e^{-t}$. Then Eqs. (40) and (41) can be put in the form [18]

$$\frac{1}{2}\frac{d}{dt}(\gamma^2 - \delta^2) = \left(j + \tfrac{1}{2}\right)(\gamma^2 + \delta^2) + 2M\gamma\delta. \tag{42}$$

Since $\gamma = r^{2j+1}\Gamma$, $\delta = r^{2j+1}\triangle$ and $\Gamma, \triangle$ are finite at $r = 0$, γ and δ vanish at $r = 0$. From (34), if $\psi^+ \in L^2$ then F, $G \sim r^{-2j-2-\epsilon}$, $\epsilon > 0$ and hence $\gamma, \delta \sim r^{-1-\epsilon}$ for $r \to \infty$. Integrating (42) therefore gives

$$\int_0^\infty \frac{dr}{r}\left[\left(j + \tfrac{1}{2}\right)(\gamma^2 + \delta^2) + 2M\gamma\delta\right] = 0. \tag{43}$$

Since $|M| < j + \frac{1}{2}$, (43) is impossible for real e. Hence the assumption that $\psi^+ \in L^2$ for $|M| < j + \frac{1}{2}$ is false.

We now turn to the negative chirality sector. From Eqs. (18), (19), and (21), $\psi^- \in L^2$ provided

$$\int_0^\infty drr^{4j+1}[f^2 + (r^2g)^2] < \infty. \tag{44}$$

From (32) negative chirality zero modes are L^2 solutions of $D\psi^- = 0$. Inserting Eqs. (18), (19), and (21) in this results in

$$\begin{aligned} &2\sqrt{j - M + \tfrac{1}{2}}f' - \sqrt{j + M + \tfrac{1}{2}}(2r^2g' + 4(j+1)g) \\ &\quad + \sqrt{j - M + \tfrac{1}{2}}af - \sqrt{j + M + \tfrac{1}{2}}r^2ag = 0, \end{aligned} \tag{45}$$

$$\begin{aligned} &2\sqrt{j + M + \tfrac{1}{2}}f' + \sqrt{j - M + \tfrac{1}{2}}(2r^2g' + 4(j+1)g) \\ &\quad - \sqrt{j + M + \tfrac{1}{2}}af - \sqrt{j - M + \tfrac{1}{2}}r^2ag = 0. \end{aligned} \tag{46}$$

There are again three cases.

Case 1 $M = -j - \frac{1}{2}$. From (46)

$$2r^2g' + 4(j+1)g - r^2ag = 0, \tag{47}$$

whose solution by inspection is

$$g(r^2) = g(r_0^2)\left(\frac{r}{r_0}\right)^{-4j-4}e^{1/2\int_{r_0^2}^{r^2} ds a(s)}. \tag{48}$$

By (44) $\psi^- \in L^2$ only if $\int_0^\infty drr^{4j+5}g^2 < \infty$, and therefore g is too singular at $r = 0$ to be in L^2.Case 2 $M = j + \frac{1}{2}$. By inspection of Eq. (45) the sign of a is reversed in (47) and (48), and hence g is too singular at $r = 0$ to be in L^2.Case 3 $|M| < j + \frac{1}{2}$. We will demonstrate that $\psi^- \notin L^2$. Let $z = r^2$,

$$\Gamma = \sqrt{\left(j + M + \tfrac{1}{2}\right)}(f + r^2g) + \sqrt{\left(j - M + \tfrac{1}{2}\right)}(r^2g - f), \tag{49}$$

$$\triangle = \sqrt{\left(j + M + \tfrac{1}{2}\right)}(f - r^2g) + \sqrt{\left(j - M + \tfrac{1}{2}\right)}(f + r^2g), \tag{50}$$

in Eqs. (45) and (46). Assuming again that a has a power series expansion about $z = 0$ one establishes that there is a solution for Γ and $\triangle$ that is finite at $z = 0$. To show that this solution is not square-integrable, let $\triangle = z^{-j-(1/2)}\delta$, $\Gamma = z^{-j-(1/2)}\gamma$ and $\lambda = \sqrt{(j + \frac{1}{2})^2 - M^2}$. Then (45) and (46) can be combined to give

$$\frac{d}{dr}(\delta^2-\gamma^2)=\frac{4\lambda}{r}(\gamma^2+\delta^2), \tag{51}$$

where

$$\delta^2-\gamma^2=4[\lambda(f^2-r^4g^2)-2Mr^2fg]r^{4j+2}, \tag{52}$$

$$\delta^2+\gamma^2=2(2j+1)[f^2+(r^2g)^2]r^{4j+2}. \tag{53}$$

From (44), if $\psi^-\in L^2$ then $f,\ r^2g\underset{r\to\infty}{\sim}r^{-2j-1-\epsilon}$, $\epsilon>0$, in which case $\lim_{r=\infty}(\delta^2-\gamma^2)=0$. Because $\triangle$, Γ are finite at $r=0$, γ, $\delta=O(r^{2j+1})$ as $r\to 0$. Hence, integration of Eq. (51) using Eq. (53) gives

$$\sqrt{\left(j+\tfrac{1}{2}\right)^2-M^2}\int_0^\infty drr^{4j+1}[f^2+(r^2g)^2]=0. \tag{54}$$

But this is impossible for $|M|<j+\frac{1}{2}$. Therefore, the assumption that $\psi^-\in L^2$ is false.

Summarizing, it has been shown that all L^2 zero modes of $\not{D}$ have positive chirality and that these only occur when $M=-j-\frac{1}{2}$ and for values of j satisfying $\nu>2j+2$.

V. SCATTERING STATES AND Δ_- AT LOW ENERGY

Having established that there are no negative chirality zero modes it cannot be concluded that the $m=0$ limit of the last term in Eq. (11) is zero. The main result of Ref. [20] is

$$\frac{\nu^2}{2}=n_+-n_-+\frac{1}{\pi}\sum_l\mu(l)[\delta_l^+(0)-\delta_l^-(0)], \tag{55}$$

where $n_\pm$ are the number of positive and negative chirality L^2 zero modes, $\delta_l^\pm(0)$ are the zero-energy scattering phase shifts for $H_\pm$ in Eq. (7), $\mu(l)$ is a weight factor, and l are the quantum numbers required to specify the phase shifts discussed below. This together with Eqs. (24) and (25) demonstrate that the zero-mass limit receives contributions from the scattering states of H_- in Eq. (7). There seems to be no alternative to actually calculating the low-energy scattering states of H_- before deciding whether the $m=0$ limit of the last term in Eq. (11) is zero.

Because $\not{D}$ is anti-Hermitian we look for eigenstates of the form $\not{D}\psi=ik\psi$. Decomposing ψ into its positive and negative chirality components and using (32) gives

$$DD^\dagger\psi^+=k^2\psi^+, \tag{56}$$

$$D^\dagger D\psi^-=k^2\psi^-. \tag{57}$$

To get the scattering states ψ^- it is easier to calculate ψ^+ and then use $D^\dagger\psi^+=-ik\psi^-$. In the representation (6) the Zeeman term, $\frac{1}{2}\sigma F$, is diagonal in the positive chirality sector, and so $DD^\dagger=H_+$ has the form

$$DD^\dagger=\begin{pmatrix}H_{1/2} & 0\\ 0 & H_{-(1/2)}\end{pmatrix}, \tag{58}$$

where the subscripts on H denote the eigenvalues of $S_3^{(1)}$ in (15). In Eqs. (17) and (20) let

$$\sqrt{\frac{2j+1}{2\pi^2}}r^{-2j-(3/2)}\rho_{\pm(1/2)}=\left(j\mp M+\tfrac{1}{2}\right)^{1/2}F \mp\left(j\pm M+\tfrac{1}{2}\right)^{1/2}G, \tag{59}$$

and decompose ψ^+ into its upper and lower components:

$$\begin{aligned}\psi^+_{1/2}&=\sqrt{\frac{2j+1}{2\pi^2}}\begin{pmatrix}D^j_{M-(1/2),m}(\hat{x})\\ 0\end{pmatrix}\frac{\rho_{1/2}(r)}{r^{3/2}},\\ \psi^+_{-(1/2)}&=\sqrt{\frac{2j+1}{2\pi^2}}\begin{pmatrix}0\\ D^j_{M+(1/2),m}(\hat{x}),\end{pmatrix}\frac{\rho_{-(1/2)}(r)}{r^{3/2}},\end{aligned} \tag{60}$$

where $\hat{x}\cdot\hat{x}=1$. Substituting Eqs. (60) in turn in Eq. (56) gives

$$\left[-\frac{d^2}{dr^2}+\frac{(2j+1)^2-\frac{1}{4}}{r^2}+(4M\pm 2)a +r^2a^2\pm r\frac{da}{dr}\right]\rho_{\pm(1/2)}=k^2\rho_{\pm(1/2)}. \tag{61}$$

Equation (61) has to be supplemented by appropriate boundary conditions. For $r>R$ these are chosen so that

$$\rho_{EjM,\pm(1/2)}(r)\underset{kr\gg 1}{\sim}\sqrt{\frac{1}{\pi k}}\cos\Big(kr-\tfrac{\pi}{2}(2j+1) +\delta^+_{jM,\pm(1/2)}(k)-\tfrac{\pi}{4}\Big), \tag{62}$$

in which case the solution of Eq. (61) for $r>R$ is

$$\rho_{E\alpha}(r)=\sqrt{\frac{r}{8}}\big(e^{i((\pi\lambda/2)-(\pi/2)(2j+1)+\delta^+_\alpha(k))}H^{(1)}_\lambda(kr) +e^{-i((\pi\lambda/2)-(\pi/2)(2j+1)+\delta^+_\alpha(k))}H^{(2)}_\lambda(kr)\big), \tag{63}$$

where $E=k^2$, $\lambda=[(2j+1)^2+4M\nu+\nu^2]^{1/2}$ and α denotes j, M, $\pm\frac{1}{2}$. The superscript on δ^+_α is a reminder that these are positive chirality phase shifts. The solutions (63) are to be joined to the solutions of Eq. (61) for $r<R$. This will determine the phase shifts. Equation (63) fixes the normalization so that $\int_0^\infty dr\rho_{E\alpha}(r)\rho_{E'\alpha}(r)=\delta(E-E')$. Then $\psi^+_{\pm 1/2}$ in Eq. (60) have the overall normalization $(\psi^+_{E\beta},\psi^+_{E'\beta'})=\delta_{\beta\beta'}\delta(E-E')$, where β represents j, M, m, $\pm\frac{1}{2}$.

Define the energy-dependent part of δ^+_α by

$$\triangle^+_\alpha(k)=\frac{\pi\lambda}{2}-\frac{\pi}{2}(2j+1)+\delta^+_\alpha(k), \qquad \mathrm{mod}\ \pi, \tag{64}$$

and denote the expansion in powers of k of the logarithmic derivative of the interior radial wave function at $r=R$ by

$$\left(\frac{r\partial_r\rho_{E\alpha}}{\rho_{E\alpha}}\right)_R=\gamma_\alpha-(kR)^2\Gamma_\alpha+O(kR)^4 \tag{65}$$

following Appendix B of Ref. [14]. The coefficients $\gamma_\alpha, \Gamma_\alpha$ are k-independent. Then for $|M| \neq j + \frac{1}{2}$

$$\tan\triangle_\alpha^+ = -\frac{\pi}{\lambda\Gamma^2(\lambda)} \frac{\gamma_\alpha - \lambda - \frac{1}{2}}{\gamma_\alpha + \lambda - \frac{1}{2}}\left(\frac{kR}{2}\right)^{2\lambda} \times (1 + O[(kR)^2, (kR)^{2\lambda}]), \tag{66}$$

with $\lambda > 1$ for $\nu > 0$. There are several special cases to consider when $|M| = j + \frac{1}{2}$. Only one need concern us here, namely, when $M = -j - \frac{1}{2}$, $\nu = 2j + 2$ with $\lambda = 1$. We will find this case important later. The result is [14]

$$\tan\triangle^+_{j,-j-(1/2),-(1/2)} = \frac{\frac{\pi}{2}(1 + O[(kR)^2 \ln(kR)])}{\ln(\frac{kR}{2}) + \gamma_E + \Gamma_{j,-j-(1/2),-(1/2)} + O[(kR)^2 \ln(kR)]}, \tag{67}$$

where γ_E is Euler's constant 0.577....

We now proceed to get the negative chirality scattering states, in particular f and g in Eq. (21) by calculating $D^\dagger \psi^+_{\pm(1/2)} = -ik\psi^-_{\pm(1/2)}$. This results in two orthogonal states [14]

$$\psi^-_{EjMm(1/2)}(x) = \frac{1}{\sqrt{2\pi^2}kr^{3/2}}\sqrt{\frac{j - M + \frac{1}{2}}{2j+1}}\begin{pmatrix} (j+m)^{1/2} D^{j-(1/2)}_{M,m-(1/2)}(\hat{x}) \\ (j-m)^{1/2} D^{j-(1/2)}_{M,m+(1/2)}(\hat{x})\end{pmatrix}\left(\frac{d}{dr} - ar + \frac{2j+\frac{1}{2}}{r}\right)\rho_{EjM(1/2)}(r)$$
$$- \frac{1}{\sqrt{2\pi^2}kr^{3/2}}\sqrt{\frac{j + M + \frac{1}{2}}{2j+1}}\begin{pmatrix} -(j-m+1)^{1/2} D^{j+(1/2)}_{M,m-(1/2)}(\hat{x}) \\ (j+m+1)^{1/2} D^{j+(1/2)}_{M,m+(1/2)}(\hat{x})\end{pmatrix}\left(\frac{d}{dr} - ar - \frac{2j+\frac{3}{2}}{r}\right)\rho_{EjM(1/2)}(r), \tag{68}$$

$$\psi^-_{EjMm,-(1/2)}(x) = \frac{1}{\sqrt{2\pi^2}kr^{3/2}}\sqrt{\frac{j + M + \frac{1}{2}}{2j+1}}\begin{pmatrix} (j+m)^{1/2} D^{j-(1/2)}_{M,m-(1/2)}(\hat{x}) \\ (j-m)^{1/2} D^{j-(1/2)}_{M,m+(1/2)}(\hat{x})\end{pmatrix}\left(\frac{d}{dr} + ar + \frac{2j+\frac{1}{2}}{r}\right)\rho_{EjM,-(1/2)}(r)$$
$$+ \frac{1}{\sqrt{2\pi^2}kr^{3/2}}\sqrt{\frac{j - M + \frac{1}{2}}{2j+1}}\begin{pmatrix} -(j-m+1)^{1/2} D^{j+(1/2)}_{M,m-(1/2)}(\hat{x}) \\ (j+m+1)^{1/2} D^{j+(1/2)}_{M,m+(1/2)}(\hat{x})\end{pmatrix}\left(\frac{d}{dr} + ar - \frac{2j+\frac{3}{2}}{r}\right)\rho_{EjM,-(1/2)}(r). \tag{69}$$

These states are normalized so that $(\psi^-_{E\beta}, \psi^-_{E'\beta'}) = \delta_{\beta\beta'}\delta(E - E')$, where β represents j, M, m, $\pm\frac{1}{2}$. Because there are no L^2 zero modes in the negative chirality sector we expect that the scattering states (68) and (69) form a complete set.

The exact negative chirality propagator is

$$\triangle_-(x, x') = \sum_\alpha \int_0^\infty dk^2 \frac{\psi^-_{E\alpha}(x)\psi^{-\dagger}_{E\alpha}(x')}{k^2 + m^2}, \tag{70}$$

with $\psi^-_{E\alpha}$ given by (68) and (69) and $\alpha = jMm, \pm\frac{1}{2}$. Now suppose $\triangle_-(x, x')$ is divided into its low and high energy parts by replacing the integral in Eq. (70) by $\int_0^{\Lambda^2} + \int_{\Lambda^2}^\infty$, with $\Lambda R \ll 1$. Then our objective is to show that the low-energy propagator has only minor deviations from the free propagator. This turns out to be the case except when $\nu = 2j + 2$ which results in a benign logarithmic mass singularity. The high energy propagator poses no obstacle to the $m = 0$ limit in (11) and is well-defined due to the assumed regularity of A_μ at the origin.

In order to proceed we replace the differential Eq. (61) with the integral equation

$$\rho_\pm(r) = A_\pm\sqrt{\frac{r}{2}}J_{2j+1}(kr) + \frac{\pi}{2}\sqrt{r}\int_0^r dr'\sqrt{r'}[J_{2j+1}(kr')Y_{2j+1}(kr) - J_{2j+1}(kr)Y_{2j+1}(kr')]V_\pm(r')\rho_\pm(r'), \tag{71}$$

where $\rho_\pm$ represents $\rho_{EjM,\pm(1/2)}$, $A_\pm$ are constants to be determined and $V_\pm = (4M \pm 2)a + r^2a^2 \pm r\frac{da}{dr}$. To fix $A_\pm$ require that $\rho_\pm$ join smoothly to the outgoing wave solution (63) at $r = R$ with δ^+_α replaced by its energy-dependent part defined in Eq. (64). Then

$$\rho_\pm(R) = \sqrt{\frac{R}{2}}(J_\lambda(kR)\cos\triangle^+_\alpha(k) - Y_\lambda(kR)\sin\triangle^+_\alpha(k)), \tag{72}$$

together with Eq. (71) at $r = R$ determine $A_\pm$.

An upper bound on $\rho_\pm(r)$ for $0 \le r \le R$ can now be obtained by iterating (71) giving

$$|\rho_\pm(r)| \le \frac{1}{(2j+1)!}\sqrt{\frac{r}{2}}\left(\frac{r}{R}\right)^{2j+1} \times \left(\frac{kR}{2}\right)^\lambda |N_\pm(j, M, \nu)| e^{(1/2)\pi Cr\int_0^R ds|V_\pm(s)|}, \tag{73}$$

for $0 \leq r \leq R$, $kR \ll 1$, $|M| \neq j + \frac{1}{2}$, with C a constant of order one and $N_\pm$ k-independent [14].

When $M = j + \frac{1}{2}$, only ρ_+ is relevant and there is no change in its overall k dependence. When $M = -j - \frac{1}{2}$ only ρ_- is relevant, and the largest modification of Eq. (73) occurs when $2j + 1 < \nu < 2j + 2$ with $0 < \lambda < 1$. Repeating the above analysis gives the same result as (73) except that the factor $(kR/2)^\lambda$ is replaced with $(kR/2)^{-\lambda}$ and N_- is replaced with a new constant $\tilde{N}_-$. The remaining cases when $M = -j - \frac{1}{2}$ result in less singular k-factors than $(kR)^{-\lambda}$.

Now it is evident that the overall k-dependence of $\rho_\pm(r)$ is not changed by differentiating it with respect to r. Therefore, the leading small k-dependence of the radial wave functions in Eqs. (68) and (69) remains $(kR)^\lambda$, $\lambda > 1$ for $M \neq -j - \frac{1}{2}$. Because of the factor k^{-1} multiplying them the negative chirality wave functions $\psi^-_{EjM,\pm(1/2)}$ fall off as $(kR)^{\lambda-1}$ as $k \to 0$ for $M \neq -j - \frac{1}{2}$, $0 \leq r \leq R$.

The case $M = -j - \frac{1}{2}$ has to be handled with care because when $2j + 1 < \nu < 2j + 2$ we have noted that ρ_- behaves as $(kR)^{-\lambda}$ as $k \to 0$ and hence one might naively conclude that $\psi^-_{EjMm,-(1/2)}$ behaves as $(kR)^{-\lambda-1}$ with $0 < \lambda < 1$ when $k \to 0$. This would induce a non-integrable singularity in the chiral propagator (70). It is shown in Ref. [14] that this does not happen.

It is now required to examine the low-energy behavior of the radial wave functions in (68) and (69) when $r > R$. The most singular behavior in k occurs when $M = -j - \frac{1}{2}$, $\nu = 2j + 2$. We need only consider ρ_- in this case. From Eqs. (14), (63), and (67) one obtains $\psi^-_{Ej,-j-(1/2),m,-(1/2)} = O(1)$. All other allowed values of M and ν result in $\psi^-_{EjMm,\pm(1/2)} = O(kR)^{|\lambda-1|}$, or less, for $kR \ll 1$ in the region between $r = 0$ and $r \gtrsim R$ [14]. For $r \gg R$ the radial wave functions in Eqs. (68) and (69) are seen from Eq. (62) to behave as in the noninteracting case except for phase shifts.

Now return to the last term in Eq. (11) and the interacting propagator (70). As noted earlier the centrifugal barrier term in Eq. (61) together with the regularity assumptions made on $a(r)$ will cause the large $j \gg \nu$ contributions to $\triangle_-(x, x')$ to approach those of the noninteracting propagator. Therefore we need only consider a finite range of j in the search for a possible mass singularity in $\triangle_-$ that would result in a nonvanishing remainder at $m = 0$.

As previously noted the largest deviation from the noninteracting case occurs when $\nu = 2j + 2$. Focus on this mode in $\triangle_-$. This mode first opens up when $\nu = 2$, the threshold value of ν for the formation of the first square-integrable zero mode in the positive chirality sector according to the discussion following Eq. (38). From Eq. (70), the second term in Eq. (69), (67), and (63) one obtains for the worst case r, $r' > R$

$$\begin{aligned}\triangle_-^{j=(1/2)\nu-1}(x, x') = \frac{M(\hat{x}, \hat{x}')}{rr'} \int_0^{\Lambda^2} \frac{dk^2}{k^2 + m^2} \bigg[& J_0(kr)J_0(kr') \\ & - \frac{\pi}{2}(Y_0(kr)J_0(kr') \\ & + Y_0(kr')J_0(kr)) \Big/ \ln\Big(\frac{kR}{2}\Big) \\ & + \frac{\pi^2}{4} Y_0(kr)Y_0(kr') \Big/ \ln^2\Big(\frac{kR}{2}\Big)\bigg] + R_\Lambda,\end{aligned} \tag{74}$$

with $\Lambda R \ll 1$. The k-independent matrix M is obtained from the second term in Eq. (69) in the calculation of $\psi^-_{E\alpha}(x)\psi^{-\dagger}_{E\alpha}(x')$, and R_Λ is the contribution to $\triangle^j_-$ from the region $k > \Lambda$. The most singular term in m in Eq. (74) occurs in the first integral which has only a logarithmic mass singularity when r, $r' \gtrsim R$

$$\begin{aligned}\triangle_-^{j=(1/2)\nu-1}(x, x') = -2\frac{M(\hat{x}, \hat{x}')}{rr'} \ln(mr_>) \\ + \text{less singular in } m,\end{aligned} \tag{75}$$

where $r_<(r_>)$ denotes the lesser (larger) of r, r'.

In summary, a mode-by-mode analysis of the exact propagator in Eq. (70) uncovers only minor deviations from the free propagator in the low-energy domain. If Π_4^- is finite at $m = 0$ so that $\lim_{m=0} m^2 \partial\Pi_4^-/\partial m^2 = 0$ and if the role of the symmetry of $F_{\mu\nu}$ at large distances in reaching this conclusion is well-understood then it should be possible to generalize this fourth-order result to $m^2\partial\Pi_6^-/\partial m^2$, etc., obtained by expanding Δ_- in Eq. (11) in a power series. We have shown in this section that its expansion is justified, considering that no nonperturbative singularities are induced in Δ_- by the scattering states that would cause the $m = 0$ limit of the last term in Eq. (11) to be nonvanishing.

VI. LARGE-MASS LIMIT OF $\mathcal{R}$

The leading term in the asymptotic expansion of $\mathcal{R}$ in Eq. (8) for large m can be calculated from the effective Lagrangian density for QED_4 in a constant-field background. This is possible provided $F_{\mu\nu}$ is assumed to be smooth enough so that a meaningful derivative expansion of $\mathcal{R}$ can be carried out. Just how smooth will be made more precise below.

The photon-photon scattering graph in $\mathcal{R}$ has been thoroughly studied by Karplus and Neuman [21]. Using their results or those of Refs. [13,22–24] one gets for large mass

$$\Pi_4 = \frac{1}{5760\pi^2 m^4}\int d^4x[4(F_{\mu\nu}F_{\mu\nu})^2 - 7({}^*F_{\mu\nu}F_{\mu\nu})^2] + O\bigg(\frac{1}{m^6}\int d^4x F_{\alpha\beta}F_{\alpha\beta}F_{\mu\nu}\partial^2 F_{\mu\nu}, \frac{1}{m^6} \times \int d^4x F_{\mu\alpha}\partial_\lambda F_{\alpha\nu}\partial_\lambda F_{\nu\beta}F_{\beta\mu}\bigg). \tag{76}$$

The sixth-order graph obtained from the expansion of $\ln\det_5$ is of order $1/m^8$. We now seek the conditions under which the leading term in $\mathcal{R}$ as $m \to \infty$, the right-hand side of (76), becomes positive.

From (76) positivity requires

$$\int d^4x(F_{\mu\nu}F_{\mu\nu})^2 > \frac{7}{4}\int d^4x({}^*F_{\mu\nu}F_{\mu\nu})^2. \tag{77}$$

From (14) and (22), ${}^*F_{\mu\nu}F_{\mu\nu} = 0$ for $r > R$ and so by (23), $F_{\mu\nu}F_{\mu\nu} = 8\nu^2/r^4$ for $r > R$. Referring again to Eqs. (22) and (23) there follows the positivity condition

$$\int_0^{R^2} dr^2 r^6 a'^2[r^4a'^2 + 4r^2aa' + a^2] > \frac{\nu^4}{R^4}, \tag{78}$$

where a prime continues to denote differentiation with respect to r^2. It is evident from (78) that one class of fields satisfying the positivity condition is characterized by a steep rise in a in the region $r \lesssim R$ where it obtains a maximum before descending as ν/r^2 in order to join smoothly with $a(r^2)$ at $r = R$. The class of admissible fields may be larger than this.

In order for the remainder term in the asymptotic expansion in Eq. (76) to be finite it is necessary that $F_{\mu\nu}$ be twice differentiable. From Eq. (12) the most singular term if $F_{\mu\nu}$ contains terms like $x_\nu M_{\mu\alpha}x_\alpha a'$ and hence the most singular term in $\partial^2 F_{\mu\nu}$ is of the form $r^2x_\nu M_{\mu\alpha}x_\alpha a'''$. Thus, the finiteness of $\int F^2F\partial^2F$ requires

$$\bigg|\int_0^R drr^7\bigg(\frac{da}{dr}\bigg)^3\frac{d^3a}{dr^3}\bigg| < \infty, \tag{79}$$

and so $a(r^2)$ must be at least 3 times differentiable. For ease of analysis we assumed in Sec. IV that $a(r^2)$ was regular at the origin, but this is not necessary. Condition (79) only requires $a \underset{r\to 0}{\sim} Cr^\beta$ with $\beta > -\frac{1}{2}$. Of course requiring $A_\mu \in \bigcap_{n>4}L^n(\mathbb{R}^4)$ rules out $\beta < 0$. Any branches in $a(r^2)$ away from $r = 0$ of the form $a(r^2) \underset{r\to r_0}{\sim} C(r^2 - r_0^2)^\alpha$, must have $\alpha > 5/4$ according to Eq. (79).

Now it may happen that a given $a(r^2)$ does not satisfy Eq. (78). This could mean that either there are no mass zeros in the remainder defined by Eq. (8) or that there are an even number of such zeros. This cannot be decided here. In our search for definite information we go back to Eq. (8) and deal only with $\ln\det_5$, treating the photon-photon graph as a subtraction like the second-order graph. If Eq. (26) is true then the $\ln m^2$ singularity is from $\ln\det_5$ alone. Then if the leading term in $\ln\det_5$'s asymptotic expansion in powers of $1/m$ is positive it certainly has at least one mass zero in the interval $0 < m < \infty$.

The leading term is the sixth-order graph given by [13,22–24]

$$\ln\det_5 = \frac{1}{40320\pi^2m^8}\int d^4x[13({}^*F_{\mu\nu}F_{\mu\nu})^2 - 8(F_{\mu\nu}F_{\mu\nu})^2]F_{\alpha\beta}F_{\alpha\beta} + O\bigg(\frac{1}{m^{10}}\int d^4xF^2F^2F_{\mu\nu}\partial^2F_{\mu\nu}, \frac{1}{m^{10}} \times \int d^4xF_{\mu_1\mu_2}\partial^2F_{\mu_2\mu_3}\ldots F_{\mu_6\mu_1}\bigg), \tag{80}$$

and hence the positivity condition is

$$\int d^4x[13({}^*F_{\mu\nu}F_{\mu\nu})^2 - 8(F_{\mu\nu}F_{\mu\nu})^2]F_{\alpha\beta}F_{\alpha\beta} > 0. \tag{81}$$

Use of Eqs. (14) and (22) results in the final positivity condition

$$\int_0^{R^2} dr^2[2r^{14}a'^6 + 12r^{12}aa'^5 + 23r^{10}a^2a'^4 + 12r^8a^3a'^3 - 19r^6a^4a'^2] < \frac{9\nu^6}{2R^8}. \tag{82}$$

The most singular terms in the remainder of the asymptotic expansion in Eq. (80) will arise from those containing ∂^2F. Following the above discussion these will be finite provided

$$\bigg|\int_0^R drr^9\bigg(\frac{da}{dr}\bigg)^5\frac{d^3a}{dr^3}\bigg| < \infty. \tag{83}$$

This requires $a \sim_{r\to 0} Cr^\beta$ with $\beta > -1/3$, at least, and any branch points in a of the form $(r^2 - r_0^2)^\alpha$ must have $\alpha > 7/6$.

It may be seen by inspection of Eq. (82) that one class of fields satisfying it are those with $a(0) \sim N\nu/R^2$, $N \gtrsim 2$ and more or less monotonically decaying to ν/R^2 at $r = R$. Such fields will not satisfy the positivity condition (78).

To summarize, when Eqs. (26), (78), and (79) are satisfied the remainder $\mathcal{R}$ in Eq. (8) has at least one zero as m varies over the interval $0 < m < \infty$. When Eqs. (26), (82), and (83) are satisfied, $\ln\det_5$ has such a zero. In this case the entire function in Eq. (4) somehow manages to reduce to unity at the mass zero(s).

VII. CONCLUSION

By choosing $O(2) \times O(3)$ symmetric background gauge fields we were able to make some provisional nonperturbative statements about the behavior of the Euclidean fermionic determinant $\det_{\rm ren}$ of QED_4 as a function of the fermionic mass. This determinant has the form $\ln\det_{\rm ren} = \Pi_2 + \Pi_4 + \ln\det_5$. The second-order term contains a charge renormalization subtraction. The remaining terms are denoted by $\mathcal{R}$ in Eq. (8). It was assumed that for $r > R$ the radial profile function $a(r^2)$ in Eq. (12) takes the form ν/r^2 for $r > R$, together with some mild regularity assumptions for $a(r^2)$ for $r < R$. With these assumptions $\det_{\rm ren}$ is free of all cutoffs, including the second-order term if on-shell charge renormalization is used. Then we showed that if the leading mass singularity of $\mathcal{R}$ as $m \to 0$ is fully determined by the chiral anomaly, then $\mathcal{R}$ has at least one zero as m varies in the interval $0 < m < \infty$, provided conditions (78) and (79) are satisfied. If not, then provided (82) and (83) are satisfied, $\ln\det_5$ has at least one such zero at which the entire function in Eq. (4) becomes unity for any fixed coupling e. Then $\ln\det_{\rm ren}$ is dominated by Π_4 for $|e| \gg 1$, which is consistent with $\det_{\rm ren}$ being an entire function of order four as discussed in Sec. II. If there is a mass zero such that $\mathcal{R}$ vanishes then $\ln\det_{\rm ren} = \Pi_2$ at this zero. If the number of mass zeros in $\mathcal{R}$ or $\ln\det_5$ is even then they will not show up in the analysis here.

This raises an interesting possibility. If $e^2 \ll 1$ then m does not have to be very large to make a meaningful $1/m$ asymptotic expansion. So, presumably, there are one or more "small" mass zeros in the weak coupling domain.

In plain language the result is this: set $e^2/\hbar c = 1/137\ldots$. Select a gauge field that satisfies (78), (79), (82), and (83). Adjust m until a mass zero appears. If m is the physical fermion mass then it probably does not coincide with a mass zero. But if, for the selected gauge field, m is near a zero then we would expect the remainder $\mathcal{R}$ or $\ln\det_5$ to be anomalously small compared to the sum of the first few graphs in their expansion. By continuity there should be a class of gauge fields for which the physical coupling and mass coincide exactly with a mass zero.

In establishing these results we also demonstrated a vanishing theorem when the field strength tensor is not (anti-)self-dual, namely, that all of the square-integrable zero modes of the Dirac operator are of one chirality. This is a generalization of the vanishing theorem of Brown, Carlitz, and Lee [25]. It would be useful to have a general vanishing theorem and to understand the physical principles underlying it.

In Sec. VI it was assumed that the expansion of $\mathcal{R}$ in powers of $1/m$ is truly an asymptotic one so that the remainder after the series is truncated is of the order of the first neglected term. A proof is needed, but for the present it is an assumption physicists accept provided the background gauge field is smooth enough.

Most of this paper deals with the question of whether it is indeed true that the leading mass singularity of $\mathcal{R}$ in Eq. (8) is determined by the chiral anomaly. We have presented evidence that it is. It is true for the case of constant **B** and **E** [15], but this is a formal result as the determinant has to be made finite by a volume cutoff. And it is also true for the QCD_4 determinant in the presence of an instanton background [26]. It is evident that the analytic, nonperturbative analysis of four-dimensional fermionic determinants is still at an early stage and may yet yield some surprises.

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