

Sigma model renormalization group flow, “central charge” action, and Perelman’s entropy

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Zamolodchikov’s *c*-theorem type argument (and also string theory effective action constructions) imply that the RG flow in 2d sigma model should be a gradient one to all loop orders. However, the monotonicity of the flow of the target-space metric is not obvious since the metric on the space of metric-dilaton couplings is indefinite. To leading (one-loop) order when the RG flow is simply the Ricci flow the monotonicity was proved by Perelman [G. Perelman, math.dg/0211159.] by constructing an “entropy” functional which is essentially the metric-dilaton action extremized with respect to the dilaton with a condition that the target-space volume is fixed. We discuss how to generalize the Perelman’s construction to all loop orders (i.e. all orders in α'). The resulting entropy is equal to minus the central charge at the fixed points, in agreement with the general claim of the *c*-theorem.

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I. INTRODUCTION

2d sigma models containing infinite number of couplings parametrized by target-space metric tensor $G_{\mu\nu}(x)$ have many interesting connections to various problems in physics and mathematics. In particular, they play an important role in string theory, describing string propagation in curved space. In order to define the quantum stress tensor of the sigma model, i.e. its renormalization on a curved 2d space, one is led to the introduction an extra scalar coupling function $\phi(x)$ or the dilaton. The corresponding metric and dilaton RG “beta-functions” β^i are then proportional to string effective equations of motion (for reviews see, e.g., [1,2]). Assuming that the *c*-theorem claim [3] should apply to the sigma model (with compact euclidean target space) one should the existence of a (local, covariant) functional of the metric such that (i) its gradient is proportional to β^i (with certain diffeomorphism terms added), (ii) it decreases along the RG flow and (iii) it is equal to the central charge at fixed points. While the gradient property of the flow is relatively easy to establish, its monotonicity is much less obvious. In particular, the direct generalization of Zamoldchikov’s proof of the *c*-theorem [4] leads to the “central charge” action which vanishes at any fixed point, instead of decreasing along the flow.

One may expect that the statement of the *c*-theorem should apply provided one considers only the $G_{\mu\nu}$ metric flow (the flow of the dilaton plays, in a sense, a secondary role). However, the technical details of the Zamolodchikov-type construction [3,4] of the “central charge” functional do not directly apply if one ignores the dependence on the dilaton. This suggests that one needs an alternative way of constructing the corresponding RG entropy. The idea of such construction was suggested by Perelman [5] (see also a review and generalizations in [6])

on the example of the Ricci flow which is the 1-loop approximation to the full sigma model RG flow. He, in turn, was inspired by the structure of the leading terms in the metric-dilaton effective action which first appeared in the string-theory context [7,8].

Below we shall present a generalization of the Perelman’s construction to all orders in sigma model loop expansion, i.e. suggest how to prove the *c*-theorem for the sigma model. We shall first review (in Sec. II) the basic facts about the structure of sigma model “beta-functions” and the associated “central charge” action. In Sec. III we shall define a modification of such action which is equal to minus the Perelman’s “ λ -entropy” and interpret the latter as a Lagrange multiplier for the fixed volume condition. We shall argue that this entropy should grow along the metric RG flow and is equal to minus the central charge at the fixed points.

II. REVIEW OF SIGMA MODEL RESULTS: WEYL ANOMALY COEFFICIENTS AND “CENTRAL CHARGE” ACTION

Let us start with a review of some known facts about renormalization of sigma model on curved 2d space. One reason to consider quantum sigma model on a curved space is to be able to define its stress tensor and its correlators which enter the standard proof of the *c*-theorem. Another is that to be able to define global quantities that may provide one with a “*c*-function” one needs to consider a 2-space of a topology of a sphere (to regularize, in particular, the IR divergences). The same problem also naturally arises in the context of the Polyakov’s approach to critical string theory when one considers propagation of a string in D-dimensional curved target space with metric $G_{\mu\nu}$ [2,7,9]. The corresponding action $I = \frac{1}{4\pi\alpha'} \int d^2z \sqrt{g} g^{ab} \partial_a x^\mu \times \partial_b x^\nu G_{\mu\nu}(x)$ is classically invariant under Weyl rescalings of the 2d metric g_{ab} , implying the decoupling of the conformal factor of this metric. Consistency of the critical string theory (where conformal factor of the 2d metric is

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not a dynamical field) then translates into the requirement of the cancellation of the Weyl anomaly. The renormalizability of the above sigma model on curved 2d background requires introduction of one extra ‘‘hidden’’ scalar (dilaton) coupling which has the same dimension as the metric term [7]

$$I = \frac{1}{4\pi\alpha'} \int d^2z \sqrt{g} [g^{ab} \partial_a x^\mu \partial_b x^\nu G_{\mu\nu}(x) + \alpha' R^{(2)} \phi(x)], \quad (2.1)$$

where $R^{(2)}$ is the curvature of g_{ab} .

The two couplings run with renormalization scale according to the corresponding beta-functions¹

$$\frac{d\varphi^i}{dt} = -\beta^i, \quad \varphi^i = (G_{\mu\nu}, \phi). \quad (2.2)$$

The operator form of the Weyl anomaly relation for the trace of the 2d stress tensor is then

$$2\pi\alpha' T_a^a = [\partial_a x^\mu \partial^a x^\nu \bar{\beta}_{\mu\nu}^G(x) + \alpha' R^{(2)} [\bar{\beta}^\phi(x)], \quad (2.3)$$

where the $\bar{\beta}^i$ are the Weyl-anomaly coefficients that differ from β^i by certain diffeomorphism terms [10,11]

$$\begin{aligned} \bar{\beta}_{\mu\nu}^G &= \beta_{\mu\nu}^G + \nabla_\mu M_\nu + \nabla_\nu M_\mu, \\ \bar{\beta}^\phi &= \beta^\phi + M^\mu \partial_\mu \phi, \end{aligned} \quad (2.4)$$

$$M_\mu = \alpha' \partial_\mu \phi + W_\mu(G). \quad (2.5)$$

W_μ is a specific covariant vector constructed out of curvature and its covariant derivatives only (it is determined from the matrix that governs mixing under renormalization of dimension 2 operators).

In dimensional regularization with minimal subtraction one finds to 2-loop order [7,10,12–14]²

$$\beta_{\mu\nu}^G = \alpha' R_{\mu\nu} + \frac{1}{2} \alpha'^2 R_{\lambda\rho\sigma} R_\nu^{\lambda\rho\sigma} + O(\alpha'^3 R^3) \quad (2.6)$$

$$\begin{aligned} \beta^\phi &= -\gamma(G)\phi + \omega(G) \\ &= c_0 - \frac{1}{2} \alpha' \nabla^2 \phi + \frac{1}{16} \alpha'^2 R_{\lambda\mu\nu\rho} R^{\lambda\mu\nu\rho} + O(\alpha'^3 R^3), \end{aligned} \quad (2.7)$$

where in critical bosonic string $c_0 = \frac{1}{6}(D - 26)$ (D being the total number of coordinates x^μ and -26 stands for the measure or ghost contribution [17]). Here ω is a scalar function of the curvature and its covariant derivatives. $\gamma(G)$ is a differential operator (scalar anomalous dimension) which in the minimal subtraction scheme has the following general form [10,13,18]

¹In our present notation (opposite to that of [1]) the RG evolution toward the IR corresponds to $t = -\ln\mu \rightarrow +\infty$ (μ is a momentum renormalization scale).

²The 3-loop α'^3 and 4-loop α'^4 corrections to beta-functions were computed, respectively, in [15,16].

$$\gamma = \Omega_2^{\mu\nu} \nabla_{(\mu} \nabla_{\nu)} + \sum_{n=3}^{\infty} \Omega_n^{\mu_1 \dots \mu_n} \nabla_{(\mu_1 \dots \mu_n)}, \quad (2.8)$$

$$\Omega_2^{\mu\nu} = \frac{1}{2} \alpha' G^{\mu\nu} + p_1 \alpha'^3 R_{\alpha\beta\gamma}^\mu R^{\nu\alpha\beta\gamma} + \dots, \quad (2.9)$$

$$\begin{aligned} \Omega_3^{\mu\nu\rho} &= q_1 \alpha'^4 D_\alpha R_{\beta}^{\mu\nu}{}^\rho R^{\alpha\beta\gamma\rho} + \dots, \\ \Omega_4^{\mu\nu\rho\lambda} &= s_1 \alpha'^4 R^{\mu\alpha\beta\nu} R^\rho_{\alpha\beta}{}^\lambda + \dots \end{aligned} \quad (2.10)$$

In the minimal subtraction scheme M_μ gets a nonzero contribution only starting with 3 loops

$$W_\mu = t_1 \alpha'^3 \partial_\mu (R_{\lambda\nu\gamma\rho} R^{\lambda\nu\gamma\rho}) + \dots \quad (2.11)$$

The 3-loop coefficients here are $p_1 = \frac{3}{16}$, $t_1 = \frac{1}{32}$ [19] and the 4-loop coefficients q_1, s_1, \dots were found in [15,20].

In general, the Weyl anomaly coefficients $\bar{\beta}_{\mu\nu}^G$ and $\bar{\beta}^\phi$ satisfy D differential identities which can be derived from the condition of nonrenormalization of the trace of the energy-momentum tensor of the sigma model [10,21]

$$\partial_\mu \bar{\beta}^\phi - (\delta_\mu^\lambda \nabla^\rho \phi + V_\mu^{\lambda\rho}) \bar{\beta}_{\lambda\rho}^G = 0, \quad (2.12)$$

where the differential operator $V_\mu^{\lambda\rho}$ depends only on $G_{\mu\nu}$. To lowest order $V_\mu^{\lambda\rho} \bar{\beta}_{\lambda\rho}^G = \frac{1}{2} \nabla^\nu (\bar{\beta}_{\mu\nu}^G - \frac{1}{2} G_{\mu\nu} G^{\lambda\rho} \bar{\beta}_{\lambda\rho}^G) + O(\alpha'^2)$. Equation (2.12) implies that once the metric conformal invariance equation is imposed, $\bar{\beta}_{\mu\nu}^G = 0$, then $\bar{\beta}^\phi = \text{const}$, and thus $\bar{\beta}^\phi = 0$ gives just one algebraic equation.

The existence of the identity (2.12) implying D conditions between $\frac{1}{2}D(D+1)+1$ functions $\bar{\beta}_{\mu\nu}^G$ and $\bar{\beta}^\phi$ would have a natural explanation if $\bar{\beta}^G$ and $\bar{\beta}^\phi$ could be obtained by variation from a covariant action functional $S(G, \phi)$ [3,14,17]

$$\frac{\delta S}{\delta \varphi^i} = \kappa_{ij} \bar{\beta}^j, \quad \bar{\beta}^i = \kappa^{ij} \frac{\delta S}{\delta \varphi^j}, \quad (2.13)$$

where κ_{ij} is a nondegenerate covariant operator. Indeed, the diffeomorphism invariance of S implies the identity

$$\nabla_\mu \frac{\delta S}{\delta G_{\mu\nu}} - \frac{1}{2} \frac{\delta S}{\delta \phi} \nabla^\nu \phi = 0 \quad (2.14)$$

which would then relate $\bar{\beta}^G$ and $\bar{\beta}^\phi$.

Indeed, such action functional is easy to find to leading order in α'

$$S = \int d^D x \sqrt{G} e^{-2\phi} [c_0 - \alpha' (\frac{1}{4} R + \partial_\mu \phi \partial^\mu \phi) + O(\alpha'^2)]. \quad (2.15)$$

Then

$$\bar{\beta}_{\mu\nu}^G = \alpha' (R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi) + O(\alpha'^2), \quad (2.16)$$

$$\bar{\beta}^\phi = c_0 - \alpha' (\frac{1}{2} \nabla^2 \phi + \nabla^\mu \phi \nabla_\mu \phi) + O(\alpha'^2) \quad (2.17)$$

follow from this action if κ in (2.13) is

$$\kappa^{ij} = \frac{1}{\sqrt{G}e^{-2\phi}} \begin{pmatrix} 4G_{\mu\lambda}G_{\nu\rho} & G_{\mu\nu} \\ G_{\lambda\rho} & \frac{1}{4}(D-2) \end{pmatrix} + O(\alpha'), \quad (2.18)$$

$$\kappa_{ij} = \sqrt{G}e^{-2\phi} \begin{pmatrix} \frac{1}{4}(G^{\mu\lambda}G^{\nu\rho} - \frac{1}{2}G^{\mu\nu}G^{\lambda\rho}) & \frac{1}{2}G^{\mu\nu} \\ \frac{1}{2}G^{\lambda\rho} & -2 \end{pmatrix} + O(\alpha'). \quad (2.19)$$

The Lagrangian in (2.15) can be written as (up to a total derivative)

$$\begin{aligned} \tilde{\beta}^\phi &\equiv \bar{\beta}^\phi - \frac{1}{4}G^{\mu\nu}\bar{\beta}_{\mu\nu}^G \\ &= c_0 - \frac{1}{4}\alpha'(R + 4\nabla^2\phi - 4\partial_\mu\phi\partial^\mu\phi) + O(\alpha'^2). \end{aligned} \quad (2.20)$$

This combination [10] may be interpreted as a ‘‘generalized central charge’’ function: it appears as the leading term in the expectation value of the trace of the stress tensor ($2\pi\langle T_a^a \rangle = \tilde{\beta}^\phi R^{(2)} + \dots$) and is equal to the central charge at the conformal point where $\bar{\beta}^G = 0$ (then $\tilde{\beta}^\phi = \bar{\beta}^\phi = \text{const}$).

Reference [4] put forward an argument (based on the idea of the proof of the c-theorem in [3]) that the ‘‘central charge’’ action³

$$\begin{aligned} S &= \int d^Dx\sqrt{G}e^{-2\phi}\tilde{\beta}^\phi(G, \phi) \\ &= \int d^Dx\sqrt{G}e^{-2\phi}(\beta^\phi - \frac{1}{4}G^{\mu\nu}\beta_{\mu\nu}^G) \end{aligned} \quad (2.21)$$

should have its equations of motion equivalent to $\bar{\beta}^G = 0$, $\bar{\beta}^\phi = 0$ to all orders in α' (provided one chooses an appropriate scheme, i.e. modulo a local redefinition of $G_{\mu\nu}$ and ϕ). This was indeed confirmed by the explicit sigma model computations up to and including the 4-loop (α'^4) order [15,19,20].

This action has a remarkable structure. In particular, it can be rewritten as

$$S = -\frac{1}{2} \left(\beta^\phi \cdot \frac{\delta}{\delta\phi} + \beta^G \cdot \frac{\delta}{\delta G_{\mu\nu}} \right) \int d^Dx\sqrt{G}e^{-2\phi}. \quad (2.22)$$

Then assuming that $G_{\mu\nu}$ and ϕ depend on the renormalization point and using (2.2) one finds [4]

$$S = \frac{1}{2} \frac{dV}{dt}, \quad V \equiv \int d^Dx\sqrt{G}e^{-2\phi}, \quad (2.23)$$

i.e. that the ‘‘central charge’’ action evaluated on the RG

³In the second equality below we used the reparametrization invariance of the action and the fact that $\tilde{\beta}^i$ differ from β^i by diffeomorphism terms. Note also that when acting on a diffeomorphism-invariant functional one has $\tilde{\beta}^i \cdot \frac{\delta}{\delta\varphi^i} = \beta^i \cdot \frac{\delta}{\delta\varphi^i}$; this relation will be used below.

running couplings is simply the RG ‘‘time’’ derivative of the generalized volume.

More generally, based on detailed study of renormalization of the sigma model Ref. [22,23] have constructed an action that reproduces the Weyl anomaly coefficients, i.e. satisfies (2.13), to any order in α' in an arbitrary (covariant) renormalization scheme. This action has the following structure

$$S_1 = \int d^Dx\sqrt{G}e^{-2\phi}(J_i\bar{\beta}^i + \rho_{ij}\bar{\beta}^i\bar{\beta}^j), \quad (2.24)$$

where $J_\phi = 1 + \dots$, $J_G^{\mu\nu} = -\frac{1}{4}G^{\mu\nu} + \dots$, etc. are functions of $\varphi_i = (\phi, G_{\mu\nu})$ determined in terms of the renormalization group quantities. The matrix ρ_{ij} is, in principle, arbitrary [Eq. (2.13) is satisfied for any ρ_{ij}] but for a specific choice of it one can show [22] that (2.24) becomes [cf. (2.22)]

$$S_1 = -\frac{1}{2} \left(\beta^\phi \cdot \frac{\delta}{\delta\phi} + \beta^G \cdot \frac{\delta}{\delta G_{\mu\nu}} \right) \int d^Dx\sqrt{G}e^{-2\phi}J_\phi(G), \quad (2.25)$$

where in the minimal subtraction scheme

$$J_\phi = 1 - \frac{1}{4}\alpha'^2 R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} + O(\alpha'^3). \quad (2.26)$$

As was pointed out in Sec. 6 of [1], if one further redefines the dilaton by $-\frac{1}{2}\ln J_\phi$ the action (2.25) reduces to the simpler-looking action (2.21) or (2.22).⁴ Thus, there should always exist a scheme in which the gradient of (2.21) reproduces $\bar{\beta}^i$ to all α' orders.⁵ References [20,22] found explicitly the renormalization scheme in which the action (2.24) of [22] reduces to the action (2.21) of [4] at orders α'^3 and α'^4 respectively.

Let us note that the representation (2.23) is closely related to the interpretation of the action whose extrema are equivalent to the vanishing of the sigma model Weyl anomaly coefficients as the string low-energy effective action for the graviton and dilaton modes. The effective action can be reconstructed from the string S-matrix. Its relation to the sigma model is explained by the observation [7] that the generating functional for the string scattering amplitudes can be interpreted as a partition function $Z = \int [dx]e^{iI}$ on the 2-sphere with the string action being the sigma model with couplings which are the string space-time fields. The realization that a renormalization of the sigma model corresponds to a subtraction of massless poles in the string scattering amplitudes and that a subtraction of the Mobius volume infinities can be done by differentiating

⁴One needs to note that under a coupling redefinition $\beta^i \cdot \frac{\delta}{\delta\varphi^i} = \beta'^i \cdot \frac{\delta}{\delta\varphi'^i}$.

⁵A possible drawback of this general argument is that the corresponding scheme choice is somewhat implicit. In the dimensional regularization with minimal subtraction the two actions begin to differ starting with α'^3 order, i.e. to relate them at 3- and higher loop orders one needs a certain field redefinition.

over the logarithm of the 2d cutoff led to the expression for the tree-level closed string effective action S in terms of the ‘‘RG time’’ derivative of the renormalised sigma model partition function Z [24]:

$$S = \frac{\partial Z}{\partial t} = \beta^i \cdot \frac{\partial Z}{\partial \varphi^i}. \quad (2.27)$$

Here we used the RG equation for Z : $\frac{dZ}{dt} = \frac{\partial Z}{\partial t} - \beta^i \cdot \frac{\partial Z}{\partial \varphi^i} = 0$. Finally, there exists a scheme choice in which the renormalized value of Z is simply proportional to the generalized volume V in (2.23) [1,24]. This explains the equivalence between the sigma model RG -motivated ‘‘central charge’’ action (2.21) and (2.23) and the string partition function-motivated effective action (2.27).

III. MONOTONICITY OF RG FLOW FOR $G_{\mu\nu}$ COUPLING

The existence of the action (2.21) and (2.25) whose derivative is proportional (2.13) to the Weyl-anomaly coefficients implies, in particular, that the RG flow of the metric $G_{\mu\nu}$ and the dilaton ϕ couplings of the sigma model (2.1) [with a specific choice (2.4) and (2.5) of the diffeomorphism terms] is a ‘‘gradient flow.’’ This flow is not, however, monotonic. Indeed, the ‘‘central charge’’ action (2.21) vanishes at the fixed points; also, $\frac{dS}{dt} = -\kappa_{ij}\tilde{\beta}^i\tilde{\beta}^j$ does not have a definite sign since the ‘‘metric’’ κ_{ij} in (2.13) and (2.18) is not sign-definite.

At the same time, one may expect (in view of the Zamolodchikov’s theorem [3] for unitary 2d theories with finite number of couplings)⁶ that if one restricts attention just to the RG flow of the metric $G_{\mu\nu}$ (and considers the case of compact Euclidean signature space) there should exist an action functional $S(G)$ whose gradient is proportional to β^G and which decreases along the $G_{\mu\nu}$ flow toward the IR.

A natural guess is that $S(G)$ should be closely related to the functional $S(G, \phi)$, e.g., it could be found by solving for the dilaton, i.e. by extremizing $S(G, \phi)$ in ϕ . The ‘‘secondary’’ role of the dilaton coupling is suggested by the existence of the identity (2.12) that expresses its beta-function in terms of the metric beta-function.

The idea of finding $S(G)$ by eliminating ϕ from $S(G, \phi)$ does not, however, work directly. At the leading order in α' the combination $\tilde{\beta}^\phi$ in (2.20) has a remarkable property that its variation over ϕ with the measure factor in (2.21) is zero. The variation of (2.21) over ϕ gives simply

⁶The proof of the Zamolodchikov’s theorem does not directly apply to the case of sigma models. If one tries to repeat the construction of [3] of the ‘‘central charge function’’ (whose gradient is the beta function) based on correlators of stress tensor [4] one needs to introduce the running dilaton coupling and then the metric on the space of couplings is indefinite (and also the value of the central charge function at the fixed point is zero).

$$\frac{\delta S}{\delta \phi} = -2\sqrt{G}e^{-2\phi}\tilde{\beta}^\phi = 0, \quad (3.1)$$

i.e. $\tilde{\beta}^\phi = 0$, and then the action S vanishes even before imposing $\tilde{\beta}^G = 0$. The same property of $\tilde{\beta}^\phi$ is true at least to order α'^4 and should be true in general in an appropriate scheme.⁷

To get a nontrivial functional $S(G)$ Perelman [5] suggested to minimize $S(G, \phi)$ in ϕ while restricting ϕ to satisfy the unit volume condition⁸:

$$V = \int d^D x \sqrt{G} e^{-2\phi} = 1. \quad (3.2)$$

Imposing this condition may be in a sense interpreted as extremizing S over the constant part of $\tilde{\beta}^\phi$ or the central charge parameter c_0 in (2.20). Indeed, adding the constraint (3.2) to the action (2.21) with the Lagrange multiplier λ we get the following functional

$$\hat{S} = \int d^D x \sqrt{G} e^{-2\phi} \tilde{\beta}^\phi + \lambda \left(\int d^D x \sqrt{G} e^{-2\phi} - 1 \right), \quad (3.3)$$

i.e. $\hat{S} = S(c_0 \rightarrow c_0 + \lambda) - \lambda$.

Let us mention in passing another relation between actions with unit volume condition and without it. Starting with $S(G, \phi)$ one may formally split the dilaton into constant and nonconstant parts as follows: $\phi(x) = \phi_0 + \tilde{\phi}(x)$, $\int d^D x \sqrt{G} e^{-2\phi} = 1$, so that $V = \int d^D x \sqrt{G} e^{-2\phi} = e^{-2\phi_0}$. Then $S(G, \phi) = e^{-2\phi_0} S(G, \tilde{\phi}) = V S(G, \tilde{\phi})$ and \hat{S} in (3.3) in which the dilaton is constrained by the volume condition can be written as [cf. (2.23)]

$$\begin{aligned} \hat{S}(G, \phi) &= S(G, \tilde{\phi}) = \frac{\int d^D x \sqrt{G} e^{-2\phi} \tilde{\beta}^\phi(G, \phi)}{\int d^D x \sqrt{G} e^{-2\phi}} \\ &= \frac{1}{2} \frac{d}{dt} \ln V. \end{aligned} \quad (3.4)$$

We shall not use this representation here.

Extremizing \hat{S} with respect to ϕ we then get [assuming that (3.1) is true to all orders in α']

$$\tilde{\beta}^\phi + \lambda = 0, \quad (3.5)$$

so that after solving for ϕ , i.e. imposing (3.5), we get

$$\hat{S} = \tilde{\beta}^\phi = -\lambda. \quad (3.6)$$

⁷The relation (3.1) is valid, in particular, if there is a scheme in which the dependence of $\tilde{\beta}^\phi$ on ϕ in (2.20) is not modified by α' corrections to all orders; this is actually true to α'^3 order but may seem to be in conflict with the α' -dependence of the operator γ in (2.7) and (2.8). But the corresponding terms can be further redefined away (or integrated by parts) at the level of the action.

⁸For an earlier closely related suggestion in specific $D = 2$ case see [25].

Thus λ has an interpretation of minus the effective central charge.

To leading order in α' the action \hat{S} (3.3) and (3.5) can be written as follows [see (2.20)]

$$\hat{S} = -\lambda + \int d^D x \sqrt{G} \Phi [\lambda + c_0 - \alpha'(-\nabla^2 + \frac{1}{4}R)] \Phi + O(\alpha'^2), \quad (3.7)$$

$$[(-\nabla^2 + \frac{1}{4}R) + O(\alpha')] \Phi = \alpha'^{-1}(c_0 + \lambda) \Phi, \quad (3.8)$$

where

$$\Phi \equiv e^{-\phi}, \quad \int d^D x \sqrt{G} \Phi^2 = 1. \quad (3.9)$$

The existence of a solution of this equation with $\Phi \equiv e^{-\phi} > 0$ requires that $c_0 + \lambda$ is the minimal eigenvalue of the operator⁹ $-\nabla^2 + \frac{1}{4}R$ which always exists on a compact space [5] (see also [25]). The corresponding eigenfunction will have no zeros and can be chosen positive which is what is required for the identification of it with $e^{-\phi}$ (or the inverse of the effective ‘‘string coupling constant’’ $g_s = e^\phi$). Thus extremizing \hat{S} in ϕ translates (for $c_0 = 0$) into choosing λ as a minimal eigenvalue of the above Laplacian.

To leading order in α' when the RG flow defined by (2.16) and (2.17) is simply the Ricci flow the Perelman’s definition of the functional whose gradient is $\bar{\beta}^G$ (i.e. β^G with an appropriate diffeomorphism term) and which grows monotonically with the RG flow toward IR ($t \rightarrow \infty$) is simply the minimal eigenvalue λ of (3.8) (for $c_0 = 0$). Thus, in view of (3.6),

$$\mathcal{S}(G) \equiv \hat{S}(G, \phi(G)) = -\lambda(G). \quad (3.10)$$

Let us extend this definition to all orders in α' . First, the variation of $\mathcal{S}(G)$ over $G_{\mu\nu}$ is the same as the $G_{\mu\nu}$ variation of $\hat{S}(G, \phi)$ or $S(G, \phi)$ with ϕ independent of $G_{\mu\nu}$: the variation over ϕ vanishes as a consequence of (3.5). Then (2.13) implies that $\frac{\delta \mathcal{S}}{\delta G_{\mu\nu}}$ is proportional to $\bar{\beta}_{\mu\nu}^G$. In addition, we may ignore the variation of \sqrt{G} since its coefficient vanishes on the equation for ϕ . Then

$$\bar{\beta}_{\mu\nu}^G = \kappa_{\mu\nu,\rho\sigma} \frac{\delta \mathcal{S}}{\delta G_{\rho\sigma}}, \quad \kappa_{\mu\nu,\rho\sigma} = \frac{4G_{\mu\rho}G_{\nu\sigma}}{\sqrt{G}e^{-2\phi}} + O(\alpha'^2). \quad (3.11)$$

Thus \mathcal{S} is a gradient function for the metric RG flow.

To study the monotonicity property of \mathcal{S} we note that

$$\begin{aligned} \frac{d}{dt} \mathcal{S} &= -\bar{\beta}_{\mu\nu}^G \cdot \frac{\delta \mathcal{S}}{\delta G_{\mu\nu}} = -\bar{\beta}_{\mu\nu}^G \cdot \frac{\delta \mathcal{S}}{\delta G_{\mu\nu}} \\ &= -\bar{\beta}_{\mu\nu}^G \cdot \kappa^{\mu\nu,\rho\sigma} \cdot \bar{\beta}_{\rho\sigma}^G. \end{aligned} \quad (3.12)$$

⁹Let us note in passing that the conformal scalar operator in D dimensions is $-\nabla^2 + \frac{D-2}{4(D-1)}R$ so that $-\nabla^2 + \frac{1}{4}R$ is conformal in the limit $D \rightarrow \infty$.

Here $\kappa^{\mu\nu,\rho\sigma}$ is the inverse of $\kappa_{\mu\nu,\rho\sigma}$ in (3.11): on the equation of motion for ϕ one need not worry about the contribution of the variation of \sqrt{G} term in the action and thus there is no extra term proportional to $-\frac{1}{2}G_{\mu\nu}G_{\rho\sigma}$ [cf. (2.19)]

$$\kappa^{\mu\nu,\rho\sigma} = \frac{1}{4}\sqrt{G}e^{-2\phi}G^{\mu\rho}G^{\nu\sigma} + O(\alpha'^2). \quad (3.13)$$

The positivity of $\kappa^{\mu\nu,\rho\sigma}$ at leading order in α' implies that \mathcal{S} monotonically decreases toward the IR ($t \rightarrow \infty$) as required of an effective central charge [3], while λ grows like an entropy [5].¹⁰ The positivity of $\kappa^{\mu\nu,\rho\sigma}$ is obvious in perturbation theory in α' , i.e. in sigma model loop expansion.¹¹ It may be possible to prove it rigorously to all orders using the general properties of renormalization of the sigma model on a curved background as discussed in [22,23].

Since the dependence on the dilaton of the beta-functions in (2.6) and (2.7) is simple (linear) the same simplicity should apply to the effective action. Starting with the action in the special scheme (2.21) we shall assume that to all orders in α' it can be put into the form similar to the leading-order action (3.7) (here we set $c_0 = 0$)

$$\hat{S} = -\lambda + \int d^D x \sqrt{G} \Phi (\lambda - \alpha' \Delta) \Phi, \quad \Phi = e^{-\phi}, \quad (3.14)$$

where

$$\Delta = -\nabla^2 + U(G), \quad (3.15)$$

$$\begin{aligned} U &= \frac{1}{4}R + \frac{1}{16}\alpha'R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} + \frac{1}{16}\alpha'^2(R_{\mu\nu\lambda\rho}R^{\mu\nu\alpha\beta}R_{\alpha\beta}^{\lambda\rho} \\ &\quad - \frac{4}{3}R_{\kappa\lambda\mu\nu}R^{\kappa\alpha\mu\beta}R_{\alpha}^{\lambda\nu\beta}) + O(\alpha'^3). \end{aligned} \quad (3.16)$$

For R^3 terms we used the result of [19,26]. We have assumed that there should exist a scheme choice in which all higher-derivative terms which may be present in the anomalous dimension operator γ in (2.7) can be integrated by parts in the action (2.21) so that Δ remains a canonical second-derivative scalar Laplacian as it was at the leading order in α' in (3.7). This is indeed what happens to order α'^4 as was explicitly verified in [15,19].

The potential function $U(G)$ is a smooth generalization of the leading-order term $\frac{1}{4}R$. Then for compact euclidean-signature space the operator Δ is again positive and its spectrum should be bounded from below. Then the eigenfunction Φ corresponding to its lowest eigenvalue λ/α'

¹⁰To leading order in α' the relation between Perelman’s entropy and the central charge was already pointed out in [6]. The same is true also in the presence of the $B_{\mu\nu}$ coupling [6]. Using the monotonicity of λ one is then able to prove the absence of periodic RG trajectories [5,6].

¹¹To become negative $\kappa^{\mu\nu,\rho\sigma}$ should go through zero but that would require $\alpha'R \sim 1$, invalidating perturbation theory expansion.

can again be chosen positive, i.e. there should exist a nonsingular solution for ϕ .¹² Combined with the positivity of the metric in (3.13), λ will then provide the generalization of the Perelman's entropy to all orders in α' , implying the irreversibility of the exact RG flow of the sigma model.

As we have seen above in (3.6), λ has also a meaning of minus the effective central charge $\tilde{\beta}^\phi$, which, at the fixed point $\tilde{\beta}^G = 0$, is equal to the usual central charge. This is in agreement with the general claim of the c-theorem. The construction of the sigma model “c-function” *a la* Ref. [3] (i.e. in terms of 2-point functions of stress-tensor components) did lead [4] to $\tilde{\beta}^\phi$, but as we explained above following Perelman's idea, to show that the RG flow of $G_{\mu\nu}$ is monotonic one is also to solve for the dilaton and restrict its constant part by the volume condition (3.2). This then confirms the validity of the c-theorem for the $G_{\mu\nu}$ RG flow of the 2d sigma model (at least to order α'^4).

To make this proof of the c-theorem rigorous (i.e. to extend it beyond α'^4 order) one is to justify our main assumption that the exact action (2.21) can be put into the form (3.14). This may be possible to achieve using the identities like (2.12) following from the renormaliza-

¹²We are grateful to S. Cherkis for a clarifying discussion of this point.

tion properties of composite operators of the sigma model [22,23].¹³

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¹³One may also try to support the “naturalness” of the form of (3.14) by the following heuristic string-theory argument. Since the λ -independent part of (3.3) and (3.14) is essentially the tree-level string effective action for the metric and dilaton and since $e^\phi = \Phi^{-1}$ is the string coupling “constant”, the scaling of the action as $\Phi^2 = e^{-2\phi}$ is the direct consequence of the structure of the string perturbation theory (i.e. of the fact that the Euler number of the 2-sphere is 2). In general, the Lagrangian in (3.14) may contain also additional terms with derivatives of ϕ , i.e. depending on $\partial_\mu(\ln\Phi)$. However, such terms “nonanalytic” in the string coupling would seem “unnatural”—one could hope that replacing the string coupling by a function of coordinates should not change the structure of the dependence of the tree-level effective action on it.

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